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Nucleon-Nucleon Interaction from Pion-Nucleon Phase-Shift Analysis

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The two-pion exchange contributions to nucleon-nucleon forces are calculated from our knowledge of pion-nucleon phase shifts and pion-pion interaction which are used as input into dispersion relations with the subtractions required by Regge asymptotic behavior. At low energies, nucleon-nucleon potentials are derived, which, with the addition of π-meson and ω-meson exchange, explain many features of the phenomenological potentials without any adjustable parameter. However, the calculated spin-orbit and central potentials have significant nonlocal components at small distances.

I. INTRODUCTION

This work is an extension and a refinement of previous work done in 1962, by two of us (W.N.C. and R.V.M.). In that paper,[1] which will be referred to as I, an attempt was made to use the Mandelstam representation in order to correlate the long- and intermediate-range nucleon-nucleon forces with known properties of π mesons, their interactions with themselves and with nucleons. In the present work as in I we confine ourselves to the longest-range part of the two nucleon interaction, in particular to the one- and two-pion–exchange contributions to the amplitude. The one-pion–exchange part is unambiguously determined in different approaches. The same cannot be said of the two-pion–exchange contribution. Many authors[2] have adopted the fundamental Lagrangian viewpoint with specific dynamical models; here, since we investigate the extent to which two-pion exchange can be understood in terms of our knowledge of pion-nucleon and pion-pion interactions, dispersion relations seem to be the appropriate framework.

Ten years ago, the $\Delta(1236)$ resonance was the only well-known feature of the pion-nucleon interaction, and the ρ meson the only established resonance in the pion-pion interaction. In paper I these features were used as input data into dispersion relations; this did give an understanding of many properties of internucleon forces, but was unsatisfactory in its predictions of the central forces.

Recently, this type of approach to the problem of internucleon forces has been revived by Brown.[3] Brown, Chemtob, Durso, and Riska[4] use essentially the same framework but improve the input by adding further baryon resonances [the $P_{11}(1400)$ and $D_{13}(1518)$], and by imposing soft-pion constraints on the pion-nucleon amplitude. Their work will be referred to as II.

From recent and more accurate pion-nucleon phase-shift analysis and various indirect studies of the pion-pion interaction, we now have much more information on these amplitudes than was available in 1962. In the present work, we pursue the task commenced in I and II of using this information as input into the Mandelstam representation in order to generate the longer-range features of the nucleon-nucleon amplitude.

Another modification in the present work concerns the number of subtractions in the dispersion relations; these are determined by the asymptotic behavior of the amplitudes. Here, the necessary number of subtractions is determined by assuming Regge asymptotic behavior and the subtraction functions related to the low angular momentum nucleon-antinucleon annihilation into two pions.

The following plan will be adopted in this paper: In Sec. II, we present the full set of dispersion equations relevant to the calculation of the two-pion–exchange amplitude. In Sec. III, we discuss the input data used in the calculation of the double spectral functions and the subtraction terms. Instead of computing the long-range phase shifts we prefer, as an initial step, to define and calculate an equivalent potential, and this we do in Sec. IV. A discussion of the results is given in Sec. V.

II. THE TWO-PION EXCHANGE

The notation of I is used throughout this paper. Goldberger *et al.*[5] demonstrated that the amplitudes associated with the Fermi invariants were in general free of kinematic constraints and plausibly satisfy the Mandelstam representation. We work, as in I, with Amati *et al.*[6] invariants. For natural-parity exchange and in particular two-pion exchange, these also plausibly satisfy the Mandelstam representation. This is because, in this case, the invariants F_2 and F_3 of Ref. 5 vanish at $t=0$.

Apart from antisymmetrization, the scattering matrix is

$$S_{21}=\delta_{21}+i\delta^4(n_1+p_1-n_2-p_2)\left(\frac{m}{2\pi E}\right)^2 M_{21}\,, \quad (2.1)$$

$$M_{21}=\sum\,[3p_i^+(w,t,\bar t)+2p_i^-(w,t,\bar t)\tau^n\cdot\tau^p]P_i\,, \quad (2.2)$$

$$p_i^\pm=\begin{bmatrix}0\\ -\dfrac{1}{2}\dfrac{g^2\delta_{i5}}{\mu^2-t}\end{bmatrix}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\rho_i^\pm(w,t')\mp(-1)^i\rho_i^\pm(\bar t,t')}{t'-t}dt'\,, \quad (2.3)$$

$$g^2/4\pi=14.0\,.$$

The one-pion–exchange term in included in Eq. (2.3) to show our normalization which is such that the differential cross section is given by

$$\frac{d\sigma}{d\Omega}=(m^2/4\pi E)^2|M_{21}\text{ antisymmetrized}|^2\,. \quad (2.4)$$

Assuming the Mandelstam representation, the weight functions $\rho_i^\pm(w,t)$ themselves satisfy relations

$$\rho_i^\pm(w,t)=\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{y_i^\pm(w',t)}{w'-w}dw'\,. \quad (2.5)$$

The dispersion relations (2.3) and (2.5) should both contain subtraction terms depending on the asymptotic behavior of the amplitude. The Mandelstam double spectral functions, $y_i^\pm(w,t)$, are related, using unitarity in the $N\bar N\to 2\pi\to N\bar N$ process and pion-nucleon fixed-t dispersion relations, to the elastic pion-nucleon scattering absorptive parts by the equations

$$\begin{aligned}
y_1^\pm(w,t)&=\frac{\pm1}{8\pi^2\sqrt{t}}\int\int ds'ds''\,K\left\{\Sigma_{AA}^\pm+m(Z+Y)\Sigma_{BA}^\pm+\frac{m^2}{4}\left[\frac{\bar t-w}{(w\bar t K)^2}+Z^2-Y^2\right]\Sigma_{BB}^\pm\right\},\\
y_2^\pm(w,t)&=\frac{\pm1}{16\pi^2\sqrt{t}}\int\int ds'ds''\,K\left\{\Sigma_{BA}^\pm(Z-Y)+\frac{m}{2}\left[-\frac{w+\bar t}{(w\bar t K)^2}+Z^2+Y^2\right]\Sigma_{BB}^\pm\right\},\\
y_3^\pm(w,t)&=\frac{\pm1}{32\pi^2\sqrt{t}}\int\int ds'ds''\,K\left[\frac{\bar t-w}{(w\bar t K)^2}+Z^2-Y^2\right]\Sigma_{BB}^\pm\\
&=2\frac{dy_4^\pm}{dw},\\
y_4^\pm(w,t)&=\frac{\pm1}{32\pi^2\sqrt{t}}\int\int ds'ds''\,K\left(-\frac{1}{w\bar t K^2}\right)\Sigma_{BB}^\pm\,,\\
y_5^\pm(w,t)&=0\,,
\end{aligned} \quad (2.6)$$

where

$$\begin{aligned}
Y&=(s'-s'')/w\,,\\
Z&=[s'+s''+t-2(\mu^2+m^2)]/\bar t\,,\\
K&=[-w(\bar t Z)^2-\bar t w(wY^2+t-4\mu^2)]^{-1/2}\,,
\end{aligned} \quad (2.7)$$

and

$$\begin{aligned}
\Sigma_{AA}^\pm&=[\sigma_A^\pm(s',t)]^*[\sigma_A^\pm(s'',t)]\,,\\
\Sigma_{BA}^\pm&=\mathrm{Re}[\sigma_B^\pm(s',t)]^*[\sigma_A^\pm(s'',t)]\,,\\
\Sigma_{BB}^\pm&=[\sigma_B^\pm(s',t)]^*[\sigma_B^\pm(s'',t)]\,,
\end{aligned} \quad (2.8)$$

where $\sigma_A^\pm(s,t)$ and $\sigma_B^\pm(s,t)$ are the absorptive parts of the usual pion-nucleon scattering amplitudes.[6]

The integrations in s' and s'' are over the connected bounded region where K^2 is positive, and therefore in physical pion-nucleon scattering energies. However, the double spectral functions are defined in the region of $t>4\mu^2$ and require an extrapolation in t from the physical pion-nucleon scattering region.

To determine the number of subtractions to be made in Eqs. (2.3) and (2.5), we assume the asymptotic behavior to be given by the Regge model. In Eq. (2.3), the asymptotic behavior in t is determined by any Regge trajectory for baryon number 2. The deuteron could lie on such a trajectory; however, the fact that for the values of w that interest us there are no dibaryon resonances with angular momentum larger than 1 implies that $\rho_i(w,t)\sim t^{\alpha(w)-1}$ with $\alpha(w)<2$. At most, one subtraction is therefore sufficient in Eq. (2.3). Such a subtraction will only influence the low partial waves or the very short-range interaction in nucleon-nucleon scattering and will not be considered further. Consider now the t-channel ($N\bar N\to N\bar N$) helicity decomposition of the $\rho_i(w,t)$, keeping only the natural-parity (two-pion in particular) contributions; we get

$$\rho_1^{\pm}(w,t)\pm\rho_1^{\pm}(\bar{t},t)=\frac{\pi q}{\sqrt{t}\,p^4}\sum_0^{\infty}(2J+1)[P_J(z)\,\mathrm{Im}F_{++}^{\pm J}-2zP_J'(z)\,\mathrm{Im}F_{-+}^{\pm J}+z(zP_J''+P_J')\,\mathrm{Im}F_{--}^{\pm J}],$$

