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Spin-3/2 Polarization Density Matrix in the Electroproduction of $\Delta(1236)$

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The production process $ep \to e\Delta$ is considered assuming one-photon exchange. Our way of parametrizing the spin-$\frac{3}{2}$ density matrix is compared with the formalism of Doncel, Michel, and Minnaert. From the explicit expression for the process in terms of three form factors, we obtain the hadronic current tensor when Δ is polarized and the density-matrix elements of Δ. It is shown that separate values of three form factors can be obtained explicitly from the differential cross section and density matrix. Further applications of this approach are suggested.

I. INTRODUCTION

In order to discuss particle reactions in general, it is not enough to consider only the differential cross section of the reactions; the complete information on the physics of the reactions also requires measurement of the particle polarization. In particular, if a particle with a high spin is produced, its decay distribution depends on its polarization and is expressed in terms of its density-matrix elements. The investigation of the density-matrix elements gives further information on the reactions.

Recently Doncel, Michel, and Minnaert[1,2] have discussed in detail the polarization density matrix and its measurement. They have also described the domain for the values of the density-matrix elements into which the experimentally measurable values must fall. However, they have not considered the detailed description of the production process.

On the other hand, we have previously shown a straightforward way to obtain the density-matrix elements for a spin-$\frac{3}{2}$ particle in specific processes.[3,4] This method can be used in any process when the transition amplitude of the reaction is expressed covariantly.

In the reaction $ep \to e\Delta(1236)$, one-photon exchange is usually assumed to dominate, and it is well known that the hadronic current contains only three independent form factors, which are related to the Coulomb, transverse-electric, and transverse-magnetic multipoles.[5,6]

Recently Jones and Scadron[7] have reconsidered the $\gamma N\Delta$ vertex and discussed the correspondence between form factors considered by various authors. Explicit values of these form factors have been predicted by Kleinert,[8] using the O(4, 2) current, and by Pritchett *et al.*[9,10] These authors have compared their values with experimental data on the differential cross section. However, the differential cross section does not give information on the two form factors corresponding to electric and magnetic multipoles separately. One of the simplest ways to obtain the explicit values of form factors experimentally is to consider the angular distribution of decay products of Δ or, in other words, the density-matrix elements of Δ.

The purpose of this paper is to connect our way of parametrizing the spin-$\frac{3}{2}$ density matrix with that of Doncel, Michel, and Minnaert[1,2], to obtain the hadronic current tensor covariantly in the process $ep \to e\Delta$ when Δ is polarized, and to express density-matrix elements of Δ in terms of momenta and form factors. The Gottfried-Jackson frame will be considered, and the result will be useful to investigate the momentum transfer dependence of form factors as well as the possible nonreality of the form factors considered by Bjorken and Walecka.[6] The same method can be applied to other production processes of Δ, $\Delta\pi$, or $\Delta\Delta$ by electrons, photons, and neutrinos.

II. DENSITY MATRIX

If the explicit form of the transition amplitude of a reaction producing a spin-$\frac{3}{2}$ particle is given covariantly, it is always possible to obtain its density matrix in terms of the form factors and momenta of particles in the form

$$\rho = \tfrac{1}{4}\left(I + \frac{2}{A}B_iS_i + \frac{1}{2A}C_{ij}S_{ij} + \frac{1}{6A}D_{ijk}S_{ijk}\right). \quad (1)$$

Here A, B_i, C_{ij}, and D_{ijk} are functions of form factors and momenta of the particles; I, S_i, S_{ij}, and S_{ijk} are 4×4 matrices and they are orthogonal in the sense that the product of any two of them is traceless. The matrices S_{ij} and S_{ijk} are symmetric with respect to the interchange of any two indices, and they are zero when two of the indices are contracted, but they are not orthogonal among themselves. A straightforward way to obtain the explicit values of A, B_i, C_{ij}, and D_{ijk} is given for specific processes in Refs. 3 and 4, but this method is applicable to any process when the transition amplitude of the reaction is expressed covariantly.

