Photon Structure Functions for the Annihilation Process $e^+e^- \to n\pi^+n\pi^- + \gamma$

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The differential cross section for the annihilation process $e^+e^- \to X + \gamma$ is compared with the cross section calculated by Terazawa for the colliding-beam process $e^+e^- \to n\pi^+n\pi^- + \gamma$, thereby approximating X to be $n\pi^+n\pi^-$. We thus obtain the contribution to the annihilation photon structure functions $\overline{W}_i^\gamma(q^2, \nu)$ due to the n pion pairs. These functions are found to be related to the structure functions $W_i^\gamma(q^2, \nu)$ for the electroproduction process $e^- + \gamma \to n\pi^+n\pi^- + e^-$ by a valid continuation.

I. INTRODUCTION

Experimental measurements performed at Frascati,[1] Novosibirsk,[2] and Orsay[3] have been reported on the colliding-beam process $e^+e^- \to$ hadrons. At $s \sim 1$ to 6 GeV^2, multibody production is realized by colliding e^+e^- beams and these data serve as a check on the validity of the one-photon exchange. The cross-section ($\sim 10^{-32}$ cm^2) behavior seems to decrease slowly with s. It is known that for $s \leq 1$ GeV^2, the cross section is described well by the vector-meson (ρ, ω, ϕ) peaks. Above this range, however, the inelastic process $e^+e^- \to X + c$, where X is anything, is of interest and annihilation structure functions may be defined for the observed hadron c.

II. CROSS SECTION AND SUM RULE

The differential cross section for the colliding-beam process

$$e^+(k) + e^-(k') \to X(p_n) + c(p)$$

may be written as

$$\frac{d\sigma(e^+e^- \to X + c)}{dp_0 d(\cos\theta)} = \frac{2\pi\alpha^2}{s^3}\left(\frac{\nu^2}{q^2} - m_c{}^2\right)^{1/2}(k^\mu k'^\nu + k^\nu k'^\mu - \tfrac{1}{2}sg^{\mu\nu})\overline{W}^c_{\mu\nu}\,, \tag{1}$$

where we have defined $\nu = p\cdot q$, $q = k + k'$, and the annihilation structure functions for particle c have been defined by

$$\begin{aligned}\overline{W}^c_{\mu\nu} &= (2\pi)^2 2p_c \sum_n (2\pi)^4\delta^4(q - p_n - p)\langle 0|\,J_\mu(0)\,|p_n p\rangle\langle p_n p|J_\nu(0)|\,0\rangle \\ &= -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\overline{W}^c_1(q^2, \nu) + \left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right)\overline{W}^c_2(q^2, \nu)\,.\end{aligned} \tag{2}$$

It follows from Eqs. (1) and (2) that the differential cross section may be expressed in the form

$$\frac{d\sigma(e^+e^- \to X + c)}{dp_0 d(\cos\theta)} = \frac{2\pi\alpha^2}{s^3}\,|\vec{p}|\left\{q^2\overline{W}^c_1(q^2, \nu) + \left[y(1-y) - \frac{m_c{}^2 s}{4\nu^2}\right]2\nu^2\overline{W}^c_2(q^2, \nu)\right\}, \tag{3}$$

where y is defined by $y = p\cdot k/\nu$ and θ is the scattering angle with respect to the incident positron beam. For one detected hadron c in the final state, the differential cross section may be written as[4] (with $z = \cos\theta$)

$$\frac{d\sigma(e^+e^- \to X + c)}{dp_0 dz} = \frac{\pi\alpha}{s^{3/2}}\left[(1 - z^2)\Gamma^c_L(q^2, p_0) + (1 + z^2)\Gamma^c_T(q^2, p_0)\right], \tag{4}$$

where $\Gamma^c_{L,T}(q^2, p_0)$ is the partial width of the (longitudinal, transverse) virtual photon into particle c plus anything. The angular integration for the inclusive process may be performed in Eq. (4) and the differential cross section is then expressed in terms of the averaged partial width $\Gamma^c(q^2, p_0)$ by[5]

$$\frac{d\sigma^c}{dp_0} = \frac{4\pi\alpha}{s^{3/2}}\Gamma^c(q^2, p_0)\,, \tag{5}$$

where

$$\Gamma^c(q^2, p_0) = \tfrac{1}{3}[\Gamma^c_L(q^2, p_0) + 2\Gamma^c_T(q^2, p_0)]\,. \tag{6}$$