$$\rho_2^{\pm}(w,t)\mp\rho_2^{\pm}(\bar{t},t)=\frac{\pi q}{M\sqrt{t}\,p^4}\sum_1^{\infty}(2J+1)[-P_J'(z)\,\mathrm{Im}F_{-+}^{\pm J}+(zP_J''+P_J')\,\mathrm{Im}F_{--}^{\pm J}],$$

$$\rho_3^{\pm}(w,t)\pm\rho_3^{\pm}(\bar{t},t)=\frac{\pi q}{M^2\sqrt{t}\,p^4}\sum_1^{\infty}(2J+1)P_J''(z)\,\mathrm{Im}F_{--}^{\pm J},\tag{2.9}$$

$$\rho_4^{\pm}(w,t)\mp\rho_4^{\pm}(\bar{t},t)=-\frac{\pi q}{M^2\sqrt{t}\,p^2}\sum_1^{\infty}(2J+1)P_J'(z)\,\mathrm{Im}F_{--}^{\pm J},$$

where

$$p^2=t/4-m^2,\quad q^2=t/4-\mu^2,\quad z=(\bar{t}-w)/4p^2.\tag{2.10}$$

The subscripts + and − refer to the nucleon-antinucleon pair having the same or opposite helicities in the initial or final states. The functions F^J are the partial-wave $N\bar{N}\to N\bar{N}$ helicity amplitudes and contain contributions from meson resonances. We associate these resonances with Regge poles and, again, the usual assumption that these poles dominate the asymptotic behavior of the $\rho_i(w,t)$ for large w determines the number of subtractions necessary in Eq. (2.5). Thus, the Regge model, with Eq. (2.9), predicts that for large w

$$\rho_1^{\pm}(w,t)\sim(w)^{\alpha(t)},\quad \rho_2^{\pm}(w,t)\sim\rho_4^{\pm}(w,t)\sim(w)^{\alpha(t)-1},\quad \rho_3^{\pm}(w,t)\sim(w)^{\alpha(t)-2},\tag{2.11}$$

where $\alpha(t)$ are the leading trajectories in the t channel. In the isospin (−) amplitudes, the leading trajectory contains the ρ and g resonances with angular momentum such that

$$\mathrm{Re}\alpha_\rho(t)\simeq0.5+0.02t.$$

Since we are only interested in the long-range forces, we are not concerned with very heavy mass exchanges. In particular, for $t<m_g^2\simeq120\mu^2$ we have $\alpha(t)<3$ and the convergent dispersion relations

$$\rho_1^-(w,t)-\rho_1^-(\bar{t},t)=S_1^-(t)(w-w_0)+\frac{(w-w_0)^3}{\pi}\int_{4m^2}^{\infty}dw'\frac{y_1'^-(w',t)}{(w'-w_0)^3}\left[\frac{1}{w'-w}+\frac{1}{w'-\bar{t}}\right]+\rho_1^B(\bar{t},t)-\rho_1^B(w,t),$$

$$\rho_2^-(w,t)+\rho_2^-(\bar{t},t)=S_2^-(t)+\frac{(w-w_0)^2}{\pi}\int_{4m^2}^{\infty}\frac{y_2'^-(w',t)}{(w'-w_0)^2}\left[\frac{1}{w'-w}+\frac{1}{w'-\bar{t}}\right]dw'-\rho_2^B(\bar{t},t)-\rho_2^B(w,t),$$

$$\rho_3^-(w,t)-\rho_3^-(\bar{t},t)=\frac{w-w_0}{\pi}\int_{4m^2}^{\infty}\frac{y_3'^-(w',t)}{(w'-w_0)}\left[\frac{1}{w'-w}+\frac{1}{w'-\bar{t}}\right]dw'+\rho_3^B(\bar{t},t)-\rho_3^B(w,t),\tag{2.12}$$

$$\rho_4^-(w,t)+\rho_4^-(\bar{t},t)=S_4^-(t)+\frac{(w-w_0)^2}{\pi}\int_{4m^2}^{\infty}\frac{y_4'^-(w',t)}{(w'-w_0)^2}\left[\frac{1}{w'-w}+\frac{1}{w'-\bar{t}}\right]dw'-\rho_4^B(\bar{t},t)-\rho_4^B(w,t).$$

In the isospin (+) amplitude the dominant trajectory, at small t, is associated with the Pomeranchukon which is roughly flat for small t and given by

$$\mathrm{Re}\alpha_P(t)\simeq1.0,$$

and for large enough t, with the f-resonance trajectory which is degenerate with the ρ and g trajectory. For $t<m_f^2\simeq80\mu^2$, $\alpha(t)<2$ and we have the following convergent dispersion relations:

$$\rho_1^+(w,t)+\rho_1^+(\bar{t},t)=S_1^+(t)+\frac{(w-w_0)^2}{\pi}\int_{4m^2}^{\infty}\frac{y_1'^+(w',t)}{(w'-w_0)^2}\left[\frac{1}{w'-w}+\frac{1}{w'-\bar{t}}\right]dw'+\rho_1^B(w,t)+\rho_1^B(\bar{t},t),$$

$$\rho_2^+(w,t)-\rho_2^+(\bar{t},t)=\frac{w-w_0}{\pi}\int_{4m^2}^{\infty}\frac{y_2'^+(w',t)}{w'-w_0}\left[\frac{1}{w'-w}+\frac{1}{w'-\bar{t}}\right]dw'+\rho_2^B(w,t)-\rho_2^B(\bar{t},t),$$

$$\rho_3^+(w,t)+\rho_3^+(\bar{t},t)=\frac{1}{\pi}\int_{4m^2}^{\infty}y_3'^+(w',t)\left[\frac{1}{w'-w}+\frac{1}{w'-\bar{t}}\right]dw'+\rho_3^B(w,t)+\rho_3^B(\bar{t},t),\tag{2.13}$$

$$\rho_4^+(w,t)-\rho_4^+(\bar{t},t)=\frac{w-w_0}{\pi}\int_{4m^2}^{\infty}\frac{y_4'(w',t)}{w'-w_0}\left[\frac{1}{w'-w}+\frac{1}{w'-\bar{t}}\right]dw'+\rho_4^B(w,t)-\rho_4^B(\bar{t},t).$$

In Eqs. (2.12) and (2.13), we have the following definitions:

$$w_0=2m^2-t/2\,, \tag{2.14}$$

$$\rho_i^B(w,t)=\frac{1}{\pi}\int_{4m^2}^{\infty} y_i^B(w',t)\left[\frac{dw'}{w'-w}\right], \tag{2.15}$$

$$y_i'^{\pm}(w,t)=y_i^{\pm}(w,t)\mp y_i^B(w,t)\,. \tag{2.16}$$

$y_i^B(w,t)$ is the contribution to $y_i^+(w,t)$ calculated from Eq. (2.6) when only the single-nucleon term is kept in the pion-nucleon amplitude. Equation (2.15) converges without subtraction, and has been separated from the other terms in order to facilitate the calculation of the two-pion-exchange potential. In Eqs. (2.12) and (2.13) we have imposed the symmetry properties of two-pion exchange under the substitution $w \leftrightarrow \bar{t}$ [or $(w-w_0)\leftrightarrow -(w-w_0)$] at a fixed t.

The amplitudes F^J are related, by unitarity, to the $N\bar{N}\to 2\pi$ helicity amplitudes[7]:

$$\mathrm{Im}F_{++}^J=(p^2q^2)^J|f_+^J|^2\,,\quad \mathrm{Im}F_{-+}^J=\frac{(p^2q^2)^J m}{[J(J+1)]^{1/2}}\mathrm{Re}(f_-^{J\,*}f_+^J)\,,\quad \mathrm{Im}F_{--}^J=\frac{(p^2q^2)^J m^2}{J(J+1)}|f_-^J|^2\,. \tag{2.17}$$

Much work has been done over the past few years on the $N\bar{N}\to 2\pi$ S- and P-wave helicity amplitudes $f_+^0(t)$, $f_+^1(t)$, and $f_-^1(t)$. We now have several reasonably consistent models which possess the correct singularity structure and contain the ρ resonance and a more or less well-defined ϵ resonance.

We use some of these models to calculate the subtraction functions $S(t)$. By combining Eqs. (2.9)–(2.17) and projecting out the S and P waves from the imaginary parts of the amplitudes as given by Eqs. (2.12) and (2.13), we get the subtraction functions as

$$S_1^+(t)=\frac{4\pi q}{\sqrt{t}\,w_0^2}(|f_+^0|^2-|f_{B+}^0|^2)-\frac{2}{\pi w_0}\int_{4m^2}^{\infty}dw'\,Q_1(x)\left[\frac{y_1'^+(w',t)}{x}-2my_2'^+(w',t)+m^2xy_3'^+(w',t)+\frac{2m^2}{w_0}y_4'^+(w',t)\right], \tag{2.18}$$

$$\begin{aligned}S_1^-(t)=&\frac{-6\pi q^3}{\sqrt{t}\,w_0^2}\left(\left|f_+^1-\frac{m}{\sqrt{2}}f_-^1\right|^2-\left|f_{B+}^1-\frac{m}{\sqrt{2}}f_{B-}^1\right|^2\right)\\&+\frac{6}{5\pi w_0^2}\int_{4m^2}^{\infty}dw'\left[2Q_3(x)\left(-\frac{y_1'^-(w',t)}{x^2}+3m\frac{y_2'^-(w',t)}{x}-2m^2y_3'^-(w',t)-3m^2\frac{y_4'^-(w',t)}{w_0x}\right)\right.\\&\left.+Q_1(x)\left(-3\frac{y_1'^-(w',t)}{x^2}+4m\frac{y_2'^-(w',t)}{x}-m^2y_3'^-(w',t)-4m^2\frac{y_4'^-(w',t)}{w_0x}\right)\right],\end{aligned} \tag{2.19}$$

