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Boundary-Condition Approach to Three-Particle Final States*

D. D. Brayshaw
Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742
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It is shown that the wave function for a three-particle system outside the range of forces may be uniquely determined by imposing a suitable set of boundary conditions. This result is expressed in terms of a one-variable integral equation with a square-integrable kernel, the solutions of which specify the three-body t matrix. The input to this equation consists of the two-particle phase shifts and two independent real-valued functions which characterize the three-body wave function in specific regions. The formalism yields an exactly unitary three-particle t matrix for arbitrary values of this input, and thus provides a practical scheme for the analysis of three-body final states.

I. INTRODUCTION

Some time ago, Feshbach and Lomon[1] demonstrated the power of the boundary-condition approach as a means of correlating a broad spectrum of N-N scattering data. This approach is based on the well-known fact that, for interactions of finite range r_0, the wave function takes on a particularly simple form at interparticle distances $r > r_0$, and may be completely characterized by stating the coefficient of the outgoing wave. This coefficient is uniquely determined in each partial wave if one specifies a value for the logarithmic derivative of ψ_l at $r = a \geq r_0$; since this value must be energy-dependent, imposing such a condition is merely an alternative to the usual description in terms of scattering phase shifts. The power of this approach lies in the empirical fact that, for the N-N system,[2] the logarithmic-derivative parameters are at most weakly dependent on the energy, and hence a few parameters are adequate to describe the scattering in the range 0–300 MeV.[1] A corresponding statement may be made for other systems of strongly interacting particles.[3] In comparison to a potential description, the computational advantages of this approach are obvious; one replaces a one-dimensional integral equation (e.g., the Lippmann-Schwinger equation[4]) by quadrature.

An analogous simplification of the three-body wave function occurs in the *exterior region*, defined by the requirement that no pair of particles is within the range of forces. Therefore, one might hope that a suitably generalized boundary condition on the exterior wave function would uniquely specify the outgoing component (i.e., determine the three-particle t matrix), resulting in a highly efficient description of three-particle final states with comparable computational advantages. Below we propose a set of boundary conditions for this purpose which determine the three-body t matrix via the solution of a one-variable integral equation. The input for this equation is cleanly separated into two-particle phase shifts and real-valued functions characterizing the three-body wave function in distinct physical regions. For any arbitrary selection of this input the formalism produces an exactly unitary three-particle t matrix. This is to be compared with an earlier approach with essentially the same motivation by Noyes.[5] In Noyes's work a one-variable equation was derived with a kernel specified in terms of the half-on-shell two-body t matrix, and a driving term involving an arbitrary expansion of the *interior* wave function. The difficulty with this formulation is that the expansion coefficients are intimately connected to the two-body phase shifts via the unitarity relation, and hence are not truly independent; selecting them arbitrarily will in general violate unitarity. In order to achieve an effective "phase shift" analysis of three-particle final states, one must require real and independent parameters; our approach satisfies this condition.

We begin in Sec. II with a brief review of the boundary-condition approach to two-particle systems, and introduce a new statement of the bound-

ary condition in Eq. (5) which is suitable for the generalization we have in mind. The three-body theory is presented in Sec. III, where we attempt to stress the basic ideas of the approach while suppressing the more complicated details. The latter are provided in Sec. IV along with background material designed to make the paper self-contained. In Sec. V we discuss the application of the resulting formalism to data analysis, and comment on some broader implications of this approach.

II. TWO-PARTICLE BOUNDARY CONDITIONS

In order to avoid unnecessary complication of the subsequent discussion we will neglect spin and isospin dependence, and will assume that in each two-body channel α there is a characteristic interaction radius a_α beyond which particles β and γ do not interact ($\alpha\beta\gamma$ cyclic). It will be helpful to first consider the boundary-condition approach in a two-body system. If $\psi_{kl}^{\text{ext}}(r)$ represents the partial-wave function for that system outside its interaction radius a, then

$$\psi_{kl}^{\text{ext}}(r) = \phi_{kl}(r) + t_l(k)\psi_{kl}^{+}(r), \tag{1}$$

with

$$\begin{aligned}\phi_{kl}(r) &= \left(\frac{2}{\pi}\right)^{1/2} i^l j_l(kr),\\ \psi_{kl}^{+}(r) &= -i\pi k M_r \left(\frac{2}{\pi}\right)^{1/2} i^l h_l(kr);\end{aligned} \tag{2}$$

k is the c.m. momentum. The coefficient t_l is determined by requiring that

$$\begin{aligned}\lim_{\epsilon\to 0} \frac{\psi_{kl}^{\text{ext}\,\prime}(a+\epsilon)}{\psi_{kl}^{\text{ext}}(a+\epsilon)} &= \lambda_l(k)\\ &= \lambda_l^{(0)} + \Delta_l(k^2).\end{aligned} \tag{3}$$

This implies that

$$\begin{aligned}t_l(k) &= \frac{N_l(k)}{D_l(k)},\\ N_l(k) &= (a\lambda_l - l)j_l(ak) + ak\, j_{l+1}(ak),\\ D_l(k) &= i\pi k M_r[(a\lambda_l - l)h_l(ak) + akh_{l+1}(ak)].\end{aligned} \tag{4}$$

In what follows we will denote by $t_l^{(0)}, N_l^{(0)}, D_l^{(0)}$ the corresponding expressions in the special case $\Delta_l = 0$. For purposes of comparison with the three-particle formulation, we note the equivalent boundary condition

$$\psi_{kl}^{\text{ext}\,\prime}(a+) - \lambda_l^{(0)}\psi_{kl}^{\text{ext}}(a+) = \left(\frac{2}{\pi}\right)^{1/2} \frac{i^l}{a} B_l(k)t_l(k). \tag{5}$$

Clearly the definition

$$B_l(k) = \pi M_r \Delta_l(k^2) N_l^{-1}(k) \tag{6}$$

establishes the equivalence; on the other hand, any real function $B_l(k) = (-1)^l B_l(-k)$ via Eq. (5) uniquely predicts a unitary two-particle t matrix (t_l).

III. BOUNDARY CONDITIONS FOR A THREE-PARTICLE SYSTEM

The wave function for a three-body system may be expressed as a function of the relative position $\vec{x}_\alpha$ of particles β and γ, and the position $\vec{y}_\alpha$ of particle α relative to the $\beta\gamma$ c.m. There are clearly three ways to so label the system; unless we wish to indicate a specific choice we will drop the subscripts and simply write $\Psi(\vec{x}, \vec{y})$. It is convenient to make the usual channel decomposition and write

$$\Psi(\vec{x}, \vec{y}) = \sum_{\beta=1}^{3} \psi^{\beta}(\vec{x}_\beta, \vec{y}_\beta), \tag{7}$$

where the channel wave function ψ^α corresponds to that part of Ψ for which, in the limit $y_\alpha \to \infty$, particle α is merely a spectator to the two-particle scattering of β and γ.[6] In what follows we take momenta $\vec{p}, \vec{q}$ conjugate to $\vec{x}, \vec{y}$, respectively, and define W to be the total energy in the three-particle c.m. system.

