1973 (unpublished).

[5]See, for instance, J. D. Bjorken, in *Particles and Fields—1972*, proceedings of the 1972 Rochester Meeting of the Division of Particles and Fields of the American Physical Society, edited by A. C. Melissinos and P. F. Slattery (A.I.P., New York, 1972); M. Bander, Phys. Rev. D 6, 164 (1972); R. C. Arnold, Phys. Rev. D 5, 1724 (1972); T. D. Lee, *ibid.* 6, 3617 (1972); K. Wilson, Cornell Report No. CLNS-131, 1970 (unpublished).

[6]C. DeTar, Phys. Rev. D 3, 128 (1971).

[7]G. F. Chew and A. Pignotti, Phys. Rev. 176, 2112 (1968).

[8]We consider only two-Reggeon branch points, e.g., $\beta=\alpha_1+\alpha_2-1$, where α_1 and α_2 are input Regge poles. Since we are leaving out the diffractive component, β_0 is therefore *not* the ***P-P*** cut. Although our present discussion does not require us to commit ourselves to a specific identification of the β_0 cut, it is our feeling that β_0 is a high-lying "effective" cut made out of two vacuum poles. A plausible candidate is the P-P' cut around $\frac{1}{2}$. When the β_0 cut appears at the end of the MP chain, it gives rise to a "leading" particle effect *within* the SRC component of the single-particle distribution. This is not to be confused with the diffraction-dissociation effect where the P-P cut is at work.

[9]See Appendix A.

[10]The choice of signs for $G_+^{(a)}$ and $G_-^{(a)}$ is in general arbitrary since they always appear in the total cross section as products, i.e., $G_+^{(a)}G_+^{(b)}$ and $G_-^{(a)}G_-^{(b)}$. Normally, we choose both $G_+^{(a)}$ and $G_-^{(a)}$ to be positive, so that the sign of the α_P-α_M term in the inclusive distribution depends on the sign of g. However, our convention is fixed by choosing U to be a proper orthogonal matrix. In this case, $G_+^{(a)}$ and g are found to be always positive, and $G_-^{(a)}$ now can take on either sign. Regardless of the convention used, the sign of $g(G_-^{(a)}/G_+^{(a)})$ is always determined by the dynamics of the model, and so is the relation between (2.28) and (2.29).

[11]See, for instance, W. Feller, *An Introduction to Probability Theory and Its Applications*. 2nd edition (Wiley, New York, 1957), Vol. 1.

[12]Equation (4.1) yields the *asymptote* of the average multiplicity $\langle n\rangle$ as a function of Y. The constant term in the asymptotic behavior, $aY+b$, normally depends on the inclusive distribution in the fragmentation region. In our picture, the leading-particle effect always contributes two particles to $\langle n\rangle$, so that the rest of b comes from the approach to scaling in the central region. Since $\langle n\rangle=2$ at $Y=0$, the sign of $(b-2)$ therefore determines the "curvature" of $\langle n\rangle$ as it approaches its asymptote. (See Fig. 2.)

[13]This is not to be confused with the threshold effect due to the minimum rapidity gap, which can occur even at the one-channel CP-model level. The latter type may be relevant for the understanding of the $\bar{p}$ and K spectra where particles with masses much larger than that of the pion are produced. Whereas both types of effects are present in nature, our present discussion is probably more relevant for pion production. Effects of the minimum rapidity gap are also discussed in Appendix A.

[14]That is, the contribution coming from the particle emitted at the end of the MP chain. However, under the CP approximation, this particle has a fixed rapidity.

[15]G. F. Chew and D. Snider, Phys. Lett. 31B, 75 (1970).

[16]M. L. Goldberger, D. Silverman, and C.-I Tan, Phys. Rev. Lett. 26, 100 (1971).

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Two-Photon Cross Section for W-Pair Production by Colliding Beams*

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We calculate the total cross section for $e^+e^-\rightarrow e^+e^-W^+W^-$, a process which proceeds via two virtual photons. If the intermediate boson ($W^\pm$) has no anomalous magnetic moment and pointlike vertices, this process can yield a larger cross section than the one-photon process $e^+e^-\rightarrow W^+W^-$ at sufficiently high energies. Otherwise, the one-photon mechanism is dominant. Numerical results for several values of m_W and the magnetic moment are presented. The effect of Z exchange in a Weinberg-type model is shown to be negligible in these results.

I. INTRODUCTION

In this note we consider the problem of colliding-electron-beam production of intermediate-boson ($W^\pm$) pairs if such bosons exist. This is an example of a fundamental process which can be studied for the first time by colliding-beam machines with high energies and luminosities that now exist or are under construction.[1] The lowest-order process which proceeds via annihilation into one virtual photon ($e^+e^-\rightarrow W^+W^-$) has been well studied.[2] Here we examine the higher-order mechanism

($ee \to \gamma^* \gamma^* ee \to eeW^+W^-$) which employs two virtual photons. Several groups[3-5] have studied such two-photon processes in other cases (e.g., pion-pair production) and found that their cross sections exceeded the one-photon processes at reasonable colliding-beam energies (at $E \sim 1$ GeV for pions). It is reasonable to ask whether such a circumstance of a large ratio of the two-photon to the one-photon process also happens in the case of vector bosons. Our result is that it can if the W boson has no anomalous magnetic moment, but not for nonzero moment. The W boson is assumed to have pointlike form factors. The effects of the Weinberg theory[6] of weak and electromagnetic interactions on this process are shown to be negligible. In Sec. II the two-photon cross section is calculated, and the results are displayed numerically for different values of m_W and the anomalous magnetic moment and are compared to the one-photon process. The modifications to the calculation due to the Weinberg theory are discussed in Sec. III.