$$\begin{aligned}S_2^-(t)=&\frac{3\sqrt{2}\,\pi q^3}{\sqrt{t}\,w_0}\mathrm{Re}\left[f_-^{1*}\left(f_+^1-\frac{m}{\sqrt{2}}f_-^1\right)-f_{B-}^1\left(f_{B+}^1-\frac{m}{\sqrt{2}}f_{B-}^1\right)\right]\\&+\frac{6}{5\pi w_0}\int_{4m^2}^{\infty}dw'[Q_3(x)-Q_1(x)]\left(\frac{y_2'^-(w',t)}{x}-my_3'^-(w',t)\right),\end{aligned} \tag{2.20}$$

$$S_4^-(t)=-\frac{3\pi q^3}{2\sqrt{t}}(|f_-^1|^2-|f_{B-}^1|^2)+\frac{6}{5\pi w_0}\int_{4m^2}^{\infty}dw'[Q_3(x)-Q_1(x)]\frac{y_4'^-(w',t)}{x}, \tag{2.21}$$

where

$$x=(w'-w_0)/w_0\,,\quad q=(t/4-\mu^2)^{1/2}\,,\quad Q_J=\text{usual Legendre functions}\,,$$

and $f_{B\pm}^J(t)$ is the one-nucleon-exchange contribution to the amplitude $f_\pm^J(t)$.

III. INPUTS AND CALCULATIONS

A. The Spectral Functions $y_i^\pm(w,t)$

Equation (2.6) shows that $y_i^\pm(w,t)$ are given in terms of the functions $\sigma_{A,B}^\pm(s,t)$, which are taken as input into this calculation. They have a single-nucleon contribution to $\sigma_B^\pm(s,t)$ of $\pi g^2\delta(s-m^2)$ and a continuum contribution. This latter is taken from phase-shift analysis of pion-nucleon scattering, extrapolated to positive-t values. Apart from this question of extrapolation, there are several different phase-shift analyses of pion-nucleon scattering; they are qualitatively similar but have significant quantitative differences. In this paper we take some account of these differences by using the CERN experimental[8] and the Glasgow A[9] solu-

TABLE I. The nonzero coefficients of the expansion $\sigma^{\pm}_{A,B}(s,t)=\sum_n C_n(s)(t/50)^n$ at $s=132\mu^2$.

		C_0	C_1	C_2	C_3	C_4
σ^+_A	CERN	−12.63	−32.53	−19.67	2.251	0.2117
	Glasgow	−8.620	−19.566	−12.62	0.3817	0.3961
σ^+_B	CERN	3.379	9.329	7.106	0.1647	0.0116
	Glasgow	2.782	7.716	7.712	1.781	0.0217
σ^-_A	CERN	−8.572	−27.74	−25.50	−1.254	0.0249
	Glasgow	−7.756	−29.39	−37.10	−9.966	0.0322
σ^-_B	CERN	1.828	6.865	6.539	0.0202	0.0014
	Glasgow	1.718	7.050	8.225	1.306	0.0018

tions.

Concerning the extrapolation, the domain of analyticity in t of $\sigma^{\pm}_{A,B}(s,t)$ is determined by the boundary of their double spectral functions. At small w, the integrals of Eqs. (2.12) and (2.13) converge very quickly and are only sensitive to $y_i'^{\pm}(w,t)$ for small values of w; these, in turn, depend only on the input functions $\sigma^{\pm}_{A,B}(s,t)$ at small values of s. For these small-s values, $\sigma^{\pm}_{A,B}(s,t)$ have a large domain of analyticity in t; in particular, neglecting pion production[10] below the ρ-resonance production threshold ($s<150\mu^2$), there is no two-pion–exchange double spectral function, and the domain of analyticity is determined by four-pion and higher-mass exchange. The four-pion double spectral boundary starts at $t=16\mu^2$; however, because of the four-pion phase-space factors it is to be expected that the effective boundaries are at much larger values of t. Phase-shift analysis gives the input functions as polynomials in t. The reliability of the extrapolation can be judged, to some extent, by examining the coefficients of these polynomials. For example in Table I, we give the nonzero coefficients c_n for $\sigma^{\pm}_{A,B}(s,t)=\sum_n c_n(t/50)^n$ at $s=132\mu^2$. The Glasgow A and the CERN experimental solutions are shown for comparison. The two solutions are similar, and the high powers of t do not give large contribution for $t<80\mu^2$. This indicates that we do have a reasonable extrapolation out to these large t values. Table II gives the same coefficients for $s=194\mu^2$. Although the high powers of t are again not large, the two solutions are not in very good accord.

We have calculated the double spectral functions $y_i'^{\pm}(w,t)$ with both phase-shift solutions in the region $w\leq 1000\mu^2$ and $t\leq 80\mu^2$. For w and t values not too large the calculations give similar results, for large w and t there can be large differences.

To illustrate these ambiguities, we show graphs of combinations of spectral functions associated with the dominant w-channel helicity amplitudes. As one would expect from Regge-pole phenomenology, we have a large helicity-nonflip combination in the isospin (+) amplitude. Figure 1(a) shows, at $t=30\mu^2$, the helicity-nonflip combination

$$y^+_{-+-+}=y_1^+-2mDy_2^++m^2D^2y_3^++Dy_4^+,$$

$$D=(w-2m^2)/2m^2.$$

In the isospin (−) amplitude, the double helicity flip is the dominant combination. This is associated with

$$y^-_{+--+}=y_1^--2my_2^-+m^2y_3^-+\frac{4m^2}{w}y_4^-,$$

and at $t=30\mu^2$ is shown in Fig. 1(b).

The functions $y_i'^{\pm}(w,t)$ as obtained above were

TABLE II. Similar to Table I, but at $s=194\mu^2$.

		C_0	C_1	C_2	C_3	C_4
σ^+_A	CERN	17.86	54.29	40.82	9.711	0.402
	Glasgow	21.10	55.82	40.61	8.494	0.058
σ^+_B	CERN	0.2176	1.1387	2.841	1.2194	0.0194
	Glasgow	−0.0112	1.4486	3.340	1.432	0.0029
σ^-_A	CERN	−8.937	−31.06	−26.81	−5.954	0.325
	Glasgow	−23.57	−69.65	−53.51	−11.73	0.0576
σ^-_B	CERN	0.6961	2.059	1.436	0.3067	0.0158
	Glasgow	2.023	5.206	3.133	0.4534	0.0028

inserted into Eqs. (2.12) and (2.13) in order to calculate the double spectral contributions to the amplitudes. As discussed in Sec. II, these equations are, in principle, convergent and we, in fact, find very rapid convergence before $w' = 1000\mu^2$. For $t < 50\mu^2$, both πN phase-shift inputs give almost the same results. For larger-t values, they begin to differ; however, these ambiguous regions give only small contributions to the high angular momentum amplitudes.

The CERN experimental phase-shift analysis extends up to $s \simeq 250\mu^2$, the Glasgow A analysis up to $s \simeq 200\mu^2$. The calculation of the functions $y_i^{\pm}(w, t)$ involves the pion-nucleon elastic amplitude only for values of s smaller than w. In particular, it can be seen from Fig. 2 that with $s \lesssim 250\mu^2$, we can obtain $y_i^{\pm}(w, t)$ for $w < 700\mu^2$ and to even larger w values for small values of t.

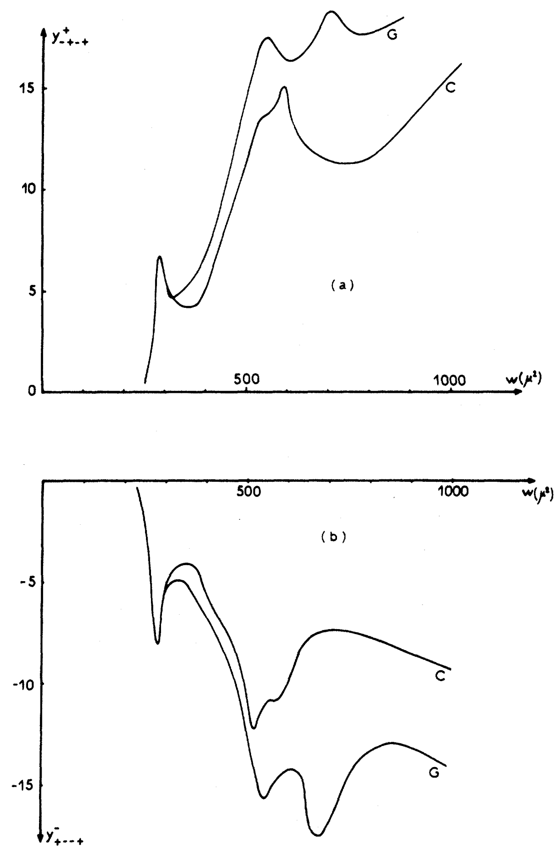

FIG. 1. The functions $y^{+}_{-+-+}(w, t)$ and $y^{-}_{+--+}(w, t)$ at $t = 30\mu^2$. G: Glasgow input; C: CERN experimental input.