We now observe that Ψ may be characterized in terms of an initial state of definite momentum $|\vec{p}_{\alpha_0}\vec{q}_{\alpha_0}\rangle$; the on-shell condition is that

$$p_{\alpha_0}^{\,2} = k_{\alpha_0}^{\,2} \equiv 2\mu_{\alpha_0}(W - q_{\alpha_0}^{\,2}/2M_{\alpha_0}), \tag{8}$$

where μ_α, M_α are the reduced masses corresponding to the momenta $\vec{p}_\alpha, \vec{q}_\alpha$, respectively. However, it is more convenient to couple the angular momenta $\vec{l}_0(\vec{p}_{\alpha_0})$ and $\vec{\lambda}_0(\vec{q}_{\alpha_0})$ to form the state $|LMl_0\lambda_0 p_{\alpha_0} q_{\alpha_0}\rangle$. Correspondingly, we expand Ψ and the ψ^β in terms of the two-direction spherical harmonics defined by Blatt and Weisskopf,[7] *viz.*,

$$\Psi^{(L,M)}(\vec{x}, \vec{y}) = \sum_{l\lambda} Y_{LMl\lambda}(\hat{x}, \hat{y})\Psi^{Ll\lambda}(x, y). \tag{9}$$

For $x_\alpha > a_\alpha$, ψ^α is that solution of the free Schrödinger equation ($H_0\psi^\alpha = W\psi^\alpha$) satisfying the boundary condition noted above as $y_\alpha \to \infty$. Thus, *exterior* to the domain in which β and γ interact, ψ^α takes the simple form[8]

$$\begin{aligned}\psi_{Ll\lambda}^{\alpha;\text{ext}}(x_\alpha, y_\alpha) &= \delta_{\alpha\alpha_0}\delta_{ll_0}\delta_{\lambda\lambda_0}\phi_{k_\alpha l}(x_\alpha)\phi_{q_\alpha\lambda}(y_\alpha)\\ &\quad + \int_0^\infty dq\, q^2 \phi_{q\lambda}(y_\alpha)\psi_{k_\alpha l}^{*}(x_\alpha)T_{Ll\lambda}^{\alpha}(q);\end{aligned} \tag{10}$$

for comparison we recall Eq. (1). The unknown quantity in Eq. (10) is the channel t matrix $T_{Ll\lambda}^{\alpha}(q)$,

which depends parametrically on $l_0, \lambda_0, q_{\alpha_0}$. In particular,

$$T^{\alpha_0}_{Ll\lambda}(q) = \delta_{ll_0}\delta_{\lambda\lambda_0}\frac{\delta(q-q_{\alpha_0})}{q^2}t_{\alpha_0 l}(k_{\alpha_0}) + M^{\alpha_0;L}_{l\lambda;l_0\lambda_0}(q, q_{\alpha_0}), \tag{11}$$

where M^α arises from multiple-scattering terms. The T^α suffice to construct the total three-body t matrix $T = \sum_\beta T^\beta$, and hence provide sufficient information to determine all scattering amplitudes of physical interest.[9]

In order to determine the T^α we will constrain Ψ at the boundary of the *exterior region*. An obvious generalization of Eq. (5) is to require that

$$\Psi^{\mathrm{ext}}_{Ll'\lambda'}(a_\alpha+, y_\alpha) - \lambda^{(0)}_{l\alpha}\Psi^{\mathrm{ext}}_{Ll\lambda}(a_\alpha+, y_\alpha) = \sum_{\beta l'\lambda'}\int_0^\infty dq\, q^2 \bar{B}^{\alpha\beta;L}_{l\lambda;l'\lambda'}(y_\alpha, q)T^\beta_{Ll'\lambda'}(q), \tag{12}$$

where $\bar{B}$ describes the dependence of the logarithmic derivative on the distance y_α of the third particle, and Ψ^{ext} represents the exterior projection of Ψ.[10] Explicitly,

$$\Psi^{\mathrm{ext}}(\vec{x}, \vec{y}) = P^{\mathrm{ext}}(\vec{x}, \vec{y})\sum_{\beta=1}^{3}\psi^{\beta;\mathrm{ext}}(\vec{x}_\beta, \vec{y}_\beta), \tag{13}$$

where

$$P^{\mathrm{ext}}(\vec{x}, \vec{y}) = \theta(x_\alpha - a_\alpha)\theta(x_\beta - a_\beta)\theta(x_\gamma - a_\gamma). \tag{14}$$

It is convenient (and involves no loss in generality) to take $\bar{B}$ in the form

$$\bar{B}^{\alpha\beta;L}_{l\lambda;l'\lambda'}(y_\alpha, q) = \sum_{l''\lambda''}P^{\mathrm{ext};L}_{l\lambda;l''\lambda''}(a_\alpha+, y_\alpha)B^{\alpha\beta;L}_{l''\lambda'';l'\lambda'}(y_\alpha, q),$$
$$B^{\alpha\beta;L}_{l\lambda;l'\lambda'}(y_\alpha, q) = \left(\frac{2}{\pi}\right)^{1/2}\frac{i^l}{a_\alpha}[\delta_{\alpha\beta}\delta_{ll'}\delta_{\lambda\lambda'}\phi_{q\lambda}(y_\alpha)B_{\alpha l}(k_\alpha) + \hat{B}^{\alpha\beta;L}_{l\lambda;l'\lambda'}(y_\alpha, q)D^{(0)}_{\beta l'}(\bar{k}_\beta)], \tag{15}$$

where $P^{\mathrm{ext};L}_{l\lambda;l'\lambda'}(a_\alpha+, y)$ is a polynomial in y which can be determined explicitly from the appropriate angular momentum projection of Eq. (14), and $B_{\alpha l}(k_\alpha)$ is the function defined in Eq. (6) with a channel index appended. By requiring that $\hat{B}/\phi_{q\lambda}(y_\alpha) \to 0$ in the limit $y_\alpha \to \infty$, we embed the two-particle boundary condition in Eq. (12) as a special case, and guarantee the correct two-body phase shifts. The (real) function $\hat{B}$ is otherwise arbitrary and represents our dynamical input. Here we have also defined $\bar{k}_\beta{}^2 = 2\mu_\beta(W_0 - q^2/2M_\beta)$, where W_0 is a negative energy parameter, and taken the (nonvanishing) function $D^{(0)}_{\beta l'}(\bar{k}_\beta)$ as an explicit factor for reasons explained below.

