Relativistic corrections to heavy quark fragmentation to S-wave heavy mesons

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The relativistic corrections of order v^2 to the fragmentation functions for the heavy quark to S-wave heavy quarkonia are calculated in the framework of the nonrelativistic quantum chromodynamics factorization formula. We derive the fragmentation functions by using the Collins-Soper definition in both the Feynman gauge and the axial gauge. We also extract them through the process $Z^0 \to Hq\bar{q}$ in the limit $M_Z/m \to \infty$. We find that all results obtained by these two different methods and in different gauges are the same. We estimate the relative size of the relativistic corrections to the fragmentation functions.

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I. INTRODUCTION

In the high energy limit, the dominant production mechanism for heavy flavored mesons with large transverse momentum is the fragmentation [1–9]. The mechanism is characterized by a universal fragmentation function which describes production probabilities of a hadron from a parton. In the limit, the differential cross section of hadron production can be factored into that of the parton production and the fragmentation function. Introduction of fragmentation functions is the consequence of the QCD factorization.

In the case of heavy-quarkonium production, a fragmentation function can further be factored into a short-distance part which describes heavy quark pair production that happened at distance $1/m$, where m is the heavy quark mass, and a long-distance part which describes the formation of the heavy quarkonium from the heavy quark pair that happened at distance $1/mv$, where v is the typical velocity of the heavy quark inside the bound state. In the nonrelativistic limit, $v \ll 1$, the nonrelativistic quantum chromodynamics (NRQCD) [10,11] factorization formula provides a systematic way to separate the short-distance effects, which happened at distance $1/m$, and long-distance effects, which happened at distance $1/mv$. Applied to the heavy-quarkonium fragmentation, the fragmentation function can be expressed as a sum of products of short-distance coefficients and NRQCD long-distance matrix elements. The short-distance coefficients can be expanded as a power series of α_s at energy scale m. The NRQCD long-distance matrix elements can be expanded in terms of v. Relativistic corrections are included automatically as contributions of the v^2 suppressed NRQCD matrix elements compared to that of the leading order one. The relativistic effects are sometimes large for the heavy mesons containing a c quark for $v^2 \sim 0.3$ [12,13].

In this paper, the relativistic corrections of order v^2 to the fragmentation functions for the heavy quark to the S-wave heavy meson are calculated in the framework of the NRQCD factorization formula. To have a consistent check, we carry out the calculations using two different methods. One is based on the Collins-Soper definition for the fragmentation function [14]. The other one is to extract the fragmentation functions through the decay width of the process $Z^0 \to Hq\bar{q}$ in the limit $M_z/m \to \infty$. In the former method, we also perform the calculations in both the Feynman gauge and the axial gauge. All these results, obtained by various methods and by different gauges, are exactly the same.

We have noticed that these relativistic corrections were calculated in Refs. [15,16]. The authors isolate contributions of the binding energy part from the total relativistic corrections. It did not follow the NRQCD factorization formula. In the NRQCD factorization language, all those relativistic correction terms correspond to the contributions of a single NRQCD long-distance matrix element. Actually, they can be related to each other by the Gremm-Kapustin relation [17]. Combining both terms together, one then expects that the short-distance coefficient of the relativistic matrix element is the identity. However, we find that our results are in disagreement with those given in Refs. [15,16] when they are expressed in the standard NRQCD factorization formula. We will compare in detail the differences between our work and theirs in Sec. IV.

This paper is organized as follows. In Sec. II, we review the Collins-Soper definition for the fragmentation function in the light-cone coordinate system and the NRQCD factorization formalism. In Sec. III, we calculate the first order relativistic corrections to the fragmentation function for a heavy quark into $^1S_0$, $^3S_1$ heavy-quarkonium states in the Feynman gauge and the axial gauge, respectively. We calculate the same fragmentation function through the process $Z^0 \to Hq\bar{q}$ in the limit $M_Z/m \to \infty$ in Sec. IV. In Sec. V, we perform the numerical analysis and present some discussions. The analytic expressions of the fragmentation functions in the leading order (LO) and the next-to-leading order (NLO) with respect to v, for a heavy quark into a heavy meson composed of two different heavy quarks, like a $\bar{b}c$ or $b\bar{c}$ bound state, are in the Appendix.

II. DEFINITION OF THE FRAGMENTATION FUNCTION AND NRQCD FACTORIZATION

In this section, we give a brief review on the definition of the fragmentation function that was presented by Collins and Soper [14] and the NRQCD factorization formula proposed by Bodwin, Braaten, and Lepage [11]. They are necessary tools for a direct calculation of the fragmentation functions that we will carry out.

A. Collins-Soper definition of fragmentation

We first recapitulate the Collins-Soper definition of the fragmentation function which has been used in calculating the quarkonium fragmentation function by Ma [18] and by Bodwin and Lee [12].

It is convenient to work in the light-cone coordinate system. In this system, a four-vector p is expressed as $p^\mu = (p^+, p^-, \mathbf{p_T})$ with $p^+ = (p^0 + p^3)/\sqrt{2}$ and $p^- = (p^0 - p^3)/\sqrt{2}$. The scalar product of two four-vectors p and q is then

$$p \cdot q = p^+ q^- + p^- q^+ - \mathbf{p_T} \cdot \mathbf{q_T}. \tag{1}$$

We also introduce a subsidiary vector n with $n^\mu = (0, 1, \mathbf{0_T})$. Then the gauge-invariant quark fragmentation function for a hadron H is defined as

