Testing the Adler and Gross–Llewellyn Smith Sum Rules in High-Energy Neutrino Reactions*

D. Cline
University of Wisconsin, Madison, Wisconsin 53706

E. A. Paschos
National Accelerator Laboratory, Batavia, Illinois 60510†
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A practical method of testing the Adler and Gross–Llewellyn Smith sum rules with high-energy neutrino beams is outlined. The test is found to rely only on a knowledge of the relative neutrino and antineutrino fluxes and does not require point-by-point knowledge of the F_2 and F_3 structure functions. This test does rely on the assumption that Bjorken scaling is satisfied.

Testing the Adler sum rule is generally recognized to be crucial to the application of presently accepted concepts of current algebra and constituent models at high energy.[1] Similarly the Gross–Llewellyn Smith (GLS) sum rule test specifically for constituents with Gell-Mann–Zweig quark quantum numbers.[2] It is generally accepted that experimentally testing these sum rules will be extremely difficult and that the sum rules may converge slowly.[3] However, a closer look indicates that, provided Bjorken scaling holds, there are simple moments of experimentally accessible quantities that are directly related to the sum rules. Furthermore, in this case, it appears that a knowledge of the flux of neutrinos and antineutrinos is not necessary, but only the weaker condition that the relative ratio of neutrinos and antineutrinos is known. Indeed, the simultaneous production of neutrinos and antineutrinos using a broad band unfocused beam allows the required knowledge of the relative flux. Using this method, we anticipate that the Gross–Llewellyn Smith sum rule can be crudely tested in the near future using neutrino carbon and antineutrino carbon interactions in large calorimeter neutrino detectors.[4] Testing the Adler sum rule will require the introduction of a hydrogen target in such experiments.[4]

Consider the ratios of the form[5,6]

$$\langle f(Q^2, E_\mu)\rangle = \int f(Q^2, E_\mu)\, \frac{d^2\sigma}{dQ^2 dE_\mu}\, dQ^2 dE_\mu \Big/ \int \frac{d^2\sigma}{dQ^2 dE_\mu}\, dQ^2 dE_\mu \quad , \tag{1}$$

where $\langle f\rangle$ can be chosen to be $\langle Q^2/2mE_\nu\rangle$, $\langle E_\mu/E_\nu\rangle$, etc. and are moments of the neutrino cross section distribution $d\sigma/dQ^2 dE_\mu$ that are related to the experimental quantities

$$\langle f\rangle = \frac{\sum f_i N_i}{\sum N_i} \quad , \tag{2}$$

where N_i is the number of events for which $f = f_i$. The quantities $\langle f\rangle$ will be independent of the neutrino flux over the neutrino energy interval ΔE in the scaling limit. Using the scale-invariant form of the neutrino and antineutrino cross section,[7]

$$\frac{d^2\sigma^{\nu,\bar\nu}}{dx\,dy} = \frac{G^2 mE}{\pi}\{F_2(x)(1-y) + \tfrac{1}{2}y^2[2xF_1(x)] \mp y(1-\tfrac{1}{2}y)[xF_3(x)]\}, \tag{3}$$

where

$$x = \frac{Q^2}{2m\nu} = \frac{1}{\omega} \text{ and } y = \frac{\nu}{E}\,;$$

F_1, F_2, and F_3 are the neutrino structure functions, and the minus sign is taken for neutrino interactions. We specify two different targets d and p, where d refers to an $I = 0$ target with equal numbers of proton and neutron targets and p refers to a hydrogen target. We then consider two classes of structure functions,

$$F_i^{\nu p}(x), \quad F_i^{\bar\nu p}(x)$$

and (4)

$$F_i^{\nu d}(x), \quad F_i^{\bar\nu d}(x),$$

and it follows that (for $d = n + p$)

$$F_i^{\nu d} = F_i^{\bar\nu d} = F_i^{\nu p} + F_i^{\nu n} = F_i^{\bar\nu p} + F_i^{\bar\nu n}\,,$$

