Crossing Symmetry and Low-Energy Pion-Pion Dynamics*

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Direct use of crossing symmetry is made to construct the s- and p-wave amplitudes in the region $0 \leq s \leq 4$ from the absorptive parts. The amplitudes constructed in this way are unique except for a single parameter, which itself is determined by unitarity; the resulting s-wave amplitudes then automatically have subthreshold zeros of the current-algebra type. When continued to the physical region by the N/D method, the p-wave produces a dynamical ρ meson without Castillejo-Dalitz-Dyson (CDD) poles. The $I = 2$ s-wave phase shifts agree well with experiment, and the $I = 0$ phase shifts, though subject to more uncertainty, also are consistent with experiment. A comparison is made with other s-wave results in the region $0 \leq s \leq 4$.

I. INTRODUCTION

There has recently been a resurgence of interest in low-energy pion-pion scattering. This interest is in part due to the availability of new experimental data, and in part to new theoretical tools such as current algebra, the Veneziano model, duality, and, more recently, the development and application of rigorous constraints arising from crossing symmetry and positivity. In particular, the constraints which apply in the gap $0 \leqslant s \leqslant 4$ between the left- and right-hand cuts have now been used in a number of calculations to study the general nature of the pion-pion amplitudes in the low-energy region and to make predictions about the s waves.[1-6] Techniques and details vary considerably, but the calculations generally produce s-wave phase shifts consistent with experiment, while linking these results to current-algebra ones through the existence of subthreshold zeros.[7]

In fact, the general success of this approach gave rise to the hope that very minimal inputs such as the existence of the ρ meson and crossing symmetry might in themselves uniquely determine the s waves. However, new studies by a number of authors indicate that this is not the case,[8] and that further inputs of some type are necessary to obtain uniquely defined solutions at any appreciable distance above threshold. Nevertheless, it is clear now that crossing symmetry is a quite relevant and important factor in low-energy pion-pion scattering.

In this paper, we use crossing symmetry in a different way to calculate the s and p waves in a completely dynamical fashion. The starting point is the observation of Roskies [9] that if the Froissart-Gribov representation for the partial-wave amplitudes,[10]

$$f_l^I(s) = \frac{4}{s-4}\frac{1}{\pi}\int_4^\infty dt A_t^I(s,t) Q_l\left(1+\frac{2t}{s-4}\right), \qquad (1.1)$$

holds for $0 \leqslant s \leqslant 4$, $l \geqslant 2$, then crossing symmetry uniquely gives the s and p waves from the absorptive parts $A_t^I(s,t)$ except for an ambiguity having two arbitrary parameters. It is generally believed, though, that (1.1) also holds for $l = 1$, an assumption that leads to some important modifications. The first consequence of this additional assumption is that the absorptive parts must satisfy certain constraint conditions, a circumstance that aids in fixing the final choice of inputs. A second is that if the constraint conditions are not exactly satisfied, then the basic Froissart-Gribov p-wave amplitude must be slightly modified in order to restore crossing symmetry in $0 \leqslant s \leqslant 4$. (Questions of convergence at $s = 4$ will be ignored here, and all expressions are assumed valid through the closed interval.) Finally, the overall ambiguity is reduced to that of a single parameter in the s waves.

The inputs to the calculation are the absorptive parts $A_t^I(s,t)$ taken basically from experiment, but adjusted somewhat so that the constraint conditions mentioned above are approximately satisfied. These inputs determine $f_l^I(s)$ in the gap and are then unitarized by the N/D method and continued into the physical region. When unitarity is imposed, it is found that the remaining parameter in the s waves is rather uniquely determined.

For the s waves, the results are of the general type reported in Refs. 3–6. Below threshold, current-algebra-type zeros appear automatically in both isospin amplitudes without any use of current-algebra results or methods being made in the calculations. In the physical region, the phase shifts are consistent with experimental determinations, small and negative for $I = 2$, resonant or near-resonant for $I = 0$. The $I = 0$ phase shifts for $s \gtrsim 15$, however, are rather sensitive to small changes of the amplitude in the unphysical region $0 \leqslant s \leqslant 4$. In this respect, our results appear to corroborate those of Ref. 8, and indicate that unique solutions in the ρ region may be difficult to obtain.

For the p waves, our chief interest is in the dynamical status of the ρ meson, which is not assumed in the s channel, and in the effects of the crossing symmetry corrections. Several recent papers have suggested that the ρ may be primarily a bound state of some higher channel and must be inserted as a CDD (Castillejo-Dalitz-Dyson) pole in a simple one-channel calculation.[11,12] Our own findings do not substantiate this conclusion, and we find a dynamically produced ρ resonance emerging quite naturally in this calculation.

In Secs. II and III, the basic equations are formulated and the form of the solutions are derived. In Secs. IV and V, numerical results are given and discussed. We also discuss here the relation of our s-wave solutions to amplitudes found by other methods. A summary and conclusions are given in Sec. VI.

II. EQUATIONS FOR s- AND p-WAVE AMPLITUDES

As was mentioned in the Introduction, it was shown by Roskies that if the Froissart-Gribov representation is assumed for $l \geq 2$, $0 \leq s \leq 4$, then the remaining partial waves f_0^0, f_0^2, f_1^1 are determined for given $A_t^I(s,t)$ by crossing symmetry up to an ambiguity involving two arbitrary parameters. We actually will assume that (1.1) holds also for $l = 1$ as suggested by Regge theory, but it is nevertheless convenient to first look at the case where this additional assumption is not made. In doing this, we can relate the two cases and make it possible to provide corrections to the pure Froissart-Gribov p-wave amplitude when the inputs violate certain conditions required by crossing symmetry. In formulating the basic equations, we follow closely the method of Roskies,[9] and the reader is referred there for further discussion.

Our notation is that the amplitudes with definite s- and t-channel isospin are written, respectively, as $A^I(s,t)$, $A^I(t,s)$ with

$$A^I(s,t) = \sum_{I'=0}^{2} \beta^{II'} A^{I'}(t,s), \tag{2.1}$$

$$\beta = \beta^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 6 & 10 \\ 2 & 3 & -5 \\ 2 & -3 & 1 \end{pmatrix}, \tag{2.2}$$

and $A_t^I(s,t)$ is the imaginary part of the amplitude in the region of interest, $t \geq 4$, $0 \leq s \leq 4$.

(a) General case, Eq. (1.1) assumed to hold only for $l \geq 2$.

Expanding the s-channel isospin amplitudes in partial waves and using (1.1), we find

$$\begin{aligned} A^{0,2}(s,t) &= \sum_{l\text{ even}} (2l+1) f_l^{0,2}(s) P_l\left(1+\frac{2t}{s-4}\right) \\ &= f_0^{0,2}(s) + \frac{4}{s-4}\,\frac{1}{\pi}\int_4^\infty dt'\, A_t^{0,2}(s,t') \sum_{\substack{l=2\\ \text{even}}}^{\infty} (2l+1) P_l\left(1+\frac{2t}{s-4}\right) Q_l\left(1+\frac{2t'}{s-4}\right), \end{aligned} \tag{2.3}$$

$$\begin{aligned} A^1(s,t) &= \sum_{l\text{ odd}} (2l+1) f_l^1(s) P_l\left(1+\frac{2t}{s-4}\right) \\ &= 3 f_1^1(s) P_1\left(1+\frac{2t}{s-4}\right) + \frac{4}{s-4}\,\frac{1}{\pi}\int_4^\infty dt'\, A_t^1(s,t') \sum_{l=3\text{ odd}}^{\infty} (2l+1) P_l\left(1+\frac{2t}{s-4}\right) Q_l\left(1+\frac{2t'}{s-4}\right). \end{aligned} \tag{2.4}$$

From the expansion

$$\frac{1}{x-y} = \sum_{l=0}^{\infty} (2l+1) Q_l(x) P_l(y)$$

and the symmetry property $P_l(-y) = (-1)^l P_l(y)$, one obtains

$$\begin{aligned} \frac{1}{x+y} + \frac{1}{x-y} &= 2\sum_{l\text{ even}} (2l+1) Q_l(x) P_l(y) \\ &= 2Q_0(x) + 2\sum_{l=2\text{ even}}^{\infty} Q_l(x) P_l(y), \end{aligned}$$

and similarly,

$$\begin{aligned} \frac{1}{x+y} - \frac{1}{x-y} &= 6P_1(x) Q_1(y) \\ &\quad + 2\sum_{l=3\text{ odd}}^{\infty} (2l+1) Q_l(x) P_l(y). \end{aligned}$$

These results can be used in (2.3) and (2.4) to give

$$A^{0,2}(s,t) = f_0^{0,2}(s) + F_{0,2}(s,t), \tag{2.5}$$

$$A^1(s,t) = 3 f_1^1(s)\left(1+\frac{2t}{s-4}\right) + F_1(s,t), \tag{2.6}$$

where $F_I(s,t)$ is expressible in terms of the absorptive parts $A_t^I(s,t)$:

$$F_I(s,t) = \frac{1}{\pi}\int_4^\infty dt'\, A_t^I(s,t') \overline{F}_I(s,t,t'), \tag{2.7}$$

$$\overline{F}_{0,2}(s,t,t') = \frac{1}{t'-t} + \frac{1}{t'+t+s-4} - \frac{4}{s-4} Q_0\left(1+\frac{2t'}{s-4}\right), \tag{2.8}$$

$$\overline{F}_1(s,t,t')=\frac{1}{t'-t}-\frac{1}{t'+t+s-4}-\frac{12}{s-4}P_1\left(1+\frac{2t}{s-4}\right)Q_1\left(1+\frac{2t'}{s-4}\right). \tag{2.9}$$

In writing (2.3) and (2.4) with only even and odd partial waves, t-u symmetry has been assumed. To enforce s-u symmetry, one requires the t-channel isospin amplitudes $A_t^I(t,s)$ to be symmetric in s and u for $I=0,2$ and antisymmetric for $I=1$. Since $s+t+u=4$, the condition $s=u$ is equivalent to $t=4-2s$, and thus

$$A_t^1(t,s)\big|_{t=4-2s}=0,\quad \frac{\partial}{\partial s}A_t^{0,2}(t,s)\Big|_{t=4-2s}=0.$$