$$\begin{aligned} D_{H/Q}(z) = {} & \frac{z^{d-3}}{4\pi}\int dx^- e^{-iP^+x^-/z}\frac{1}{3}\mathrm{Tr}_{\mathrm{color}}\frac{1}{2}\mathrm{Tr}_{\mathrm{Dirac}} \\ & \times\left\{n\cdot\gamma\langle 0|Q(0)\bar{\mathrm{P}}\exp\left\{-ig_s\int_0^\infty d\lambda n\cdot A^T(\lambda n^\mu)\right\}\right. \\ & \times a_H^\dagger(P^+, \mathbf{0_T})a_H(P^+, \mathbf{0_T})\mathrm{P}\exp \\ & \times\left.\left\{ig_s\int_{x^-}^\infty d\lambda n\cdot A^T(\lambda n^\mu)\right\}\bar{Q}(0, x^-, \mathbf{0_T})|0\rangle\right\}, \end{aligned} \tag{2}$$

where $Q^\dagger$ and $a_H^\dagger$ are the creating operators of the heavy quark and the produced hadron, respectively, A_μ is the gluon field, P is the momentum of the hadron H, $\mathrm{P}\exp\{\cdots\}$ means the path-order gauge link, and d is the dimension of the space-time, which is taken to be 4 in the following calculations. Equation (2) is taken as an average over the colors and the spins of the initial heavy quark Q. The function $D_{H/Q}$ is interpreted as the probability of a quark with momentum k to decay into the hadron H with momentum component $P^+ = zk^+$.

The gauge independence of the definition of Eq. (2) implies that the calculation can be carried out in any gauge. In practice, we calculate the fragmentation function in both the Feynman gauge and the axial gauge to have a gauge-independent check.

B. NRQCD factorization formalism

The physics of heavy quarkonium involves several momentum scales: the mass m, the typical 3-momentum mv, and the typical kinetic energy of the heavy quark mv^2. In the nonrelativistic limit, $v \ll 1$, these scales satisfy a relation: $mv^2 \ll mv \ll m$. NRQCD factorization separates the short-distance effects of the annihilation or the production of the heavy quark and antiquark pair from the long-distance effects of the quarkonium formation by an almost on-shell heavy quark pair. The former one can be calculated using perturbation theory in α_s, while the latter one is described by the long-distance NRQCD matrix elements.

According to the NRQCD factorization formula, the fragmentation function of the heavy quark into the heavy-quarkonium H can be written in the form [11]

$$\begin{aligned} D_{H/Q}(z) = {} & \sum_n (F^n(z)\langle 0|\mathcal{O}^H(n)|0\rangle + G^n(z)\langle 0|\mathcal{P}^H(n)|0\rangle) \\ & + O(v^4), \end{aligned} \tag{3}$$

where $\mathcal{O}^H(n)$ and $\mathcal{P}^H(n)$ are NRQCD operators, and $F^n(z)$ and $G^n(z)$ are the short-distance coefficients. The index n represents the overall quantum number of the operator. We take the leading Fork state approximation in which the quantum numbers of the hadron are the same as that of the operators. We are interested only in the fragmentation of two S-wave heavy-quarkonium states. For our purpose, we need only two operators for each hadron state,

$$\begin{aligned} \mathcal{O}^H({}^1S_0) &= \frac{1}{N_c}\chi^\dagger\psi\sum_X |H+X\rangle\langle H+X|\psi^\dagger\chi, \\ \mathcal{P}^H({}^1S_0) &= \frac{1}{2N_c}\left[\chi^\dagger\left(\frac{i\overleftrightarrow{\mathbf{D}}}{2}\right)^2\psi\sum_X|H+X\rangle\right. \\ & \quad \times\left.\langle H+X|\psi^\dagger\chi + \mathrm{H.c.}\right], \end{aligned} \tag{4}$$

for a hadron H with quantum number ${}^1S_0$, and

$$\begin{aligned} \mathcal{O}^H({}^3S_1) &= \frac{1}{N_c}\chi^\dagger\sigma^i\psi\sum_X |H+X\rangle\langle H+X|\psi^\dagger\sigma^i\chi, \\ \mathcal{P}^H({}^3S_1) &= \frac{1}{2N_c}\left[\chi^\dagger\sigma^i\left(\frac{i\overleftrightarrow{\mathbf{D}}}{2}\right)^2\psi\sum_X|H+X\rangle\right. \\ & \quad \times\left.\langle H+X|\psi^\dagger\sigma^i\chi + \mathrm{H.c.}\right], \end{aligned} \tag{5}$$

for a hadron H with quantum number ${}^3S_1$.

The factorization formula (3) holds not only for hadron states H, but also for on-shell free quark-antiquark states with the same quantum numbers. When it is applied to the free quark-antiquark states, the matrix elements are different but the short-distance coefficients are the same. The short-distance coefficients $F^n(z)$ and $G^n(z)$ can then be determined by matching the free quark-antiquark state fragmentation.

III. FRAGMENTATION FUNCTIONS BY THE COLLINS-SOPER DEFINITION

We now calculate the fragmentation functions via the Collins-Soper definition (2) in the framework of the NRQCD factorization formula. As mentioned in the last section, we determine the short-distance coefficients by matching the production of the on-shell free quark-antiquark spin-singlet and spin-triplet states. We do calculations in both the Feynman gauge in the first subsection and the axial gauge in the second one.

A. Calculating the fragmentation functions in the Feynman gauge

Following [19], we introduce projection operators [19] for the spin and color states of a quark-antiquark pair as follows:

$$\Pi_0 = \frac{(\not{P}/2 - \not{q} - m)\gamma_5(\not{P} + 2E)(\not{P}/2 + \not{q} + m)}{8\sqrt{2N_c}E^2(E+m)},$$
$$\Pi_1 = \frac{(\not{P}/2 - \not{q} - m)\gamma_\mu(\not{P} + 2E)(\not{P}/2 + \not{q} + m)}{8\sqrt{2N_c}E^2(E+m)}, \quad (6)$$

where the subscript indices 0 and 1 refer to spin singlet and spin triplet, respectively, $N_c = 3$, and $E = \sqrt{m^2 - q^2} = \sqrt{m^2 + \mathbf{q}^2}$. The total momentum P and the half-relative momentum q are related to the four-momentum of the quark p_1 and that of antiquark p_2 as

$$p_1 = \tfrac{1}{2}P + q, \qquad p_2 = \tfrac{1}{2}P - q. \quad (7)$$

We also define $v^2 \equiv \mathbf{q}^2/m^2$ for later use.