The explicit appearance of P^{ext} in Eqs. (13) and (15) ensures that our boundary condition [Eq. (12)] is only applied to that part of Ψ which can be constructed from the exterior representation of the ψ^β; this is in fact necessary since we have no equivalent representation in the interior. As a consequence, however, we observe that there will in general be some minimal distance y^0_α such that both sides of Eq. (12) vanish identically for $y_\alpha < y^0_\alpha$. That is, for $x_\alpha = a_\alpha+$, y^0_α corresponds to particle α being close enough to β or γ to interact; P^{ext} vanishes for y_α less than this value. The geometry for the special case of equal masses and radii is shown in Fig. 1; in this case $y^0_\alpha = \frac{1}{2}\sqrt{3}a$. As a result, Eq. (12) is without content for $y_\alpha < y^0_\alpha$. Unfortunately, such a constraint is not adequate to specify the T^α uniquely, and we thus require an auxiliary boundary condition for this purpose.

The form of this auxiliary constraint is essentially determined by the requirements of unitarity, and was derived by the present author in the context of a particular case of this formalism (singular cores)[11]; this corresponds to setting Δ_l and the input functions defined below identically equal to zero. If we denote the second (integral) term of Eq. (10) by $\psi^{\alpha;\mathrm{out}}_{Ll\lambda}(x_\alpha, y_\alpha)$, this condition can be expressed in the form

$$\psi^{\alpha;\mathrm{out}}_{Ll'\lambda'}(a_\alpha+, y_\alpha) - \lambda^{(0)}_{l\alpha}\psi^{\alpha;\mathrm{out}}_{Ll\lambda}(a_\alpha+, y_\alpha) = \left(\frac{2}{\pi}\right)^{1/2}\frac{i^l}{a_\alpha}\sum_{\beta l'\lambda'}\int_0^\infty dq\, q^2\bar{C}^{\alpha\beta;L}_{l\lambda;l'\lambda'}(y_\alpha, q)T^\beta_{Ll'\lambda'}(q) \tag{16}$$

for $y_\alpha < y^0_\alpha$, where we take $\bar{C}$ in the form

$$\bar{C}^{\alpha\beta;L}_{l\lambda;l'\lambda'}(y_\alpha, q) = \delta_{\alpha\beta}\delta_{ll'}\delta_{\lambda\lambda'}\phi_{q\lambda}(y_\alpha)R_{\alpha l}(q)D^{(0)}_{\alpha l}(k_\alpha) + \hat{C}^{\alpha\beta;L}_{l\lambda;l'\lambda'}(y_\alpha, q)D^{(0)}_{\beta l'}(\bar{k}_\beta),$$
$$R_{\alpha l}(q) = 1 - \frac{D^{(0)}_{\alpha l}(\bar{k}_\alpha)}{D^{(0)}_{\alpha l}(k_\alpha)}. \tag{17}$$

Here $\hat{C}$ is an arbitrary real function.

The appearance of the explicit factor $D^{(0)}_{\beta l'}(\bar{k}_\beta)$ in Eqs. (15) and (17) serves two purposes. In the first place, it provides sufficient convergence to the resulting integral equation to guarantee a square-integrable (L_2) kernel, provided only that $\hat{B}$ and $\hat{C}$ are also L_2. Secondly, the dependence on W_0 as a parameter has a useful physical interpretation. For example, if the three-body system has a bound state of energy W_B and we choose $W_0 = W_B$, then the energy-independent parts of $\hat{B}$ and $\hat{C}$ could in principle be determined from a knowledge of the

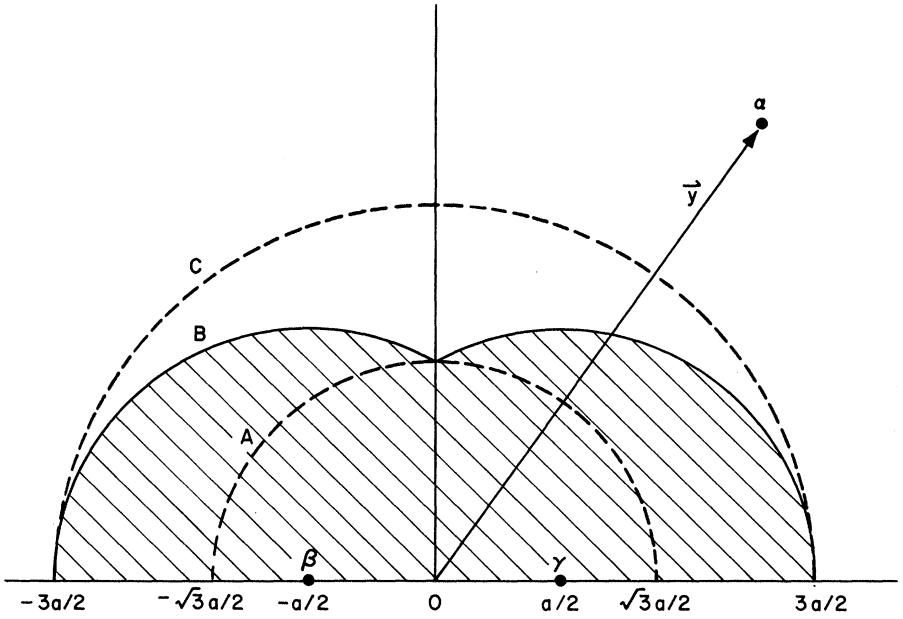

FIG. 1. Geometry defining the exterior and interior regions in the special case of equal masses and radii. The interior region is cross-hatched and separated from the exterior by curve B. Curve A bounds the domain $y_\alpha < y_\alpha^0$ for which particle α is close enough to β and/or γ to interact, while curve C corresponds to the minimal distance $y_\alpha^{\max}$ such that α is in the exterior independent of the angle $\hat{x}\cdot\hat{y}$.

bound-state wave function through Eqs. (12) and (16); i.e., in this case the ψ's would be known.