and for a C^{12} target

$$F_i^{\nu C} = 6F_i^{\nu d}\ .$$

In the scaling limit the Adler and Gross–Llewellyn Smith sum rules can be expressed as

$$\int \frac{d\omega}{\omega}(F_2^{\nu n}-F_2^{\nu p})=2 \quad \text{(Adler)}$$
$$=\int \frac{d\omega}{\omega}(F_2^{\bar{\nu}p}-F_2^{\nu p})=2 \quad \text{(Adler)}, \tag{5}$$

$$\int F_3^{\nu d}dx=\int \frac{d\omega}{\omega}(F_3^{\nu p}+F_3^{\nu n})$$
$$=\int \frac{d\omega}{\omega^2}(F_3^{\nu d})=-6 \quad \text{(GLS)}. \tag{6}$$

We consider the GLS sum rule first and form the mean values

$$\langle\omega\rangle_{\bar{\nu}d,\nu d}=\int \frac{d^2\sigma^{\bar{\nu}d,\nu d}}{dx\,dy}\frac{dx\,dy}{x}\Big/\sigma_{\bar{\nu}d,\nu d}\,. \tag{7}$$

Subtracting the quantity $(\sigma_{\bar{\nu}d}/\sigma_{\nu d})\langle\omega\rangle_{\bar{\nu}d}$ from $\langle\omega\rangle_{\nu d}$ gives [see Eq. (22)]

$$\frac{\sigma_{\bar{\nu}d}}{\sigma_{\nu d}}\langle\omega\rangle_{\bar{\nu}d}-\langle\omega\rangle_{\nu d}=\frac{1}{\sigma_{\nu d}}\int\left[\frac{1}{x}\left(\frac{d\sigma^{\bar{\nu}d}}{dx\,dy}-\frac{d\sigma^{\nu d}}{dx\,dy}\right)\right]dx\,dy$$
$$=\frac{2\left[\int y(1-\frac{1}{2}y)dy\right]\left\{\int F_3^{\nu d}dx\right\}}{\left[\frac{1}{2}+\frac{1}{2}\langle L\rangle-\frac{1}{6}\langle R\rangle\right]\int F_2^{\nu d}dx}$$
$$=\frac{2}{3}\frac{1}{\frac{1}{2}+\frac{1}{2}\langle L\rangle-\frac{1}{6}\langle R\rangle}\int F_3^{\nu d}dx, \tag{8}$$

where $\langle L\rangle$ and $\langle R\rangle$ are defined by Bjorken and Paschos,[8] and $\sigma_{\nu d}$, $\sigma_{\bar{\nu}d}$ are the total cross sections.

In the scaling limit the ratio of total cross sections $\sigma_{\bar{\nu}d}/\sigma_{\nu d}$ becomes constant. Present CERN data suggest that this value is

$$\frac{\sigma_{\bar{\nu}d}}{\sigma_{\nu d}}=\tfrac{1}{3}(1+\epsilon), \tag{9}$$

where ϵ is less than 0.15.[9] For simplicity we derive illustrative results based on $\epsilon\to 0$, keeping in mind that if ϵ is found to be larger at high energies the equations can be appropriately modified. The resulting equation is

$$\frac{\sigma_{\bar{\nu}d}}{\sigma_{\nu d}}\langle\omega\rangle_{\bar{\nu}d}-\langle\omega\rangle_{\nu d}=\tfrac{2}{3}\int_0^1 F_3^{\nu d}dx\,. \tag{10}$$