The total cross section for e^+e^- annihilation into hadrons is expressible in terms of the decay width $\Gamma(\gamma\to\text{hadrons})$ of a virtual photon of mass $s^{1/2}$ into hadrons by[4]

$$\sigma_{\text{tot}}(e^+e^-\to\gamma\to\text{hadrons})=\frac{4\pi\alpha}{s^{3/2}}\Gamma(\gamma\to\text{hadrons}). \tag{7}$$

The multiplicity $\overline{N}_c$ of hadrons of type c is then defined by an integration of the energy of particle c and is

$$\int_{m_c}^{\infty}\frac{4\pi\alpha}{(q^2)^{3/2}}\Gamma(q^2,p_0)dp_0=\overline{N}_c(s)\sigma_{\text{tot}}(s). \tag{8}$$

Furthermore, from energy conservation it follows that by summing over all different types of hadrons

$$\sum_i\int_{m_i}^{\infty}\frac{4\pi\alpha}{(q^2)^{3/2}}\Gamma^i(q^2,p_0)p_0dp_0=s^{1/2}\sigma_{\text{tot}}(s). \tag{9}$$

We assume that the annihilation structure functions are related by[6,7]

$$\overline{W}_1^c(q^2,\nu)=-\frac{\nu^2}{q^2}\overline{W}_2^c(q^2,\nu). \tag{10}$$

This relation is analogous to similar relationships between the proton structure functions[8] obtained from deep-inelastic electron-proton scattering and also is seen to hold for the annihilation pion structure functions[6] for $e^+e^-\to\pi^\pm+\text{hadrons}$. Performing the θ integration in Eq. (3) and using Eq. (10), we find from Eq. (5) that

$$\Gamma^c(q^2,\nu)=\frac{\alpha}{(q^2)^{3/2}}\left(\frac{\nu^2}{q^2}-m_c^2\right)^{1/2}\left(\frac{2}{3}+\frac{m_c^2q^2}{2\nu^2}\right)\times\left[-\nu^2\overline{W}_2^c(q^2,\nu)\right], \tag{11}$$

where we have used $p_0=\nu s^{-1/2}$. Equation (11) establishes the connection between the structure functions and the partial width.

It is possible to write a sum rule of the Bloom and Gilman[9] type. We assume, defining $\omega=-2\nu/q^2$ [with $m_i(q^2)^{1/2}\le\nu\le\frac{1}{2}q^2$], that

$$(q^2)^{1/2}\int_{m_i}^{(p_0)_{\max}}\frac{4\pi\alpha}{(q^2)^{3/2}}\Gamma^i(q^2,p_0)p_0dp_0=\pi\alpha\int_{-1}^{0}\Gamma^i(\omega)\omega d\omega, \tag{12}$$

which may hold in the Bj (Bjorken) scaling-limit sense.

The partial-width expression, Eq. (11), assumes the scaling-limit form

$$\Gamma^c(q^2,\nu)\xrightarrow[\nu\to\infty]{q^2\to\infty}-\tfrac{1}{6}\alpha\omega^2\nu\overline{W}_2^c(\omega)\equiv\Gamma^c(\omega), \tag{13}$$

and therefore from the sum rule, Eq. (12), and Eqs. (13) and (9), we are able to write

$$\sum_i\int_{m_i}^{\infty}\frac{4\pi\alpha}{(q^2)^{3/2}}\Gamma^i(q^2,p_0)p_0dp_0=-\frac{\pi\alpha^2}{6s^{1/2}}\sum_i\int_{-1}^{0}\omega^3\nu\overline{W}_2^i(\omega)d\omega=s^{1/2}\sigma_{\text{tot}}(s). \tag{14}$$

Our derivation of the sum rule, Eq. (14), relates the integral of $\omega^3\nu\overline{W}_2^i(\omega)$ over ω to the total cross section for colliding e^+e^- beams. Layssac and Renard[7] have used finite-energy sum rules (FESR) and the duality concept for pion structure functions integrated over the physical region.