The consequences of our boundary conditions can be expressed as a set of coupled one-variable integral equations, which for brevity we express as an operator equation in terms of the states $|\alpha l\lambda q\rangle$, defined (for fixed L) such that

$$\langle \beta l'\lambda' q'|\alpha l\lambda q\rangle = \delta_{\alpha\beta}\delta_{ll'}\delta_{\lambda\lambda'}\frac{\delta(q'-q)}{q^2}. \tag{18}$$

It is useful to define an operator X such that

$$T^{\alpha}_{Ll'\lambda}(q) = -\frac{t_{\alpha l}(k_\alpha)}{N^{(0)}_{\alpha l}(k_\alpha)}\langle \alpha l\lambda q|X|\alpha_0 l_0\lambda_0 q_{\alpha_0}\rangle; \tag{19}$$

then

$$\begin{aligned} &X = \Omega + KX, \\ &K = \theta + K^{(0)}\rho + [(1-\theta)\hat{B} + \theta(\hat{C}-1)](1-R)\rho. \end{aligned} \tag{20}$$

Here $K^{(0)}$ and Ω are the kernel and driving term, respectively, obtained for the special case considered in SCI, while $\hat{B}, \hat{C}, R$ correspond in an obvious way to the functions defined above. The operator θ corresponds to the projection operator $\theta(y_\alpha^0 - y_\alpha)$ in this basis, while ρ is diagonal with the value

$$\rho_{\alpha l}(q) = \frac{t_{\alpha l}(k_\alpha)}{t^{(0)}_{\alpha l}(k_\alpha)}. \tag{21}$$

Thus ρ contains the two-body bound-state poles (if any). The real input functions $\hat{B}$ and $\hat{C}$ may have arbitrary matrix elements in this basis; provided they are L_2, the kernel K is also L_2, and hence X is well defined. We note that the poles of $(1-K)^{-1}$ for negative values of W determine the energies of the three-particle bound states. The only information our approach cannot provide is the internal structure of these states, but if this information is somehow available to us (deduced from electron scattering experiments, for example), it can be used as indicated above to determine some of our parameters.

IV. DETAILS OF THE FORMALISM

In this section we present the requisite detail for application of our formalism to the analysis of three-particle final states. In particular, we give the matrix elements of $K^{(0)}$ and Ω in the above basis, and discuss the relation of X to the physical observables. As in the preceding discussion, we restrict ourselves to the absence of spin and isospin for simplicity. We first note that if m_α denotes the mass of particle α, the reduced masses μ_α, M_α are given by

$$\mu_\alpha^{-1} = m_\beta^{-1} + m_\gamma^{-1}, \quad M_\alpha^{-1} = m_\alpha^{-1} + (m_\beta + m_\gamma)^{-1}, \tag{22}$$

where $(\alpha\beta\gamma)$ are permutations of (123).

In what follows we will refer to $q_{\alpha_0}, k_{\alpha_0}$ simply as q_0, k_0; no confusion should arise from this practice since the index α_0 always appears explicitly. It is also convenient to introduce the function[12]

$$G^L_{l\lambda l'\lambda'}(\phi_1,\phi_2,\phi_3)=\frac{4\pi^{3/2}(2\lambda+1)^{1/2}}{2L+1}\sum_{m,m'}C(\lambda lL;0m)C(\lambda' l'L;m-m',m')Y^*_{lm}(\phi_1,0)Y_{l'm'}(\phi_2,0)Y_{\lambda' m-m'}(\phi_3,0), \tag{23}$$

where the sum ranges over the values of m, m' allowed by l, l', respectively. The (total) three-particle t matrix $T^{\alpha\alpha_0;L}_{l\lambda l_0\lambda_0}(q,q_0;W)$ may then be expressed in terms of the channel t matrices via the relation[13]

$$T^{\alpha\alpha_0;L}_{l\lambda l_0\lambda_0}(q,q_0;W)=T^{\alpha}_{Ll\lambda}(q)+\sum_{\beta l'\lambda'}\int_{-1}^{1}dz\,G^L_{l\lambda l'\lambda'}(\theta,\tilde\phi_{\alpha\beta},\tilde\psi_{\alpha\beta})T^{\beta}_{Ll'\lambda'}(\tilde Q_{\alpha\beta}), \tag{24}$$

where $\cos\theta=z$ and $\tilde\phi_{\alpha\beta}, \tilde\psi_{\alpha\beta}, \tilde Q_{\alpha\beta}$ are defined by the expressions below for the on-shell value $p=k_\alpha$:

$$\begin{aligned}
P_{\alpha\beta}&\equiv P_{\alpha\beta}(p,q,z)=\left(\frac{\mu_\beta^2}{m_\gamma^2}\rho^2\mp\frac{2\mu_\beta^2}{m_\gamma M_\alpha}pqz+\frac{\mu_\beta^2}{M_\alpha^2}q^2\right)^{1/2},\\
Q_{\alpha\beta}&=\left(p^2\pm\frac{2\mu_\alpha}{m_\gamma}pqz+\frac{\mu_\alpha^2}{m_\gamma^2}q^2\right)^{1/2},\\
\cos\phi_{\alpha\beta}&=\left(-\frac{\mu_\beta}{m_\gamma}pz\pm\frac{\mu_\beta}{M_\alpha}q\right)\Big/P_{\alpha\beta},\qquad \cos\psi_{\alpha\beta}=\left(\mp pz-\frac{\mu_\alpha}{m_\gamma}q\right)\Big/Q_{\alpha\beta}.
\end{aligned} \tag{25}$$

Here the upper (lower) sign corresponds to $\alpha\beta\gamma$ $(\beta\alpha\gamma)$ cyclic. Thus, for example, $\tilde Q_{\alpha\beta}=Q_{\alpha\beta}(k_\alpha,q,z)$.

In order to illustrate the relation between T and the physical amplitudes, we will assume for simplicity that there is but a single bound state in each two-body channel of definite l corresponding to the binding energy $-\nu_{\alpha l}$. As the energy $s_\alpha=k_\alpha^2/2\mu_\alpha$ of the two-body subsystem tends to $\nu_{\alpha l}$,

$$t_{\alpha l}(k_\alpha)\sim\frac{g_{\alpha l}^2}{s_\alpha-\nu_{\alpha l}}, \tag{26}$$

where $g_{\alpha l}$ is the on-shell value of the form factor associated with the bound state. The three-body t matrix may then be expressed in the form

$$\begin{aligned}
T^{\alpha\alpha_0;L}_{l\lambda l_0\lambda_0}(q,q_0;W)=&\frac{g_{\alpha l}}{s_\alpha-\nu_{\alpha l}}e^{\alpha\alpha_0;L}_{l\lambda l_0\lambda_0}(W)\frac{g_{\alpha_0 l_0}}{s_{\alpha_0}-\nu_{\alpha_0 l_0}}+\frac{g_{\alpha l}}{s_\alpha-\nu_{\alpha l}}c^{\alpha\alpha_0;L}_{l\lambda l_0\lambda_0}(q_0;W)+b^{\alpha\alpha_0;L}_{l\lambda l_0\lambda_0}(q;W)\frac{g_{\alpha_0 l_0}}{s_{\alpha_0}-\nu_{\alpha_0 l_0}}\\
&+f^{\alpha\alpha_0;L}_{l\lambda l_0\lambda_0}(q,q_0;W).
\end{aligned} \tag{27}$$