Using the crossing condition (2.1) for $A_t^I(t,s)$ together with (2.5) and (2.6), these give three equations for the unknown partial-wave amplitudes f_0^0, f_0^2, f_1^1. With the notation

$$F_I(s)=F_I(s,4-2s),\qquad \Delta_s F_I(s)=\frac{\partial}{\partial s}F_I(s,t)\Big|_{t=4-2s}, \tag{2.10}$$

$$f_0=f_0^0,\ f_1=f_1^1,\ f_2=f_0^2, \tag{2.11}$$

these conditions become

$$2f_0(s)-5f_2(s)+9k_1f_1(s)=-2F_0(s)+5F_2(s)-3F_1(s), \tag{2.12a}$$

$$f_0'(s)+5f_2'(s)+9k_1f_1'(s)-18k_2f_1(s)=-\Delta_sF_0(s)-5\Delta_sF_2(s)-3\Delta_sF_1(s), \tag{2.12b}$$

$$2f_0'(s)+f_2'(s)-9k_1f_1'(s)+18k_2f_1(s)=-2\Delta_sF_0(s)-\Delta_sF_2(s)+3\Delta_sF_1(s), \tag{2.12c}$$

$$k_1(s)=\frac{4-3s}{s-4},\quad k_2(s)=\frac{4-2s}{(s-4)^2},\quad f_I'(s)=\frac{df_I(s)}{ds}. \tag{2.13}$$

As shown by Roskies, Eqs. (2.12a)–(2.12c) determine f_0, f_1, f_2 up to the ambiguity

$$\begin{aligned}f_0(s)&\to f_0(s)+5a+2b(3s-4),\\ f_1(s)&\to f_1(s)-b(4-s),\\ f_2(s)&\to f_2(s)+2a-b(3s-4),\end{aligned} \tag{2.14}$$

where a and b are arbitrary constants. The solution of these equations will be deferred to Sec. III.

(b) The case when the Froissart-Gribov expression is assumed also for $l=1$.

In (2.4), the term $3f_1^1(s)P_1(1+2t/(s-4))$ is no longer separated out. Then proceeding as before, one can write

$$A^1(s,t)=F_1^{\rm FG}(s,t), \tag{2.15}$$

$$F_1^{\rm FG}(s,t)=\frac{1}{\pi}\int_4^\infty dt'A_t^1(s,t')\overline{F}_1^{\rm FG}(s,t,t'), \tag{2.16}$$

$$\overline{F}_1^{\rm FG}(s,t,t')=\frac{1}{t'-t}-\frac{1}{t'+t+s-4}. \tag{2.17}$$

$F_1(s,t)$ and $F_1^{\rm FG}(s,t)$ are simply related by

$$\begin{aligned}F_1(s,t)&=F_1^{\rm FG}(s,t)-\frac{12}{s-4}P_1\left(1+\frac{2t}{s-4}\right)\frac{1}{\pi}\times\int_4^\infty dt'A_t^1(s,t')Q_1\left(1+\frac{2t'}{s-4}\right)\\&=F_1^{\rm FG}(s,t)-3\left(1+\frac{2t}{s-4}\right)f_1^{\rm FG}(s).\end{aligned} \tag{2.18}$$

Equations similar to (2.12a)–(2.12c) are derived with the terms $f_1(s)$ missing:

$$2f_0(s)-5f_2(s)=-2F_0(s)+5F_2(s)-3F_1^{\rm FG}(s), \tag{2.19a}$$

$$f_0'(s)+5f_2'(s)=-\Delta_sF_0(s)+5\Delta_sF_2(s)-3\Delta_sF_1^{\rm FG}(s), \tag{2.19b}$$

$$2f_0'(s)+f_2'(s)=-2\Delta_sF_0(s)-\Delta_sF_2(s)+3\Delta_sF_1^{\rm FG}(s). \tag{2.19c}$$

It is easily seen that there is now only one arbitrary constant, and the amplitudes f_0, f_2 are determined except for the ambiguity

$$\begin{aligned}f_0(s)&\to f_0(s)+5a,\\ f_2(s)&\to f_2(s)+2a.\end{aligned} \tag{2.20}$$

Since (2.19a)–(2.19c) are three equations for the two unknowns f_0, f_2, there is no guarantee that they are consistent. That is, the assumptions of $f_1=f_1^{\rm FG}$ and crossing symmetry together imply restrictions on $A_t^I(s,t)$. The consistency condition is easily found by differentiating (2.19a) and substituting in the values of f_0', f_2' as found from (2.19b) and (2.19c). In doing this, it is convenient to introduce the quantity

$$\Delta_tF_I(s)=\frac{\partial F_I}{\partial t}(s,t)\Big|_{t=4-2s} \tag{2.21}$$

so that

$$\frac{d}{ds}F_t(s)\equiv\frac{d}{ds}F_I(s,4-2s)=\Delta_sF_I(s)-2\Delta_tF_I(s). \tag{2.22}$$

Now when (2.19a) is differentiated and the expressions for f_0', f_2' are inserted, the terms $\Delta_sF_{0,2}$ cancel out and leave the condition

$$6\mathcal{R}(s) \equiv 2\Delta_t F_0(s) + 3\Delta_t F_1^{\mathrm{FG}}(s) - 5\Delta_t F_2(s) - 6\Delta_s F_1^{\mathrm{FG}}(s) = 0 . \quad (2.23)$$

Using explicit expressions for $\Delta_t F_0$, etc., in (2.23) gives[13]

$$\mathcal{R}(s) = (4-3s)R(s) = 0 , \quad (2.24)$$

$$R(s) = \frac{1}{\pi}\int_4^\infty \frac{dt}{(t-4+2s)(t-s)} \left\{ \frac{[2A_t^0(s,t) + 3A_t^1(s,t) - 5A_t^2(s,t)](2t+s-4)}{6(t-4+2s)(t-s)} - \frac{\partial}{\partial s} A_t^1(s,t) \right\} . \quad (2.25)$$

The general condition that the Froissart-Gribov expression for $l=1$ be consistent with crossing symmetry is therefore that the symmetry function $R(s)$ vanish for $0 \le s \le 4$, where $R(s)$ is given by (2.25). This is a highly nontrivial condition and places severe restrictions on the absorptive parts $A_t^I(s,t)$. In particular, the contribution to $R(s)$ from the crossed-channel ρ meson is everywhere positive, which means that there must be strong contributions from higher partial waves and Regge terms (s waves do not contribute to R) if crossing symmetry is not to be violated. If proper account is taken, however, of these other contributions, it is possible to find absorptive parts, consistent with experimental data, which approximately satisfy (2.24). This has been discussed elsewhere,[14] and the reader is referred there for full details. In this paper, our assumption is that the Froissart-Gribov expression holds for $l=1$, but we deal with the fact that in actual calculations, approximations lead to a small but nonvanishing symmetry function. In this case, $R(s)$ provides a convenient vehicle for enforcing crossing symmetry in the region $0 \le s \le 4$.

III. SOLUTIONS FOR f_0, f_1, f_2

We wish to relate the s- and p-wave amplitudes for the cases (a) and (b) discussed in Sec. II. This will be done be relating the p-wave solution of (2.12a)–(2.12c) to f_1^{FG}, given by (1.1), and the s-wave solutions of (2.12a)–(2.12c) to those of (2.19a)–(2.19c). First, consider the p waves. From (2.12b) and (2.12c), one obtains

$$2f_0' - 5f_2' + 3(9k_1 f_1' - 18k_2 f_1) = -2\Delta_s F_0 + 5\Delta_s F_2 + 9\Delta_s F_1 , \quad (3.1)$$

and from (2.12a),

$$2f_0' - 5f_2' + 9k_1' f_1 + 9k_1 f_1' = -2F_0' + 5F_2' - 3F_1' . \quad (3.2)$$

Eliminating $2f_0' - 5f_2'$ gives a single first-order differential equation for f_1. Using (2.22), this can be written as

$$f_1' - \frac{1}{s-4} f_1 = \frac{s-4}{18(4-3s)} [2\Delta_t F_0 - 5\Delta_t F_2 + 3\Delta_t F_1 - 6\Delta_s F_1] . \quad (3.3)$$

This equation can of course be solved directly, but it is convenient to first rewrite the right-hand side in such a way as to introduce F_1^{FG} in place of F_1. From the relation (2.18), one can verify

$$\Delta_t F_1 - 2\Delta_s F_1 = \Delta_t F_1^{\mathrm{FG}} - 2\Delta_s F_1^{\mathrm{FG}} + 6(4-3s)\frac{d}{ds}\left[\frac{1}{s-4} f_1^{\mathrm{FG}}(s)\right] . \quad (3.4)$$

Then, using the definition of $\mathcal{R}(s)$, (2.23),

$$2\Delta_t F_0 - 5\Delta_t F_2 + 3\Delta_t F_1 - 6\Delta_s F_1 = 18(4-3s)\frac{d}{ds}\left[\frac{1}{s-4} f_1^{\mathrm{FG}}(s)\right] + 6\mathcal{R}(s) . \quad (3.5)$$

Equation (3.3) therefore becomes, using (2.24),

$$f_1' - \frac{1}{s-4} f_1 = (s-4)\frac{d}{ds}\left[\frac{1}{s-4} f_1^{\mathrm{FG}}\right] + \frac{s-4}{3} R(s) , \quad (3.6)$$

with a solution

$$f_1(s) = \frac{s-4}{3}\int \frac{ds'}{4-3s'} \times \left\{ 3(4-3s')\frac{d}{ds'}\left[\frac{1}{s-4} f_1^{\mathrm{FG}}(s')\right] + (4-3s')R(s') \right\} ,$$

$$f_1(s) = f_1^{\mathrm{FG}}(s) + \frac{s-4}{3}\int_{4/3}^{s} ds' R(s') \quad (3.7)$$

where the integration is taken such that $f_1(\frac{4}{3}) = f_1^{\mathrm{FG}}(\frac{4}{3})$. Thus, if the symmetry condition (2.24) is satisfied, $f_1(s) = f_1^{\mathrm{FG}}(s)$, with $b=0$ in (2.14). If, however, due to approximations, $R(s)$ does not vanish identically, then the term involving $R(s)$ gives a correction so as to make $f_1(s)$ consistent with crossing symmetry. The choice of solution $f_1(\frac{4}{3})=f_1^{FG}(\frac{4}{3})$ is taken because, according to (2.24), $f_1^{FG}(s)$ is always compatible with crossing symmetry at $s=\frac{4}{3}$.