B. The Subtraction Functions

The subtraction functions occurring in Eqs. (2.12) and (2.13) have been calculated from Eqs. (2.18)–(2.21) using various models for $f_0^{+}(t)$ and $f_1^{\pm}(t)$:

(i) the resonant $\pi\pi$ S-wave solution of Ref. 11 called BD1 in Ref. 12 and referred to here as R;

(ii) the nonresonant $\pi\pi$ S-wave solution of Ref. 13 called here NR;

(iii) the $\pi\pi$ P-wave solution of Ref. 11 called here N and of Ref. 14 called here H.

References 11, 13, and 14 contain forms for $f_0^{+}(t)$ and $f_1^{\pm}(t)$ for $t < 45\mu^2$. The resulting subtrac-

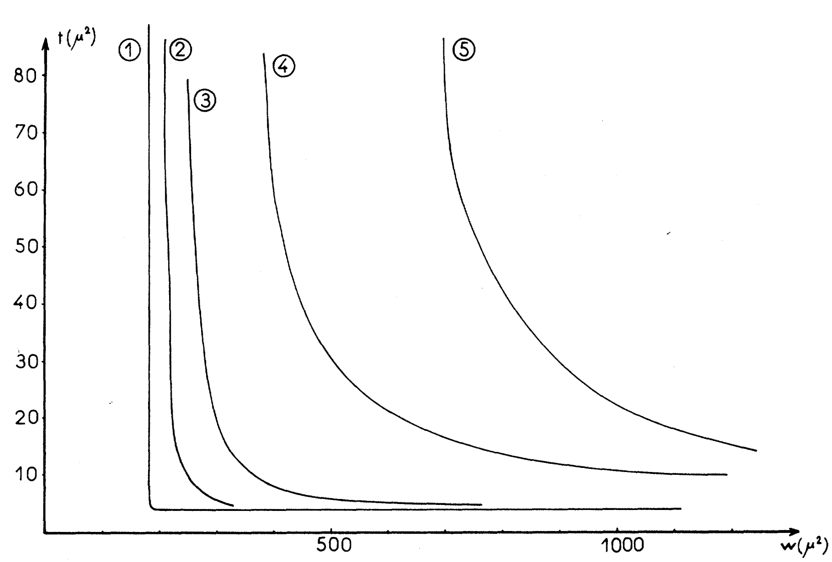

FIG. 2. Nucleon-nucleon double spectral boundaries. The spectral functions are zero to the left of the boundaries. Curve 1 corresponds to $s' = s'' = m^2 = 45.54\mu^2$. Curve 2 corresponds to $s' = m^2$, $s'' = (m+\mu)^2 = 59.8\mu^2$. Curve 3 corresponds to $s' = m^2$, $s'' = m_\Delta^2 = 79.33\mu^2$. Curve 4 corresponds to $s' = s'' = m_\Delta^2$. Curve 5 corresponds to $s' = m^2$, $s'' = 248\mu^2$.

tion functions $S_1^+(t)$ and $S_1^-(t)$ are shown in Fig. 3. The contributions from the double spectral functions $y_i'^{\pm}(w, t)$ in Eqs. (2.18)–(2.21) were found to be very small, which explains the independence of $S_1^+(t)$ and $S_1^-(t)$ from the Glasgow or CERN πN phase-shift models. It can be seen also that the subtractions associated with the ρ meson are similar for the two models. The parameters of ρ exchange are well established; in fact, we obtain similar results for the low-energy amplitude by treating ρ-meson exchange as a simple particle exchange, taking, for example, the old coupling-constant estimations of Ref. 15. From Fig. 3, the S-wave subtraction is not so well determined.

With these ingredients one can calculate directly, by standard techniques, from the amplitude M_{21} the nucleon-nucleon phase shifts, at least for higher angular momentum values. Detailed work on this program is currently under way. The essential features of two-pion–exchange nucleon-nucleon forces can also be expressed in terms of equivalent potentials. This is done in the next sections.

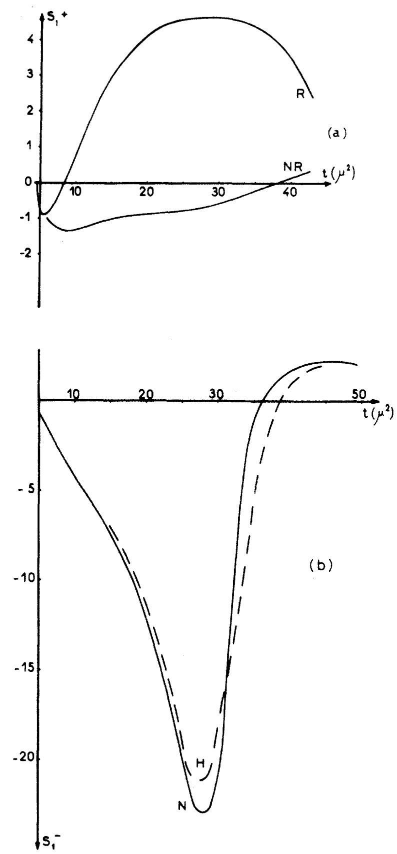

FIG. 3. The subtraction functions. (a) $S_1^+(t)$. R is calculated from $f_+^0(t)$ of Ref. 11. It is constructed from a pion-pion phase shift which passes through 90° in the ϵ mass region. NR is calculated from $f_+^0(t)$ of Ref. 13. In this model the pion-pion phase shift does not pass through 90°. (b) $S_1^-(t)$. N is calculated from $f_\pm^1(t)$ of Ref. 11. H is calculated from $f_\pm^1(t)$ of Ref. 14. $S_2^-(t)$ and $S_4^-(t)$ have very similar shape and size with both models. $S_2^-(t)$ is smaller by a factor of 0.16, and $S_4^-(t)$ is larger by a factor of 1.2.

IV. DEFINITION OF THE POTENTIAL

While the previous sections are devoted to the derivation of the two-pion–exchange contribution to the relativistic two-nucleon S matrix, we would like to derive here an equivalent potential. Of course, such a potential is not necessary when one is only concerned with the two-nucleon system and could even be ill defined, especially at high energies. However, a potential which has a reasonably sound basis outside the core region would be useful in various nuclear calculations. This is especially true since the various phenomenological potentials which fit the low-energy two-nucleon data differ even at rather large distances. Also, a potential, which is reliable for distances smaller than those where the one-pion–exchange potential dominates, would help in reducing the degree of arbitrariness of the core region.

As in papers I and II, we wish to define a potential which, when inserted into a Schrödinger equation, gives, in an energy range sufficiently below the meson production threshold, the same T matrix as the scattering amplitude derived in the previous sections. The T matrix is related to the potential V by the Lippmann-Schwinger equation

$$\langle \vec{p}_2 | T | \vec{p}_1 \rangle = \langle \vec{p}_2 | V | \vec{p}_1 \rangle - \int d^3q \langle \vec{p}_2 | V | \vec{q} \rangle \frac{m}{\vec{q}^{\,2} - \vec{p}^{\,2} - i\epsilon} \langle \vec{q} | T | \vec{p}_1 \rangle , \tag{4.1}$$

and is normalized by

$$\frac{d\sigma}{d\Omega} = (2\pi)^6 \left(\frac{m}{4\pi} \right)^2 | T_{21} |^2 ; \tag{4.2}$$

we therefore require

$$\langle \vec{p}_2 | T | \vec{p}_1 \rangle = -\frac{1}{(2\pi)^3} \frac{m}{E} \bar{u}(p_2) \chi_{p_2}^\dagger \bar{u}(n_2) \chi_{n_2}^\dagger M_{21} u(n_1) \chi_{n_1} u(p_1) \chi_{p_1} , \tag{4.3}$$

where M_{21} is the field-theoretic causal matrix defined by Eqs. (2.1) and (2.2).

As we are concerned with the definition of a nonrelativistic potential, it is appropriate to express the causal amplitude M_{21} in terms of the usual nonrelativistic invariants[16] $\tilde{\Omega}_\alpha$ [α = central, spin-orbit, tensor, spin-spin, and (spin-orbit)^2] rather than the relativistic invariants P_i which we have used in the previous sections. The transformation matrix X such that

$$P_i = \sum_\alpha X_{i\alpha} \tilde{\Omega}_\alpha \tag{4.4}$$

can be obtained by direct calculation and is

$$X_{i\alpha} = \begin{bmatrix} u^2 & -\frac{u}{m(E+m)} & 0 & 0 & -\frac{1}{[2m(E+m)]^2} \\ -2muv & -\frac{2v'}{E+m} & 0 & 0 & -\frac{2(2E+m)}{[2m(E+m)]^2} \\ m^2v^2 & \frac{2E+m}{E+m}v & 0 & 0 & -\left[\frac{2E+m}{2m(E+m)}\right]^2 \\ u'^2+\frac{t+4p^2}{4m^2} & \frac{u'}{m(E+m)}+\frac{1}{m^2} & -\frac{1}{12m^2} & \frac{t}{6m^2} & -\frac{1}{[2m(E+m)]^2} \\ 0 & 0 & \frac{1}{12m^2} & \frac{t}{12m^2} & 0 \end{bmatrix}, \tag{4.5}$$

where

$$u = 1 - \frac{t}{4m(E+m)}, \quad u' = \frac{E}{m} + \frac{t}{4m(E+m)}, \quad v = -1 + \frac{2E^2}{m^2} + \frac{t(2E+m)}{4m^2(E+m)}, \quad v' = 1 + \frac{E}{m} - \frac{E^2}{m^2} - \frac{t(2E+m)}{4m^2(E+m)}.$$

(The transformation matrix used in I is the adiabatic limit of the one given here.)