In the Feynman gauge, there are four diagrams responsible for the fragmentation process at the LO in α_s, as shown in Fig. 1. Denote their contributions to the fragmentation function $D_{H/Q}(z)$ as D_{11}, D_{12}, D_{21}, and D_{22}, respectively. D_{ij} can then be expressed as

$$D_{ij} = 4E\frac{z}{24\pi}\int\frac{dq_1^+ d^2\mathbf{q}_{1\mathbf{T}}}{(2\pi)^3 2q_1^+}2\pi\delta(k^+ - P^+ - q_1^+)F_c \times \mathrm{Tr}[\not{n}R_i(\not{q}_1 + m)L_j], \quad (8)$$

where F_c represents the color factor

$$F_c = \mathrm{Tr}[T^aT^aT^bT^b] = \frac{16}{3}, \quad (9)$$

and L_i and R_i read

$$R_1 = \frac{1}{(P/2 - q_1 + q)^2}\frac{\not{P} + \not{q}_1 + m}{(P + q_1)^2 - m^2}\gamma^\mu\Pi\gamma^\mu,$$
$$L_1 = \frac{1}{(P/2 - q_1 + q)^2}\gamma^\nu\Pi^\dagger\gamma^\nu\frac{\not{P} + \not{q}_1 + m}{(P + q_1)^2 - m^2}, \quad (10)$$
$$R_2 = \Pi\not{n}, \qquad L_2 = \not{n}\Pi^\dagger$$

where k and q_1 are the momenta of the initial and the final free quark, respectively. L_i and R_i represent the amplitude arising from the left and right sides of the cut line in Fig. 1, respectively. A factor of $4E$ has been included in the phase space of Eq. (8) in order to cancel the relativistic normalization of H in Eq. (2)

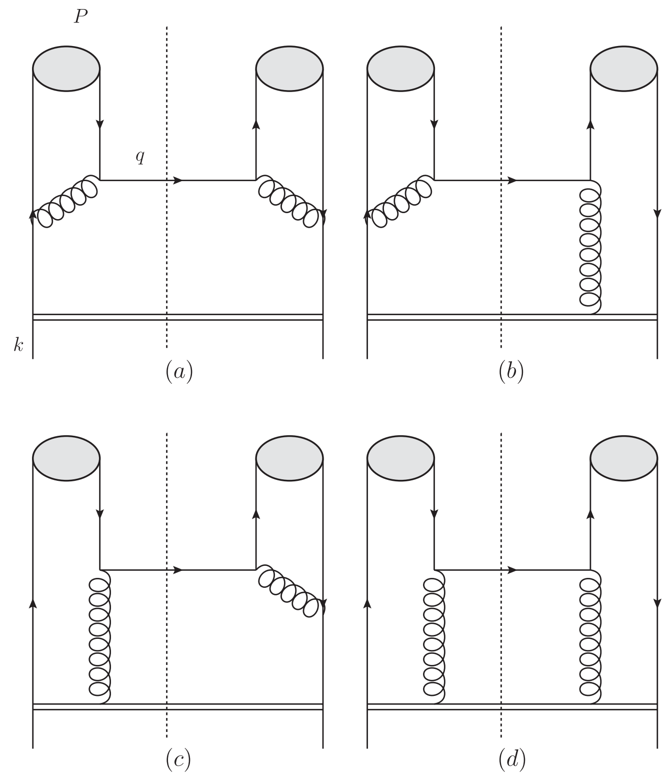

FIG. 1. Feynman diagrams for fragmentation of a heavy quark into an S-wave heavy-quarkonium at LO in α_s. The double line denotes the eikonal line and the shaded blob denotes the S-wave quarkonium.

Now, we pick out the S-wave part of D_{ij}. For a generic amplitude $\mathcal{M}$, the S-wave part of $\mathcal{M}$ is [19]

$$\mathcal{M}_{S\text{-wave}} = \mathcal{M}_0 + \frac{\mathbf{q}^2}{m^2}\mathcal{M}_2 + \mathcal{O}\left(\frac{\mathbf{q}^4}{m^4}\right), \quad (11)$$

where the first two terms on the right-hand side of Eq. (11) are LO and NLO contributions in the v^2 expansion. Here,

$$\mathcal{M}_0 = \mathcal{M}|\mathbf{q} \to 0, \qquad \mathcal{M}_2 = \frac{m^2 I^{\alpha\beta}}{6}\frac{\partial^2\mathcal{M}}{\partial q^\alpha \partial q^\beta}\bigg|_{\mathbf{q}\to 0}, \quad (12)$$

where

$$I^{\alpha\beta} = -g^{\alpha\beta} + \frac{P^\alpha P^\beta}{4E^2}. \quad (13)$$

The LO fragmentation function in v can easily be obtained by setting $q = 0$ and $E = m$ in Eqs. (8) and (10). For the $^1S_0$ state, it reads

$$D^{(0)}_{^1S_0/Q}(z) = \sum_{ij} D^{(0)}_{ij}(z) = \frac{32\alpha_s^2 z(1-z)^2(48+8z^2-8z^3+3z^4)}{81m^3(2-z)^6}, \quad (14)$$

where the superscript (0) denotes the contribution from the LO in v.