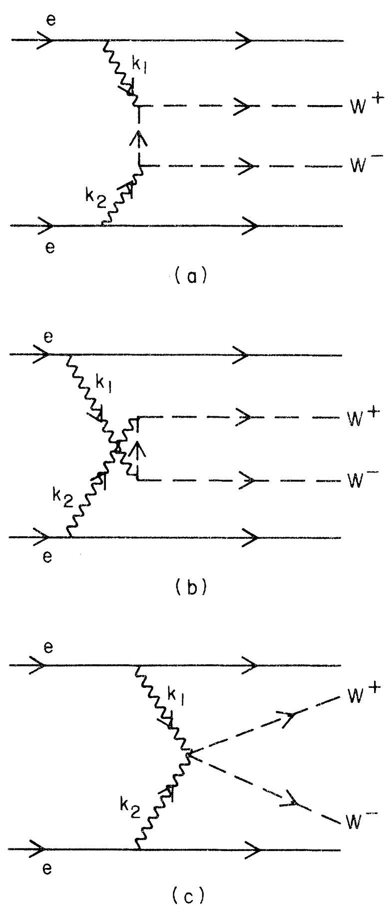

FIG. 1. Diagrams for the two-photon process $ee \to eeW^+W^-$: (a) direct, (b) crossed, (c) seagull.

II. TWO-PHOTON CROSS SECTION FOR W-PAIR PRODUCTION

In Fig. 1 are shown the diagrams which contribute to $ee \to eeW^+W^-$ to order α^4 in the cross section. One contrasts them with the lowest-order diagrams for $e^+e^- \to W^+W^-$, shown in Fig. 2, which proceeds via one virtual photon. Very simple considerations reveal the striking difference between these two processes. In Fig. 2 the photon has $k^2 = 4E^2 \geq 4m_W{}^2 > 0$, which is large and timelike; however, in Fig. 1 for either photon

$$-k^2 \cong 2EE'(1-\cos\theta') + m_e{}^2(E-E')^2/EE',$$

where θ' is the angle between the initial and final electrons, and E (E') is the initial (final) electron energy in the lab. If the electrons are detected very close to the forward direction, $\cos\theta' \cong 1$ and $k^2 \cong -m_e{}^2$, which is small and spacelike. In fact, the photons are essentially real ($k^2 \cong 0$) and one can consider the calculation of these diagrams in two parts: (a) Study the spectrum of "almost real" photons emitted by the electrons; (b) calculate the process $\gamma\gamma \to W^+W^-$. This general approach, clearly the Weizsäcker-Williams[7] approximation in the context of relativistic quantum theory, has been investigated by many people.[3-5] In particular, Brodsky, Kinoshita, and Terazawa[3] have given an exhaustive discussion of the general two-photon process $ee \to ee\gamma^*\gamma^* \to eeX$, and have compared exact calculations with calculations in this "equivalent photon" approximation (e.g., for $ee \to ee\pi^0$). They have shown that the approximation is a very good one (erring by the order of 10%) and becomes better for a more massive final state X. This is reasonable since the equivalent-photon approach is, roughly speaking, an expansion in $k^2/m_X{}^2 \cong m_e{}^2/m_X{}^2$.

Here we simply draw upon the general results of Brodsky *et al.* applied to the situation at hand. Their central result is that

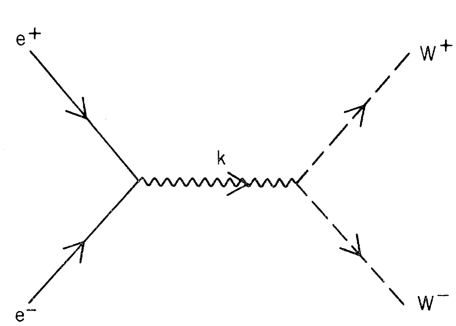

FIG. 2. Diagram for the one-photon process $e^+e^- \to W^+W^-$.