Because of its analytic structure M_{21} can be written as

$$M_{21} = M_{21}^{\mathrm{OPEC}} + M_{21}^{\mathrm{TPEC}} + \cdots, \tag{4.6}$$

where M_{21}^{OPEC} and M_{21}^{TPEC}, as functions of t, have respectively a pole at $t=\mu^2$ and a cut beginning at $t=4\mu^2$ due to the two-pion-exchange contribution; the other terms of this sum correspond to three-pion and further cuts.

On the other hand, it has been shown[17] that for superpositions of Yukawa potentials, the T matrix defined by (4.1) is an analytic function of t which has a pole at $t=\mu^2$ and branch points at $t=4\mu^2, 9\mu^2, \ldots$; we therefore can write T as

$$T_{21} = T_{21}^{\mathrm{OPEP}} + T_{21}^{\mathrm{TPEC}} + \cdots \tag{4.7}$$

where again T_{21}^{OPEP} and T_{21}^{TPEC} are associated, respectively, the pole at μ^2 and the cut with branch point at $4\mu^2$, etc. We make the same decomposition of the potential, namely,

$$V = V^{\mathrm{OPEP}} + V^{\mathrm{TPEP}} + \cdots. \tag{4.8}$$

Taking into account the proper normalization factors as given by Eqs. (2.4) and (4.2), and identifying terms having the same singularities yields

$$\langle \vec{p}_2 | T^{\mathrm{OPEP}} | \vec{p}_1 \rangle = -\frac{1}{(2\pi)^3}\frac{m}{E} M_{21}^{\mathrm{OPEC}} \text{ for the pion pole} \tag{4.9}$$

and

$$\langle \vec{p}_2 | T^{\mathrm{TPEC}} | \vec{p}_1 \rangle = -\frac{1}{(2\pi)^3}\frac{m}{E} M_{21}^{\mathrm{TPEC}} \text{ for the two-pion branch point.} \tag{4.10}$$

The corresponding one-pion-exchange potential is

$$V^{\text{OPEP}}(\vec{\Delta}) = -\frac{1}{(2\pi)^3}\frac{m}{E} M_{21}^{\text{OPEC}}$$
$$= \frac{1}{(2\pi)^3}\frac{g^2}{m^2}\frac{m}{E}\frac{(\vec{\sigma}^p\cdot\vec{\Delta})(\vec{\sigma}^n\cdot\vec{\Delta})}{\mu^2 - t}\vec{\tau}^n\cdot\vec{\tau}^p . \tag{4.11}$$

Since

$$\langle\vec{p}_2|T^{\text{TPEC}}|\vec{p}_1\rangle = \langle\vec{p}_2|V^{\text{TPEP}}|\vec{p}_1\rangle - \langle\vec{p}_2|V^{\text{OPEP},2}|\vec{p}_1\rangle , \tag{4.12}$$

where

$$\langle\vec{p}_2|V^{\text{OPEP},2}|\vec{p}_1\rangle = \int d^3q\,\frac{\langle\vec{p}_2|V^{\text{OPEP}}|\vec{q}\rangle m\langle\vec{q}|V^{\text{OPEP}}|\vec{p}_1\rangle}{\vec{q}^{\,2} - \vec{p}^{\,2} - i\epsilon} , \tag{4.13}$$

we define

$$\langle\vec{p}_2|V^{\text{TPEP}}|\vec{p}_1\rangle = -\frac{m}{E}\frac{1}{(2\pi)^3}M_{21}^{\text{TPEC}} + \langle\vec{p}_2|V^{\text{OPEP},2}|\vec{p}_1\rangle . \tag{4.14}$$

This equation gives the two-pion–exchange potential in terms of our M_{21}^{TPEC} with the iterated one-pion–exchange potential $\langle\vec{p}_2|V^{\text{OPEP},2}|\vec{p}_1\rangle$ subtracted out.

Since in Eq. (4.13) $|\vec{q}|$ is not necessarily equal to $|\vec{p}|$, the calculation of $\langle\vec{p}|V^{\text{OPEP},2}|\vec{p}_1\rangle$ requires an off-energy-shell extrapolation of $\langle\vec{p}|V^{\text{OPEP}}|\vec{q}\rangle$. This was done in I by taking the adiabatic limit ($E/m = 1$) of V^{OPEP} given by Eq. (4.11), which amounts to treating it as a strictly local potential. The two-pion–exchange potential V^{TPEP} as defined by Eq. (4.14) then has a singularity, as a function of w, at $w = 4m^2$. The associated cut vanishes in the limit of $E/m = 1$, and was discarded in I as part of the adiabatic limiting procedure. However, it has been pointed out by Partovi and Lomon[18] that this is a bad approximation for the calculation of the central potentials. If V^{TPEP} is to be strictly real and not have a singularity as a function of energy at $w = 4m^2$, then relativistic phase space suggests that the kinematic factor of m/E in V^{OPEP} must be interpreted as $[m/E(p)]^{1/2}[m/E(q)]^{1/2}$. In the Schrödinger-equation framework this means that the one-pion–exchange potential is somewhat energy-dependent, which implies a certain nonlocality in coordinate space.

The extrapolation to be inserted into Eq. (4.13) is suggested by the γ_5 pion-nucleon interaction,

$$\langle\vec{p}|V^{\text{OPEP}}|\vec{q}\rangle = -\frac{1}{(2\pi)^3}\frac{g^2}{4m^2}\left(\frac{m}{E(p)}\right)^{1/2}\left(\frac{m}{E(q)}\right)^{1/2}\frac{[E(p)+m][E(q)+m]}{\mu^2 + (\vec{p}-\vec{q})^2}$$
$$\times\left(\frac{\vec{\sigma}^n\cdot\vec{p}}{E(p)+m} - \frac{\vec{\sigma}^n\cdot\vec{q}}{E(q)+m}\right)\left(\frac{\vec{\sigma}^p\cdot\vec{p}}{E(p)+m} - \frac{\vec{\sigma}^p\cdot\vec{q}}{E(q)+m}\right)\vec{\tau}^n\cdot\vec{\tau}^p . \tag{4.15}$$

In doing so, we get the same iterated Yukawa potential as the one derived in II and Ref. 18 through the Blankenbecler-Sugar-Logunov-Tavkhelidze equation.

If one recasts this iterated Yukawa potential in the form

$$\langle\vec{p}_2|V^{\text{OPEP},2}|\vec{p}_1\rangle = \frac{1}{(2\pi)^3}\frac{m}{E}\frac{1}{\pi}\sum_\alpha\tilde{\Omega}_\alpha\int_{4\mu^2}^{\infty}dt'\,\frac{3\xi_\alpha^+(w,t') + 2\xi_\alpha^-(w,t')\tau^n\tau^p}{t'-t} \tag{4.16}$$

(relevant formulas can be found in II or in Appendix A of Ref. 18, for example), the two-pion–exchange potential can be rewritten

$$\langle\vec{p}_2|V^{\text{TPEP}}|\vec{p}_1\rangle = -\frac{1}{(2\pi)^3}\sum_\alpha\tilde{\Omega}_\alpha\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{3\eta_\alpha^+(w,t') + 2\eta_\alpha^-(w,t')\tau^n\tau^p}{t'-t}dt' , \tag{4.17}$$

with

$$\eta_\alpha^\pm(w,t') = \frac{m}{E}\sum_i\{\pm\rho_i^B(w,t') + \rho_i^{\pm\prime}(w,t') \mp (-)^i[\pm\rho_i^B(\bar{t},t') + \rho_i^{\pm\prime}(\bar{t},t')]\}X_{i\alpha} - \frac{m}{E}\xi_\alpha^\pm(w,t') \tag{4.18}$$

and

$$\rho_i^{\pm\prime}(w,t) = \rho_i^\pm(w,t) \mp \rho_i^B(w,t) .$$

In Eq. (4.18), $\rho_i^B(w,t')$ and $\xi_\alpha^\pm(w,t')$, as functions of w, both have a cut starting at $w = 4m^2$. However, with the prescription of Eq. (4.15), the discontinuities across this cut exactly cancel in the combinations which appear in Eq. (4.18). These particular combinations have only faraway left-hand cuts at $w=-4\mu(2m+\mu)$ [from $\xi_\alpha^\pm(w,t)^{19}$] and at $w=-t$ [from $\rho_i^B(\bar{t},t)$] which give rise to very weak energy dependence for values of w relevant to nuclear physics. We shall hereafter refer to the contributions to $\eta_\alpha^\pm(w,t')$ from ρ_i^B and $\xi_\alpha^\pm$ as the "fourth-order contributions." Because of the cancellations discussed above, the energy dependence of these fourth-order contributions is very weak. The remaining terms $\rho_i^{\pm\prime}$ in Eq. (4.18) which come from the double spectral functions $y_i^{\pm\prime}$ have closer singularities (at the pion-production thresholds), and they are responsible for most of the energy dependence of our potentials. In the following, we shall call them the "double spectral contributions."