To gain the NLO result, we first expand the amplitudes R_1, R_2, L_1, and L_2 in terms of Eq. (11). Substituting them into (10) and keeping only $\mathbf{q}^2$ terms, we get the NLO relativistic correction terms to the fragmentation function of the $^1S_0$ quark-antiquark final state. The result reads

$$D^{(2)}_{^1S_0/Q}(z) = \sum_{ij} D^{(2)}_{ij}(z) = \frac{16\alpha_s^2 z(1-z)^2(-2112+2496z-80z^2+128z^3-268z^4+148z^5-15z^6)v^2}{243m^3(2-z)^8}. \quad (15)$$

In a similar way, the fragmentation function of a heavy quark into the $^3S_1$ quark-antiquark final state can be calculated. We list the obtained results as follows:

$$D^{(0)}_{^3S_1/Q}(z) = \sum_{ij} D^{(0)}_{ij}(z) = \frac{32\alpha_s^2 z(1-z)^2(16-32z+72z^2-32z^3+5z^4)}{27m^3(2-z)^6}, \quad (16)$$

for the LO, and

$$D^{(2)}_{^3S_1/Q}(z) = \sum_{ij} D^{(2)}_{ij}(z) = \frac{16\alpha_s^2 z(1-z)^2(-1344+5184z-14416z^2+18176z^3-8924z^4+2092z^5-183z^6)v^2}{243m^3(2-z)^8}, \quad (17)$$

for the NLO, in v^2. In Eqs. (16) and (17), we have summed over the polarizations of the final $^3S_1$ state.

B. Calculating the fragmentation functions in the axial gauge

In the axial gauge, the Wilson line operators in the definition of Eq. (8) disappear for $n \cdot A = 0$; consequently, only Fig. 1(a) contributes to the fragmentation function. With $n = (0, n^-, \mathbf{0_T})$, the gluon propagator in this gauge is

$$D^{\mu\nu}(k) = \frac{i}{k^2}\left(-g^{\mu\nu} + \frac{k^\mu n^\nu + k^\nu n^\mu}{k \cdot n}\right). \quad (18)$$

Similar to the last subsection, the fragmentation function for a heavy quark Q into a quark-antiquark, the $^1S_0(^3S_1)$ state reads

$$D_{^1S_0(^3S_1)/Q} = 2E \int \frac{dq_1^+ d^2\mathbf{q_{1T}}}{(2\pi)^3 2q_1^+} 2\pi\delta(k^+ - P^+ - q_1^+) F_c \,\mathrm{Tr}[\not{n} R(\not{q}_1 + m) L], \quad (19)$$

where

$$R = \frac{1}{(P/2 - q_1 + q)^2} \frac{\not{P} + \not{q}_1 + m}{(P+q_1)^2 - m^2} \times \gamma^\mu \Pi_{0(1)} \gamma^\nu \left(-g^{\mu\nu} + \frac{k^\mu n^\nu + k^\nu n^\mu}{k \cdot n}\right),$$

$$L = \frac{1}{(P/2 - q_1 + q)^2} \gamma^\alpha \Pi^\dagger_{0(1)} \gamma^\beta \frac{\not{P} + \not{q}_1 + m}{(P+q_1)^2 - m^2} \times \left(-g^{\alpha\beta} + \frac{k^\alpha n^\beta + k^\beta n^\alpha}{k \cdot n}\right). \quad (20)$$

Again, we take L and R to denote the contributions from the left and the right side of the cut line in Fig. 1, respectively.

Repeating the procedure of the last subsection, the LO and the NLO results can be obtained easily. We find that they are exactly the same as Eqs. (14) and (15) for the quark-antiquark $^1S_0$ state and Eqs. (16) and (17) for the $^3S_1$ state. We, therefore, have verified the gauge independence of our results.

C. Matching the short-distance coefficients

In order to match the short-distance coefficients, we need to calculate the same fragmentation functions by using the NRQCD factorization formula. It is straightforward to calculate the NRQCD matrix elements. With the standard normalization, they read

$$\langle \mathcal{O}^{Q\bar{Q}}(^1S_0)\rangle = 2N_c, \qquad \langle \mathcal{O}^{Q\bar{Q}}(^3S_1)\rangle = 2(d-1)N_c,$$
$$\langle \mathcal{P}^{Q\bar{Q}}(n)\rangle = \frac{\mathbf{q}^2}{m^2}\langle \mathcal{O}^{Q\bar{Q}}(n)\rangle, \quad (21)$$

where $n = {}^1S_0, {}^3S_1$. Factors 2 and N_c on the right-hand side of the first two equations within Eq. (21) arise from the spin and the color factors for normalized heavy-quark states, and factor $d-1$ with $d=4$ arises from the summation over the polarizations in the spin-triplet final state. Consequently, the fragmentation functions in the NRQCD factorization formula read

$$D_{Q\bar{Q}(^1S_0)/Q} = 2N_c(F^{^1S_0} + \mathbf{q^2} G^{^1S_0}),$$
$$D_{Q\bar{Q}(^3S_1)/Q} = 2N_c(d-1)(F^{^3S_1} + \mathbf{q^2} G^{^3S_1}). \quad (22)$$

The short-distance coefficients can then be determined by matching Eqs. (14)–(17) with Eq. (22):

$$\begin{aligned}
F^{^1S_0} &= \frac{16\alpha_s^2 z(1-z)^2(48+8z^2-8z^3+3z^4)}{243m^3(2-z)^6},\\
G^{^1S_0} &= \frac{8\alpha_s^2 z(1-z)^2(-2112+2496z-80z^2+128z^3-268z^4+148z^5-15z^6)}{729m^5(2-z)^8},\\
F^{^3S_1} &= \frac{16\alpha_s^2 z(1-z)^2(16-32z+72z^2-32z^3+5z^4)}{243m^3(2-z)^6},\\
G^{^3S_1} &= \frac{8\alpha_s^2 z(1-z)^2}{2187m^5(2-z)^8}(-1344+5184z-14\,416z^2+18\,176z^3-8924z^4+2092z^5-183z^6).
\end{aligned} \tag{23}$$

The expressions in Eq. (23) constitute the key formulas of this work. Our results for $F^{(^1S_0)}$ and $F^{^3S_1}$ are in agreement with those in Refs. [5,8].