This density matrix can also be expressed in terms of the multipole parameters t_L^M as[11]

$$\rho = \frac{1}{(2S+1)} \sum_{L=0}^{2S} \sum_{M=-L}^{L} (2L+1)\, t_L^{M*} T_L^M . \qquad (2)$$

Doncel, Michel, and Minnaert[1,2] prefer to introduce a set of Hermitian matrices $Q_M^{(L)}$ and a set of real parameters $r_M^{(L)}$ and write Eq. (2) as

$$\rho = \frac{1}{2S+1} I + \frac{2S}{2S+1} \sum_{L=1}^{2S} \sum_{M=-L}^{L} Q_M^{(L)} r_M^{(L)} . \qquad (3)$$

For $S=\frac{3}{2}$, $Q_M^{(L)}$'s are 4×4 matrices and they are orthogonal in the sense that

$$\mathrm{Tr}\{Q_M^{(L)} Q_{M'}^{(L')}\} = \tfrac{4}{3}\, \delta_{LL'} \delta_{MM'} . \qquad (4)$$

The matrices S_i, S_{ij}, and S_{ijk} are related to $Q_M^{(1)}$, $Q_M^{(2)}$, and $Q_M^{(3)}$, respectively. Equation (1) has some advantage especially when the production process is considered.

Because A, B_i, C_{ij}, and D_{ijk} contain all the information on the reaction from the explicit form of the covariant transition amplitude of a production process, and because the $r_M^{(L)}$ are fully discussed by Doncel, Michel, and Minnaert in connection with decay processes, it is useful to connect them explicitly. The result is

$$\begin{aligned}
r_1^{(1)} &= (\tfrac{5}{3})^{1/2} \frac{B_1}{A}, \quad r_{-1}^{(1)} = (\tfrac{5}{3})^{1/2} \frac{B_2}{A}, \quad r_0^{(1)} = (\tfrac{5}{3})^{1/2} \frac{B_3}{A}, \\
r_2^{(2)} &= \frac{1}{2A}(C_{11} - C_{22}), \quad r_{-2}^{(2)} = \frac{1}{2A} C_{[12]}, \\
r_1^{(2)} &= \frac{1}{2A} C_{[13]}, \quad r_{-1}^{(2)} = \frac{1}{2A} C_{[23]}, \\
r_0^{(2)} &= \frac{1}{2\sqrt{3}A}(2C_{33} - C_{11} - C_{22}), \\
r_3^{(3)} &= \frac{1}{4A}(\tfrac{3}{2})^{1/2}(D_{111} - D_{[122]}), \\
r_{-3}^{(3)} &= \frac{1}{4A}(\tfrac{3}{2})^{1/2}(D_{[112]} - D_{222}), \\
r_2^{(3)} &= \frac{1}{4A}(D_{[113]} - D_{[223]}), \quad r_{-2}^{(3)} = \frac{1}{4A} D_{[123]}, \\
r_1^{(3)} &= \frac{1}{4\sqrt{10}A}(4D_{[113]} - D_{[122]} - 3D_{111}), \\
r_{-1}^{(3)} &= \frac{1}{4\sqrt{10}A}(4D_{[233]} - D_{[112]} - 3D_{222}), \\
r_0^{(3)} &= \frac{1}{4A}(\tfrac{3}{5})^{1/2}(2D_{333} - D_{[113]} - D_{[223]}),
\end{aligned} \qquad (5)$$

where the bracket [ ] in the subscripts implies the sum over the permutation of subscripts, e.g.,

$$D_{[123]} = D_{123} + D_{132} + D_{213} + D_{231} + D_{312} + D_{321} . \qquad (6)$$

The explicit form[4] of the density-matrix elements can be written as follows:

$$\begin{aligned}
\rho_{3/2,3/2(-3/2,-3/2)} &= \tfrac{1}{4} \pm \tfrac{3}{4}(\tfrac{3}{5})^{1/2} r_0^{(1)} + \tfrac{1}{4}\sqrt{3}\, r_0^{(2)} \pm \tfrac{1}{4}(\tfrac{3}{5})^{1/2} r_0^{(3)}, \\
\rho_{1/2,1/2(-1/2,-1/2)} &= \tfrac{1}{4} \pm \tfrac{1}{4}(\tfrac{3}{5})^{1/2} r_0^{(1)} - \tfrac{1}{4}\sqrt{3}\, r_0^{(2)} \pm \tfrac{3}{4}(\tfrac{3}{5})^{1/2} r_0^{(3)}, \\
\rho_{3/2,1/2(-1/2,-3/2)} &= \frac{3}{4\sqrt{5}}(r_1^{(1)} - i r_{-1}^{(1)}) \pm \tfrac{1}{4}\sqrt{3}\,(r_1^{(2)} - i r_{-1}^{(2)}) + \tfrac{1}{4}(\tfrac{6}{5})^{1/2}(r_1^{(3)} - i r_{-1}^{(3)}), \\
\rho_{3/2,-1/2(1/2,-3/2)} &= \tfrac{1}{4}\sqrt{3}\,(r_2^{(2)} - i r_{-2}^{(2)} \pm r_2^{(3)} \mp i r_{-2}^{(3)}), \\
\rho_{1/2,-1/2} &= \tfrac{1}{2}(\tfrac{3}{5})^{1/2}(r_1^{(1)} - i r_{-1}^{(1)}) - \tfrac{3}{4}(\tfrac{2}{5})^{1/2}(r_1^{(3)} - i r_{-1}^{(3)}), \\
\rho_{3/2,-3/2} &= \tfrac{3}{4}(\tfrac{2}{3})^{1/2}(r_3^{(3)} - i r_{-3}^{(3)}),
\end{aligned} \qquad (7)$$