$$\sigma_{ee\to eeX}(E)\cong 2\left(\frac{\alpha}{\pi}\right)^2[\ln(E/m_e)]^2 \times\int_{s_{\rm th}}^{4E^2}\frac{ds}{s}\sigma_{\gamma\gamma\to X}(s)f\left(\frac{\sqrt{s}}{2E}\right), \quad (1)$$

where $s_{\rm th}$ is the threshold value of s and

$$f(x)=(2+x^2)^2\ln(1/x)-(1-x^2)(3+x^2)\ . \quad (2)$$

Our problem, then, is to determine $\sigma_{\gamma\gamma\to W^+W^-}(s)$. We assume that the W boson obeys the standard quantum electrodynamics of massive vector bosons as given by Lee and Yang[8] and has no strong interactions (i.e., pointlike form factors). The W boson has a magnetic moment $\mathfrak{M}=1+\kappa$ in units of $e/2m_W$; the quadrupole moment is here not arbitrary but given by $Q=-e\kappa/m_W^2$. Figure 3 gives the relevant diagrams and notation.[9] Note that with the bosons of momenta k_1, k_2, q_1, q_2 are associated polarization vectors $\epsilon_1^\mu, \epsilon_2^\nu, \eta_1^\alpha, \eta_2^\beta$, respectively. The photon-photon center-of-mass frame is chosen for convenience; as usual, $s=(k_1+k_2)^2=(q_1+q_2)^2$, and θ is the angle between $\vec{q}_1$ and $\vec{k}_1$ in this frame. E is the colliding-beam energy ($p_1^0=p_2^0=E$) in the lab frame.

The S-matrix element is then

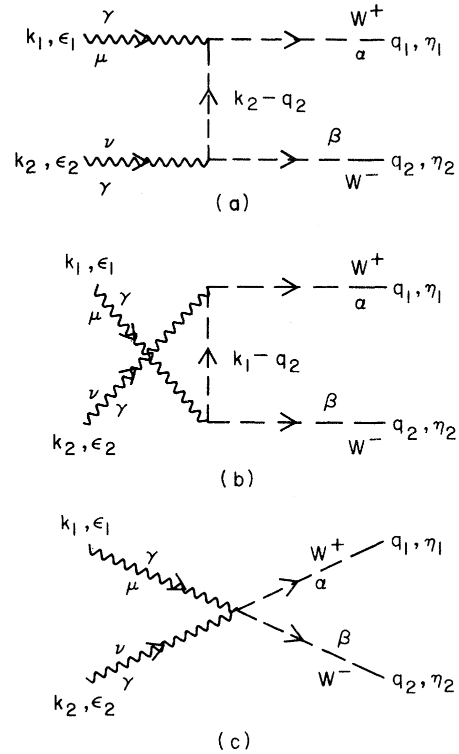

FIG. 3. Diagrams for the process $\gamma\gamma\to W^+W^-$: (a) direct, (b) crossed, (c) seagull.

$$S_{fi}=\frac{-ie^2}{(2k_1^0 2k_2^0 2q_1^0 2q_2^0)^{1/2}}(2\pi)^4\delta^{(4)}(q_1+q_2-k_1-k_2)\epsilon_1^\mu\epsilon_2^\nu\eta_1^\alpha\eta_2^\beta M_{\mu\nu\alpha\beta}, \quad (3)$$

where

$$M_{\mu\nu\alpha\beta}=M^d_{\mu\nu\alpha\beta}+M^c_{\mu\nu\alpha\beta}+M^{sg}_{\mu\nu\alpha\beta}$$

and

$$M^d_{\mu\nu\alpha\beta}=\{g_{\alpha\rho}(q_1+k_2-q_2)_\mu-g_{\alpha\mu}[q_1\mathfrak{M}-(\mathfrak{M}-1)(k_2-q_2)]_\rho-g_{\rho\mu}[(k_2-q_2)\mathfrak{M}-(\mathfrak{M}-1)q_1]_\alpha\}$$
$$\times\frac{-g^{\rho\sigma}+(k_2-q_2)^\rho(k_2-q_2)^\sigma/m_W^2}{(k_2-q_2)^2-m_W^2}\{g_{\beta\sigma}(k_2-2q_2)_\nu-g_{\sigma\nu}[(k_2-q_2)\mathfrak{M}+(\mathfrak{M}-1)q_2]_\beta$$
$$+g_{\beta\nu}[q_2\mathfrak{M}+(\mathfrak{M}-1)(k_2-q_2)]_\sigma\} \quad (4)$$

is the direct term [Fig. 3(a)]. Also,

$$M^c_{\mu\nu\alpha\beta}=M^d_{\mu\nu\alpha\beta}(q_1\leftrightarrow q_2,\ \alpha\leftrightarrow\beta)$$

is the crossed term [Fig. 3(b)] and

$$M^{sg}_{\mu\nu\alpha\beta}=2g_{\mu\nu}g_{\alpha\beta}-g_{\mu\alpha}g_{\nu\beta}-g_{\mu\beta}g_{\nu\alpha}$$

is the seagull contribution [Fig. 3(c)] required by Bose statistics and gauge invariance, $k_1^\mu M_{\mu\nu\alpha\beta}=k_2^\nu M_{\mu\nu\alpha\beta}=0$. This requirement is explicitly satisfied by the above tensor.