IV. EXTRACTING FRAGMENTATION FUNCTION FROM $Z^0 \to Q\bar{Q}H$

In this section, we extract the heavy quark fragmentation functions from the decay width of the process $Z^0 \to Q\bar{Q}H$ in the limit of $\frac{M_Z}{m} \to \infty$, where M_Z and m are the masses of the gauge boson Z^0 and the heavy quark Q, respectively. This approach has been used to derive the LO order fragmentation functions in Refs. [5,8]. We now use the method to extend the calculations to the NLO fragmentation functions in v^2. There are four Feynman diagrams responsible for this process. However, if the calculation is carried out in the axial gauge, there is only one diagram, shown in Fig. 2, contributing to the fragmentation function.

Following [8], we isolate the heavy quark fragmentation probability $\int_0^1 dz D_{H/Q}(z)$ by dividing the decay rate Γ_1 of $Z^0 \to HQ\bar{Q}$ by the decay rate Γ_0 of $Z^0 \to Q\bar{Q}$ in the limit $\frac{M_Z^2}{m^2} \to \infty$,

$$\begin{aligned}
\Gamma_0 &= \frac{1}{2M_z}\int \frac{d^3q_2}{(2\pi)^3 2q_2^0}\frac{d^3k}{(2\pi)^3 2k^0}(2\pi)^4\\
&\quad\times \delta^4(Z-q_2-k)\frac{1}{3}\sum |A_0|^2,\\
\Gamma_1 &= \frac{4E}{2M_z}\int \frac{d^3q_2}{(2\pi)^3 2q_2^0}\frac{d^3P}{(2\pi)^3 2P^0}\frac{d^3q_1}{(2\pi)^3 2q_1^0}(2\pi)^4\\
&\quad\times \delta^4(Z-q_2-P-q_1)\frac{1}{3}\sum |A_1|^2,
\end{aligned} \tag{24}$$

where Z is the momentum of the initial Z^0 boson and A_0, A_1 are the amplitudes of the two processes, which will be given below. We use the same notations for other particles as in the last section. Similar to Eq. (8), a factor $4E$ has been included in Eq. (24) to compensate for the relativistic normalization of the final-state hadron. To facilitate the extraction of the fragmentation probability, one can rewrite the three-body phase space integral for the outgoing particles in an iterated form by introducing integrals over k and s,

$$\begin{aligned}
&\int \frac{d^3q_2}{(2\pi)^3 2q_2^0}\frac{d^3P}{(2\pi)^3 2P^0}\frac{d^3q_1}{(2\pi)^3 2q_1^0}(2\pi)^4\delta^4(Z-q_2-P-q_1)\\
&= \int\frac{ds}{2\pi}\int \frac{d^3q_2}{(2\pi)^3 2q_2^0}\frac{d^3k}{(2\pi)^3 2k^0}(2\pi)^4\delta^4(Z-q_2-k)\\
&\quad\times \int \frac{d^3q_1}{(2\pi)^3 2q_1^0}\frac{d^3P}{(2\pi)^3 2P^0}(2\pi)^4\delta^4(k-P-q_1)\\
&= \int\frac{ds}{2\pi}\int \frac{d^3q_2}{(2\pi)^3 2q_2^0}\frac{d^3k}{(2\pi)^3 2k^0}(2\pi)^4\delta^4(Z-q_2-k)\frac{1}{8\pi}\\
&\quad\times \int_0^1 dz\theta\left(s-\frac{4E^2}{z}-\frac{m^2}{1-z}\right),
\end{aligned} \tag{25}$$

where $s \equiv k^2$, $z = P^+/k^+$, as given before.

With the standard model Feynman rule, the amplitudes A_0 and A_1 read

$$\begin{aligned}
A_0 &= -i\epsilon_\nu(Z)\bar{u}(k)\Gamma^\nu v(\bar{q}_2),\\
A_1 &= i\epsilon_\nu(Z)\frac{g^2C_F}{(s-m^2)}D_{\alpha\beta}(q_1+P/2)\bar{u}(q_1)\\
&\quad\times \gamma^\alpha\Pi_{0(1)}\gamma^\beta(\not{k}+m)\Gamma^\nu v(q_2),
\end{aligned} \tag{26}$$

where C_F equals $\frac{4}{3}$, Γ^ν is the interaction vertex between Z^0 and $Q\bar{Q}$, and $\Pi_{0(1)}$ is the spin projection operator given by Eq. (6).

We use Eqs. (11) and (12) to pick out the S-wave part of the amplitude. The LO and the NLO contributions can then be separated by using the same approach as used in Sec. III.

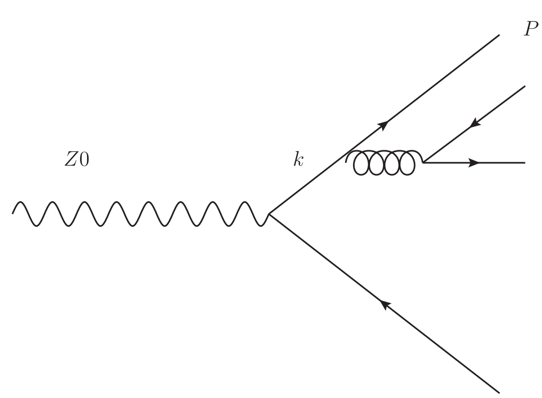

FIG. 2. Feynman diagram for $Z^0 \to H + Q\bar{Q}$ in LO in α_s.