where the lower signs in the equation correspond to the values of ρ with the subscripts in parentheses. Equation (7) gives all the relations between the density-matrix elements of the spin-$\frac{3}{2}$ particle and the parameter $r_M^{(L)}$ tabulated in Ref. 1, where even and odd polarizations are treated separately.

Subscripts in B_i, C_{ij}, and D_{ijk} imply the x, y, z coordinates in the Δ rest frame, and they are independent of the choice of a coordinate system. The relations of $r_M^{(L)}$ in the t-channel helicity (Gottfried-Jackson) and transversality coordinates in Ref. 1 can be proved immediately after changing indices from 1, 2, 3 in the helicity frame to 1, 3, 2, respectively, and putting an extra minus sign when the subscripts contain an odd number of 2 in the helicity frame, e.g.,

$$\begin{aligned}
{}^H r_3^{(3)} &= \frac{1}{4A}(\tfrac{3}{2})^{1/2}({}^H D_{111} - {}^H D_{[122]}) \\
&= \frac{1}{4A}(\tfrac{3}{2})^{1/2}({}^T D_{111} - {}^T D_{[133]}) \\
&= -\tfrac{1}{4}\sqrt{15}\, {}^T r_1^{(3)} + \tfrac{1}{4}\, {}^T r_3^{(3)} .
\end{aligned} \qquad (8)$$

All the other relations between ${}^H r_M^{(L)}$ and ${}^T r_M^{(L)}$ tabulated in Ref. 1 can be obtained similarly. The relations between $r_M^{(L)}$ in other channels are listed in Ref. 1, and these can be used to relate A, B_i, C_{ij}, and D_{ijk} in different channels.

If the parity is conserved in a reaction, the real part of the products of form factors does not appear in B_i and D_{ijk}. Therefore, if form factors are assumed to be real, as is the usual case of time-reversal invariance, B_i and D_{ijk} do not appear at all in the density matrix. This gives the even polarization density matrix in Ref. 1. In this case, one can see from Eqs. (6) and (7) that

$$\rho_{3/2,-3/2}=\rho_{1/2,-1/2} =\rho_{-1/2,1/2} =\rho_{-3/2,3/2} =0\,, \tag{9}$$

and the treatment of density-matrix elements is much simplified.

From the above discussion, one can see that an important point in the treatment of the density matrix is to obtain explicitly A, B_i, C_{ij}, and D_{ijk} as given in Refs. 3 and 4. In Sec. III, the electroproduction of $\Delta(1236)$ will be discussed. The method is simplified and can be extended to the arbitrary spin case.

III. ELECTROPRODUCTION OF Δ

Recently the electroproduction process $ep\to e\Delta$ has been considered[6-10] to investigate the mass spectrum of nucleons and the $\gamma N\Delta$ vertex. The transition of $p\to\Delta$ can be assumed to take place through one-photon exchange. It is characterized by one Coulomb, one electric and one magnetic multipole, and therefore the relativistic $\gamma N\Delta$ vertex contains three independent form factors. Many authors[6,7,12,13] have used a somewhat different form for the form factors. Recently Jones and Scadron[7] have discussed the relation between these three form factors and compared the predictions based on different models. The $\gamma N\Delta$ vertex introduced by Jones and Scadron is convenient because it simplifies the calculation of the density matrix.