Here e is the amplitude for scattering from the initial (α_0, l_0) bound state to the final (α, l) bound state (rearrangement amplitude); in the special case $\alpha=\alpha_0$, $l=l_0$ it is the elastic amplitude for scattering of particle α_0 from that bound state. The amplitude c corresponds to the capture of β by γ ($\alpha\beta\gamma$ cyclic) to form the outgoing (α, l) bound state, while b is the amplitude for breakup of the initial (α_0, l_0) bound state to a state of three outgoing particles. Finally, f is the amplitude for scattering from an initial state of three free particles to a final state of three free particles. In the absence of experiments involving three-particle initial states, e and b are the quantities of greatest interest. We note that in Eqs. (24) and (27), q (q_0) takes on either the continuous range of values for which k_α (k_0) are real, or the discrete values corresponding to vanishing of the bound-state denominators. Given the trivial relation between $T^{\alpha}_{Ll\lambda}$ and X stated in Eq. (19), the determination of X by Eq. (20) leads in a straightforward manner to the physical amplitudes via the above relations.

By employing standard methods involving rotation functions analogous to those of Ref. 13, one may easily deduce an expression for the $P^{\mathrm{ext};L}_{l\lambda;l'\lambda'}(x_\alpha,y_\alpha)$ representation of the P^{ext} operator defined in Eq. (14). Taking the limit $x_\alpha\to a_\alpha+$, we obtain

$$\begin{aligned}
P^{\mathrm{ext};L}_{l\lambda;l'\lambda'}(a_\alpha+,y_\alpha)=&\sum_{L'}\frac{(2L'+1)}{4\pi}(-)^{L'-L+l'+\lambda'}[(2\lambda'+1)(2l'+1)]^{1/2}C(L'l'l;00)C(L'\lambda'\lambda;00)\\
&\times W(l\lambda l'\lambda';LL')\theta^{\alpha}_{L'}(y_\alpha),
\end{aligned} \tag{28}$$

where $W(l\lambda l'\lambda'; LL')$ is the Racah coefficient as defined in Ref. 12, and

$$\theta^{\alpha}_{L'}(y_\alpha) = \frac{2\pi}{2L'+1}\left[P_{L'+1}(z) - P_{L'-1}(z)\right]^{z^+_\alpha}_{z^-_\alpha}. \tag{29}$$

Here $P_L(z)$ is the usual Legendre polynomial (with the convention $P_{-1}\equiv 1$), and

$$z^+_\alpha = z^+_\alpha(y_\alpha)$$
$$= \min\left\{1, \left(\frac{\mu_\alpha^2}{m_\epsilon^2}a_\alpha^2 + y_\alpha^2 - a_\sigma^2\right)\Big/\left(\frac{2\mu_\alpha}{m_\epsilon}a_\alpha y_\alpha\right)\right\}, \tag{30}$$
$$z^-_\alpha = z^-_\alpha(y_\alpha)$$
$$= \max\left\{-1, \left(\frac{\mu_\alpha^2}{m_\sigma^2}a_\alpha^2 + y_\alpha^2 - a_\epsilon^2\right)\Big/\left(\frac{-2\mu_\alpha}{m_\sigma}a_\alpha y_\alpha\right)\right\},$$

and $\alpha\sigma\epsilon$ are cyclic.

The value y^0_α discussed in Sec. III corresponds to that value of y_α for which $z^+_\alpha = z^-_\alpha$:

$$y^0_\alpha = \left(\frac{\mu_\alpha}{m_\epsilon}a_\epsilon^2 + \frac{\mu_\alpha}{m_\sigma}a_\sigma^2 - \frac{\mu_\alpha^2}{m_\epsilon m_\sigma}a_\alpha^2\right)^{1/2}, \tag{31}$$

providing the expression in parentheses is positive ($y^0_\alpha = 0$ otherwise). For $y_\alpha < y^0_\alpha$, $P^{\rm ext}$ vanishes identically. Similarly, there is a value $y^{\max}_\alpha$ for which $z^\pm_\alpha = \pm 1$ for $y_\alpha > y^{\max}_\alpha$; it follows that

$$\theta^{\alpha}_{L'}(y_\alpha) = 4\pi\delta_{L'0},$$
$$P^{{\rm ext};L}_{l\lambda;l'\lambda'}(a_\alpha+, y_\alpha) = \delta_{ll'}\delta_{\lambda\lambda'}, \tag{32}$$

for $y_\alpha > y^{\max}_\alpha$. For example, in the special case of equal masses and radii, $y^0_\alpha = \frac{1}{2}\sqrt{3}\,a$, $y^{\max}_\alpha = \frac{3}{2}a$. If we specialize still further to only $l=0$ channels, the relevant quantity is

$$P^{{\rm ext};L}_{0\lambda;0\lambda'}(a, y) = \delta_{\lambda L}\delta_{\lambda' L}\begin{cases} 1, & y \geq \frac{3}{2}a; \\ \dfrac{y^2 - \frac{3}{4}a^2}{ay}, & \frac{1}{2}\sqrt{3}\,a \leq y \leq \frac{3}{2}a; \\ 0, & y \leq \frac{1}{2}\sqrt{3}\,a. \end{cases} \tag{33}$$

For subsequent use below we shall introduce the function

$$Q^{\alpha L}_{l\lambda l'\lambda'}(y_\alpha) = i^{l'+\lambda'-l-\lambda}P^{{\rm ext};L}_{l\lambda;l'\lambda'}(a_\alpha+, y_\alpha), \tag{34}$$

which is also a real polynomial in y_α by virtue of the Clebsch-Gordan coefficients in Eq. (28).

We now consider the operator $K^{(0)}$ required in Eq. (20). We first note that an operator $\bar{Q}$ exists such that

$$\sum_{l''\lambda''}\bar{Q}^{\alpha L}_{l\lambda l''\lambda''}(y_\alpha)Q^{\alpha L}_{l''\lambda''l'\lambda'}(y_\alpha) = \delta_{ll'}\delta_{\lambda\lambda'}\theta(y_\alpha - y^0_\alpha); \tag{35}$$

this follows quite trivially from the nature of $P^{\rm ext}$.[14] It is useful to introduce a length $b_\alpha > y^{\max}_\alpha$, and to represent the matrix elements of $K^{(0)}$ in the form

$$\langle\alpha l\lambda q|K^{(0)}|\beta l'\lambda' q'\rangle = \frac{N^{\alpha\beta;L}_{l\lambda l'\lambda'}(q,q')}{D^{(0)}_{\beta l'}(k'_\beta)},$$
$$N^{\alpha\beta;L}_{l\lambda l'\lambda'}(q,q') = \bar{N}^{\alpha\beta;L}_{l\lambda l'\lambda'}(q,q') + \int_{y^0_\alpha}^{b_\alpha} dy\, y^2 j_\lambda(yq)N^{\alpha\beta;L}_{l\lambda l'\lambda'}(y,q'). \tag{36}$$

The point here is simply that if we took $b_\alpha \to \infty$, the integral term would contribute all of N. However, the numerical evaluation of the integral in that limit would be quite difficult. This problem is avoided by the explicit form for $\bar{N}$ given below.