III. SOFT-PION MODEL, $X=n\pi^+n\pi^-$

We next consider a specific model, namely, the process shown in Fig. 1. Terazawa,[10] using the soft-pion technique, partial conservation of the axial-vector current, and current-algebra methods, has developed a gauge-invariant matrix element applicable to the process $\gamma+\gamma\to n\pi^+n\pi^-$. This matrix element has been utilized in a number of related processes.[10] We shall consider the differential cross section for the annihilating beam process

$$e^+(k)+e^-(k')\to n\pi^+(p_i)n\pi^-(q_i)+\gamma(p).$$

It reads[10]

$$\frac{d\sigma(e^+e^-\to n\pi^+n\pi^-+\gamma)}{dp_0d(\cos\theta)}=\frac{\pi^2(4\pi\alpha)^3}{4E^4}p_0(1-\tfrac{1}{2}\sin^2\theta)\left(\frac{3}{F_\pi^2}\right)^{2(n-1)}\frac{1}{(n!)^2}\left[\frac{1}{F_\pi^2}\int dm^2\frac{\rho_V(m^2)-\rho_A(m^2)}{m^2-2Ep_0}\right]^2$$
$$\times\int\prod_{i=1}^{n}\frac{d^3p_i}{(2\pi)^32p_{i0}}\frac{d^3q_i}{(2\pi)^32q_{i0}}\delta^4\Big(q-p-\sum_i p_i-\sum_i q_i\Big), \tag{15}$$

where E is the energy of each one of the lepton beams, and $q=(2E,\vec{0})$, $p=(p_0,0,p_0\sin\theta,\ p_0\cos\theta)$ are the four-momenta of the exchanged and emitted photons, respectively, as shown in Fig. 1. The vector and axial-vector spectral functions in Eq. (15) may be approximated by pole-dominance forms, that is, $\rho_V(m^2)=g_\rho^2\delta(m^2-m_\rho^2)$ and $\rho_A(m^2)=g_A^2\delta(m^2-m_A^2)$. From Weinberg's[11] first and second sum rules and the

Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin[12] (KSRF) relation $g_\rho^2 F_\pi^{-2} = 2m_\rho^2$, it follows that $g_\rho^2 = g_A^2$ and $m_A^2 = 2m_\rho^2$. The factor in square brackets in Eq. (15) containing the spectral functions then may be expressed as

$$\left[\frac{1}{F_\pi^2}\int dm^2 \frac{\rho_V(m^2) - \rho_A(m^2)}{m^2 - 2Ep_0}\right]^2 = \left[\frac{2m_\rho^4}{(m_\rho^2 - \nu)(2m_\rho^2 - \nu)}\right]^2, \tag{16}$$

where we have set $\nu = p\cdot q = 2Ep_0$.

The multipion phase-space integration in Eq. (15) may be accomplished due to a method developed by Bjorken and Brodsky.[13] We find

$$\int \prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3 2p_{i0}} \frac{d^3q_i}{(2\pi)^3 2q_{i0}} (2\pi)^4 \delta\Big(q - p - \sum_i p_i - \sum_i q_i\Big) = \frac{1}{8\pi} \frac{2n-1}{[(2n-1)!]^2} \left[\frac{(q-p)^2}{16\pi^2}\right]^{2(n-1)}. \tag{17}$$

It is convenient to define $y = (p\cdot k)/(p\cdot q)$ to specify the scattering angle θ of the emitted photon with respect to the incident e^+ beam. It follows that $y = \sin^2(\theta/2)$. The differential cross section in Eq. (15) may be brought, using Eqs. (16) and (17), into the form

$$\frac{d\sigma(e^+e^- \to n\pi^+ n\pi^- + \gamma)}{dp_0 d(\cos\theta)} = \frac{2\pi\alpha^2}{s^3} p_0 \frac{4\alpha}{\pi} \frac{1}{(n!)^2} \frac{2n-1}{[(2n-1)!]^2} \left[\frac{m_\rho^4}{(m_\rho^2 - \nu)(2m_\rho^2 - \nu)}\right]^2 \left[\frac{3(q^2 - 2\nu)}{16\pi^2 F_\pi^2}\right]^{2(n-1)} [q^2 - 2y(1-y)q^2]. \tag{18}$$