We have studied the w dependence of the potential functions $\eta_\alpha^\pm(w,t)$, for w less than the pion-production threshold, and found that $\eta_T^\pm(w,t)$, $\eta_{SS}^\pm(w,t)$, and $\eta_{SO}^-(w,t)$ have only small w dependence implying that their associated potentials should be reasonably local. This is not the case for the functions $\eta_C^\pm(w,t)$ and, to a lesser extent, for $\eta_{SO}^+(w,t)$. However, for the calculations presented in Sec. V, we define a local nucleon-nucleon potential in momentum space by

$$V^{\text{TPEP}}(\vec{\Delta})=-\frac{1}{(2\pi)^3}\sum_\alpha\tilde{\Omega}_\alpha\frac{1}{\pi}\int_{4\mu^2}^\infty\frac{dt'}{t'-t}\left[3\eta_\alpha^+(w,t')+2\tau^n\tau^p\eta_\alpha^-(w,t')\right], \tag{4.19}$$

where w has a fixed value. Hereafter, we define

$$\eta_\alpha^\pm(t')=\eta_\alpha^\pm(w,t') .$$

The form for the potential, in configuration space, was given in I. For completeness, we list below the useful formulas:

$$V^{\text{TPEP}}(\vec{x})=\sum_\alpha\Omega_\alpha\left[3U_\alpha^+(r)+2U_\alpha^-(r)\tau^n\cdot\tau^p\right],$$

$$U_C^\pm(r)=-\frac{1}{(2\pi)^2}\int_{4\mu^2}^\infty\eta_C^\pm(t)\frac{e^{-rt^{1/2}}}{r}\,dt,$$

$$U_{SO}^\pm(r)=+\frac{1}{(2\pi)^2}\int_{4\mu^2}^\infty\eta_{SO}^\pm(t)\frac{e^{-rt^{1/2}}}{r^2}\,t^{1/2}\left(1+\frac{1}{rt^{1/2}}\right)dt\ ,$$

$$U_T^\pm(r)=-\frac{1}{(2\pi)^2}\int_{4\mu^2}^\infty\eta_T^\pm(t)\frac{e^{-rt^{1/2}}}{r}\,t\left(1+\frac{3}{rt^{1/2}}+\frac{3}{r^2t}\right)dt,$$

$$U_{SO2}^\pm(r)=\frac{1}{(2\pi)^2}\int_{4\mu^2}^\infty\eta_{SO2}^\pm(t)\frac{e^{-rt^{1/2}}}{r^3}\times t\left(1+\frac{3}{rt^{1/2}}+\frac{3}{r^2t}\right)dt, \tag{4.20}$$

$$U_{SS}^\pm(r)=-\frac{1}{(2\pi)^2}\int_{4\mu^2}^\infty\eta_{SS}^\pm(t)\frac{e^{-rt^{1/2}}}{r}\,dt\ .$$

Because of our limited knowledge of the spectral functions, different cutoffs had to be introduced into Eq. (4.20). The "fourth-order contributions" were cut off at $t=600\mu^2$, which, for our purposes, is essentially the same as ∞. The "double spectral contributions" were cut off at $t=80\mu^2$ for the isospin (+) amplitude and $t=120\mu^2$ for the isospin (−) amplitude. The subtraction function contributions $S_i^+(t)$ were cut off at $t=45\mu^2$ corresponding to the limitations of the input models. As we have discussed in Sec. II, our scheme of calculation does not include exchange masses as large as the f and g mesons, since in Eq. (4.20) contributions to the potentials from large t are suppressed by the exponential factors $e^{-\sqrt{t}r}$. For example, at $r>0.6\mu^{-1}$, this factor for g-meson exchange is less than 4% that for ρ-meson exchange.

Finally, we add the well-known one-pion-exchange potential and, as part of the three-pion exchange, a one-boson-exchange potential corresponding to the exchange of the ω meson. For the ω coupling constants we take

$$G_\omega^T/G_\omega^V=-0.12\ .$$

The smallness of this ratio is established from many different phenomenological sources. The actual size of the coupling is not well established, and if we take the ratio of the vector coupling to protons of the ω and ρ to be

$$G_\omega^V/G_\rho^V=2.3\ ,$$

G_ω^V is then the same as that of II, which facilitates the comparison of our results with those of II.

We do obtain reasonable fits to the phenomenological potentials with this ratio; however, we do not include ϕ-meson exchange in our calculation, and some of our ω-exchange potential could be ascribed to the exchange of the ϕ meson.

V. RESULTS AND DISCUSSION

The inputs of these calculations come from several different sources, and we consider that some are better determined than others. To remind the reader of these inputs we list them below in order of decreasing reliability:

(i) The "fourth-order contributions." They are unaffected by the present uncertainties on the $\pi\pi$ and πN phase shifts. In the calculation of the potentials, the modifications suggested by Ref. 18 to the method of Charap and Tausner have been taken into account.

(ii) The "double spectral contributions." These contributions along with (i) contain the exchange of two pions in D waves and above. They depend upon the πN phase-shift analysis, and they give rise to most of the energy dependence of the potentials.

(iii) The "subtraction functions in the isospin (−) state." These are related to the exchange of two pions in a P state, which is dominated by the ρ meson.

(iv) The "subtraction function in the isospin (+) state," which is related to the exchange of two pions in an S state.

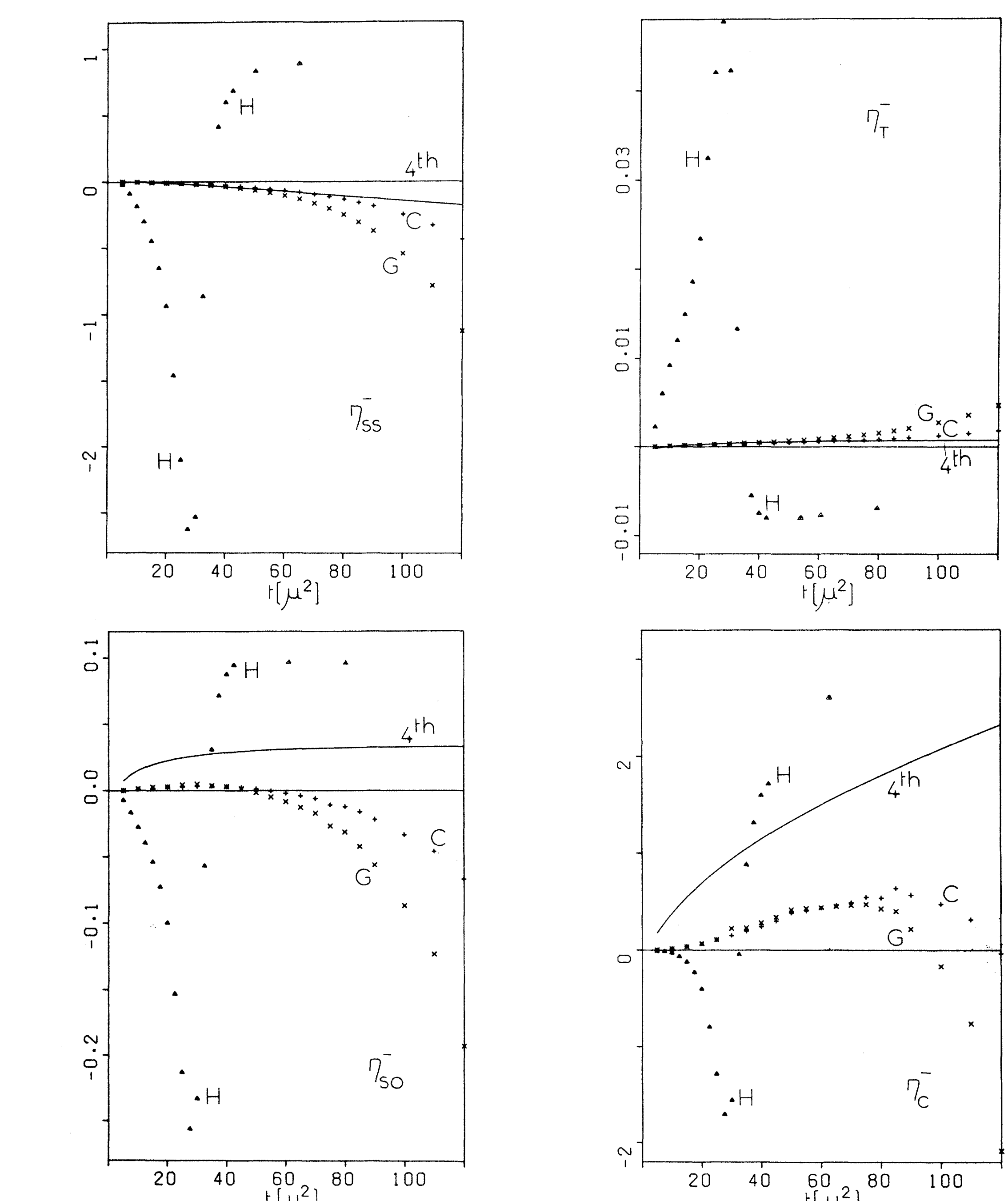

FIG. 4. The potential spectral functions $\eta_\alpha^-(w,t)$ for fixed $w=4m^2$: H is the contribution of the isospin (−) subtraction functions with the $\pi\pi$ P wave of Ref. 14. 4th is the "fourth-order contribution." G and C are the "double spectral contribution" from Glasgow and CERN solutions, respectively.

(v) The ω exchange. It represents part of the three-pion exchange. However, the over-all coupling of the ω to nucleons is not well determined. This exchange only contributes to the isospin (+) potentials.

We have displayed in Figs. 4 and 5 the contributions (i)–(iv) to the potential spectral functions $\eta_\alpha^\pm(w=4m^2, t)$ to show their relative importance.