For Gell-Mann–Zweig quark constituents we find

$$\langle\omega\rangle_{\nu d}-\frac{\sigma_{\bar{\nu}d}}{\sigma_{\nu d}}\langle\omega\rangle_{\bar{\nu}d}=4, \tag{11}$$

and for the Sakata model

$$\langle\omega\rangle_{\nu d}-\frac{\sigma_{\bar{\nu}d}}{\sigma_{\nu d}}\langle\omega\rangle_{\bar{\nu}d}=\tfrac{4}{3}, \tag{12}$$

implying a considerable difference between the first moment of ω for the two cases. Verification of relation (11) requires only a knowledge of the ratio $\sigma_{\bar{\nu}d}/\sigma_{\nu d}$ and the moments of the ω distribution for $\bar{\nu}d$ and νd scattering, and, of course, that the sum rule converge in the accessible neutrino energy range.[10]

Turning to the Adler sum rule and assuming that the Callan-Gross relation[11,12]

$$F_2=2xF_1 \tag{13}$$

holds, we derive

$$\tfrac{2}{3}\int_0^1[F_2^{\nu n}-F_2^{\nu p}]\frac{dx}{x}+\tfrac{1}{2}R=\frac{\sigma_{\nu p}}{\sigma_{\nu d}}\left[\langle\omega\rangle_{\bar{\nu}p}\left(\frac{\sigma_{\bar{\nu}p}}{\sigma_{\nu p}}\right)-\langle\omega\rangle_{\nu p}\right], \tag{14}$$

where

$$R=\frac{\sigma_{\bar{\nu}d}}{\sigma_{\nu d}}\langle\omega\rangle_{\bar{\nu}d}-\langle\omega\rangle_{\nu d}\,. \tag{15}$$

Experimental measurement of R as well as the ratios $\sigma_{\nu p}/\sigma_{\nu d}$ and $\sigma_{\bar{\nu}p}/\sigma_{\nu p}$ and the two moments of ω, $\langle\omega\rangle_{\bar{\nu}p}$ and $\langle\omega\rangle_{\nu p}$, allows a test of the Adler sum rule. Assuming both the GLS and Adler sum rules, relation (14) can be rewritten as

$$\frac{\sigma_{\nu p}}{\sigma_{\nu d}}\left[\langle\omega\rangle_{\nu p}-\left(\frac{\sigma_{\bar{\nu}p}}{\sigma_{\nu p}}\right)\langle\omega\rangle_{\bar{\nu}p}\right]=\tfrac{2}{3}\;. \tag{16}$$

A direct measurement of $\sigma_{\nu p}/\sigma_{\nu d}$ can be made independent of neutrino flux if a hydrogen target and a carbon target are simultaneously exposed to the same neutrino beam, whereas the ratio $\sigma_{\bar{\nu}p}/\sigma_{\nu p}$ requires a hydrogen target exposed to a mixed neutrino-antineutrino beam of known ratio.

There are also an infinite number of sum rules given by

$$\tfrac{2}{3}\int x^n(F_2^{\nu d})dx+\tfrac{1}{3}\int x^{n+1}(F_3^{\bar{\nu}p}-F_3^{\nu p})dx$$
$$=\frac{\sigma_{\nu p}}{\sigma_{\nu d}}\left[\langle x^n\rangle_{\bar{\nu}p}(\sigma_{\bar{\nu}p}/\sigma_{\nu p})+\langle x^n\rangle_{\nu p}\right]. \tag{17}$$

Using the Llewellyn Smith relation[13] (which assumes nonintegral charged quarks) with the Callan-Gross relation[11]

$$12(F_1^{\gamma p}-F_1^{\gamma n})=\frac{6}{x}(F_2^{\gamma p}-F_2^{\gamma n})$$
$$=(F_3^{\nu p}-F_3^{\bar{\nu}p}), \tag{18}$$

relation (17) can be rewritten as

$$\tfrac{2}{3}\int x^n(F_2^{\nu d})dx+2\int x^n(F_2^{\gamma n}-F_2^{\gamma p})dx$$
$$=\frac{\sigma_{\nu p}}{\sigma_{\nu d}}\left[\langle x^n\rangle_{\bar{\nu}p}\left(\frac{\sigma_{\bar{\nu}p}}{\sigma_{\nu p}}\right)+\langle x^n\rangle_{\nu p}\right]. \tag{19}$$