Although the derivation of $K^{(0)}$ from Eq. (12) and the representation for $\psi^{\alpha;{\rm ext}}$ given in Eq. (10) is straightforward, the mechanics of this procedure are quite tedious and we shall merely quote the result. One finds that $\bar{N}^{\alpha\alpha;L}$ vanishes, while for $\beta\neq\alpha$,

$$\bar{N}^{\alpha\beta;L}_{l\lambda l'\lambda'}(q,q') = -2\mu_\beta\int_{-1}^{1}dz\,\frac{Q_{\alpha\beta}}{R_{\alpha\beta}}\,\frac{N^{(0)}_{\alpha l}(K_{\alpha\beta})f_\lambda(q, b_\alpha, Q_{\alpha\beta})}{q^2 - Q_{\alpha\beta}^2 - i\epsilon} \times G^L_{l\lambda l'\lambda'}(\gamma_{\alpha\beta}, \delta_{\alpha\beta}, \theta), \tag{37}$$

where

$$R_{\alpha\beta} = \left[2\mu_\beta W - \left(\frac{\mu_\beta}{\mu_\alpha} - \frac{\mu_\beta^2}{m_\gamma^2}z^2\right)q'^2\right]^{1/2},$$
$$Q_{\alpha\beta} = \frac{-\mu_\beta}{m_\gamma}zq' + R_{\alpha\beta},$$
$$K_{\alpha\beta} = [2\mu_\alpha(W - Q_{\alpha\beta}^2/2M_\alpha)]^{1/2},$$
$$\cos\gamma_{\alpha\beta} = \mp\left(\frac{\mu_\alpha}{M_\beta}q'z + \frac{\mu_\alpha}{m_\gamma}R_{\alpha\beta}\right)\Big/K_{\alpha\beta}, \tag{38}$$
$$\cos\delta_{\alpha\beta} = \pm R_{\alpha\beta}/k'_\beta,$$
$$\cos\theta = z.$$

Here (and in what follows) square roots are to be taken with the cut along the positive real axis so that $\mathrm{Im}(x)^{1/2} \geq 0$, and the upper (lower) sign is taken for $\alpha\beta\gamma$ ($\beta\alpha\gamma$) cyclic. As in SCI,

$$f_l(p,a,k) = iak[akh_{l+1}(ak)j_l(ap) - h_l(ak)apj_{l+1}(ap)]. \tag{39}$$

We note that for $W<0$, the integrand of Eq. (37) evaluated at z is $(-1)^{l'}$ times the complex conjugate of its value at $-z$; combined with the properties of $D^{(0)}_{\beta l'}(k'_\beta)$ this implies that the $\bar{N}$ contribution to the kernel is real for negative W.[15]

The remainder of the kernel can be evaluated

via the function

$$N_{l\lambda l'}^{\alpha\beta;L}{}_{\lambda'}(y,q') = \sum_{l''\lambda''} \bar{Q}_{l\lambda l''}^{\alpha L}{}_{\lambda''}(y)\hat{N}_{l''}^{\alpha\beta;L}{}_{\lambda''l'\lambda'}(y,q'), \tag{40}$$

where

$$\hat{N}_{l\lambda l'}^{\alpha\alpha;L}{}_{\lambda'}(y,q') = 2i\mu_\alpha a_\alpha k'_\alpha h_{l'}(a_\alpha k'_\alpha)(\lambda_{l'\alpha}^{(0)} - \lambda_{l\alpha}^{(0)})Q_{l\lambda l'}^{\alpha L}{}_{\lambda'}(y)j_{\lambda'}(yq'). \tag{41}$$

For $\beta \neq \alpha$,

$$\hat{N}_{l\lambda l'}^{\alpha\beta;L}{}_{\lambda'}(y,q') = 2ia_\alpha\mu_\beta k'_\beta i^{l'+\lambda'-l-\lambda}\int_{z_\alpha^-}^{z_\alpha^+} dz\left(\frac{\partial}{\partial a_\alpha} - \lambda_{l\alpha}^{(0)}\right)[j_{\lambda'}(Y_{\alpha\beta}q')h_{l'}(X_{\alpha\beta}k'_\beta)G_{l\lambda l'\lambda'}^{L}(\theta,\eta_{\alpha\beta},\xi_{\alpha\beta})], \tag{42}$$

where

$$\begin{aligned} X_{\alpha\beta} &= \left(\frac{\mu_\alpha^2}{m_\gamma^2}a_\alpha^2 \mp \frac{2\mu_\alpha}{m_\gamma}a_\alpha yz + y^2\right)^{1/2}, \quad Y_{\alpha\beta} = \left(\frac{\mu_\beta^2}{M_\alpha^2}a_\alpha^2 \pm \frac{2\mu_\beta^2}{m_\gamma M_\alpha}a_\alpha yz + \frac{\mu_\beta^2}{m_\gamma^2}y^2\right)^{1/2}, \\ \cos\eta_{\alpha\beta} &= \left(\frac{-\mu_\alpha}{m_\gamma}a_\alpha z \pm y\right)\Big/X_{\alpha\beta}, \quad \cos\xi_{\alpha\beta} = \left(\mp\frac{\mu_\beta}{M_\alpha}a_\alpha z - \frac{\mu_\beta}{m_\gamma}y\right)\Big/Y_{\alpha\beta}, \quad \cos\theta = z. \end{aligned} \tag{43}$$

Since the functions in the square brackets are elementary, the indicated differentiation can be performed explicitly. However, the resulting expression is rather unwieldy in comparison with the above.