Proceeding to the cross section in the standard way, one gets

$$\sigma_{\gamma\gamma\to W^+W^-}(s)=\frac{\pi\alpha^2}{2s}\left(1-\frac{4m_W^2}{s}\right)^{1/2}\int_{-1}^{1}d\cos\theta\,|\overline{\mathfrak{M}}_{fi}|^2, \quad (5)$$

where

$$|\overline{\mathfrak{M}}_{fi}|^2=\frac{1}{4}\sum_{\rm spins}\epsilon_1^\mu\epsilon_2^\nu\eta_1^\alpha\eta_2^\beta\epsilon_1^{\mu'}\epsilon_2^{\nu'}\eta_1^{\alpha'}\eta_2^{\beta'}\times M_{\mu\nu\alpha\beta}M_{\mu'\nu'\alpha'\beta'} \quad (6)$$

has been summed over final spins and averaged over initial spins. This can be expressed as

$$|\mathfrak{M}_{fi}|^2=\tfrac{1}{4}(g^{\alpha\alpha'}-q_1^\alpha q_1^{\alpha'}/m_W^2)(g^{\beta\beta'}-q_2^\beta q_2^{\beta'}/m_W^2)\times(\tilde{M}^{\mu\nu}_{\alpha\beta}\tilde{M}_{\mu\nu\alpha'\beta'}-8g_{\alpha\beta}g_{\alpha'\beta'}) , \quad (7)$$

where $\tilde{M}$ denotes that the supplementary conditions $q_1\cdot\eta_1=q_2\cdot\eta_2=0$ have been used to set terms proportional to q_1^α or q_2^β to zero, and also, for the photons, $k_1\cdot\epsilon_1=k_2\cdot\epsilon_2=0$ have been used to set terms with k_1^μ and k_2^ν to zero. There is a subtlety here responsible for the extra term, which is explained further in the Appendix. Essentially, the condition $k\cdot\epsilon=0$ cannot be naively applied to both photons in a two-photon process without some care being taken to obtain a correct and truly gauge-invariant result.

To obtain an explicit expression for $|\mathfrak{M}_{fi}|^2$, the algebraic computer program REDUCE by Hearn[10] was used. The necessity of this is evident from the complexity of Eq. (4). The reader is referred to the work of Kim and Tsai[5] for the complete expression for $\sigma_{\gamma\gamma\to W^+W^-}$. Here it suffices to note that

$$\sigma_{\gamma\gamma\to W^+W^-}\sim\frac{\alpha^2 s}{m_W^2}\quad(\kappa\neq1) ,$$
$$\sigma_{\gamma\gamma\to W^+W^-}\sim\frac{\alpha^2}{m_W^2}\quad(\kappa=1) \quad (8)$$

to illustrate the behavior of the cross section. The full result will be used in the numerical integration required to derive the accurate cross sections.

Now the expression for $\sigma_{\gamma\gamma\to W^+W^-}(s)$, Eq. (5), can be put into Eq. (1) to give the complete expression for the cross section of $ee\to eeW^+W^-$. We would like this answer in the lab frame for convenience. This is not difficult, since $\sigma_{\gamma\gamma\to W^+W^-}(s)$, even though calculated in the photon-photon center-of-mass frame, is a function only of s and is therefore Lorentz-invariant. Thus

$$\sigma_{ee\to eeW^+W^-}(E)\cong\frac{\alpha^4}{\pi}[\ln(E/m_e)]^2\times\int_{4m_W^2}^{4E^2}\frac{ds}{s^2}f\left(\frac{\sqrt{s}}{2E}\right)\left(1-\frac{4m_W^2}{s}\right)^{1/2}\times\int_{-1}^{1}d\cos\theta\,|\mathfrak{M}_{fi}|^2 , \quad (9)$$

where $f(x)$ was given in Eq. (2). This is the desired result. To investigate its behavior, it is necessary to integrate out the complicated expression for $|\mathfrak{M}_{fi}|^2$ obtained earlier. However, to get a qualitative idea, we use the leading dependence given in Eq. (8). This yields ($\kappa\neq1$)

$$\sigma_{T2\gamma}(E)\sim\alpha^4[\ln(E/m_e)]^2\frac{E^2}{m_W^4}I(m_W/E)+\text{terms of lower order in } E, \quad (10)$$

where $\sigma_{T2\gamma}$ is the total cross section for this two-photon process and where

$$I(y)=\int_y^1 x(1-y^2/x^2)^{1/2}f(x)\,dx . \quad (11)$$

If $E\gg m_W$, $I(m_W/E)\cong I(0)\cong0.4$. Note that, except for a factor of $[\ln(E/m_e)]^2/m_W^2$, this is a function only of E/m_W. This should be compared with the one-photon prediction of

$$\sigma_{ee\to W^+W^-}(E)=(\pi\alpha^2\beta^3/3\gamma^2m_W^2)\times[\gamma^4\kappa^2+(\kappa^2+3\kappa+1)\gamma^2+\tfrac{3}{4}] \quad (12)$$

as given by Tsai and Hearn,[2] where $\gamma\equiv E/m_W$, $\beta\equiv(1-\gamma^{-2})^{1/2}$. Note that, if $\kappa=0$, the total cross section for this one-photon process $\sigma_{T1\gamma}\sim\pi\alpha^2/3m_W^2$, a constant at high energy, whereas if $\kappa\neq0$, $\sigma_{T1\gamma}\sim\pi\alpha^2E^2\kappa^2/3m_W^4$. From Eq. (10) it seems likely that $\sigma_{T2\gamma}$ would have no chance to overtake $\sigma_{T1\gamma}$ unless $\kappa=0$. This surmise must be examined quantitatively, of course. The integrations were done using the multidimensional Monte Carlo integration routine SHEP by G. C. Sheppey. The results are shown in Figs. 4–6 for various values of m_W and κ. Note that, if $\kappa=0$ and $m_W\sim2$ GeV, then $\sigma_{T2\gamma}$ overtakes $\sigma_{T1\gamma}$ at $E\sim30$ GeV and is a significant fraction at lower energies. But for nonzero κ, $\sigma_{T2\gamma}$ is always a couple of orders of magnitude lower than $\sigma_{T1\gamma}$.