After doing this, the squared amplitude can be obtained by multiplying the amplitude by its complex conjugate. Averaging the spins and colors over the initial state and summing them over the final state, we obtain

$$\frac{1}{3}\sum|A_0|^2=\frac{1}{3}\Big(-g^{\mu\nu}+\frac{Z^\mu Z^\nu}{M_z^2}\Big)\mathrm{Tr}[\Gamma_\mu(\not{q}_2-m)\Gamma_\nu(\not{k}+m)],$$
$$\frac{1}{3}\sum|A_1|^2=\frac{16g^4}{9}\frac{1}{(s-m^2)^2}\frac{1}{(q_1+P/2)^4}\Big(-g^{\mu\nu}+\frac{Z^\mu Z^\nu}{M_z^2}\Big)\times\mathrm{Tr}[\Gamma_\mu(\not{q}_2-m)\Gamma_\nu C], \tag{27}$$

where C is a product of Dirac matrices. In the limit of $\frac{M_Z}{m}\to\infty$, for simplicity, we keep only those terms with the leading order contributions in $\frac{m^2}{M_Z^2}$. For this process, one has the following scaling relations:

$$\not{k}\sim\frac{1}{z}\not{P}\sim\frac{1}{1-z}\not{q}_1\sim\mathcal{O}(M_z),$$
$$P\cdot P\sim k\cdot k\sim k\cdot P\sim\not{k}\not{P}\sim m^2. \tag{28}$$

Thus, in the LO of $\mathcal{O}(\frac{m^2}{M_z^2})$, one can expect that $C\sim c\not{k}$, where c is the function of scalar products. After dividing Γ_1 by Γ_0 and integrating over s, we obtain $\int_0^1 dzD_{H/Q}(z)$ immediately. The calculation is lengthy but straightforward. We have omitted the detailed steps. A careful comparison shows that the results obtained by this method are exactly the same with those presented in Sec. III, for both the spin singlet and spin triplet.

We noticed that the author in Ref. [16] also has computed the heavy quark fragmentation functions of the S-wave pseudoscalar and the vector heavy quarkonia. The author derived the fragmentation functions by using the approach in Ref. [8] as we have done in this section. Contrary to the NRQCD factorization formula, he divided the relativistic corrections into a piece caused by the binding energy w and another piece proportional to $\langle\mathbf{p}^2\rangle$. With the Gremm-Kapustin relation [17], these two pieces can be combined into one by setting $w=M_H-2m=mv^2$. A comparison shows that Eq. (13) in Ref. [16], which is the amplitude of the process $Z^0\to Q\bar{Q}H$, agrees with the amplitude in our paper up to $\mathcal{O}(v^2)$, except for the wave function and some other normalization constant factors, which do not contribute to the NLO relativistic correction. Applying the Gremm-Kapustin relation to the NLO fragmentation functions, we then expect that the results in Ref. [16] should be in agreement with ours up to a normalization constant. However, we find that our result in Eq. (23) of this paper is different from theirs after adding their Eqs. (44) and (45) together for the pseudoscalar state and after adding their Eqs. (21) and (28) together for the vector state, and by setting $\langle\mathbf{p}^2\rangle$ with m^2v^2 in their paper [16].

To have a detailed comparison, we list our results and theirs for the contributions to the NLO fragmentation function arising from the binding energy parts. Our results for the spin singlet and spin triplet read

$$D^w_{{}^1S_0/Q}(z)=\frac{16w\alpha_s^2z(1-z)^2(-576+768z-432z^2+352z^3-204z^4+80z^5-9z^6)}{81(2-z)^8m^4},$$
$$D^w_{{}^3S_1/Q}(z)=\frac{16w\alpha_s^2z(1-z)^2(-576+1920z-4464z^2+4640z^3-2092z^4+520z^5-53z^6)}{81(2-z)^8m^4}, \tag{29}$$

where their results read [16]

$$D^w_{{}^1S_0/Q}(z)=\frac{16w\alpha_s^2|\psi(0)|^2z(1-z)^2(-576+768z-432z^2+272z^3-132z^4+52z^5-3z^6)}{81(2-z)^8m^4},$$
$$D^w_{{}^3S_1/Q}(z)=\frac{32w\alpha_s^2|\psi(0)|^2z(1-z)^2}{81(2-z)^8m^4}(-288+960z-2232z^2+2356z^3-1160z^4+419z^5-100z^6+9z^7). \tag{30}$$

We see that some terms are the same while some terms are different.

For the contributions excluding those from the binding energy part, we have noticed that instead of using Eq. (12) in our paper, they used Eq. (26) from their paper to derive their result. This may cause the difference between our result and theirs in the remainder part of the NLO fragmentation functions. It is certainly incorrect in contracting their Eq. (26) with $g^{\mu\nu}$, because both sides of that equation are not equal. We have also noticed that the authors in another paper [15] also calculated the same relativistic corrections to the fragmentation functions. Our results are also in disagreement with theirs when comparing Eqs. (12) and (19) in their paper with Eq. (23) in our paper.

V. NUMERICAL RESULTS AND DISCUSSIONS

We have computed the contributions of the LO in v^2 and the relativistic corrections of relative order v^2 to the fragmentation functions for a heavy quark fragmentation into the S-wave spin-singlet and spin-triplet heavy-quarkonium states. We now use these analytic results to estimate the relative sizes of the relativistic corrections for the heavy quark fragmentation into the S-wave hadron. To this end, we estimate the ratio of the NLO NRQCD matrix element to the LO one using the Gremm-Kapustin [17] relation as taken in Ref. [12]. The simple relation between the NRQCD matrix elements $\langle\mathcal{O}^H\rangle$ and $\langle\mathcal{P}^H\rangle$ reads [12]

$$v^2 = \frac{\langle \mathcal{P}^H \rangle}{m^2 \langle \mathcal{O}^H \rangle}, \tag{31}$$

where M and $m_{\rm pole}$ are the masses of heavy quarkonium and heavy quark, respectively. When this relation is applied to determine the relativistic corrections, the large ambiguities arise from the uncertainty of the values of the pole mass $m_{\rm pole}$ [20], while the masses of the heavy quarkonia are precisely determined by experiments [21]:

$$\begin{aligned} M_{\eta_c} &= 2.980\ \text{GeV}, & M_{\eta_b} &= 9.300\ \text{GeV}, \\ M_{J/\psi} &= 3.097\ \text{GeV}, & M_{\Upsilon(1S)} &= 9.460\ \text{GeV}. \end{aligned} \tag{32}$$