The transition amplitude for $ep\to e\Delta$ is

$$\langle f|T|i\rangle=\frac{e^2}{q^2}\,\bar u(k')\gamma_\mu u(k)\langle\Delta|J_\mu|p\rangle\,, \tag{10}$$

where the leptonic current $\bar u(k')\gamma_\mu u(k)$ is well known and the hadronic current is

$$\langle\Delta|J_\mu|p\rangle=-\frac{(m'+m)}{m\Delta(q)}\,\bar u_\alpha(p')\{2G_C q_\alpha(q^2p_\mu-q\cdot pq_\mu)\gamma_5+4G_E R_{\alpha\mu}{}^2\gamma_5+(G_M-G_E)[q^2+(m'-m)^2]R_{\alpha\mu}\}u(p)\,. \tag{11}$$

Here k (ϵ), k' (ϵ'), p (E), and p' (E') are momenta (energies) of the incoming and the outgoing electron, proton, and Δ, respectively. m and m' are masses of the proton and Δ. We define in Eq. (11)

$$q=p'-p=k-k', \tag{12a}$$

$$\Delta(q)=[q^2+(m'-m)^2][q^2+(m'+m)^2]\,, \tag{12b}$$

$$R_{\alpha\beta}=\epsilon_{\alpha\beta\lambda\tau}p_\lambda q_\tau\,, \tag{12c}$$

and

$$R_{\alpha\beta}{}^2=R_{\alpha\lambda}R_{\lambda\beta}\,. \tag{12d}$$

The form factors G_C, G_E, and G_M are actually the same as those defined by Kleinert[8] and by Ash *et al.*,[13] but they are normalized differently. These authors compared their form factors with those of Bjorken and Walecka,[6] and the form factors in Eq. (11) are related to g_1, g_2, and g_3 in Ref. 6 as follows:

$$G_M(q^2)=\frac{m\Delta(q)}{4(m'+m)}\,(3g_2+g_3)\,, \tag{13a}$$

$$G_E(q^2)=\frac{m\Delta(q)}{4(m'+m)}\,(g_2-g_3)\,, \tag{13b}$$

$$G_C(q^2)=\frac{m\Delta(q)}{2(m'+m)}\,g_1\,. \tag{13c}$$

The absolute square of the transition amplitude, after averaging over the spin states of initial particles and summing over spin states of the final electron (but not summing over spin states of Δ), is

$$|\langle f|T|i\rangle|^2=\frac{e^4}{q^4}\,t_{\mu\nu}T_{\mu\nu}\,. \tag{14}$$

The leptonic current tensor $t_{\mu\nu}$ is

$$t_{\mu\nu}=\frac{1}{2\epsilon\epsilon'}\,(k_\mu k'_\nu+k_\nu k'_\mu+\tfrac12 q^2\delta_{\mu\nu})\,. \tag{15}$$

In order to obtain the hadronic current tensor $T_{\mu\nu}$ when the spin states of Δ are not summed, the following two formulas are most useful:

$$\frac{1}{2}\sum_{\text{spin }1/2}\bar{u}_\mu(p')u(p)\bar{u}(p)u_\nu(p')=\frac{(mm'-p\cdot p')}{12EE'}\left[I_{\mu\nu}+\frac{5(K_1+K_2+K_3)+3K_1K_2K_3}{4|K|}\left(\frac{1}{2m'}\right)\epsilon_{\mu\nu\lambda\tau}\eta_\lambda p'_\tau - \frac{(K_1K_2+K_2K_3+K_3K_1)}{4|K|}\eta_{\mu\nu}\right] \tag{16}$$

and

$$\frac{1}{2}\sum_{\text{spin }1/2}\bar{u}_\mu(p')u(p)\bar{u}(p)\gamma_5u_\nu(p')=\frac{1}{24EE'}\left\{R_{\mu\nu}+\frac{5(K_1+K_2+K_3)+3K_1K_2K_3}{4|K|}\left(\tfrac{1}{5}m'\right)[4p\cdot\eta I_{\mu\nu}-\eta_\mu I_\nu(p)-\eta_\nu I_\mu(p)] + \frac{(K_1K_2+K_2K_3+K_3K_1)}{2|K|}\epsilon_{\mu\nu\lambda\tau}\eta_{\lambda\rho}p_\rho p'_\tau-\frac{K_1K_2K_3}{|K|}m'\eta_{\mu\nu\lambda}p_\lambda\right\}, \tag{17}$$

where our notation in Ref. 3 is used except that the spin-$\frac{3}{2}$ spinor u_μ is normalized[14] as