Two different extreme extrapolation procedures have been employed in the derivation of Eq. (18). The soft-pion limit has been used in the development of Eq. (15) and the hard-pion limit has been taken in the phase factor in Eq. (17). The differential cross section, Eq. (18), therefore has a questionable reliability. Nevertheless, we shall use it to extract the annihilation photon structure functions. If we assume that the n-pion-pair hadron state may largely contribute to the photon structure functions, this approximation is realized by comparing the two cross sections, Eqs. (18) and (3), when the mass m_c is set to zero. We find

$$\overline{W}_1^\gamma(q^2, \nu) = \frac{4\alpha}{\pi} \frac{2n-1}{[(2n-1)!]^2} \left[\frac{m_\rho^4}{(m_\rho^2 - \nu)(2m_\rho^2 - \nu)}\right]^2 \times \left[\frac{3(q^2 - 2\nu)}{16\pi^2 F_\pi^2}\right]^{2(n-1)} \tag{19}$$

and

$$\nu\overline{W}_2^\gamma(q^2, \nu) = \frac{-4\alpha}{\pi} \frac{2n-1}{[(2n-1)!]^2} \left[\frac{m_\rho^4}{(m_\rho^2 - \nu)(2m_\rho^2 - \nu)}\right]^2 \times \frac{q^2}{\nu}\left[\frac{3(q^2 - 2\nu)}{16\pi^2 F_\pi^2}\right]^{2(n-1)}. \tag{20}$$

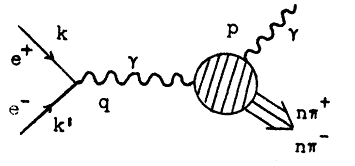

FIG. 1. The n-pion-pair contribution to the photon structure functions by colliding e^+e^- beams.

It may be noted that Eqs. (19) and (20) are in agreement with the relationship between the photon structure functions $\overline{W}_i^\gamma$, Eq. (10), for the case $c = \gamma$.

It is interesting to compare the structure functions W_i^γ for the electroproduction process[14]

$$e^- + \gamma \to n\pi^+ n\pi^- + e^-$$

with those obtained in this paper, Eqs. (19) and (20). It follows that

$$\overline{W}_i^\gamma(q^2, \nu) = W_i^\gamma(q^2, -\nu). \tag{21}$$

Therefore, in this model, the continuation of $W_i^\gamma(q^2, -\nu)$ with $q^2 < 0$ to $\overline{W}_i^\gamma(q^2, \nu)$ with $q^2 > 0$ seems to be valid. However, in general, this continuation is not valid. This problem has recently been investigated by Gatto and Preparata[15] and seems to be model-dependent.

The Bjorken limit[16] of the photon structure functions, Eqs. (19) and (20), may be taken. Writing the scaling-limit functions in terms of the scaling variable ω, this scaling condition is found to hold only for the case $n = 3$ as was also the case in the electroproduction process.[14] It follows that ($q^2 \to \infty$, $\nu \to \infty$)

$$\overline{W}_1^\gamma(q^2, \nu) \to \frac{64\alpha}{\pi} \frac{5}{(5!)^2} \left(\frac{3m_\rho^2}{16\pi^2 F_\pi^2}\right)^4 \frac{1}{\omega^4}(\omega + 1)^4 \tag{22}$$

and

$$\nu\overline{W}_2^\gamma(q^2, \nu) \to \frac{128\alpha}{\pi} \frac{5}{(5!)^2} \left(\frac{3m_\rho^2}{16\pi^2 F_\pi^2}\right)^4 \frac{1}{\omega^5}(\omega + 1)^4 \tag{23}$$

in the scaling limit with ω fixed.

IV. CONCLUSION

The applicability of the soft-pion result given in Ref. 10 seems somewhat limited because it was originally derived for $p_i \simeq q_i \simeq 0$ and therefore we would have $q \simeq p$. However, Yan[17] has considered the process

$$ee \to e\,e + \gamma(q_1) + \gamma(q_2)$$

with

$$\gamma(q_1) + \gamma(q_2) \to \pi^+(k_1) + \pi^-(k_2)$$

and he has extended the soft-pion result to the region where q_1^2 and q_2^2 are large. Hence, it is unknown and rather doubtful that we can take the soft-pion amplitude seriously for q^2 large and $p^2 = 0$ as we have done.

Another comment worth noting is that the saturation of the spectral function integral in Eq. (15) can be accepted for q^2 small, but is definitely contrary to the philosophy of taking the Bj (Bjorken) limit.

The photon structure functions derived in this paper are seen to have Bj scaling behavior for the case $n = 3$. We have shown that the continuation condition, Eq. (21), holds in this model. Because of the assumptions mentioned and those made for the phase-space integral, the number $n = 3$ seems to be purely accidental.

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