Next, we obtain the s-wave solutions of (2.19a)–(2.19c), denoting them by f_0^{FG}, f_2^{FG}. As noted before, the equations are not consistent unless the symmetry condition $R(s)=0$ is satisfied, and f_0^{FG}, f_2^{FG} may therefore not be uniquely defined. It is nevertheless convenient to formally obtain solutions of these equations in a way that can be simply related to the solutions of (2.12a)–(2.12c). Again, because (2.19b) and (2.19c) are differential equations, the solutions are defined only up to additive constants, and the restriction is made that $f_0^{FG}(s)$, $f_2^{FG}(s)$ satisfy (2.19a) at $s=\frac{4}{3}$. Since from (2.16), $F_1^{FG}(\frac{4}{3})=0$, this can be done conveniently by constructing solutions such that

$$f_0^{FG}(\tfrac{4}{3})=-F_0(\tfrac{4}{3}),\quad f_2^{FG}(\tfrac{4}{3})=-F_2(\tfrac{4}{3}). \tag{3.8}$$

From (2.5) and the definition of $F_I(s)$ in (2.10), it is seen that this is equivalent to taking $A^0(\frac{4}{3},\frac{4}{3})=A^2(\frac{4}{3},\frac{4}{3})=0$. Recalling the relation for the Chew-Mandelstam coupling constant[15]

$$A^I(\tfrac{4}{3},\tfrac{4}{3})=-\begin{pmatrix}5\\0\\2\end{pmatrix}\lambda\ ,$$

it is seen that our canonical solutions correspond to $\lambda=0$, and the parameter a in (2.20) is just $a=-\lambda$. We can therefore rewrite (2.20) as

$$f_0(s)\to f_0(s)-5\lambda,\quad f_2(s)\to f_2(s)-2\lambda. \tag{3.9}$$

Instead of working with f_0, f_2 directly, it is convenient to use the combinations

$$f(s)=f_0(s)+2f_2(s),\quad g(s)=2f_0(s)-5f_2(s). \tag{3.10}$$

From (2.19b) and (2.19c), one obtains

$$\frac{df^{FG}}{ds}(s)=-\Delta_s F_0(s)-2\Delta_s F_2(s) = -F_0'(s)-2F_2'(s) +2[-\Delta_t F_0(s)-\Delta_t F_2(s)], \tag{3.11}$$

$$f^{FG}(s)=-F_0(s)-2F_2(s) -2\int_{4/3}^{s} ds'[\Delta_t F_0(s')+2\Delta_t F_2(s')]. \tag{3.12}$$

The other independent solution, $g^{FG}(s)$, can be taken directly from (2.19a):

$$g^{FG}(s)=-2F_0(s)+5F_2(s)-3F_1^{FG}(s). \tag{3.13}$$

Turning now to (2.12b) and (2.12c), one obtains again

$$\frac{df}{ds}(s)=-\Delta_s F_0(s)-2\Delta_s F_2(s),$$

which is the same as (3.11) so that

$$f(s)=f^{FG}(s). \tag{3.14}$$

On the other hand, (2.12a) can be written in the form

$$g(s)=-2F_0(s)+5F_2(s)-3F_1(s)-9k_1 f_1(s), \tag{3.15}$$

which becomes, using (2.18),

$$g(s)=-2F_0(s)+5F_2(s)-3F_1^{FG}(s) +\frac{9(4-3s)}{s-4}f_1^{FG}(s)-\frac{9(4-3s)}{s-4}f_1(s). \tag{3.16}$$

Comparison with (3.13) and (3.7) gives

$$g(s)=g^{FG}(s)-3(4-3s)\int_{4/3}^{s} ds' R(s'). \tag{3.17}$$

Collecting these results,

$$f_1(s)=f_1^{FG}(s)+\frac{s-4}{3}\int_{4/3}^{s} ds' R(s'), \tag{3.18a}$$

$$f(s)=f^{FG}(s), \tag{3.18b}$$

$$g(s)=g^{FG}(s)-3(4-3s)\int_{4/3}^{s} ds' R(s'), \tag{3.18c}$$

and it is seen explicitly that the general solutions reduce to the Froissart-Gribov ones when the symmetry condition (2.24) is satisfied and $R(s)=0$ for $0\le s\le 4$. However, as noted, $R(s)$ will not vanish identically in actual calculations due to the approximate nature of the inputs, and in this case, it is necessary to retain the extra terms in (3.18a)–(3.18c) to enforce crossing symmetry. On the other hand, we take $b=0$ in (2.14), leaving only the one-parameter ambiguity of (3.9). The amplitudes constructed in this way then agree with the Froissart-Gribov ones at $s=\frac{4}{3}$, where (2.19a)–(2.19c) are always compatible, but are modified elsewhere to satisfy crossing symmetry.

We note here that the amplitudes f and g are of interest in themselves, f being essentially the neutral pion amplitude, $f=3f_0^{00}$, and g having the important property that it is determined completely by the absorptive parts and is free of the ambiguity (3.9). The s-wave amplitudes having definite isospin are of course found by inverting (3.10):

$$f_0(s)=\tfrac{1}{9}[5f(s)+2g(s)],\quad f_2(s)=\tfrac{1}{9}[2f(s)-g(s)]. \tag{3.19}$$

In solving for f^{FG} and g^{FG} from (3.12) and (3.13), one need only know $F_{0,2}(s)$, $\Delta_t F_{0,2}(s)$, and $F_1^{FG}(s)$. Explicit expressions for these quantities are

$$F_I(s)=\frac{1}{\pi}\int_4^\infty dt A_t^I(s,t)\bar{F}_I(s,t), \tag{3.20}$$

$$\bar{F}_0(s,t)=\bar{F}_2(s,t)$$
$$=\frac{1}{t-4+2s}+\frac{1}{t-s}-\frac{2}{s-4}\ln\frac{t+s-4}{t}, \tag{3.21}$$

$$\bar{F}_1^{FG}(s,t)=\frac{1}{t-4+2s}-\frac{1}{t-s}\quad, \tag{3.22}$$

$$\Delta_t F_I(s)=\frac{1}{\pi}\int_4^\infty dt A_t^I(s,t)\Delta_t\bar{F}_I(s,t), \tag{3.23}$$

$$\Delta_t\bar{F}_0(s,t)=\Delta_t\bar{F}_2(s,t)$$
$$=\frac{1}{(t-4+2s)^2}-\frac{1}{(t-s)^2}\ . \tag{3.24}$$

IV. NUMERICAL RESULTS: AMPLITUDES FOR $0\leqslant s\leqslant 4$

The amplitudes in (3.18a)–(3.18c) are determined except for the ambiguity (3.9) once $A_t^I(s,t)$ are specified. The basic inputs in our actual calculations were crossed-channel partial waves through $l'=3$,

$$A_t^I(s,t)=\sum_{l',I'}\beta^{II'}(2l'+1)\,\mathrm{Im}f_{l'}^{I'}(t)P_{l'}\left(1+\frac{2s}{t-4}\right), \tag{4.1}$$

and for large t, $t>t_0$, s-channel Regge forms

$$A_t^I(s,t)=\gamma_I(s)t^{\alpha_I(s)}\ , \tag{4.2}$$

with t_0 taken above the g resonance, $t_0=160$.

Resonant partial waves (ϵ,ρ,f,g resonances) were parametrized by the unitary form

$$f_l^I(t)=\frac{\bar{\Gamma}\rho^{2l}}{H(t)(t_R-t)-i\bar{\Gamma}\rho^{2l+1}}\quad, \tag{4.3}$$

where

$$\rho(t)=\left(\frac{t-4}{t}\right)^{1/2},\quad H(t)=\frac{t_R+C_1}{t_R+C_2}\frac{t+C_2}{t+C_1}\ ,$$

and $\bar{\Gamma}$ is related to the experimental width Γ by

$$\bar{\Gamma}=\frac{\sqrt{t_R}\,\Gamma}{[\rho(t_R)]^{2l+1}}\ .$$

The values of the parameters were taken as far as possible from experiment, but where freedom exists, were chosen so as to minimize $R(s)$ in accordance with the symmetry condition (2.24). As noted previously, the inputs used have appreciable contributions to R from $l'\geqslant 2$ partial waves and Regge terms to cancel those of the $l'=1$ partial wave. The importance of terms other than s and p waves was first noticed by Lovelace[16] and more recently emphasized by Basdevant *et al.*[17] This was also discussed in detail in Ref. 14, and the parameters used in the present calculation are the same as given there; for convenience, the values of the parameters are given again in Table I. In this table, a_l^I are the scattering lengths defined (in the s channel) by

$$a_l^I=\lim_{s\to 4}\frac{f_l^I(s)}{v^l},\quad v=\frac{s-4}{4}\ . \tag{4.4}$$

For simplicity, we will also often denote the s- and p-wave scattering lengths by the isospin label

$$a_0=a_0^0,\quad a_2=a_0^2,\quad a_1=a_1^1\ . \tag{4.5}$$

In addition to the resonant partial-wave contributions, we have also used a nonresonant $I=2$ s wave with scattering length $a_2=-0.05$, dropping to $s_0^2=-15°$ at $E=765$ MeV, and $I=0$, $I=1$ Regge terms (Pomeron and ρ) with

$$\alpha_p(s)=1+0.00568s\ ,$$
$$\gamma_p(s)=0.0447e^{0.0161s}\ ,$$
$$\alpha_\rho(s)=0.57+0.0143s\ ,$$
$$\gamma_\rho(s)=0.0462\ .$$

These values are also discussed in Ref. 14. The explicit form for the resulting contributions to the amplitudes is given in Appendix A, and with these results, the amplitudes f_0, f_1, f_2 are readily calculated in the region $0\leqslant s\leqslant 4$.