III. EFFECT OF WEINBERG THEORY

In Fig. 7 are shown some of the diagrams contributed in lowest order, in addition to those of Fig. 1, by the Weinberg theory.[6] This is a gauge theory with spontaneously broken symmetries which renders the weak interactions finite and unifies them with electromagnetic interactions, but at the price of additional massive neutral vector (Z) and scalar (ϕ) fields. Here only the Z bosons contribute since the coupling of ϕ to the leptons is proportional to the lepton mass. A detailed calculation of the cross section is not our purpose here, but simply to show that these corrections are negligible.

The point can be made with any part of the Weinberg corrections, say, Fig. 7(a), which contributes a term proportional to

$$\tfrac{1}{2}e^2g^2[\bar{u}(p_1')\gamma_\mu u(p_1)][\bar{u}(p_2')\gamma_\nu(dP_-+cP_+)u(p_2)]\times\frac{g^{\mu\mu'}}{k_1^2}\frac{g^{\nu\nu'}-k_2^\nu k_2^{\nu'}/m_Z^2}{k_2^2-m_Z^2}M_{\mu'\nu'\alpha\beta}\eta_1^\alpha\eta_2^\beta , \quad (13)$$

where $g^2/8m_W=G/\sqrt{2}$, $P_\pm=\frac{1}{2}(1\pm\gamma_5)$, and c,d are

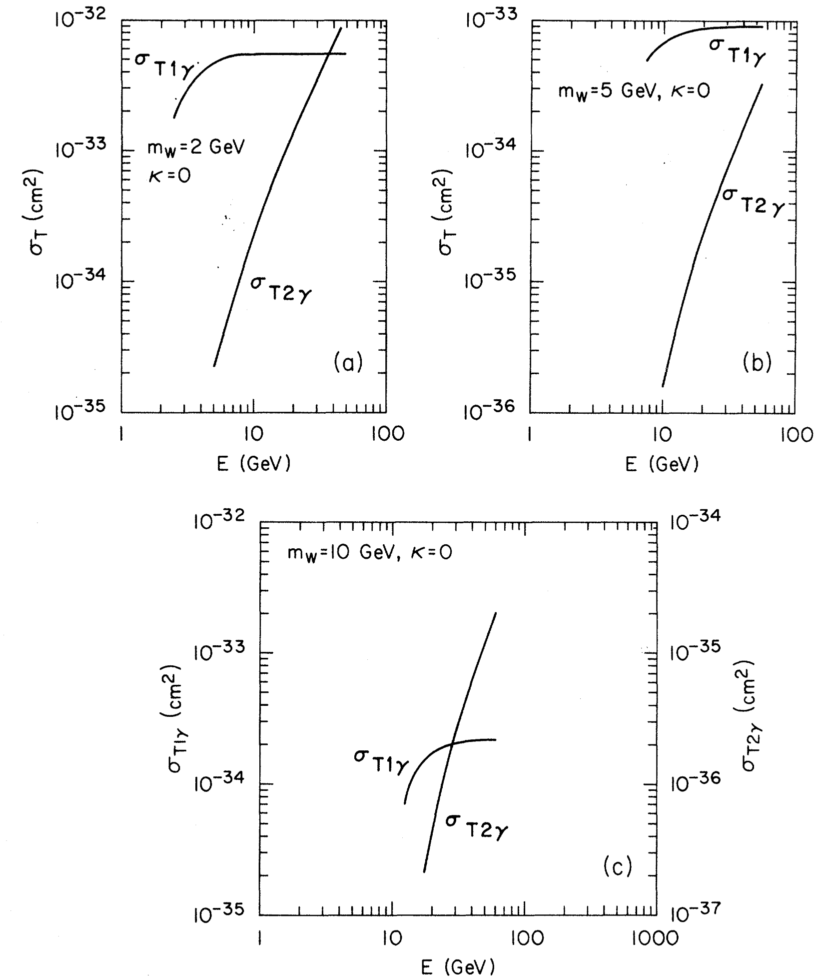

FIG. 4. Total cross sections for $\sigma_{T1\gamma}$ and $\sigma_{T2\gamma}$ when $\kappa=0$: (a) $m_W=2$ GeV, (b) $m_W=5$ GeV, (c) $m_W=10$ GeV. Note: In (c) the scale on the left refers to $\sigma_{T1\gamma}$ and the scale on the right refers to $\sigma_{T2\gamma}$.

constants $[c=2(1-R),\ d=1-2R;\ R=m_W^2/m_Z^2]$. Evidently, if $k_2^2/m_Z^2 \ll 1$, this term is of order $1/m_Z^2$ compared to purely electromagnetic ones. A simple estimate of the ratio of differential cross sections of scattered electrons forward versus at large angle shows that nearly all events involve electron scattering forward and so $|k_2^2| \cong m_e^2$; also $m_Z > m_W \sim 40$ GeV in the Weinberg theory. Thus these additional terms do not contribute significantly.