Finally, we consider the driving term Ω of our integral equation. In analogy to Eqs. (36) and (40) we write

$$\begin{aligned} \langle \alpha l\lambda q|\Omega|\alpha_0 l_0\lambda_0 q_0\rangle &= \bar{\Omega}_{Ll\lambda}^{\alpha}(q) + \int_{y_\alpha^0}^{b_\alpha} dy\, y^2 j_\lambda(yq)\Omega_{Ll\lambda}^{\alpha}(y), \\ \Omega_{Ll\lambda}^{\alpha}(y) &= \sum_{l'\lambda'} \bar{Q}_{l\lambda l'}^{\alpha L}{}_{\lambda'}(y)\hat{\Omega}_{Ll\lambda}^{\alpha}(y). \end{aligned} \tag{44}$$

One then finds that

$$\begin{aligned} \bar{\Omega}_{Ll\lambda}^{\alpha}(q) = &-\delta_{\alpha\alpha_0}\delta_{ll_0}\delta_{\lambda\lambda_0}N_{\alpha l}^{(0)}(k_0)\left[\frac{\delta(q-q_0)}{q^2} - \theta_\lambda(q,q_0;b_\alpha)\right] \\ &- \int_{-1}^{1} dz\, N_{\alpha l}^{(0)}(P_{\alpha_0\alpha})\left[\frac{\delta(q-Q_{\alpha_0\alpha})}{q^2} - \theta_\lambda(q,Q_{\alpha_0\alpha};b_\alpha)\right]G_{l_0\lambda_0 l\lambda}^{L}(\theta,\phi_{\alpha_0\alpha},\psi_{\alpha_0\alpha}), \end{aligned} \tag{45}$$

where $P_{\alpha_0\alpha}$, $Q_{\alpha_0\alpha}$, $\phi_{\alpha_0\alpha}$, $\psi_{\alpha_0\alpha}$ are as in Eq. (25) with $p = k_0$ and $q = q_0$, and $\cos\theta = z$. Here we have also employed the (partial-wave) momentum-space representation of the unit step function $\theta(b_\alpha - y_\alpha)$:

$$\theta_\lambda(q,q';R) = \frac{2R^2}{\pi}\left[\frac{qj_{\lambda+1}(Rq)j_\lambda(Rq') - q'j_{\lambda+1}(Rq')j_\lambda(Rq)}{q^2-q'^2}\right]. \tag{46}$$

The remaining information required is the explicit form of $\hat{\Omega}$, which can be expressed as

$$\begin{aligned} \hat{\Omega}_{Ll\lambda}^{\alpha}(y) = &\frac{-2}{\pi}\delta_{\alpha\alpha_0}Q_{l\lambda l_0\lambda_0}^{\alpha L}(y)j_{\lambda_0}(yq_0)[N_{\alpha l_0}^{(0)}(k_0) + a_\alpha(\lambda_{l\alpha}^{(0)} - \lambda_{l_0\alpha}^{(0)})j_{l_0}(a_\alpha k_0)] \\ &+ \frac{2a_\alpha}{\pi}i^{l_0+\lambda_0-l-\lambda}\int_{z_\alpha^-}^{z_\alpha^+} dz\left(\frac{\partial}{\partial a_\alpha} - \lambda_{l\alpha}^{(0)}\right)[j_{l_0}(k_0X_{\alpha\alpha_0})j_{\lambda_0}(q_0Y_{\alpha\alpha_0})G_{l\lambda l_0\lambda_0}^{L}(\theta,\eta_{\alpha\alpha_0},\xi_{\alpha\alpha_0})]. \end{aligned} \tag{47}$$

Here $X_{\alpha\alpha_0}$, $Y_{\alpha\alpha_0}$, $\eta_{\alpha\alpha_0}$, $\xi_{\alpha\alpha_0}$ are given by Eq. (43), $\cos\theta = z$, and the indicated differentiation can again be performed explicitly.

The above expressions provide sufficient information to construct the kernel K of Eq. (20), and to use its solution (X) in order to calculate actual cross sections. It should be pointed out that the apparent complexity of some of the formulas is a general affliction of the three-body problem, and not a particular consequence of our approach. In essence all of our expressions involve only linear combinations of elementary functions.

V. DISCUSSION

It is generally accepted that reactions such as (d, np) or $(p, 2p)$ are a potential source of new information concerning aspects of the basic nuclear interaction not manifested in np or pp elastic scattering. The prospect of investigating such "off-energy-shell" effects has stimulated a number of exact three-body calculations employing simple models for the nuclear force. These calculations

indicate that, while on-shell effects apparently dominate to the extent that even simple off-shell mechanisms provide reasonable agreement with the bulk of the data, a number of experimental parameters appear sufficiently sensitive to the details to lend encouragement to this general program.[16] However, this particular line of approach has several major drawbacks. In the first place, the computations involved are sufficiently formidable to restrict one in practice to unrealistically simple potential models. As a consequence, such calculations are best suited to tell us the extent to which the data are *not* sensitive to the details of the interaction; repeating the calculation with different potentials can provide only crude insight at best given our limited possibility of choice.

Moreover, even granting that improved numerical techniques may eliminate this difficulty, the nature of this approach is such that the on-shell and off-shell effects are inextricably linked in the parameters characterizing the potential; in order to test some aspect of the off-shell behavior it is necessary to choose a new shape for the potential, refit its parameters to maintain agreement with the two-body data, and then solve the new three-body problem presented. On a more fundamental level, one may question the extent to which a potential description is adequate in characterizing the short-range behavior of *any* strongly interacting system. Related assumptions such as the neglect of relativistic effects and three-body forces may also prejudice the conclusions gleaned from this approach.

On the other hand, the boundary-condition approach described in the preceding sections affords an alternative mode of analysis which is particularly well suited for the extraction of off-shell information from experiment. By separating the on-shell information (two-body phase shifts) from the off-shell behavior characterized by the $\hat{B}, \hat{C}$, input functions, one is in a position to investigate the relative sensitivity of the available phase space to either on- or off-shell effects. A systematic and efficient procedure for this purpose would be to expand $\hat{B}$ and $\hat{C}$ in some reasonable complete set which is separable in the variables q, q'. We could then represent Eq. (20) in the form

$$\begin{aligned} &X = \Omega + KX, \\ &K = K_1 + AK_2, \\ &K_1 = K^{(0)}\rho + \theta - \theta(1-R)\rho, \\ &K_2 = (1-R)\rho, \\ &A = (1-\theta)\hat{B} + \theta\hat{C} \\ &\quad = \sum_{n,m} A_{nm} |\phi_n\rangle\langle\psi_m|. \end{aligned} \tag{48}$$

It follows that

$$X = \left[1 + Z_1 \sum_n |\phi_n\rangle\langle X_n|K_2\right] Z_1\Omega, \tag{49}$$

where $Z_1 = (1-K_1)^{-1}$, and $\langle X_n|$ satisfies

$$\begin{aligned} &\langle X_n| = \langle \tilde{\psi}_n| + \sum_m K_{nm}\langle X_m|, \\ &\langle \tilde{\psi}_n| = \sum_m A_{nm}\langle \psi_m|, \\ &K_{nm} = \langle \tilde{\psi}_n|K_2 Z_1|\phi_m\rangle. \end{aligned} \tag{50}$$

We note that the operators Ω, K_1, K_2 are completely specified by the two-body on-shell parameters, and hence one can construct Z_1 numerically by solving an integral equation with only on-shell input. Given a specific choice for the sets $|\phi_n\rangle, \langle\psi_m|$, one can then tabulate $Z_1|\phi_n\rangle$ and the set of complex numbers $\langle\psi_n|K_2 Z_1|\phi_m\rangle$. With this information any choice of the real numbers A_{nm} specifies a value for X via the solution of the coupled *algebraic* equations represented by Eq. (50). That is, one need solve an integral equation only *once* for each value W of the total energy; analysis of the off-shell parameters then reduces to linear algebra.