Before discussing our results, we note that since the total amplitudes satisfy crossing symmetry by construction, and since our input absorptive parts are positive, nearly all the constraints imposed by crossing symmetry and posi-

TABLE I. Input parameters; the constants c_1, c_2 are determined from the scattering length a_I and phase shifts at the energy E_1.

l	I	$m=\sqrt{t_R}$ (MeV)	Γ (MeV)	a_I	E_1 (MeV)	$\delta(E_1)$	c_1	c_2
0	0	900	400	0.16	600	35°	−3.380	−1.921
1	1	765	115	0.035	600	13°	−0.5873	60.91
2	0	1264	150	0.0016	765	2°	−2.487	−196.2
3	1	1670	150	0.000 05	765	1°	−3.948	420.4

tivity[1] are automatically satisfied. We have, for example, tested our final results using the five Roskies relations relating the s and p waves.[9] These can all be written in the form $\int_0^4 ds\, \psi(s)=0$, and using as a measure the quantity $[\int_0^4 ds\, \psi(s)]/\int_0^4 ds\, |\psi(s)|$, we find crossing symmetry to be valid to about one part in 10^4. The remaining discrepancy is associated with numerical integration errors and use of asymptotic forms for the Regge terms. Also, we have examined some of the relations involving the neutral-pion amplitude $f_0^{00}=\frac{1}{3}f$, and find that they are always satisfied. The computed amplitude $f(s)$ is shown in Fig. 1, where most of the simple inequalities, e.g.,[18]

$$f(4) \geq f(0) \geq f(3.19),$$

$$f(3.205) \geq f(0.2134) \geq f(2.9863),$$

can be verified by inspection. The minimum of $f(s)$ occurs at $s \simeq 1.64$, rather near its maximum value $s=1.697$.

The one important exception is a recent constraint of Yen and Roskies[19] which combines crossing symmetry and the positivity of individual partial waves, $\text{Im} f_l^I(t)$, a stronger condition than the positivity of the total absorptive parts $A_t^I(t, s)$. In our calculations, all explicit partial-wave contributions have $\text{Im} f_l^I(t) \geq 0$, but we also use the Regge terms which may not be equivalent to such a condition. In addition, the neutral amplitude (or f) used in the Yen-Roskies constraints is decoupled from the ρ, which means that the $I=0$ Pomeron term plays an important role in this particular amplitude. It is therefore not particularly surprising that the final Yen-Roskies condition (6.5) of Ref. 19, which is known to be particularly stringent,[19,3] is violated. It would be of interest to see what sort of modifications of our Pomeron input are necessary to satisfy this test, but we have not looked at this question. In any event, the effect of the violation on the isospin amplitudes should not be serious since f, used in this test, is generally more than an order of magnitude smaller than g (see Fig. 1). For completeness, we give here the computed values for this test: $\bar{x}=-0.125$, $\bar{y}=-0.0022$.

The amplitudes g and f_1 both have the property that they are determined completely by the assumed inputs and are independent of the parameter λ in (3.9). These amplitudes are shown in Figs. 1 and 2 and have the associated threshold constants[20] $L=\frac{1}{6}(2a_0-5a_2)=0.098$, $a_1=0.0347$. The value of L is in good agreement with that of the "universal curve",[21] $L\simeq 0.1$, while a_1 is consistent with recent experimental estimates[21,22] and the assumed input value $a_1=0.035$ (Table I). Note also that the relation $L=3a_1$ is nearly satisfied, indicating approximately linear amplitudes.[7]

The s-wave isospin amplitudes f_0, f_2, on the other hand, depend on the value of λ, which is not determined by crossing symmetry. Thus, the question of the existence of zeros in the region $0 \leq s \leq 4$, a feature predicted by current algebra,[7] cannot be discussed without an additional requirement to fix λ. It has been argued on general grounds, i.e., independent of current algebra, that such zeros should actually appear in the physical amplitudes,[23,24] and this idea can be easily checked here by requiring reasonable scattering lengths and then looking to see if the zeros indeed appear.

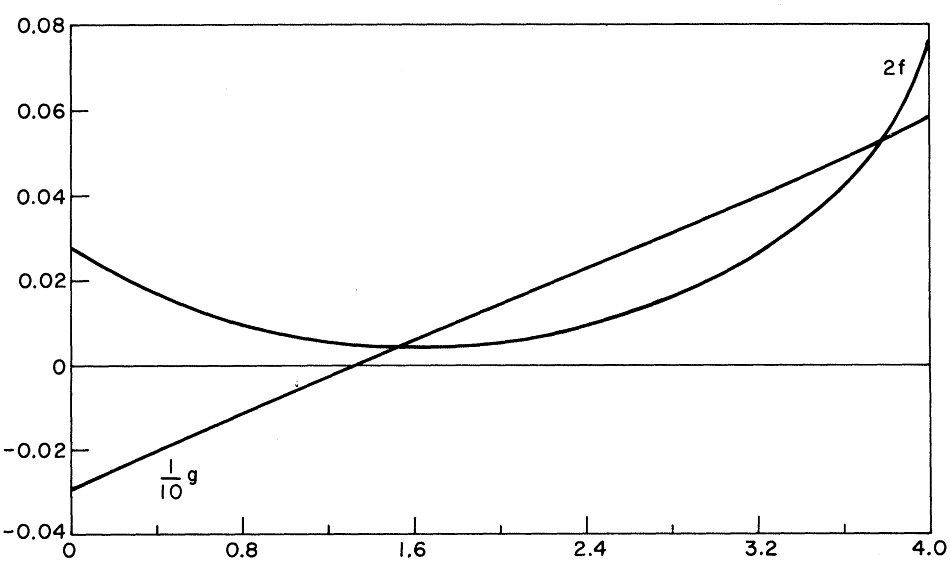

FIG. 1. The amplitudes $f(s)$, $g(s)$ for $0 \leq s \leq 4$ $(m_\pi=1)$.

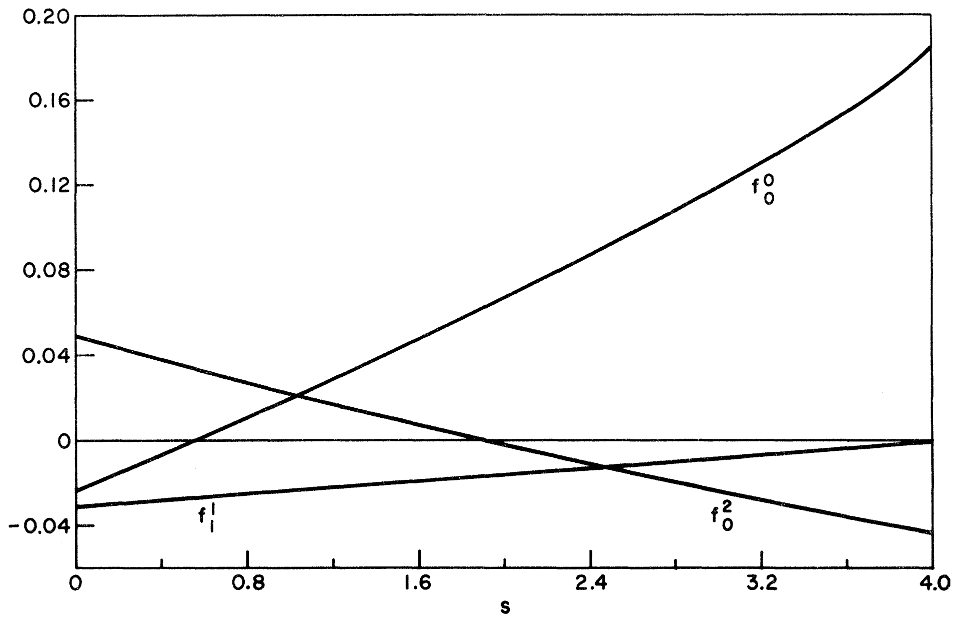

FIG. 2. The amplitudes $f_0^0(s), f_1^1(s), f_0^2(s)$ for $0 \le s \le 4$ ($m_\pi = 1$).

The computed values of f_0 and f_2 at threshold are $f_0(4) = 0.152$, $f_2(4) = -0.057$. This corresponds to $\lambda = 0$, so that from (3.9), one has the possible scattering lengths

$$a_0 = 0.152 - 5\lambda\,,\qquad a_2 = -0.057 - 2\lambda\,. \tag{4.6}$$

The characteristic features of the amplitudes are indicated in Table II for a range of a_0 of physical interest,[21] $0.14 \le a_0 \le 0.21$. For these values, both the $I = 0$ and $I = 2$ s-wave amplitudes have subthreshold zeros s_0, s_2 which furthermore agree well with the Pennington-Pond sum rule[23] $4s_0 + 5s_2 = 12$.

In Sec. V, we will actually fix λ by using unitarity to find a preferred solution. Here we merely report the result:

$$a_0 = 0.185\,,\quad a_2 = -0.043\,,\quad \lambda = -0.00664 \qquad s_0 = 0.56\,,\quad s_2 = 1.91\,, \tag{4.7}$$

which does indeed have reasonable scattering lengths and subthreshold zeros.