In order to isolate some of the more ambiguous parts of our calculations, the $\pi\pi$ S-wave exchange, and the ω exchange, we first compare our isospin (–) and (+) potentials with the (–) and (+) combina-

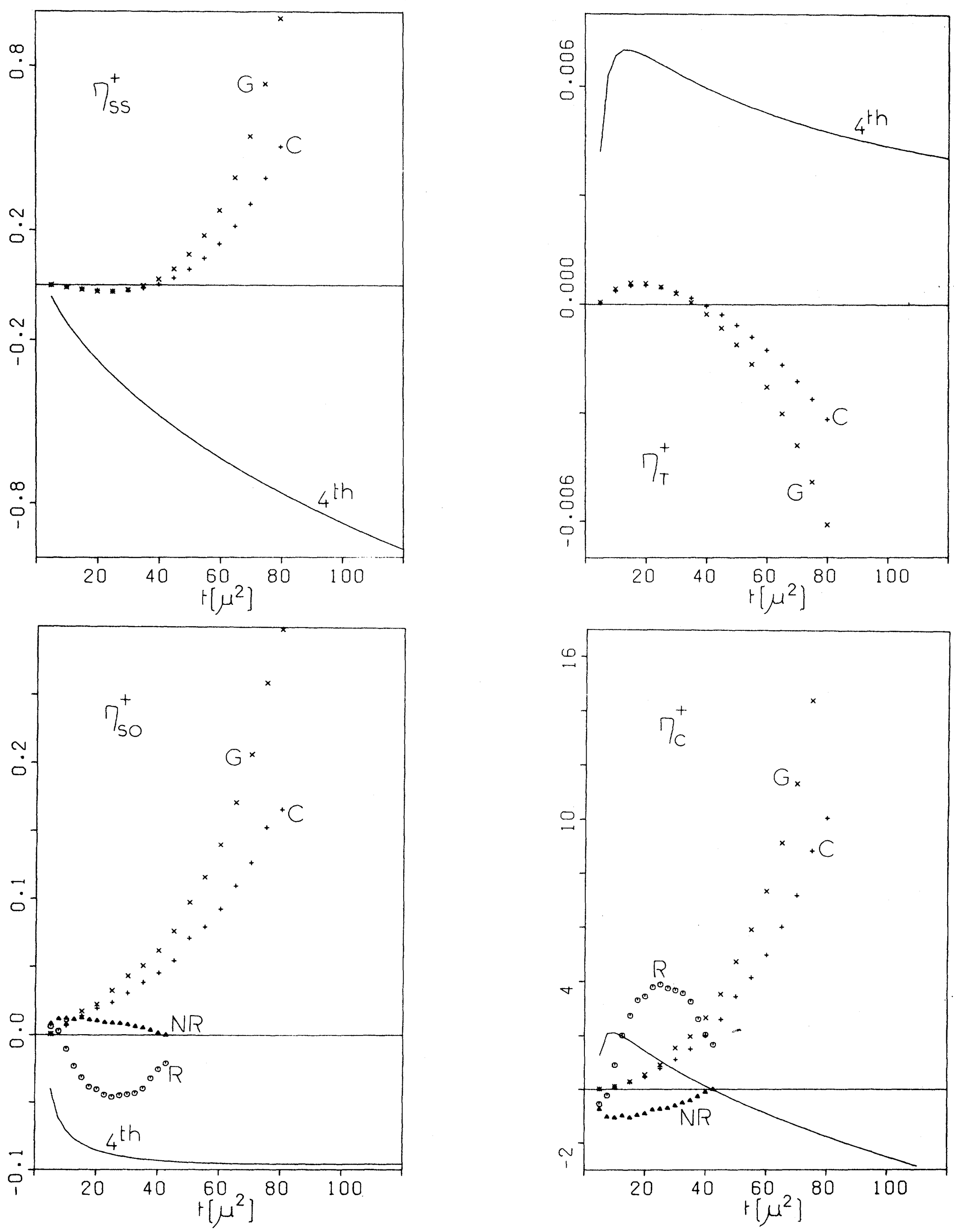

FIG. 5. The potential spectral functions $\eta_\alpha^+(w,t)$ for fixed $w=4m^2$: R and NR are the contributions of the isospin (+) subtraction function with a resonant and nonresonant $\pi\pi$ S wave, the other notations have the same meaning as in Fig. 4.

tions of the Yale[20] and Hamada-Johnston[21] phenomenological potentials.

First then let us consider the $\eta_{\alpha}^{-}(w, t)$ potential spectral functions and the potentials $U_{\alpha}^{-}(r)$. It can be seen from Fig. 4 that the "double spectral contributions" to the $\eta_{\alpha}^{-}(w, t)$, for $t<50\mu^2$, are small and almost the same with both phase-shift analyses, but they become for $t>50\mu^2$ large and depend on the particular phase-shift analysis. This means that our potentials $U_{\alpha}^{-}(r)$ have very little ambiguity for large r (say $r>0.5\mu^{-1}$), but become progressively more dependent on the input models and more energy-dependent since most of the energy dependence, as discussed in Sec. IV, is due to the "double spectral contributions." As for the P-wave subtraction functions $S_i^{-}(t)$, they are derived from P-wave $\pi\pi$ interaction of Refs. 13 and 14 up to $t=45\mu^2$ only. For $t\sim 45\mu^2$, the largest contribution to these subtraction functions comes from the function $f_{B}^{1}\text{-}(t)$ [see Eq. (2.21)]. In Fig. 4, we assume that this term dominates at the larger t values, and the corresponding contributions to the subtraction functions are also plotted for $45\mu^2<t<90\mu^2$.

The $U_{\rm SS}^{-}(r)$ and $U_{\rm T}^{-}(r)$ potentials are mainly due to the π and $\pi\pi$ P-wave exchange, the "fourth-order" and the "double spectral" contributions being very small. The main uncertainty in these calculated potentials comes from the P-wave subtraction function S_4^{-} above the cutoff at $t=45\mu^2$. Our potentials have been calculated with the assumption that S_4^{-} is dominated by $f_{B}^{1}\text{-}(t)$ at the larger t values. The amount of such contributions to the results of $U_{\rm SS}^{-}(r)$ and $U_{\rm T}^{-}(r)$ is less than 20% at $r=0.6\mu^{-1}$, and, at least for $r>0.6\mu^{-1}$, our results are in good agreement with the phenomenological potentials.

For all of our input models, the potential $U_{\rm SO}^{-}(r)$ (Fig. 6), although qualitatively correct, is too small by a factor of about 3. This is a little surprising to us. Here also, the main uncertainty in the calculation is the cutoff at $t=45\mu^2$ in the P-wave subtraction contribution. The model, described in the previous paragraph, for continuing this contribution to larger t values has been again used in the calculation of our potential $U_{\rm SO}^{-}(r)$. The A_1 is the next heavy boson ($t=59\mu^2$) to contribute to this potential, and it gives a contribution of the wrong sign. If the phenomenological potentials are correct, then this is the only clear example for the necessity of including uncorrelated three-pion-exchange effects in the theoretical potential.

The "double spectral contributions" to the function $\eta_{\rm C}^{-}(w, t)$ are not as small as they are in the previous cases, inducing therefore some energy dependence of the $U_{\rm C}^{-}(r)$ potential. We regard the quantitative fit of our calculation with phenomenology as significant (Fig. 6), but we are aware of the energy dependence of $U_{\rm C}^{-}(r)$.

Let us turn now to the isospin (+) combinations. The potentials $U_{\rm SS}^{+}(r)$ and $U_{\rm T}^{+}(r)$ have no contributions from $\pi\pi$ S- or P-wave exchange, the dominant parts come from the "fourth-order" terms, the "double spectral contributions" and the ω-meson exchange are small (Fig. 5). Figure 7 shows good agreement with phenomenology. We have not included η-meson exchange in this calculation. Its inclusion, with a small coupling constant as in Refs. 4 and 18 would bring our results closer to phenomenology. $U_{\rm SO}^{+}(r)$ is the first potential in this discussion to be influenced by the $\pi\pi$ S-wave exchange, and although this is ambiguous, it is small. Our $U_{\rm SO}^{+}(r)$ has also a 50% contribution from our model of ω exchange. The main feature of this potential is the largeness of the "double spectral contributions" (see Fig. 5) with their non-negligible energy dependence inducing a significant nonlocal component to the potential. Such nonlocality should not change the qualitative features of the potential shown in Fig. 6; the sign and size are in agreement with phenomenology. As can be seen from Fig. 5, there is no dominant contribution to $\eta_{\rm C}^{+}(w, t)$. The "double spectral contributions" are large. There are also $\pi\pi$ S-wave and ω-exchange parts. Here ω exchange is repulsive and $\pi\pi$ S-wave exchange attractive. With the ω-nucleon coupling-constant value given in Sec. IV, both the Yale and Hamada-Johnston potentials favor the nonresonant $\pi\pi$ S-wave model (Fig. 6). Although the S-wave resonant model is more attractive than phenomenology, an increase of our model ω coupling could restore reasonable agreement with phenomenology. However, this $U_{\rm C}^{+}(r)$ potential is the most model-dependent potential of this work.