$$\bar{u}_\mu u_\mu = m'/E'. \tag{18}$$

Here η_μ is the polarization four vector defined in Ref. 3 as

$$\eta_\mu=\left(\vec{s}+\frac{\vec{p}'\vec{p}'\cdot\vec{s}}{m'(E'+m')},\, i\,\frac{\vec{p}'\cdot\vec{s}}{m'}\right), \tag{19a}$$

$$\eta_{\mu\nu}=3\eta_\nu\eta_\nu - I_{\mu\nu}, \tag{19b}$$

$$\eta_{\mu\nu\lambda}=\tfrac{9}{2}\eta_\mu\eta_\nu\eta_\lambda-\tfrac{9}{10}(\eta_\mu I_{\nu\lambda}+\eta_\nu I_{\lambda\mu}+\eta_\lambda I_{\mu\nu}), \tag{19c}$$

and

$$I_{\mu\nu}=\delta_{\mu\nu}+\frac{p'_\mu p'_\nu}{m'^2}, \tag{19d}$$

$$I_\mu(p)=I_{\mu\nu}p_\nu. \tag{19e}$$

$I_{\mu\nu}$, η_μ, $\eta_{\mu\nu}$, and $\eta_{\mu\nu\lambda}$ are relativistic generalizations of δ_{ij}, s_i, s_{ij}, and s_{ijk} given in Ref. 3. The former coincides with the latter at the Δ rest frame. Equation (16) is a simplified form of Eq. (A1) in Ref. 4. Using Eqs. (16) and (17) and neglecting the antisymmetric part in $\mu\nu$, one obtains

$$\begin{aligned}
T_{\mu\nu}=&\frac{(m'+m)^2}{24EE'm^2m'^2\Delta(q)}[q^2+(m'-m)^2]\\
&\times\Big\{|G_C|^2(q^2p_\mu-q\cdot pq_\mu)(q^2p_\nu-q\cdot pq_\nu)+m'^2(|G_M|^2+3|G_E|^2){R_{\mu\nu}}^2\\
&+\frac{5(K_1+K_2+K_3)+3K_1K_2K_3}{4|K|}\left(\frac{m'}{10}\right)[-G_C(G_M^*+9G_E^*)(q^2p_\mu-q\cdot pq_\mu)R_\nu(\eta)+G_C^*(G_M+9G_E)(q^2p_\nu-q\cdot pq_\nu)R_\mu(\eta)]\\
&+\frac{(K_1K_2+K_2K_3+K_3K_1)}{|K|}\left(\frac{m'^2}{\Delta(q)}\right)\eta_{\alpha\beta}[4m'^2\,\mathrm{Re}(G_EG_M^*)p_\alpha p_\beta {R_{\mu\nu}}^2-|G_C|^2p_\alpha p_\beta(q^2p_\mu-q\cdot pq_\mu)(q^2p_\nu-q\cdot pq_\nu)\\
&\qquad+G_C(G_M^*+G_E^*)(q^2p_\mu-q\cdot pq_\mu){R_{\nu\alpha}}^2p_\beta+G_C^*(G_M+G_E)(q^2p_\nu-q\cdot pq_\nu){R_{\mu\alpha}}^2p_\beta\\
&\qquad+\tfrac{1}{4}\Delta(q)(|G_M-G_E|^2-4|G_E|^2)R_{\alpha\mu}R_{\beta\nu}]\\
&+\frac{2K_1K_2K_3}{|K|}\left(\frac{m'^3}{\Delta(q)}\right)\eta_{\alpha\beta\gamma}p_\gamma[G_C(G_M^*-G_E^*)p_\alpha R_{\beta\nu}(q^2p_\mu-q\cdot pq_\mu)\\
&\qquad-G_C^*(G_M-G_E)p_\alpha R_{\beta\mu}(q^2p_\nu-q\cdot pq_\nu)-2G_EG_M^*{R_{\alpha\mu}}^2R_{\beta\nu}+2G_E^*G_MR_{\alpha\mu}{R_{\beta\nu}}^2]\Big\},
\end{aligned} \tag{20}$$

where $R_\nu(\eta)$ implies $R_{\nu\lambda}\eta_\lambda$. The unpolarized hadronic current tensor is four times the terms which do not contain K's in Eq. (20). Using the relation

$${R_{\mu\nu}}^2=\tfrac{1}{4}\Delta(q)\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)+q^2\left(p_\mu-\frac{q\cdot p}{q^2}q_\mu\right)\left(p_\nu-\frac{q\cdot p}{q^2}q_\nu\right), \tag{21}$$

one obtains

$$T_{\mu\nu}(\text{unpol}) = \frac{(m'+m)^2}{6EE'm^2m'^2[q^2+(m'+m)^2]} \times\left\{\tfrac{1}{4}m'^2\Delta(q)(|G_M|^2+3|G_E|^2)\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) + \left[[q^4|G_C|^2+q^2m'^2(|G_M|^2+3|G_E|^2)]\left(p_\mu-\frac{q\cdot p}{q^2}q_\mu\right)\left(p_\nu-\frac{q\cdot p}{q^2}q_\nu\right)\right]\right\}, \tag{22}$$

from which the structure functions W_1 and W_2 can be obtained explicitly.