TABLE II. Characteristics of s-wave amplitudes for $0 \le s \le 4$.

a_0	a_2	a_0/a_2	s_0	s_2	$4s_0 + 5s_2$	λ
0.14	−0.061	−2.28	1.55	1.15	11.9	0.002 36
0.16	−0.053	−3.80	1.22	1.40	11.9	−0.000 64
0.18	−0.045	−3.97	0.67	1.82	11.8	−0.005 64
0.20	−0.037	−5.35	0.21	2.17	11.7	−0.009 64
0.21	−0.033	−6.30	0.00	2.35	11.7	−0.011 64

In comparing this result with some amplitudes recently found by other methods,[3-6,25] it is useful to divide them into groups, depending on whether or not they have values of a_1 and L similar to our solutions. As noted previously, these quantities are completely determined by the absorptive parts, and can be considered to give a simple measure of an effective strength of the dynamics. We have done this in Table III where our preferred solution (first entry) is compared with other amplitudes. The second entry comes from a recent current-algebra calculation in which unitarity corrections were systematically included[25]; this solution is, in fact, very similar to our own in all respects, including the positions of the zeros. (A recent phenomenological analysis, R. C. Johnson, Durham report, 1972, also produces similar s-wave characteristics, $L = 0.103$, $a_0 = 0.183$, $a_2 = -0.050$, $s_0 = 0.60$, $s_2 = 1.90$; the p-wave scattering length, however, is larger than ours, $a_1 = 0.05$.) The next two entries are inverse amplitude calculations[3,4], where the physical ρ was enforced by means of a subtraction. These solutions, because of the differences in s_0, s_2, and a_0/a_2, appear to be quite different, but this appearance is actually misleading because the quantities in question are rather sensitive to small changes in λ. The fifth entry shows a solution of our own with an adjusted value of λ, $\lambda = 0.00236$, and this is seen to closely resemble the previous two solutions. Since the required change in λ, $\Delta\lambda < 0.01$, is small com-

TABLE III. Comparison of preferred s-wave amplitudes with other determinations.

Solution	$L=\frac{1}{6}(2a_0-5a_2)$	a_1	a_0	a_2	a_0/a_2	s_0	s_2	$4s_0+5s_2$
Preferred ($\lambda=-0.006\,64$)	0.098	0.035	0.185	−0.043	−4.3	0.56	1.91	11.8
Morgan and Shaw[21]	0.106	0.036	0.211	−0.043	−4.9	0.42	2.0	11.7
Carrotte and Johnson[3]	0.085	0.04	0.136	−0.047	−2.9	1.5	1.2	12.0
Widder[4]	0.103	0.036	0.146	−0.065	−2.2	1.625	1.1	12.0
($\lambda=0.002\,36$)	0.098	0.035	0.140	−0.061	−2.3	1.55	1.15	11.95
Le Guillou *et al.*[5]	0.130	0.045	0.206	−0.073	−2.8	1.4	1.27	11.95
Bonnier and Gauron[6]	0.157	0.052	0.22	−0.10	−2.2	1.75	1.00	12.0

pared to the scattering lengths, it would appear that all the solutions having similar values of a_1 and L are actually quite similar.

Two additional calculations are also listed in the second section of Table III.[4,5] These also impose the physical ρ using a K-matrix-type formulation, but nevertheless have over-all dynamical strengths, as measured by a_1 and L, considerably stronger than ours. These solutions cannot be generated from our own by any choice of λ, and we would have to change our basic inputs to obtain equivalent solutions. It would seem then that a precise determination of the parameters L and a_1 is of great importance.[26]

V. UNITARIZATION: PHYSICAL REGION RESULTS

Unitarization is done using the N/D method as described in a previous paper.[27] The left-hand cut is calculated for $-32 \leq s \leq 0$ using

$$\mathrm{Im} f_l^I(s)=\frac{2}{s-4}\sum_{I'=0}^{2}\beta^{II'}\int_4^{4-s} dt\,A_t^{I'}(t,s)\times P_l\left(1+\frac{2t}{s-4}\right), \quad (5.1)$$

where $A_t^I(t,s)$ are the crossed-channel absorptive parts used in the gap calculation. For $s<-32$, the unknown left-hand cut is replaced by poles whose positions and residues are adjusted so that the amplitude computed by the N/D method most closely matches that found in Sec. IV for $0 \leq s \leq 4$. The number of poles used is somewhat arbitrary, and we have relied heavily on our previous experience with known test functions which indicate that two poles are adequate to give good results.[27] Both one- and two-pole calculations were made to check that drastic changes did not occur in going from one case to the other, but our final reported results are generally for two poles, the N/D output taken to agree exactly with the amplitude at the values $s=0$, $s=+2$, and the pole positions adjusted to give a best over-all fit. The parameter taken to determine a best fit was

$$Z=\frac{\sum_{i=1}^{51}[f_l^I(s_i)-N(s_i)/D(s_i)]^2}{\sum_{i=1}^{51}[f_l^I(s_i)]^2},$$

where $f_l^I(s_i)$ is the appropriate gap amplitude computed in Sec. IV at equally spaced points s_i in $0 \leq s \leq 4$. Again, there is a degree of arbitrariness in defining a best fit by this particular method, but it is a reasonable one which was found to give good results for test functions. For definiteness, we will quote a specific result obtained by the procedure described above, but it should probably be thought of as a typical rather than unique result.

The unitarization method is similar to that of Kang and Lee,[28] but the treatment of the left-hand cut is quite different. Kang and Lee use what we call a "pure pole" model, i.e., the entire left-hand cut is represented by poles. The calculation has been criticized by Tryon,[29] who points out that the residues used by Kang and Lee are not consistent with the left-hand cut as determined from the crossed channels, several of the residues even having the wrong sign. This problem does not occur in our method since the entire region of the left-hand cut $-32 \leq s \leq 0$ which is determined by crossing is explicitly retained. Furthermore, by keeping this part of the cut, the matching and stability problems that plague pure pole methods[30] are largely avoided.[27]

A. $l=1$

In the p wave, our interest centers on the effects of the symmetry term in (3.18a) and on the existence of the ρ meson as a dynamical resonance. We performed two separate calculations, one using the Froissart-Gribov amplitude f_1^{FG} alone, and another with the complete amplitude f_1. We find the following general results:

(a) The output using the symmetry term is appreciably better than that coming from the pure Froissart-Gribov amplitude.

(b) The complete amplitude leads to a dynamical ρ resonance with parameters near experimentally observed ones.

The phase shifts for each case are shown in Fig.

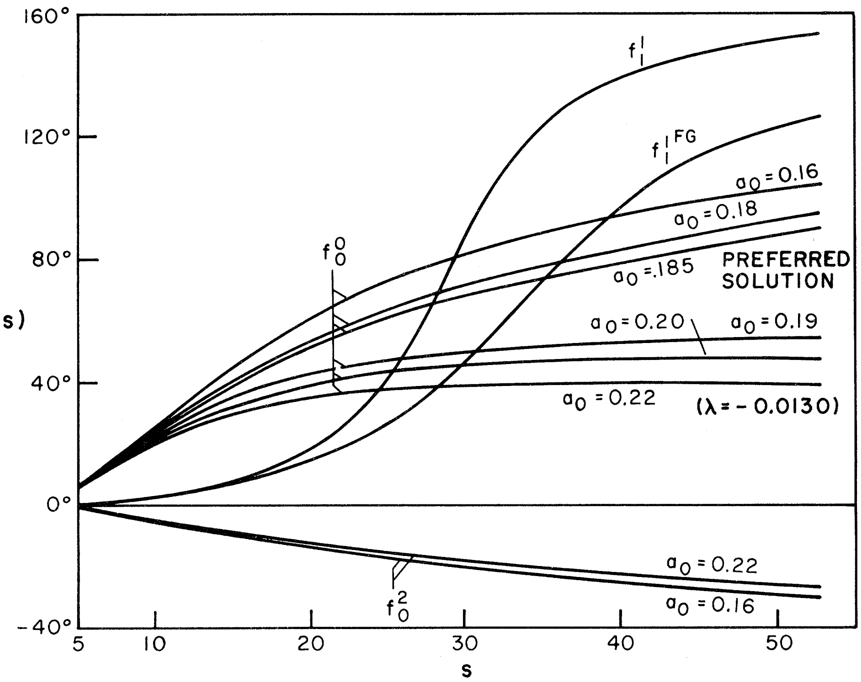

FIG. 3. s- and p-wave phase shifts ($m_\pi = 1$).

3 and some relevant parameters are listed in Table IV. As mentioned previously, some variation in the output is allowed due to inherent ambiguity in the method, but tests made in the present calculations indicate that the qualitative statements (a) and (b) above are not affected by any reasonable changes in procedure. Three general types of test were made:

(1) Changes of pole positions. The pole positions are chosen to give an optimal fit in the gap, but one can use other positions which lead to fits nearly as good. For small variations of pole position, only small changes in output result, e.g., changes in position of 10% typically lead to output variations $\Delta m, \Delta\Gamma \lesssim 5$ MeV. If large variations in pole positions are made, then a wider variety of outputs can be found, but the qualitative results are not altered as long as Z is kept comparable to the optimal case. For calculations retaining the symmetry term, a resonance was consistently obtained with $m_\rho \lesssim 800$ MeV, $\Gamma_\rho \lesssim 200$ MeV.

(2) Changes in number of poles. We have tested the stability of the two-pole optimal solutions by adding a third pole having an arbitrary residue, subject only to the condition that the match in the gap, i.e., Z, be comparable to that of the original solution. Again, the output changes slowly as the residue of the third pole is varied from zero, increases of Z by 1% corresponding to changes Δm, $\Delta\Gamma \lesssim 5$ MeV, and increases of Z by 20% corre-

TABLE IV. Characteristics of s and p waves in the physical region.

p-wave solutions		m (MeV)	Γ (MeV)	Z	a_I	δ(765 MeV)
Froissart-Gribov		869	363	3.5×10^{-8}	0.035	
f_1 (F. G. plus symmetry)		777	168	2.1×10^{-8}	0.035	
$b = -0.0002$		765	148	2.2×10^{-8}	0.035	
s-wave solutions						
$\lambda = -0.0016$	$I=0$	802	879	4.8×10^{-5}	0.160	85°
	$I=2$	···	···	5.3×10^{-5}	−0.053	−19.2°
$\lambda = -0.0066$	$I=0$	1025	1244	1.0×10^{-6}	0.185	65°
(preferred)	$I=2$	···	···	1.2×10^{-6}	−0.043	−18.7°
$\lambda = -0.0136$	$I=0$	···	···	4.5×10^{-5}	0.220	40°
	$I=2$	···	···	8.8×10^{-7}	−0.029	−17.0°

sponding to changes Δm, $\Delta\Gamma \lesssim 20$ MeV. A single pole also leads to a resonant output, though with much poorer results.