Taking $m_{c\,\rm pole} = 1.4$ GeV $m_{b\,\rm pole} = 4.6$ GeV, as commonly quoted by most authors, we immediately obtain the v^2 for different heavy-quarkonium states by Eq. (31) as

$$\begin{aligned} v^2_{\eta_c} &= 0.13, & v^2_{\eta_b} &= 0.02, \\ v^2_{J/\psi} &= 0.21, & v^2_{\Upsilon(1S)} &= 0.06. \end{aligned} \tag{33}$$

One should regard this as only a rough estimate of the size of v^2. The relativistic corrections to the fragmentation functions for the charmonium like the η_c and the J/ψ are considerable, while they are negligible for the bottomonium like the η_b and the $\Upsilon(1S)$, owing to smaller relative velocity for bottomonium. We plot the distributions vs the longitudinal momentum fraction z of the LO and the NLO fragmentation functions of the η_c and the J/ψ productions in Fig. 3. The distribution diagram for the b quark fragmentation has a similar shape to that of the c quark fragmentation, so we do not display it repeatedly. The numerical result of the ratios of the integrated NLO relativistic corrections to the fragmentation functions to the leading ones are shown in Table I. Varying the values of v^2 within a quite large range, we find that the contributions of the relativistic corrections are negative. However, they may become positive in some range of the pole mass of c quark, which corresponds to a negative v^2.

TABLE I. The ratios for the integrated NLO relativistic corrections to the leading ones. $D^{(0)}(z)$ is the leading order fragmentation function, and $D^2(z)$ is NLO relativistic correction.

Relative corrections	η_c	J/ψ	η_b	$\Upsilon(1S)$
$\int dz D^{(2)}(z) / \int dz D^{(0)}(z)$	-13.2%	-13.1%	-2.4%	-3.7%

We see that the estimated relativistic corrections are not negligible in comparison with the LO in v^2 for the charmonium. The cross sections of the fragmentation production for a heavy quarkonium will change considerably when the NLO contributions are taken into account. Our results are compatible with the natural expectation based on the velocity scaling rule in the NRQCD factorization formula.

In summary, we have calculated the relativistic corrections of order v^2 to the fragmentation functions for the heavy quark to the S-wave heavy meson in the framework of the NRQCD factorization formula. We have carried out the calculations by two different methods. One is based on the Collins-Soper definition for the fragmentation function. The other one is to extract the fragmentation functions through the decay width of the process $Z^0 \to Hq\bar{q}$ in the limit $M_z/m \to \infty$. In the former method, we also have performed the calculations in both the Feynman gauge and the axial gauge. All these results, obtained by different methods and by different gauges, are exactly the same. They are in disagreement with those obtained in Refs. [15,16]. We have used our analytic results to estimate

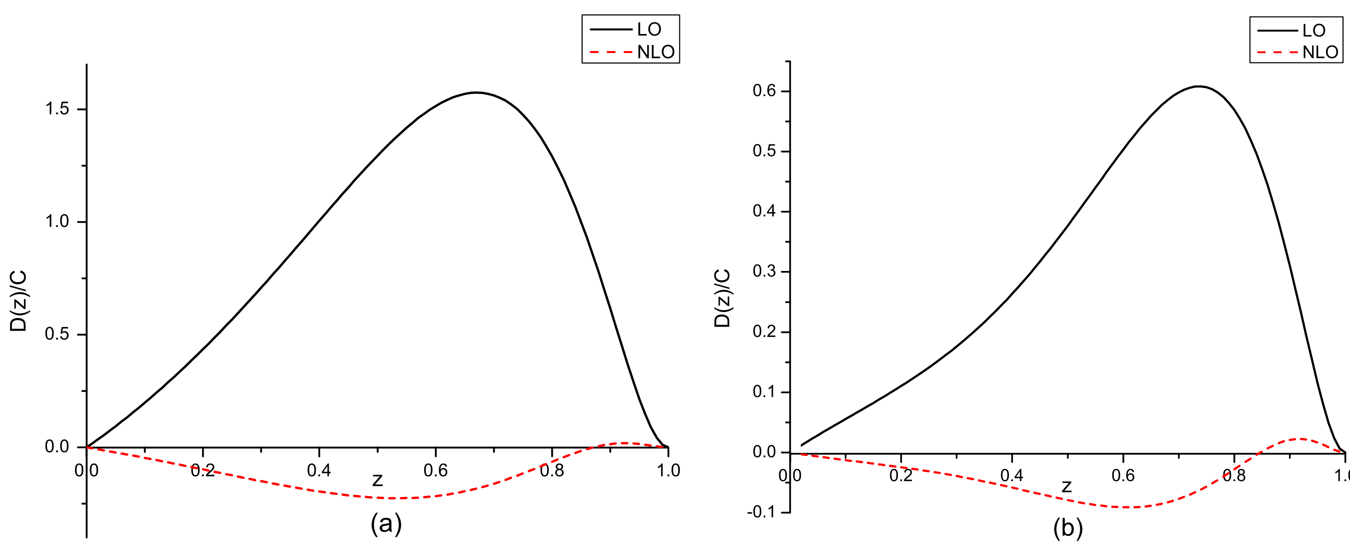

FIG. 3 (color online). The distributions vs z of the fragmentation functions $D(c \to \eta_c)$ and $D(c \to J/\psi)$. (a) is for η_c and (b) is for J/ψ. The thick solid line shows the leading order contribution, and the dashed line shows the relativistic correction. The factor C in the diagrams represents $10^{-2}\alpha_s^2\langle\mathcal{O}\rangle$.

the relative size of the relativistic corrections to the fragmentation functions.