Using Eqs. (15) and (20) one can write Eq. (14) as

$$|\langle f|T|i\rangle|^2 = \frac{e^4(m'+m)^2}{24q^4\epsilon\epsilon' EE'm^2m'^2[q^2+(m'+m)^2]} \times\Bigg(-|G_C|^2q^2R(k)\cdot R(k)+(|G_M|^2+3|G_E|^2)m'^2[\tfrac{1}{8}q^2\Delta(q)-R(k)\cdot R(k)]$$
$$-\frac{5(K_1+K_2+K_3)+3K_1K_2K_3}{4|K|}\left(\frac{im'}{10}\right)\text{Im}[G_C^*(G_M+9G_E)]\,q^2p\cdot(k+k')R(k)\cdot\eta$$
$$+\frac{(K_1K_2+K_2K_3+K_3K_1)}{|K|}\left[\frac{m'^2}{\Delta(q)}\right]\eta_{\alpha\beta}\{|G_C|^2q^2R(k)\cdot R(k)p_\alpha p_\beta + 4\,\text{Re}(G_EG_M^*)p_\alpha p_\beta m'^2[\tfrac{1}{8}q^2\Delta(q)-R(k)\cdot R(k)] + \text{Re}[G_C(G_M^*+G_E^*)]\,q^2p\cdot(k+k')R_\alpha{}^2(k)p_\beta + \tfrac{1}{4}\Delta(q)(|G_M-G_E|^2-4|G_E|^2)[R_\alpha(k)R_\beta(k)-\tfrac{1}{4}q^2R_{\alpha\beta}{}^2]\}$$
$$+\frac{K_1K_2K_3}{|K|}\left[\frac{2im'^3}{\Delta(q)}\right]\eta_{\alpha\beta\gamma}p_\gamma[\text{Im}[G_C(G_M^*-G_E^*)]\,q^2p\cdot(k+k')p_\alpha R_\beta(k)+4\,\text{Im}(G_E^*G_M)R_\alpha(k)R_\beta{}^2(k)]\Bigg)\,. \tag{23}$$

So far we have treated the reaction covariantly.

The unpolarized differential cross section in the laboratory frame can be obtained from the terms which do not contain K's in Eq. (23):

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} = \frac{\alpha^2\cos^2(\frac{1}{2}\theta)}{4\epsilon^2\sin^4(\frac{1}{2}\theta)[1+(2\epsilon/m)\sin^2(\frac{1}{2}\theta)][1+q^2/(m'+m)^2]} \times\frac{1}{6m^2m'^2}\{q^4|G_C|^2+[q^2+\vec{q}^{\,2}\tan^2(\tfrac{1}{2}\theta)]m'^2(|G_M|^2+3|G_E|^2)\}, \tag{24}$$

where every value in Eq. (24) is evaluated in the laboratory frame. Equation (24) corresponds to the Rosenbluth formula for the elastic scattering.

In order to obtain the polarization density matrix, we consider the Δ rest frame and obtain

$$|\langle f|T|i\rangle|^2 = \frac{e^4(m'+m)^2}{24q^4\epsilon\epsilon' EE'm^2m'^2[q^2+(m+m')^2]} \times\left[A+\frac{5(K_1+K_2+K_3)+3K_1K_2K_3}{4|K|}B_is_i+\frac{(K_1K_2+K_2K_3+K_3K_1)}{4|K|}C_{ij}s_{ij}+\frac{K_1K_2K_3}{4|K|}D_{ijk}s_{ijk}\right], \tag{25}$$

where

$$A=-|G_C|^2q^2R(k)\cdot R(k)+(|G_M|^2+3|G_E|^2)m'^2[\tfrac{1}{8}q^2\Delta(q)-R(k)\cdot R(k)], \tag{26a}$$