(3) Changes in input. We find that our calculations are stable with respect to input changes, i.e., small changes of input lead to only small changes in output. For example, altering the $l=0$, $I=2$ phase shift at the ρ mass by 5% leads to a negligible alteration of output, and eliminating the $l=0$, $I=2$ input completely changes the resonance parameters $\Delta m \approx 10$ MeV, $\Delta\Gamma \approx 15$ MeV.

We find, then, that the qualitative results are independent of details of the method and that a dynamical ρ meson is a persistent feature of the calculation. This finding is contrary to many suggestions that the ρ must probably be inserted as a CDD pole.[11,12] Most of the pessimistic conclusions of other authors regarding the ρ simply reflect the fact that in particular calculations, a satisfactory resonance could not be obtained. The paper of Tryon, however, is somewhat different in that his conclusions are based on rigorously derived sum rules of the form[12]

$$\frac{c_n}{\pi}\int_{-\infty}^{1}\frac{d v\,\mathrm{Im} f_1^1(v)}{v^{n+1}} = \frac{1}{3\pi}\int_0^{\infty}\frac{d v\, F_n(v)}{(v+1)^{n+1}}\,, \qquad (5.2)$$

where $v=(s-4)/4$, c_n are constants, and $F_n(v)$ are functions computed from absorptive parts, resonances for small v, Regge terms for large v; explicit forms are given in Ref. 12 for $n=1, 2, 3$.

Unfortunately, although rigorous, these sum rules are extremely difficult to use in practice because of large cancellations. The problem can be seen in Table V where the right-hand sides coming from our own inputs and the left-hand sides corresponding to the calculation of Table IV are given. On the left-hand side, the contributions (5.2) of the cut $-32 \le s \le 0$ come almost completely from the s and p waves, and are nearly balanced by equal contributions to the right-hand side. Thus, if one tries to determine the residues of the poles which replace the left-hand cut for $s<-32$ by matching the left- and right-hand sides of the sum rules, these residues will be determined almost completely by the high-energy region above the ρ where there is substantial inelasticity and amplitudes are poorly determined. In our calculations, the residues actually used lead to quite satisfactory agreement for $n=2, 3$, and for $n=1$ are about half the needed magnitude to bring the two sides into agreement. Noting the large uncertainties of the calculation, the discrepancy is not unreasonable. We should point out here that our own method of determining the pole residues, i.e., matching amplitudes in the gap, does not suffer from such cancellation difficulties. All contributions except the $I=2$ s-wave one are additive, and those coming from energies above the ρ are only about a third of the total amplitude.

In Tryon's calculations, the distant parts of the left-hand cut are estimated by demanding agreement of the right- and left-hand sides of the sum rules, the right-hand sides being respectively (for his inputs) 0.0096, −0.0032, 0.0052 for $n=1, 2, 3$. Because of the cancellations mentioned above, this procedure is clearly dangerous unless a large latitude in high-energy inputs is allowed. Unfortunately, Tryon allows no variation of inputs at all and takes all contributions above the f^0 meson from the Veneziano model. Tryon himself mentions other estimates that lead to substantially smaller high-energy inputs, but does not use them in his actual calculations, claiming that the left-hand cut is already "too weak" to generate a ρ, and that other such estimates lead only to a further weakening of the cut. However, consistency

TABLE V. Contributions to Tryon sum rules.

	Left-hand side			Right-hand side			
	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	
f_0^0	0.0057	−0.0034	0.0040	0.0071	−0.0037	0.0041	f_0^0
f_0^2	−0.0007	0.0003	−0.0003	−0.0007	0.0003	−0.0003	f_0^2
f_1^1	−0.0088	0.0026	−0.0013	−0.0096	0.0027	−0.0015	f_1^1
cut($-32 \le s \le 0$)	−0.0038	−0.0005	0.0024	−0.0032	−0.0007	0.0023	$f_0^0+f_0^2+f_1^1$
				0.0032	−0.0003		f_2^0
				0.0026	0.0005		f_3^1
poles ($s<-32$)	0.0053	−0.0005	0.0001	0.0048	−0.0001		Regge
Total left-hand side	0.0015	−0.0010	0.0025	0.0074	−0.0006	0.0023	Total right-hand side

conditions aside, the fact that the much "weaker" cut used in our own calculations does generate a ρ certainly shows this argument to be incorrect.

A second deficiency of Tryon's calculations is that he uses only a single pole to represent the entire distant part of the left-hand cut. In a situation such as the present one where pole contributions are extremely important (see Table V), such a restriction is rather severe. As a test, we did a calculation by Tryon's method, i.e., determining the pole residues from the sum rules (using our own inputs), but allowing two poles instead of one. More explicitly, the pole residues were taken so as to satisfy the $n=1$ and $n=2$ sum rules exactly (the poles contribute negligibly to the $n=3$ sum rule), the pole positions then being free parameters. As mentioned previously, we do not feel that this is as good a procedure as that of matching amplitudes in the gap, but did it simply to see if a resonance could be generated. We found, in fact, that we could generate a resonance at nearly any desired energy. Though the outputs are by no means satisfactory (the widths are much too large), they do provide a counterexample to Tryon's claim that cuts consistent with the sum rules can give rise only to small output phase shifts.

If the results of our calculations are correct and not simply in some way fortuitous, then it is clear that retaining crossing symmetry is extremely important in producing the ρ. Even with the inputs taken so as to make $R(s)$ very small (the amplitude f_1^{FG} is indistinguishable from f_1 on the scale of Fig. 2), the symmetry term causes a noticeable improvement in the amplitude when continued to the physical region.

The importance of the symmetry term is perhaps made more plausible by considering a greatly simplified analytic situation. Suppose that there is only a ρ meson in the crossed channel, and further, that it is represented in zero-width approximation, $\mathrm{Im}A_1^1(t)=\pi\bar{\Gamma}\delta(t-t_R)$. The Froissart-Gribov and symmetry terms in (3.18) become

$$f_1^{FG}(s)=\frac{6}{s-4}\bar{\Gamma}P_1\left(1+\frac{2s}{t_R-4}\right)Q_1\left(1+\frac{t_R}{s-4}\right),$$

$$f_1^S(s)=\frac{(s-4)(3s-4)\bar{\Gamma}}{(t_R-4)(3t_R-4)(t_R-s)},$$

and $f_1(s)=f_1^{FG}(s)+f_1^S(s)$. In the region $0\leq s\leq 4$, we have $s\ll t_R$, $P_1\simeq 1$, $Q_1\simeq\frac{1}{3}[(s-4)/2t_R]^2$, and f_1^S is small throughout the gap,

$$\frac{f_1^S}{f_1^{FG}}\simeq\frac{2}{3}\frac{(3s-4)}{t_R}\lesssim 0.1 .$$

Nevertheless, if the amplitude is directly continued into the physical region, it is the symmetry term $f_1^S(s)$ which contains the pole at $s=t_R$.

The situation for the real problem is of course much more complex, both because of contributions other than the ρ which reduce considerably the magnitude of the symmetry term and because the continuation to the physical region is done numerically in a much more indirect manner. In Table VI, we show how the addition of the symmetry term affects the way in which the amplitude is built up from N and D in the gap. As can be seen, although the N/D approximation to the Froissart-Gribov amplitude is nearly equal to the symmetry-corrected one, the same is not true of the separate values of N and D. In particular, as s increases, the values of $D(s)$ fall more rapidly for the symmetry corrected case. It is chiefly this steeper decrease in D that leads to the lower resonance energy and reduced width of the ρ meson in the physical region.

Finally, we note that it is possible to improve the output further by relaxing the condition $b=0$ and treating b as a free parameter. Again, exact results vary somewhat depending on details, but for any given set of inputs and matching procedure, the results can always be improved by allowing b to vary. For the particular case reported in Table IV and Fig. 3, taking $b=-0.0002$ gives a minimum Z solution with $m_\rho=765$ MeV, $\Gamma_\rho=148$ MeV (see Table IV).

TABLE VI. Effect of symmetry corrections in the gap.

	Froissart-Gribov			Symmetry corrected		
s	N/D	N	D	N/D	N	D
0.00	−0.031 64	−0.031 64	1.000	−0.031 69	−0.031 69	1.000
0.64	−0.026 59	−0.026 14	0.983	−0.026 61	−0.026 03	0.978
1.28	−0.021 75	−0.021 01	0.966	−0.021 75	−0.020 80	0.956
1.92	−0.016 89	−0.016 01	0.948	−0.016 88	−0.015 76	0.934
2.56	−0.011 91	−0.011 08	0.930	−0.011 90	−0.010 84	0.911
3.20	−0.006 76	−0.006 16	0.912	−0.006 75	−0.006 00	0.888
3.84	−0.001 38	−0.001 23	0.892	−0.001 38	−0.001 19	0.864
4.00	0.000 00	0.000 00	0.887	0.000 00	0.000 00	0.858

B. $l=0$

After crossing symmetry is enforced, the s-wave amplitudes still have the ambiguity (3.9). The procedure we used to resolve this ambiguity was to unitarize f_0 by the N/D method, using different values of λ. As λ was changed, the compatibility of unitarity and the calculated left-hand cut with the amplitude in the gap was measured by noting the change in the matching parameter Z. The results are shown in Fig. 4 where it is seen that Z goes through a clear minimum with a scattering length $a_0 \simeq 0.185$ ($\lambda \simeq -0.00664$). For this value of λ, the N/D calculation produces a high-mass resonance ($m_\epsilon \simeq 1025$ MeV) of the superbroad type ($\Gamma_\epsilon \simeq 1244$ MeV), and the corresponding $I=2$ phase shifts drop to about $-19°$ at the ρ mass. Unfortunately, however, the nature of the $I=0$ phase shifts is changing very rapidly just as Z goes through this minimum, so that small changes in the gap correspond to rather large changes in the ρ region. A further difficulty is that we can find no values of λ which lead to values of Z as small as those obtained in the p-wave case, which indicates some further residual uncertainty. Thus, although our output is certainly consistent with the near-resonant or resonant results currently favored by experiment,[31-33] we cannot make any strong statement at present about the $I=0$ phase shifts. In order to exhibit the rapid variation of the solution, we show phase shifts for the range of scattering lengths $0.16 \leq a_0 \leq 0.22$ in Fig. 3, and list some parameters in Table IV.