By requiring the A_{nm} to provide a detailed fit to the data, it should thus be possible to characterize the off-shell content of the experiment in terms of this set of real numbers; this constitutes the equivalent of a "phase shift" analysis for the three-particle final state. Specific choices for the sets $|\phi_n\rangle, \langle\psi_m|$ will ultimately depend on the experience gained through numerical studies with our formalism, but qualitatively we would expect $\hat{B} \propto e^{-\mu y_\alpha}$ for large y_α, for example, suggesting a choice of the form $\langle q|\phi_n\rangle = (q^2+\mu_n^2)^{-1}$ for $\lambda = 0$, with appropriate generalizations for higher λ.

A more ambitious approach would be to construct models for $\hat{B}$ and $\hat{C}$. The former, for example, would attempt to characterize the behavior of the logarithmic derivative as the third particle is brought in from infinity (but outside the range of forces). Viewing such a program as an alternative to potential theory, one should keep in mind that potential models capable of fitting data are at most weakly connected to an understanding of the underlying dynamics. Such an approach, of course, would be immensely easier to calculate with one-variable instead of two-variable integral equations. Finally, we note that this formulation can be easily extended to take into account relativistic kinematics, and can be generalized in a straightforward fashion to systems of four or more particles. Numerical investigations of the formalism are now under way and will be discussed in subsequent articles.

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[2]In applying this analysis to such a system one of course violates the finite-range assumption. However, neglected terms fall off like $e^{-\mu a}$ and should not be important for $a > \mu^{-1}$.
[3]See, for example, W. W. S. Au and E. L. Lomon, Phys. Lett. **4,** 327 (1963); and other references cited therein.
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[6]For greater detail as to our conventions, see D. D. Brayshaw, Phys. Rev. Lett. **26,** 659 (1971).
[7]J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (Wiley, New York, 1952), p. 790.
[8]This follows as a simple consequence of the coordinate-space formulation of the Faddeev equations; see, for example, Ref. 5.
[9]Explicit formulas for this purpose are provided in Sec. IV.
[10]Here the sum over l' runs over as many partial waves as one believes are necessary to take into account in each channel; the sum over λ' is of course finite for fixed l' and L. Equation (12) is to be interpreted as a separate equation for each distinct set of the relevant indices $(\alpha l \lambda)$.
[11]D. D. Brayshaw, Phys. Rev. D **7,** 1835 (1973); hereafter we shall refer to this paper as SCI.
[12]Here and in what follows we use the notation of M. E. Rose in *Elementary Theory of Angular Momentum* (Wiley, New York, 1957).
[13]The derivation of this and similar expressions to follow involves standard manipulations with rotation functions similar to those employed by A. Ahmadzadeh and J. A. Tjon, Phys. Rev. **139,** B1085 (1965).
[14]In practice one may easily construct $\overline{Q}$ analytically; a general formula is neither necessary nor illuminating.
[15]The $\overline{N}$ portion of our kernel is essentially identical in structure to kernels which arise in the potential formulation of the three-body problem under the assumption of separable interactions. In particular, the denominator $(q^2 - Q_{\alpha\beta}{}^2)$ in Eq. (37) can vanish for $W > 0$, necessitating contour deformation methods of the type developed by J. H. Hetherington and L. H. Schick, Phys. Rev. **137,** B935 (1965); and extended by R. Aaron and R. D. Amado, Phys. Rev. **150,** 857 (1966), to the type of calculations we have in mind.
[16]The current status of this work has been summarized by I. Šlaus in *Few Particle Problems in the Nuclear Interaction* (North-Holland, Amsterdam, 1972); earlier references are cited therein.

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How to Avoid $\Delta Y = 1$ Neutral Currents*

Julian Schwinger
University of California at Los Angeles, Los Angeles, California 90024
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The problem is posed of exhibiting a mechanism that avoids $\Delta Y = 1$ neutral currents without invoking experimentally unknown types of particles. The proposed solution rejects the Cabibbo rotation in favor of a mixing, between two types of unit-spin mesons, that is produced by the SU_3-symmetry-breaking interaction. One quantitative prediction that is well satisfied is the identity of the strong-interaction coupling constants appearing in $\pi^{\pm}$ decay and in $\rho^0 \to e^+ + e^-$.

Unified theories of electromagnetic and weak interactions generally face a problem with hadronic neutral currents that change hypercharge. Such currents are strikingly suppressed in nature, but are usually implied by the Cabibbo rotation that introduces the $\Delta Y = 1$ charged currents. This has led to several suggestions, of varying degrees of charm, which are uniformly couched in the language of hypothetical subnuclear constituents.[1] The number of the latter has thereby been increased, from three, to four, five, seven, ... The phenomenological orientation of source theory[2] invites a more conservative attempt. Can one exhibit a mechanism for avoiding unwanted neutral currents that refers only to experimentally recognized types of particles? This note sketches an affirmative answer.

First we must review the archetypal treatment of the leptons.[3] These particles are grouped into leptonic charge triplets,[4] $L = +1$: μ^+, ν, e^-, and the chiral charge-bearing currents represented by

$$j^{\mu}_{ab} = \tfrac{1}{2}\psi\gamma^0\gamma^{\mu}T_{ab}\psi\,, \quad ab = 12, 21\,. \tag{1}$$

Here we have introduced the antisymmetrical matrices

$$T_{ab} = \tfrac{1}{2}(t_{ab} + i\gamma_5\{t_3, t_{ab}\}), \quad ab = 12, 21 \tag{2}$$

where[5]

$$\sqrt{2}\,t_{12} = t_1 + it_2, \quad \sqrt{2}\,t_{21} = t_1 - it_2, \tag{3}$$

and the t_a, $a = 1, 2, 3$ are the 3×3 imaginary, antisymmetrical matrices of unit isotopic spin. The T matrices obey the commutation relations of the