$$B_i=-\tfrac{1}{10}(im')\,\text{Im}[G_C^*(G_M+9G_E)]\,q^2p\cdot(k+k')R_i(k), \tag{26b}$$

$$C_{ij}=\frac{4m'^2}{\Delta(q)}\{|G_C|^2q^2R(k)\cdot R(k)p_ip_j+4\,\text{Re}(G_EG_M^*)m'^2[\tfrac{1}{8}q^2\Delta(q)-R(k)\cdot R(k)]p_ip_j + \text{Re}[G_C(G_M^*+G_E^*)]\,q^2p\cdot(k+k')R_i{}^2(k)p_j+\tfrac{1}{4}\Delta(q)(|G_M-G_E|^2-4|G_E|^2)[R_i(k)R_j(k)-\tfrac{1}{4}q^2R_{ij}{}^2]\}, \tag{26c}$$

$$D_{ijk}=\frac{8im'^3}{\Delta(q)}\{\text{Im}[G_C(G_M^*-G_E^*)]\,q^2p\cdot(k+k')p_ip_jR_k(k)+4\,\text{Im}(G_E^*G_M)p_iR_j(k)R_k{}^2(k)\}. \tag{26d}$$

Finally, the density-matrix elements can be obtained from Eq. (7) using Eqs. (5) and (26a)–(26d), or from Eq. (3) in Ref. 3 and Eqs. (26a)–(26d). In the Gottfried-Jackson frame,[15] $\vec{p}$, $\vec{R}(k)$, and $\vec{R}^2(k)$ have the components $(0,0,p)$, $(0,-im'pk_x,0)$, and $(m'^2p^2k_x,0,0)$, respectively, and the density-matrix elements are explicitly

$$\rho_{3/2,3/2}=\frac{3}{16A}m'^4p^2(q^2+2k_x^2)|G_M+G_E|^2, \qquad (27a)$$

$$\rho_{3/2,1/2}=\frac{\sqrt{3}}{8A}m'^2q^2pk_xp\cdot(k+k')G_C^*(G_M+G_E), \qquad (27b)$$

$$\rho_{3/2,-1/2}=\frac{\sqrt{3}}{8A}m'^4p^2k_x^2[|G_M-G_E|^2-4|G_E|^2-4i\,\mathrm{Im}(G_E^*G_M)], \qquad (27c)$$

$$\rho_{1/2,-1/2}=\frac{i}{4A}m'^2q^2pk_xp\cdot(k+k')\times\mathrm{Im}[G_C(G_M^*-3G_E^*)], \qquad (27d)$$

$$\rho_{3/2,-3/2}=0, \quad \text{etc.} \qquad (27e)$$

Here p is the magnitude of the nucleon momentum at the Δ rest frame, and k_x^2 and p^2 can be replaced by $R(k)\cdot R(k)/\Delta(q)$ and $\Delta(q)/4m'^2$ covariantly.

The decay distribution of Δ in the Gottfried-Jackson frame depends on real values of the density-matrix elements $\rho_{3/2,3/2}$, $\mathrm{Re}\rho_{3/2,1/2}$, and $\mathrm{Re}\rho_{3/2,-1/2}$. While the differential cross section in Eq. (24) gives information on $|G_C|^2$ and $|G_M|^2+3|G_E|^2$, one can see that experimental measurement of these density-matrix elements gives information on $|G_M|^2-|G_E|^2$ and $\mathrm{Re}(G_M^*G_E)$ from the following equations:

$$\frac{4A}{3m'^4p^2}\left(\frac{2\rho_{3/2,3/2}}{q^2+2k_x^2}+\frac{\sqrt{3}\,\mathrm{Re}\rho_{3/2,1/2}}{k_x^2}\right)=|G_M|^2-|G_E|^2, \qquad (28a)$$

$$\frac{2A}{3m'^4p^2}\left(\frac{2\rho_{3/2,3/2}}{q^2+2k_x^2}-\frac{\sqrt{3}\,\mathrm{Re}\rho_{3/2,1/2}}{k_x^2}\right)=|G_E|^2+\mathrm{Re}(G_M^*G_E). \qquad (28b)$$

Therefore, it is obvious that separate values of $|G_C|^2$, $|G_M|^2$, and $|G_E|^2$ can be obtained, and theoretical models [8–10] which predict explicit values of these form factors can be checked more rigorously. Also a possible nonreality of the form factors can be checked from Eqs. (27a)–(27e).