On the other hand, the $I=2$ phase shifts are very slowly changing with λ in this region[34] (Fig. 3 and Table IV) and agree well with recent experimental results.[31] Another satisfying point is that Z becomes small for $I=2$ at about the same values of λ as it does for $I=0$ (Fig. 4). The agreement is not perfect, but the results seem quite satisfactory in view of the fact that there is no automatic reason why there should by any correlation between the two minima. The reason for the flat nature of the Z curve on the left side is presumably that the scattering length a_2 is becoming small enough to make unitarity no longer very restrictive.

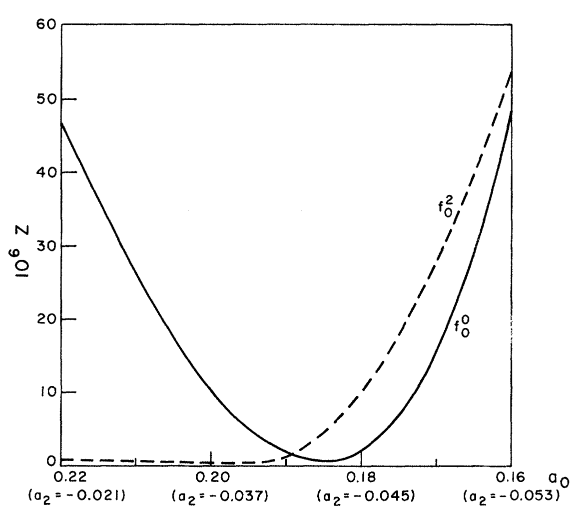

FIG. 4. Matching in the gap for f_0^0, f_0^2.

VI. SUMMARY AND CONCLUSIONS

In the calculations outlined in this paper, we have made strong use of crossing symmetry in two different ways, first by using basic inputs which were adjusted to approximately satisfy physical region constraints, and then by directly imposing crossing symmetry in the region $0 \leq s \leq 4$ when constructing the s and p waves. Unitarity then leads to a determination of the single free parameter, the Chew-Mandelstam coupling constant, and to an output in qualitative agreement with experiment in both the s and p waves.

For the p waves, a reasonably good ρ meson emerges naturally as a dynamical pion-pion resonance without any recourse to CDD poles, cutoffs, or other adjustable parameters. In view of the many negative findings by others, one must of course admit the possibility that our results are somehow accidental or fortuitous. We can find no evidence of this in any of the tests made, however, and the appearance of the ρ seems to be a very stable and persistent consequence of the inputs and method of calculation described. Though the question needs further study, our own results suggest that the failure of other efforts to obtain the ρ dynamically is primarily due to the use of inputs and approximations which violate crossing symmetry.

For the s waves, the most interesting result is that the subthreshold zeros are obtained in a quite natural way using a direct calculation from the crossed channel absorptive parts. As has been noted, the amplitudes are quite similar to those obtained by quite different methods, current-algebra techniques on one hand, and the existence of the ρ (in the direct channel) plus crossing symmetry and positivity on the other. Nevertheless, one must evidently make a clear distinction between results obtained in the gap, which are rather uniquely determined by the inputs and unitarity, and results at appreciable distances above threshold. For $I=2$, there appears to be no problem, i.e., the phase shifts are stable with respect to small changes in the gap, and moreover agree well with experiment. For $I=0$, however, small variations in the gap correspond to rather large changes of the phase shifts in the ρ region, so that while the results are clearly consistent with experiment, no very strong statement can be made.

There are two ways in which these calculations can be improved: through better choice of absorptive parts in the inputs, and through the introduction of inelasticity. The inputs actually used, though reasonable, are certainly not optimal ones, particularly at high energies. Possible improvement here involves the use of more experimental information, and the adjustment of absorptive parts in accordance with crossing symmetry constraints [minimizing $R(s)$], the $\bar{x}, \bar{y}$ test of Yen and Roskies, and ultimately, the self-consistency of output with the input and the Tryon sum rules. The other factor is the use of inelasticity, which has not been introduced at all in these calculations. It is evident now that there is appreciable inelasticity above the $K\bar{K}$ threshold[31,32] and this could be quite important in understanding the s waves.[35] The fact, though, that the major features of the low-energy region have already appeared without the use of inelasticity is highly encouraging.

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APPENDIX A: EXPLICIT FORMS FOR RESONANCE AND REGGE CONTRIBUTIONS

The basic form for the symmetry function is given by (2.25)

$$R(s)=\frac{1}{\pi}\int_4^\infty \frac{dt}{(t-4+2s)(t-s)}\left\{\frac{[2A_t^0(s,t)+3A_t^1(s,t)-5A_t^2(s,t)](2t+s-4)}{6(t-4+2s)(t-s)}-\frac{\partial A_t^1(s,t)}{\partial s}\right\}\quad . \tag{A1}$$

The s-channel isospin amplitudes in (A1) can be written in terms of the t-channel isospin amplitudes via the crossing relation (2.1) as

$$R(s)=\frac{1}{\pi}\int_4^\infty \frac{dt}{(t-4+2s)(t-s)}\left\{\frac{(2t+s-4)A_t^1(t,s)}{(t-4+2s)(t-s)}-\frac{\partial}{\partial s}\sum_{I'=0}^{2}\beta^{1I'}A_t^{I'}(t,s)\right\}\quad . \tag{A2}$$

Inserting the t-channel partial-wave expansion

$$A_t^{I'}(t,s)=\sum_{l'=0}^{\infty}(2l'+1)\,\mathrm{Im}\,A_{l'}^{I'}(t)P_{l'}\left(1+\frac{2s}{t-4}\right), \tag{A3}$$

where $l'+I'$ is even, the $l'=0$ terms in (A2) cancel and leave an expansion for $R(s)$:

$$R(s)=\sum_{I'=0}^{2}\sum_{l'=1}^{\infty}(2l'+1)\beta^{1I'}\int_4^\infty dt\,\mathrm{Im}\,A_{l'}^{I'}(t)R_{l'}^{I'}(s,t), \tag{A4}$$

$$\int_{4/3}^{s}ds'R(s')=\sum_{I'=0}^{2}\sum_{l'=1}^{\infty}(2l'+1)\beta^{1I'}\int_4^\infty dt\,\mathrm{Im}\,A_{l'}^{I'}(t)\int_{4/3}^{s}ds'R_{l'}^{I'}(s',t). \tag{A5}$$

Explicit expressions for the first few l' are

$$R_1^1(s,t)=\frac{2}{(t-4)(t-s)^2}\quad , \tag{A6}$$

$$\int_{4/3}^{s}ds'R_1^1(s',t)=\frac{3s-4}{(t-4)(t-s)(3t-4)}\quad , \tag{A7}$$

$$R_2^{0,2}(s,t)=-\frac{6}{(t-4)^2(t-s)}\quad , \tag{A8}$$

$$\int_{4/3}^{s}ds'R_2^{0,2}(s',t)=\frac{6}{(t-4)^2}\ln\frac{3(t-s)}{3t-4}\quad , \tag{A9}$$

$$R_3^1(s,t)=\frac{2}{(t-s)^2(t-4)^3}[20s^2+5(t-12)s+16+12t-4t^2]\,, \tag{A10}$$

$$\int_{4/3}^{s}ds'R_3^1(s',t)=\frac{1}{(t-4)^3}\left[\frac{(21t^2-48t+16)(3s-4)}{(t-s)(3t-4)}+\frac{20(3s-4)}{3}+(45t-60)\ln\frac{3(t-s)}{3t-4}\right]\quad . \tag{A11}$$

In actual use, the t integrals are cut off at a finite value $t=t_0$, and for $t>t_0$, s-channel Regge poles

$$A_t^I(s,t)=\gamma_I(s)t^{\alpha_I(s)} \tag{A12}$$

are used. Keeping $I=0$ (Pomeron) and $I=1$ (ρ) terms in (A1) gives a large-t contribution:

$$R^H(s)\simeq\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt}{t^2}\left\{\frac{2\gamma_P(s)t^{\alpha_P(s)}+3\gamma_\rho(s)t^{\alpha_\rho(s)}}{3t}-\gamma_\rho'(s)t^{\alpha_\rho(s)}-\gamma_\rho(s)\alpha_\rho'(s)t^{\alpha_\rho(s)}\ln t\right\}$$

$$=\frac{2}{3\pi}\frac{\gamma_P(s)}{(2-\alpha_P)t_0^{2-\alpha_P}}+\frac{1}{\pi t_0^{1-\alpha_\rho}}\left\{\frac{\gamma_\rho}{(2-\alpha_\rho)t_0}-\frac{1}{1-\alpha_\rho}\left[\gamma_\rho'+\gamma_\rho\alpha_\rho'\left(\ln t_0+\frac{1}{1-\alpha_\rho}\right)\right]\right\}\ . \tag{A13}$$

Because of the additional factor $1/t_0$ in the first term, R^H is dominated by the ρ contribution and gives an important cancellation to keep $R(s)$ small.