The tensor $M_{\mu\nu\alpha\beta}$ has the same form for ZW^+W^- and γW^+W^- vertices, except in the Weinberg theory κ is constrained to be unity, that is, a Yang-Mills[11] type vertex. We therefore expect the two-photon process to be small compared to the one-photon process in any gauge theory. It should be noted that, at infinitely high energies, the additional diagrams will probably prevent the cross section from violating the unitarity bound since the ZW^+W^- and γW^+W^- vertices have opposite signs. In the one-photon case, Weinberg showed[6] that $\sigma_{T1\gamma} \propto 1/E^2$ eventually. To resolve this question in the present case will not be essayed here.

IV. CONCLUSION

In the experimental quest for the elusive intermediate boson the two-photon process here discussed may not be without significance. For the special case of $\kappa=0$, we have noted its role. In this case, a luminosity of 10^{32} cm^{-2} sec^{-1} would give ~10^{-1} counts sec^{-1} of these events at $E\sim 20$ GeV and $m_W=2$ GeV. Admittedly, for a particle with such a large mass (if, that is, it exists) not to have an anomalous moment is hard to believe. Indeed, as Weinberg[6] and Kim and Tsai[12] have remarked, $\kappa=1$ would assure that the W Compton scattering would satisfy a Drell-Hearn[13] sum rule.

This leads to an interesting point. If the W indeed does have strong interactions and is described by form factors that decrease rapidly with q^2, the two-photon process might indeed dominate the one-photon by virtue of its soft photons. On the other

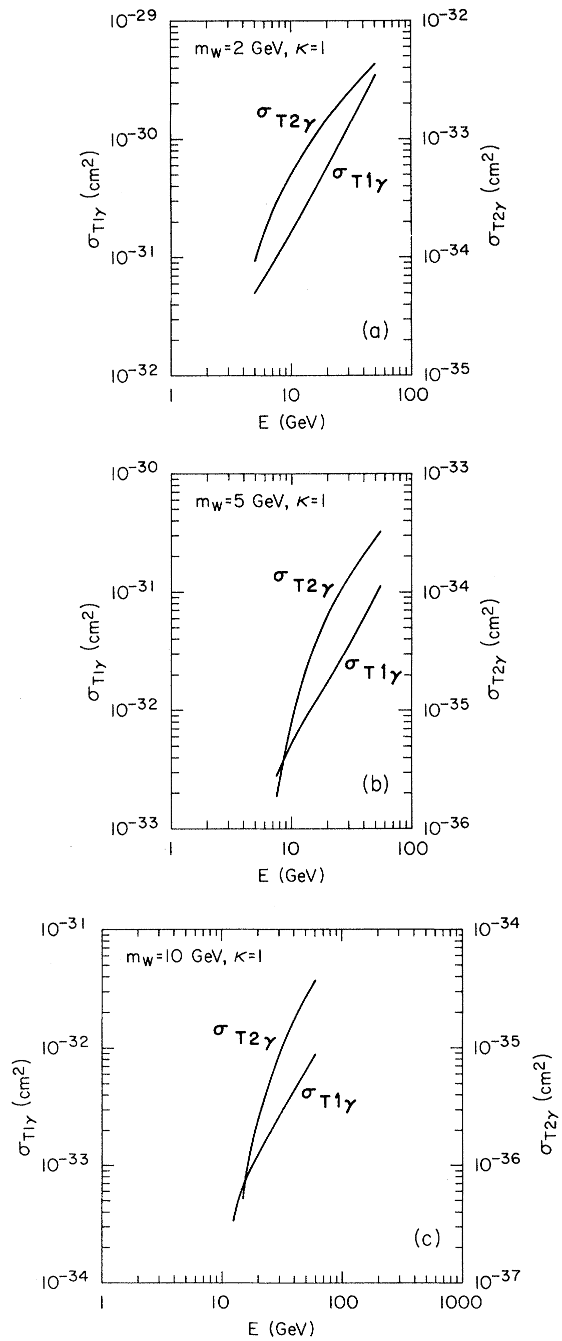

FIG. 5. Total cross sections for $\sigma_{T1\gamma}$ and $\sigma_{T2\gamma}$ when $\kappa=1$: (a) $m_W=2$ GeV, (b) $m_W=5$ GeV, (c) $m_W=10$ GeV. Note: The scale on the left refers to $\sigma_{T1\gamma}$ and the scale on the right refers to $\sigma_{T2\gamma}$.