Relation (19) implies that a direct experimental relation should obtain between moments of x measured in charged lepton scattering and the moments measured in neutrino interactions. The first two moments ($n=0$ and $n=1$) can be evaluated giving, for $n=0$,

$$\tfrac{2}{3}\int F_2^{\nu d}dx+2\int [F_2^{\gamma n}-F_2^{\gamma p}]dx=\frac{\sigma_{\bar{\nu}p}}{\sigma_{\nu d}}+\frac{\sigma_{\nu p}}{\sigma_{\nu d}}\ , \tag{20}$$

and for $n=1$

$$\tfrac{2}{3}\int xF_2^{\nu d}dx+2\int x(F_2^{\gamma n}-F_2^{\gamma p})dx =\frac{\sigma_{\nu p}}{\sigma_{\nu c}}\left[\langle x\rangle_{\bar{\nu}p}\left(\frac{\sigma_{\bar{\nu}p}}{\sigma_{\nu p}}\right)+\langle x\rangle_{\nu p}\right]. \tag{21}$$

Using the best estimates for the neutrino integrals[6,14]

$$\int F_2^{\nu d}dx=2(0.49\pm0.07) =0.98\pm0.14 \tag{22}$$

and

$$\int xF_2^{\nu d}dx\simeq 2(0.12)=0.24 \tag{23}$$

and the electroproduction integrals[15]

$$\int (F_2^{\gamma p}-F_2^{\gamma n})dx=0.05 \tag{24}$$

and

$$\int x(F_2^{\gamma p}-F_2^{\gamma n})dx=0.016, \tag{25}$$

we find

$$\frac{\sigma_{\bar{\nu}p}}{\sigma_{\nu d}}+\frac{\sigma_{\nu p}}{\sigma_{\nu d}}=\tfrac{2}{3}(0.98\pm0.14)+2\int (F_2^{\gamma n}-F_2^{\gamma p})dx \approx 0.55\pm0.09 \tag{26}$$

and

$$\frac{\sigma_{\nu p}}{\sigma_{\nu d}}\left[\langle x\rangle_{\bar{\nu}p}\left(\frac{\sigma_{\bar{\nu}p}}{\sigma_{\nu p}}\right)+\langle x\rangle_{\nu p}\right]\simeq 0.16+2\int x(F_2^{\gamma n}-F_2^{\gamma p})dx \simeq 0.13. \tag{27}$$

Experimental verification of relations (26) and (27) would constitute a partial test of the Llewellyn Smith relation, testing the point-by-point relation (18) between the structure functions measured in electromagnetic and weak lepton inclusive interactions.

Although the testing of the Adler and GLS sum rules undoubtedly be extremely difficult, it is not necessary to separate the structure function point by point, but instead only the first moments of the distribution and some relative cross sections measurements are required. In addition, there exists an infinite set of sum rules relating the moments of the x distributions measured in electromagnetic and weak scattering processes, provided the Llewellyn Smith relation holds. Testing these sum rules would constitute an important test of the nonintegral charge quark model. Even if relation (18) were found to be incorrect experimentally, it is still expected that a similar relation between the weak and electromagnetic structure functions might hold and that a relation analogous to (19) would be discovered. Clearly a parametrization of the experimental data in terms of moments would be important in this regard.

Note added in proof. Equation (10) can be easily generalized. Using the hypothesis that the V-A interference is maximal, one obtains

$$\frac{\sigma^{\bar{\nu}d}}{\sigma^{\nu d}}\langle x^n\rangle_{\bar{\nu}d}-\langle x^n\rangle_{\nu d}=-\tfrac{2}{3}\int x^nF_2^{\bar{\nu}d}(x)dx\ .$$

Then appealing to the quark-parton relation

$$F_2^{\nu d}(x)\approx 4F_2^{\gamma d}(x),$$

one can evaluate the moments on the right-hand side. The $n=0$ and 1 moments are given in the text.