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APPENDIX

In this Appendix, we will extend our calculations to the heavy quark fragmentation into the heavy meson with two different heavy quark masses, such as the B_c meson.

For the flavored heavy meson, the spin-singlet and the spin-triplet projection operators are given by

$$\Pi_0 = \frac{(\not p_2 - m_2)\gamma^5(\not P + E_1 + E_2)(\not p_1 + m_1)}{4\sqrt{2N_c}(E_1+E_2)\sqrt{E_1E_2(E_1+m_1)(E_2+m_2)}},$$
$$\Pi_1 = \frac{(\not p_2 - m_2)\gamma^\mu(\not P + E_1 + E_2)(\not p_1 + m_1)}{4\sqrt{2N_c}(E_1+E_2)\sqrt{E_1E_2(E_1+m_1)(E_2+m_2)}}, \tag{A1}$$

where m_1 and m_2 are the masses of the two quarks in the meson, P, q, p_1, and p_2 are the four-momentum of the meson, the half-relative momentum of the two quarks in the meson, the momentum of the quark with mass m_1, and the momentum of the quark with mass m_2, respectively. E_1 and E_2 are defined as $E_1 = \sqrt{\mathbf{q}^2 + m_1^2}$ and $E_2 = \sqrt{\mathbf{q}^2 + m_2^2}$, respectively. In addition, we have $p_1 = rP + q$ and $p_2 = (1-r)P - q$, where r is defined as $\frac{E_1}{E_1+E_2}$.

The NRQCD operators for the flavored meson are still the same as the operators defined in Eqs. (4) and (5), except that the ψ and χ now describe different heavy quark fields.

Using the same methods as we did in context of the paper, we obtain the short-distance coefficients for the heavy quark with mass m_1 fragmenting into the S-wave spin-singlet and the spin-triplet heavy flavored mesons as

$$F(^1S_0) = \frac{4(1-z)^2 z\alpha_s^2}{243M^3(y_1-1)^2(y_1z-1)^6}[3y_1^2(2y_1^2-2y_1+1)z^4 - 2y_1(18y_1^2-17y_1+5)z^3 + (68y_1^2-62y_1+15)z^2 + (18-36y_1)z + 6], \tag{A2}$$

$$G(^1S_0) = \frac{-\alpha_s^2}{729M^5(y_1-1)^4y_1^2(y_1z-1)^8}[(108y_1^8 - 264y_1^7 + 372y_1^6 - 240y_1^5 + 57y_1^4)z^9 + (-216y_1^8 - 432y_1^7 + 1460y_1^6 - 2108y_1^5 + 1352y_1^4 - 320y_1^3)z^8 + (108y_1^8 + 1656y_1^7 - 1032y_1^6 - 1780y_1^5 + 4311y_1^4 - 2996y_1^3 + 690y_1^2)z^7 + (-960y_1^7 - 3804y_1^6 + 7068y_1^5 - 5098y_1^4 - 1048y_1^3 + 2460y_1^2 - 664y_1)z^6 + (3004y_1^6 + 836y_1^5 - 6650y_1^4 + 9908y_1^3 - 4950y_1^2 + 612y_1 + 45)z^5 + (-3776y_1^5 + 4248y_1^4 - 3472y_1^3 - 60y_1^2 + 588y_1 - 36)z^4 + (1780y_1^4 - 1688y_1^3 + 1716y_1^2 - 344y_1 - 45)z^3 + (-384y_1^3 + 108y_1^2 - 204y_1 + 18)z^2 + (36y_1^2 + 12y_1 + 18)z], \tag{A3}$$

$$F(^3S_1) = \frac{4z(1-z)^2\alpha_s^2}{243M^3(y_1-1)^2(y_1z-1)^6}[y_1^2(2y_1^2-2y_1+3)z^4 - 2y_1(2y_1^2-3y_1+5)z^3 + 3(4y_1^2-6y_1+5)z^2 - 2(2y_1+1)z + 2], \tag{A4}$$

$$G(^3S_1) = \frac{-\alpha_s^2}{2187M^5(y_1-1)^4y_1^2(y_1z-1)^8}[(156y_1^8 - 360y_1^7 + 624y_1^6 - 492y_1^5 + 171y_1^4)z^9 + (-312y_1^8 + 192y_1^7 + 284y_1^6 - 2188y_1^5 + 2324y_1^4 - 960y_1^3)z^8 + (156y_1^8 + 696y_1^7 - 924y_1^6 + 1376y_1^5 + 2657y_1^4 - 4084y_1^3 + 2070y_1^2)z^7 + (-528y_1^7 - 1500y_1^6 + 3636y_1^5 - 7642y_1^4 + 2984y_1^3 + 1676y_1^2 - 1992y_1)z^6 + (1516y_1^6 - 188y_1^5 - 1970y_1^4 + 10020y_1^3 - 10038y_1^2 + 4320y_1 + 135)z^5 + (-2144y_1^5 + 3592y_1^4 - 8384y_1^3 + 7188y_1^2 - 2868y_1 - 288)z^4 + (868y_1^4 + 952y_1^3 - 1232y_1^2 + 708y_1 + 189)z^3 + (-528y_1^3 + 252y_1^2 - 132y_1 - 54)z^2 + (84y_1^2 - 36y_1 + 18)z], \tag{A5}$$

where $M = m_1 + m_2$ and $y_1 = \frac{m_1}{m_1+m_2}$.

We have computed the short-distance coefficients using the two ways described in the paper, which give the same results. One also can easily check that Eqs. (A2)–(A5) reduce to Eq. (23), in the equal mass limit. Our results for $F(^1S_0)$ and $F(^3S_1)$ agree with those obtained in Refs. [6,18].

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