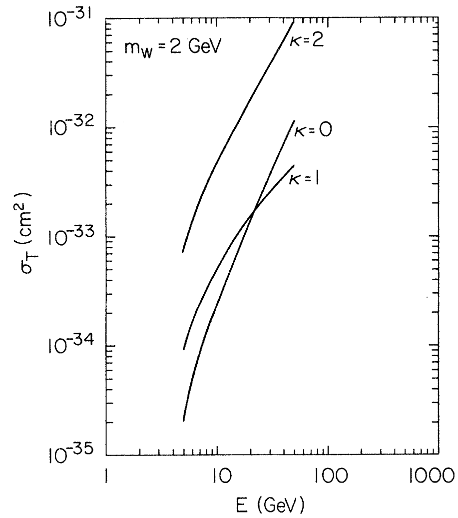

FIG. 6. Total cross sections for $\sigma_{T2\gamma}$ when $m_W=2$ GeV and $\kappa=0,1,2$.

hand, one would expect such W's to be, for instance, electroproduced off protons; Kogut[14] has shown that one could probe in this way up to m_W ~5 GeV at SLAC energies. If these W's are more massive still, the two-photon process would play a useful role in searching for them and in setting limits on their mass and strong interactions.

ACKNOWLEDGMENTS

I would like to thank Professor Sidney D. Drell for suggesting this investigation and for much valuable advice and encouragement. I thank also Dr. Stanley J. Brodsky for many helpful discussions.

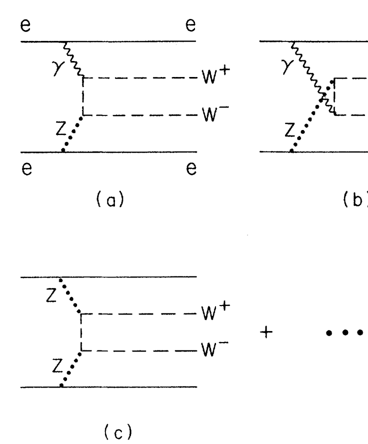

FIG. 7. Some of the diagrams contributed to the two-photon process by the Weinberg theory.

APPENDIX

Consider, for simplicity, the case of the production of a pair of spin-0 bosons.[15] The amplitude is proportional to $\epsilon_1^\mu \epsilon_2^\nu M_{\mu\nu}$, where

$$M_{\mu\nu} = -\frac{(2q_1 - k_1)_\mu (k_2 - 2q_2)_\nu}{2k_2 \cdot q_2} - \frac{(2q_1 - k_2)_\nu (k_1 - 2q_2)_\mu}{2k_1 \cdot q_2} - 2g_{\mu\nu}, \tag{A1}$$

which satisfies $k_1^\mu M_{\mu\nu} = k_2^\nu M_{\mu\nu} = 0$ explicitly. Now apply the subsidiary conditions $k_1 \cdot \epsilon_1 = k_2 \cdot \epsilon_2 = 0$; then (dropping an over-all factor of 2)

$$\tilde{M}_{\mu\nu} = \frac{q_{1\mu} q_{2\nu}}{k_2 \cdot q_2} + \frac{q_{1\nu} q_{2\mu}}{k_1 \cdot q_2} - g_{\mu\nu} \tag{A2}$$

and

$$M_{\mu\nu} = \tilde{M}_{\mu\nu} - \tfrac{1}{2} k_{2\nu} \left(\frac{q_{1\mu}}{k_2 \cdot q_2} + \frac{q_{2\mu}}{k_1 \cdot q_2} \right) - \tfrac{1}{2} k_{1\mu} \left(\frac{q_{1\nu}}{k_1 \cdot q_2} + \frac{q_{2\nu}}{k_2 \cdot q_2} \right) + \tfrac{1}{4} k_{1\mu} k_{2\nu} \left(\frac{1}{k_2 \cdot q_2} + \frac{1}{k_1 \cdot q_2} \right) \tag{A3}$$

so that

$$k_1^\mu \tilde{M}_{\mu\nu} = k_{2\nu} , \quad k_2^\nu \tilde{M}_{\mu\nu} = k_{1\mu} , \tag{A4}$$

which is, by itself, "pseudo-gauge-invariant" even though $\epsilon_1^\mu k_2^\nu \tilde{M}_{\mu\nu} = \epsilon_2^\nu k_1^\mu \tilde{M}_{\mu\nu} = 0$. Now $M_{\mu\nu} M^{\mu\nu} = \tilde{M}_{\mu\nu} \tilde{M}^{\mu\nu} - 2$, so dropping *both* $k_{1\mu}$ and $k_{2\nu}$ (i.e., using *both* subsidiary conditions) results in an error in the cross section. It is easily verified that setting *either* $k_{1\mu}$ *or* $k_{2\nu}$ to zero will give the correct answer.

Alternatively, use the full expression for the summation

$$\sum_{\text{spins}} \epsilon_1^\mu \epsilon_1^{\mu'} \epsilon_2^\nu \epsilon_2^{\nu'} M_{\mu\nu} M_{\mu'\nu'} = \left(-g^{\mu\mu'} - \frac{k_1^\mu k_1^{\mu'}}{(k_1 \cdot \eta)^2} + \frac{k_1^\mu \eta^{\mu'} + k_1^{\mu'} \eta^\mu}{k_1 \cdot \eta} \right) \left(-g^{\nu\nu'} - \frac{k_2^\nu k_2^{\nu'}}{(k_2 \cdot \eta)^2} + \frac{k_2^\nu \eta^{\nu'} + k_2^{\nu'} \eta^\nu}{k_2 \cdot \eta} \right) M_{\mu\nu} M_{\mu'\nu'}$$