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[4]See the NAL proposals for E1A (the Harvard-Pennsylvania-Wisconsin Neutrino Experiment at NAL) and for E21.

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[9]Preliminary Results of the Gargamelle Collaboration, P. Heusse, in *Proceedings of the Sixteenth International Conference on High Energy Physics, National Accelerator Laboratory, Batavia, Ill., 1972*, Ref. 3, Vol. 2, p. 206. In estimating $\sigma_{\nu d}$ in Eqs. (8), (10), and subsequent equations, we adopt the view that linear rise has already settled in.

[10]Recently O. Nachtmann [Phys. Rev. D 7, 3340 (1973)] has shown that within the parton model, both the Adler and GLS sum rules cannot converge for $\omega < 20$ and probably require $\omega > 40$.

[11]C. G. Callan and D. J. Gross, Phys. Rev. Lett. 22, 156 (1969).

[12]This relation is consistent with $\sigma_S/\sigma_T \sim$ small in the SLAC data, and is implied by spin-$\frac{1}{2}$ parton models and the ratio of $\sigma_{\bar{\nu}d}/\sigma_{\nu d} \sim \frac{1}{3}$ measured at low energy.

[13]C. H. Llewellyn Smith, Nucl. Phys. B17, 277 (1970).

[14]D. H. Perkins, in *Proceedings of the Sixteenth International Conference on High Energy Physics, National Accelerator Laboratory, Batavia, Ill., 1972*, Ref. 3, Vol. 4, p. 189.

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Errata

General Treatment of the Breaking of Chiral Symmetry and Scale Invariance in the SU(3) σ Model, J. Schechter and Y. Ueda [Phys. Rev. D 3, 2874 (1971)].

1. In Eq. (4.7) for $g_{KK\eta'}$ replace $(\kappa^2 - \eta^2)$ by $(\kappa^2 - \eta'^2)$.

2. In Eq. (5.11″) the sign of the third (i.e. last) term should be changed from + to −. This has the consequence that, although the physics of the situation remains the same, Table I on page 2888 should be replaced by the following:

TABLE I. Predicted width.

ϵ^2 $(\pi_0{}^2)$	$\Gamma(\eta' \to \eta\,\pi^+\pi^-)$ (MeV)	$g_{\sigma'\eta\eta'}$ (π_0)
35	134	−828
45	9.3	−137
50.4	4.4	−78
75	0.94	−8.2
100	0.37	+10.8
200	0.09	+25.8

3. Add the following at the end of footnote 31: The properly covariant energy-momentum tensor is

$$\tilde{\Theta}_{\mu\nu} = \Theta_{\mu\nu} - \tfrac{3}{2}\,\delta_{\mu\nu} \sum_{a=1}^{3} \alpha_a A_a \,,$$

which satisfies $\langle \tilde{\Theta}_{\mu\nu} \rangle_0 \simeq 0$.

Pion-Deuteron Scattering at High Energies, Deepinder P. Sidhu and C. Quigg [Phys. Rev. D 7, 755 (1973)]. In Fig. 1, the labels Magnetic and Quadrupole should be interchanged.

The second term on the right-hand side of Eq. (18) should read

$$\phi_b(q')[B(\tfrac{1}{4}q^2 - q'^2\cos\alpha)/10].$$

Inclusive Vector-Meson Production at Small t in the Dual Resonance Model, J. Randa [Phys. Rev. D 7, 2236 (1973)]. There are over-all sign errors in Eqs. (B1), (B2), and (B4). In each case the minus sign preceding the integral or summation should be deleted. Equations (B3) and (B5), as well as the expressions in the tables, are correct. Also, the sixth parameter of I in Eq. (3.6) is $\alpha_3 + \alpha_9 - \alpha_7$, not $\alpha_2 + \alpha_9 - \alpha_7$.