We also consider here the contributions to the s-wave Froissart-Gribov solutions $f^{\rm FG}$, $g^{\rm FG}$. For small t, the t-channel partial wave expansions (A3) are again used. For $f^{\rm FG}$, given by (3.12) and (3.20)–(3.24), one finds that the $I=1$ isospin terms cancel:

$$f^{\rm FG}(s)=\frac{1}{\pi}\int_4^\infty dt\sum_{I'=0}^{2}\sum_{l'=0}^{\infty}(2l'+1)\,{\rm Im}f_{l'}^{I'}(t)(-\beta^{0I'}-2\beta^{2I'})\left[\overline{R}_0(s,t)P_{l'}\left(1+\frac{2s}{t-4}\right)+2I_{l'}-2J_{l'}\right]$$

$$=\frac{1}{\pi}\int_4^\infty dt\sum_{l'=0\ \rm even}^{\infty}(2l'+1)[-{\rm Im}f_{l'}^0(t)-2\,{\rm Im}f_{l'}^2(t)]\left[\overline{R}_0(s,t)P_{l'}\left(1+\frac{2s}{t-4}\right)+2I_{l'}-2J_{l'}\right]\ , \tag{A14}$$

$$I_{l'}(s,t)=\int_{4/3}^{s}ds'\,\frac{P_{l'}\left(1+\frac{2s'}{t-4}\right)}{(t-4+2s)^2}\ , \tag{A15}$$

$$J_{l'}(s,t)=\int_{4/3}^{s}ds'\,\frac{P_{l'}\left(1+\frac{2s'}{t-4}\right)}{(t-s)^2} \tag{A16}$$

In our calculations, only $l'=0, 2$ were used,

$$I_0-J_0=-\frac{(3s-4)^2}{(3t-4)(t-4+2s)(t-s)}\ , \tag{A17}$$

$$I_2-J_2=-\frac{3}{2}\frac{(3s-4)(4t-s-4)}{(t-4)^2(t-s)}-\frac{6(3t-4)}{(t-4)^2}\ln\frac{3(t-s)}{3t-4}-\tfrac{1}{2}(I_0-J_0)\,. \tag{A18}$$

$g^{\rm FG}$ is easily found from (3.13):

$$g^{\rm FG}(s)=\frac{1}{\pi}\int_4^\infty dt\left\{\sum_{l'\rm odd}(2l'+1)\,{\rm Im}f_{l'}^1(t)P_{l'}\left(1+\frac{2s}{t-4}\right)\left[\frac{-6}{t-4+2s}-\frac{3}{t-s}+\frac{27}{6}\frac{2}{s-4}\ln\frac{t+s-4}{t}\right]\right.$$

$$+\sum_{l'\rm even}(2l'+1)P_{l'}\left(1+\frac{2s}{t-4}\right)\left[\left(\frac{2}{t-s}-\frac{2}{s-4}\ln\frac{t+s-4}{t}\right){\rm Im}f_{l'}^0(t)\right.$$

$$\left.\left.+\left(\frac{-5}{t-s}+\frac{15}{6}\frac{2}{s-4}\ln\frac{t+s-4}{t}\right){\rm Im}f_{l'}^2(t)\right]\right\}. \tag{A19}$$

Again, in actual use, Regge forms are used for $t>t_0$:

$$f_H^{\rm FG}(s)=\frac{1}{\pi}\int_{t_0}^\infty dt[-A_t^0(s,t)\overline{F}_0(s,t)]+\frac{2}{\pi}\int_{t_0}^\infty dt\int_{4/3}^{s}ds'[-A_t^0(s',t)\Delta_t\overline{F}_0(s',t)]\ ,$$

and with the Pomeron in (A12) and large t limits for $\overline{F}_0, \Delta_t\overline{F}_0$,

$$f_H^{\rm FG}(s)\simeq\frac{-\gamma_P(s)(16-32s+13s^2)}{3\pi[2-\alpha_P(s)]t_0^{2-\alpha_P}}-\frac{4}{\pi}\int_{4/3}^{s}ds'\,\frac{(4-3s')\gamma_P(s')}{[2-\alpha_P(s')]t_0^{2-\alpha_P(s')}}. \tag{A20}$$

In the same way, from (3.13),

$$g_H^{\rm FG}(s)=\frac{1}{\pi}\int_{t_0}^\infty dt[-2A_t^0(s,t)\overline{F}_0(s,t)-3A_t^1(s,t)\overline{F}_1^{\rm FG}(s,t)]$$

$$\simeq-\frac{2}{3\pi}\frac{(16-32s+13s^2)\gamma_P}{(2-\alpha_P)t_0^{2-\alpha_P}}-\frac{3}{\pi}\frac{(4-3s)\gamma_\rho}{(1-\alpha_\rho)t_0^{1-\alpha_P}}. \tag{A21}$$

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[1]For a review of the rigorous constraints, see A. Martin, *Scattering Theory: Unitarity, Analyticity and Crossing* (Springer, New York, 1969); for a more recent review see G. Wanders, in *Springer Tracts in Modern Physics*, edited by G. Höhler (Springer, New York, 1971), Vol. 57, p. 22.

[2]G. Wanders and O. Piguet, Nuovo Cimento 56A, 417 (1968); G. Auberson, O. Piguet, and G. Wanders, Phys. Lett. 28B, 41 (1968); R. H. Graham and R. C. Johnson, Phys. Rev. 188, 2362 (1969); J. L. Basdevant and B. W. Lee, Phys. Rev. D 2, 1680 (1970); R. Baier, H. Kühnelt, and F. Widder, Acta Phys. Austriaca 32, 65 (1970); B. Bonnier, Nucl. Phys. B10, 467 (1969); B17, 525 (1970); B. Bonnier and P. Gauron, *ibid.* B21, 465 (1970); H. Kühnelt, Z. Phys. 246, 448 (1971).

[3]J. B. Carrotte and R. C. Johnson, Durham report, 1972 (unpublished).

[4]F. Widder, Acta Phys. Austriaca 34, 264 (1971).

[5]J. C. Le Guillou, A. Morel, and H. Navelet, Nuovo Cimento 5A, 659 (1971).

[6]B. Bonnier and P. Gauron, Nucl. Phys. B36, 11 (1972).

[7]S. Weinberg, Phys. Rev. Lett. 17, 616 (1966).

[8]E. P. Tryon, Phys. Lett. 36B, 470 (1971); A. Piguet and G. Wanders, Nucl. Phys. B46, 295 (1972); J. L. Basdevant, C. D. Froggatt, and J. L. Petersen, Phys. Lett. 41B, 172 (1972); 41B, 178 (1972); B. Bonnier and P. Gauron, Nucl. Phys. B52, 506 (1973); A. Arneodo and J. T. Donohue, Université de Nice Report No. NTH 72-25, 1972 (unpublished).

[9]R. Roskies, Nuovo Cimento 65A, 467 (1970).

[10]Throughout, we use units with $m_\pi = 1$ unless MeV is specifically stated. The normalization of the amplitudes is such that unitarity for the partial waves is $\text{Im} f_l(s) = [(s-4)/s]^{1/2} |f_l(s)|^2$.

[11]R. Majumdar and V. S. Varma, Nucl. Phys. B16, 364 (1970); R. C. E. Devenish, J. C. Eilbeck, and D. H. Lyth, *ibid.* B37, 313 (1971); B. R. Webber, Phys. Rev. D 3, 1971 (1971); C. B. Lang and P. Urban, Graz report, 1972 (unpublished). A notable exception is the recent calculation of S. C. Prasad and J. J. Brehm [Phys. Rev. D6, 3216 (1972)] using current algebra, crossing symmetry, and unitarity.

[12]E. P. Tryon, Phys. Lett. 38B, 527 (1972).

[13]This condition can also be derived from the work of G. Wanders, Nuovo Cimento 63A, 108 (1969), Eq. (2.6).

[14]J. Dilley, Nucl. Phys. B46, 275 (1972). The definition of $R(s)$ differs by a factor π from the present paper.

[15]G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

[16]C. Lovelace, Nuovo Cimento 21, 305 (1961).

[17]J. L. Basdevant, J. C. Le Guillou, and H. Navelet, Nuovo Cimento 7A, 363 (1972).

[18]A. Martin, Nuovo Cimento 47A, 265 (1967); 58A, 303 (1968); A. K. Common, *ibid.* 53A, 946 (1968); G. Auberson, *ibid.* 68A, 281 (1970).

[19]H. C. Yen and R. Roskies, Phys. Rev. D 4, 1873 (1971).

[20]The fact that a_1 and L are completely determined by the absorptive parts has also been pointed out by S. Krinsky, Phys. Rev. D 4, 1046 (1971).

[21]D. Morgan and G. Shaw, Phys. Rev. D 2, 520 (1970).

[22]M. G. Olsson, Phys. Rev. 162, 1338 (1967); J. H. Scharenguivel *et al.*, Nucl. Phys. B22, 16 (1971).

[23]M. R. Pennington and P. Pond, Nuovo Cimento 3A, 548 (1971).

[24]G. W. Bennett and R. C. Johnson, Phys. Lett. 36B, 483 (1971); J. Dilley, Nucl. Phys. B25, 227 (1970).

[25]D. Morgan and G. Shaw, Nucl. Phys. B43, 365 (1972). The solution listed in Table III is their "dual quartic" one.

[26]A further discussion of the importance of these parameters is given in the "mini-rapporteur" talk of D. Morgan at the Amsterdam International Conference on Elementary Particles, 1971 [see Rutherford Report No. RPP/C/30 (unpublished)].

[27]T. Gibbons and J. Dilley, Phys. Rev. D 3, 1196 (1971).

[28]J. S. Kang and B. W. Lee, Phys. Rev. D 3, 2814 (1971).

[29]E. P. Tryon, Phys. Rev. D 5, 1039 (1972).

[30]M. L. Mehta and P. K. Srivastava, Phys. Rev. 137, B423 (1965); A. H. Bond, *ibid.* 147, 1058 (1966); M. R. Williamson and A. E. Everett, *ibid.* 147, 1074 (1966); A. F. Antippa and A. E. Everett, *ibid.* 178, 2443 (1969); 186, 1571 (1969).

[31]J. P. Baton, G. Laurens, and J. Reignier, Phys. Lett. 33B, 528 (1970).

[32]B. Y. Oh *et al.*, Phys. Rev. D1, 2494 (1970).

[33]P. Baillon *et al.*, Phys. Lett. 38B, 555 (1972).

[34]For $I = 2$, only one pole was used in the N/D calculations. As seen in Table IV, the match is as good as for the two poles in the $I = 0$ case, and when two poles were tried for $I = 2$ the phase shifts became unstable and often fell rapidly through $-90°$, indicating the presence of ghost states. With one pole, the matching parameter and output changed smoothly with λ.

[35]C. Lovelace and F. Wagner, Nucl. Phys. B28, 141 (1971); M. Davier and H. Harari, Phys. Lett. 35B, 239 (1971); M. Alston-Garnjost *et al.*, *ibid.* 36B, 152 (1971); W. Beusch, in *Proceedings of the Philadelphia Conference on Meson Spectroscopy*, edited by C. Baltay and A. Rosenfeld (Columbia Univ. Press, New York, 1970).