The method used here can be applied to any cases which contain a Δ particle, e.g., production processes of Δ, $\Delta\pi$, or $\Delta\Delta$ by electrons, photons, and neutrinos whether the target is polarized or not. For example, in the case of the neutrino production of Δ,[3,16] eight form factors are contained in the transition amplitude in general. It can be reexpressed in a form like Eq. (11), and one can calculate density-matrix elements using only Eqs. (16) and (17). Formulas like Eqs. (16) and (17) can be obtained in more complicated cases.

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Electron-Deuteron Scattering and Two-Photon Exchange*

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The general expression for the electron-deuteron scattering amplitude is derived within the framework of Glauber theory. An approximation to this expression, used earlier, is shown to be valid at large, but not at small, momentum transfers. Two-photon exchange effects are somewhat smaller than previously thought. A simple extrapolation of the data does *not* indicate two-photon exchange dominance at large momentum transfers. Existing data indicate that because of interferences, two-photon exchange effects can change the cross section by $\sim 10\%$ for $-t \approx 1$ $(\text{GeV}/c)^2$.

It has been known for some time that double scatterings dominate in collisions of high-energy hadrons with deuterons at large momentum transfers.[1] The methods used to analyze such collisions are usually based upon Glauber theory.[2] The amplitude F_2 for double scattering is a two-dimensional integral, over momentum transfers $\hbar\vec{q}$, involving the hadron-proton and hadron-neutron strong-interaction elastic scattering amplitudes $f_p(\vec{q})$ and $f_n(\vec{q})$ and the deuteron wave function. It takes the form[3]

$$F_2(\vec{q}) = \frac{i}{2\pi k}\int f_n(\tfrac{1}{2}\vec{q}+\vec{q}\,')f_p(\tfrac{1}{2}\vec{q}-\vec{q}\,')S(\vec{q}\,')d^2q', \quad (1)$$

where $S(\vec{q})$ is the deuteron form factor and $\hbar\vec{k}$ is the incident momentum. Since the deuteron is considerably larger than the range of the hadron-nucleon strong interaction, $S(q')$ decreases much more rapidly with increasing q' near $q'=0$ than do $f_{n,p}(\frac{1}{2}\vec{q} \pm \vec{q}\,')$. Hence it is usually a good approximation to replace the amplitudes f_n and f_p in Eq. (1) by their values at $q'=0$. This leads to an approximation to F_2 given by

$$F_2(q) \approx i f_n(\tfrac{1}{2}q) f_p(\tfrac{1}{2}q)\langle r^{-2}\rangle/k, \quad (2)$$

where $\langle r^{-2}\rangle$ is the expectation value, in the deuteron ground state, of the inverse-square neutron-proton separation. The intensity for double scattering is then given by

$$|F_2(q)|^2 \approx \frac{1}{k^2}\frac{d\sigma_n(\frac{1}{2}q)}{d\Omega}\frac{d\sigma_p(\frac{1}{2}q)}{d\Omega}\langle r^{-2}\rangle^2. \quad (3)$$

Double scattering is typically smaller than single scattering near the forward direction. However the single-scattering intensity contains a factor $S^2(\frac{1}{2}q)$ which decreases rapidly with q near $q=0$. We see that in Eq. (3) the structure of the deuteron appears only via $\langle r^{-2}\rangle$, a constant. Consequently, double scattering does not decrease so rapidly with q, and eventually dominates single scattering. Its q dependence is insensitive to the structure of the deuteron.

Equation (3) has been recently used by Gunion and Stodolsky[4] to describe electron-deuteron scattering, an electromagnetic interaction. Since Eqs. (1)–(3) were derived for strong interactions, it is necessary to derive an equivalent expression for the electromagnetic case. We see, for example, that for e-d scattering the integral in Eq. (1) diverges since $f_p(q) \propto q^{-2}$ for small q. Hence Eqs. (2) and (3) will clearly not be valid for small q. The need for a special derivation was recognized in Ref. 4.

Let the e-p and e-n scattering amplitudes, f_p and f_n, be written as

$$f_{p,n}(\vec{q}) = \frac{ik}{2\pi}\int [1 - e^{i\chi_{n,p}(\vec{b})}]e^{i\vec{q}\cdot\vec{b}}d^2b, \quad (4)$$

where $\vec{b}$ is an impact-parameter vector and $\chi_{n,p}(\vec{b})$ are phase-shift functions. In Glauber theory, the e-d elastic scattering amplitude will take the form[2,3]