$$= M_{\mu\nu} M^{\mu\nu}, \tag{A5}$$

where $\eta = (1, 0, 0, 0)$ is a unit timelike vector. On the right-hand side, tedious calculation verifies that

$$M^{\mu\nu} M_{\mu\nu} = \left(-g^{\mu\mu'} - \frac{k_1^\mu k_1^{\mu'}}{(k_1 \cdot \eta)^2} + \frac{k_1^\mu \eta^{\mu'} + k_1^{\mu'} \eta^\mu}{k_1 \cdot \eta} \right) \left(-g^{\nu\nu'} - \frac{k_2^\nu k_2^{\nu'}}{(k_2 \cdot \eta)^2} + \frac{k_2^\nu \eta^{\nu'} + k_2^{\nu'} \eta^\nu}{k_2 \cdot \eta} \right) \tilde{M}_{\mu\nu} \tilde{M}_{\mu'\nu'} \tag{A6}$$

so that the gauge terms in $\sum \epsilon^\mu \epsilon^{\mu'}$ "compensate" for the "gauge terms" omitted in $\tilde{M}$. It is precisely the conservation of the current that enforces this. No such problem exists for the massive vector field since the gauge freedom has been removed. Precisely analogous results to the above are found for $M_{\mu\nu\alpha\beta}$, where now $k_1^\mu \tilde{M}_{\mu\nu\alpha\beta} = -2k_{2\nu} g_{\alpha\beta}$ and $k_2^\nu \tilde{M}_{\mu\nu\alpha\beta} = -2k_{1\mu} g_{\alpha\beta}$ are the analogs of the "pseudo-gauge-invariance" statements in Eq. (A4). Then follows

$$\sum_{\text{spins}} \epsilon_1^\mu \epsilon_1^{\mu'} \epsilon_2^\nu \epsilon_2^{\nu'} M_{\mu\nu\alpha\beta} M_{\mu'\nu'\alpha'\beta'} = \tilde{M}^{\mu\nu}_{\alpha\beta} \tilde{M}_{\mu\nu\alpha'\beta'} - 8 g_{\alpha\beta} g_{\alpha'\beta'} \ .$$

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Boundary-Condition Approach to Three-Particle Final States*

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It is shown that the wave function for a three-particle system outside the range of forces may be uniquely determined by imposing a suitable set of boundary conditions. This result is expressed in terms of a one-variable integral equation with a square-integrable kernel, the solutions of which specify the three-body t matrix. The input to this equation consists of the two-particle phase shifts and two independent real-valued functions which characterize the three-body wave function in specific regions. The formalism yields an exactly unitary three-particle t matrix for arbitrary values of this input, and thus provides a practical scheme for the analysis of three-body final states.

I. INTRODUCTION

Some time ago, Feshbach and Lomon[1] demonstrated the power of the boundary-condition approach as a means of correlating a broad spectrum of N-N scattering data. This approach is based on the well-known fact that, for interactions of finite range r_0, the wave function takes on a particularly simple form at interparticle distances $r > r_0$, and may be completely characterized by stating the coefficient of the outgoing wave. This coefficient is uniquely determined in each partial wave if one specifies a value for the logarithmic derivative of ψ_l at $r = a \geq r_0$; since this value must be energy-dependent, imposing such a condition is merely an alternative to the usual description in terms of scattering phase shifts. The power of this approach lies in the empirical fact that, for the N-N system,[2] the logarithmic-derivative parameters are at most weakly dependent on the energy, and hence a few parameters are adequate to describe the scattering in the range 0–300 MeV.[1] A corresponding statement may be made for other systems of strongly interacting particles.[3] In comparison to a potential description, the computational advantages of this approach are obvious; one replaces a one-dimensional integral equation (e.g., the Lippmann-Schwinger equation[4]) by quadrature.

An analogous simplification of the three-body wave function occurs in the *exterior region*, defined by the requirement that no pair of particles is within the range of forces. Therefore, one might hope that a suitably generalized boundary condition on the exterior wave function would uniquely specify the outgoing component (i.e., determine the three-particle t matrix), resulting in a highly efficient description of three-particle final states with comparable computational advantages. Below we propose a set of boundary conditions for this purpose which determine the three-body t matrix via the solution of a one-variable integral equation. The input for this equation is cleanly separated into two-particle phase shifts and real-valued functions characterizing the three-body wave function in distinct physical regions. For any arbitrary selection of this input the formalism produces an exactly unitary three-particle t matrix. This is to be compared with an earlier approach with essentially the same motivation by Noyes.[5] In Noyes's work a one-variable equation was derived with a kernel specified in terms of the half-on-shell two-body t matrix, and a driving term involving an arbitrary expansion of the *interior* wave function. The difficulty with this formulation is that the expansion coefficients are intimately connected to the two-body phase shifts via the unitarity relation, and hence are not truly independent; selecting them arbitrarily will in general violate unitarity. In order to achieve an effective "phase shift" analysis of three-particle final states, one must require real and independent parameters; our approach satisfies this condition.

We begin in Sec. II with a brief review of the boundary-condition approach to two-particle systems, and introduce a new statement of the bound-