We have studied explicitly the energy dependence of both our potentials spectral functions $\eta_{\alpha}^{\pm}(w, t)$ and our potentials $U_{\alpha}^{\pm}(r)$. Figures 8 and 9 show, for different values of w between the elastic threshold $4m^2$ and the pion production threshold $(2m+\mu)^2$, the variation of our potentials obtained from the Glasgow A phase-shift solution and the nonresonant $\pi\pi$ S-wave model. The $U_{\rm SS}^{-}$, $U_{\rm T}^{-}$, $U_{\rm SO}^{-}$, $U_{\rm SS}^{+}$, and $U_{\rm T}^{+}$ possess the desirable feature of being very weakly energy-dependent, especially for large $r>0.6\mu^{-1}$. The approximate 10% energy dependence of $U_{\rm T}^{-}$, for $r<0.6\mu^{-1}$, comes from the strong cancellation between the π and the $\pi\pi$ P-wave exchanges in this region giving rise to the turnover of the potential. As expected from the importance of the "double spectral contributions," the energy dependence of the central potentials $U_{\rm C}^{\pm}(r)$ and, to a lesser extent, of the spin-orbit $U_{\rm SO}^{+}$ is significant.

In Fig. 10, we show, in the isospin 0 and 1 com-

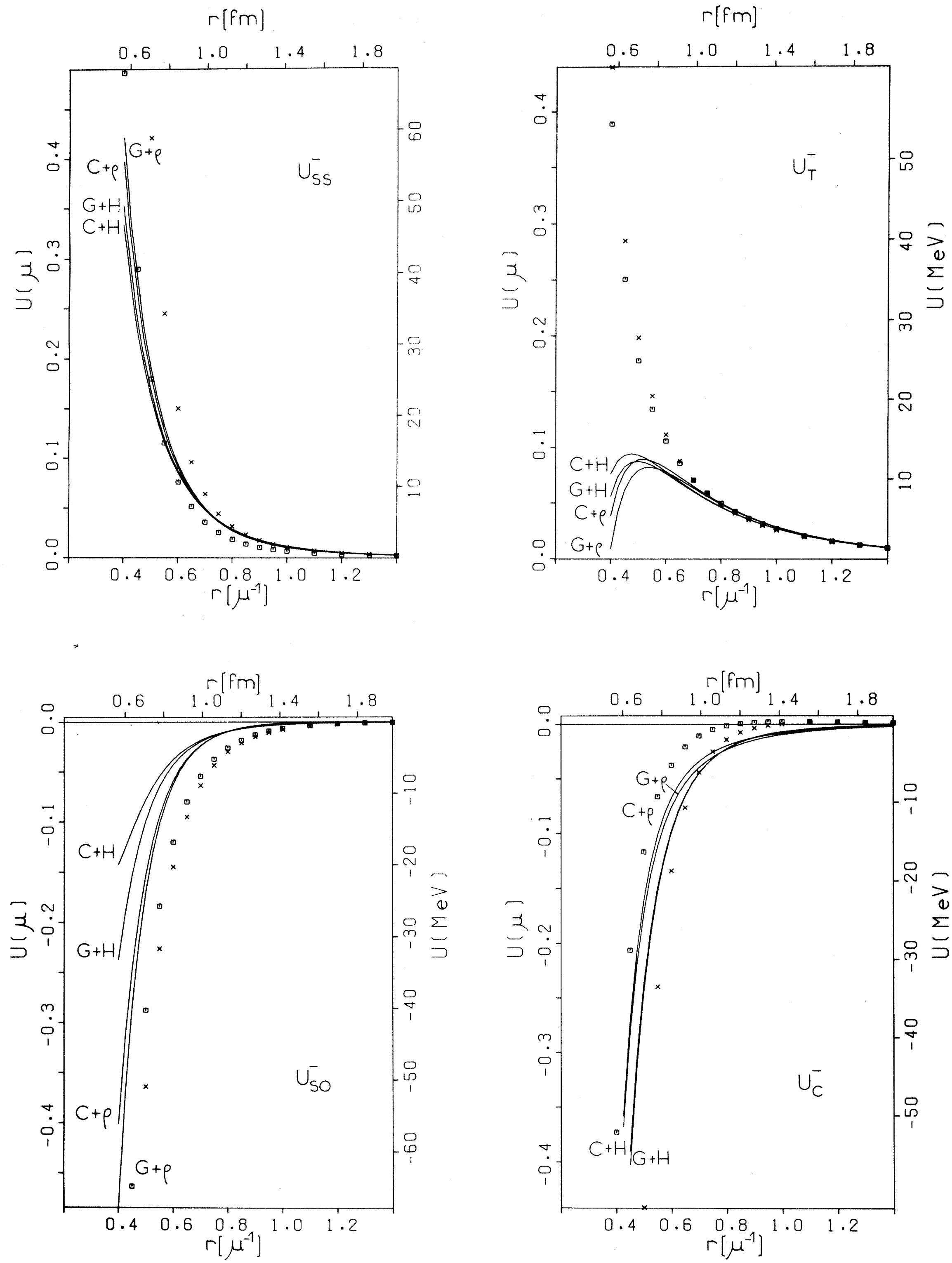

FIG. 6. The isospin (−) potentials for $w=4m^2$: $C+\rho$ = CERN πN phase shifts + a simple ρ pole in the subtraction functions S^-. $G+\rho$ = Glasgow πN phase shifts + a simple ρ pole in the subtraction functions S^-. $C+H$ = CERN πN phase shifts + H model in the subtraction functions S^-. $G+H$ = Glasgow πN phase shifts + H model in the subtraction functions S^-. The Yale potentials appear as ×; the Hamada-Johnston potentials appear as ⊡.

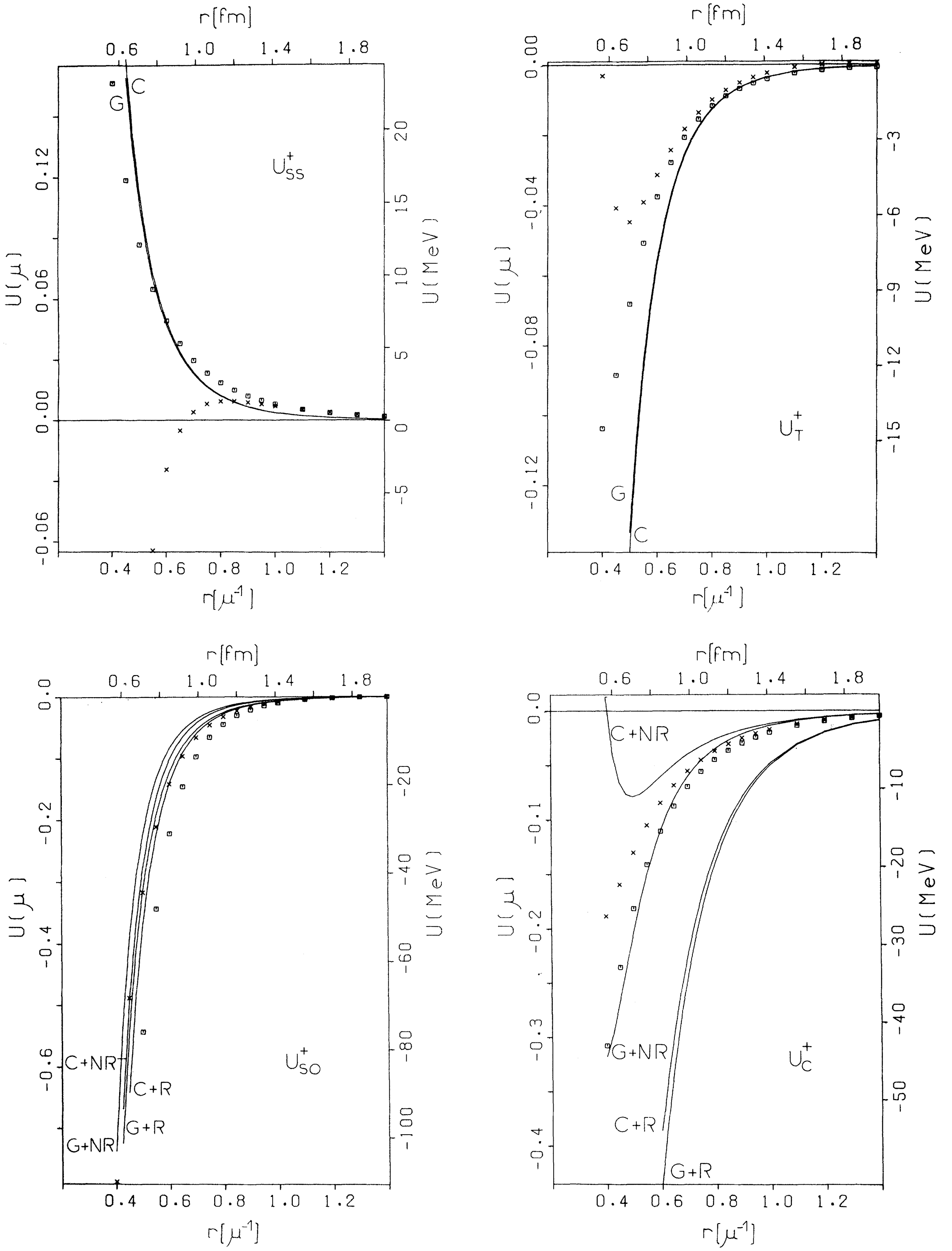

FIG. 7. The isospin (+) potentials for $w = 4m^2$. $C+R$ = CERN πN phase shifts + resonant $\pi\pi$ S wave in the subtraction function S_1^+. $G+R$ = Glasgow πN phase shifts + resonant $\pi\pi$ S wave in the subtraction function S_1^+. $C+NR$ = CERN πN phase shifts + nonresonant $\pi\pi$ S wave in the subtraction function S_1^+. $G+NR$ = Glasgow πN phase shifts + nonresonant $\pi\pi$ S wave in the subtraction functions S_1^+. The Yale potentials appear as ×, the Hamada-Johnston potentials appear as ⊡.

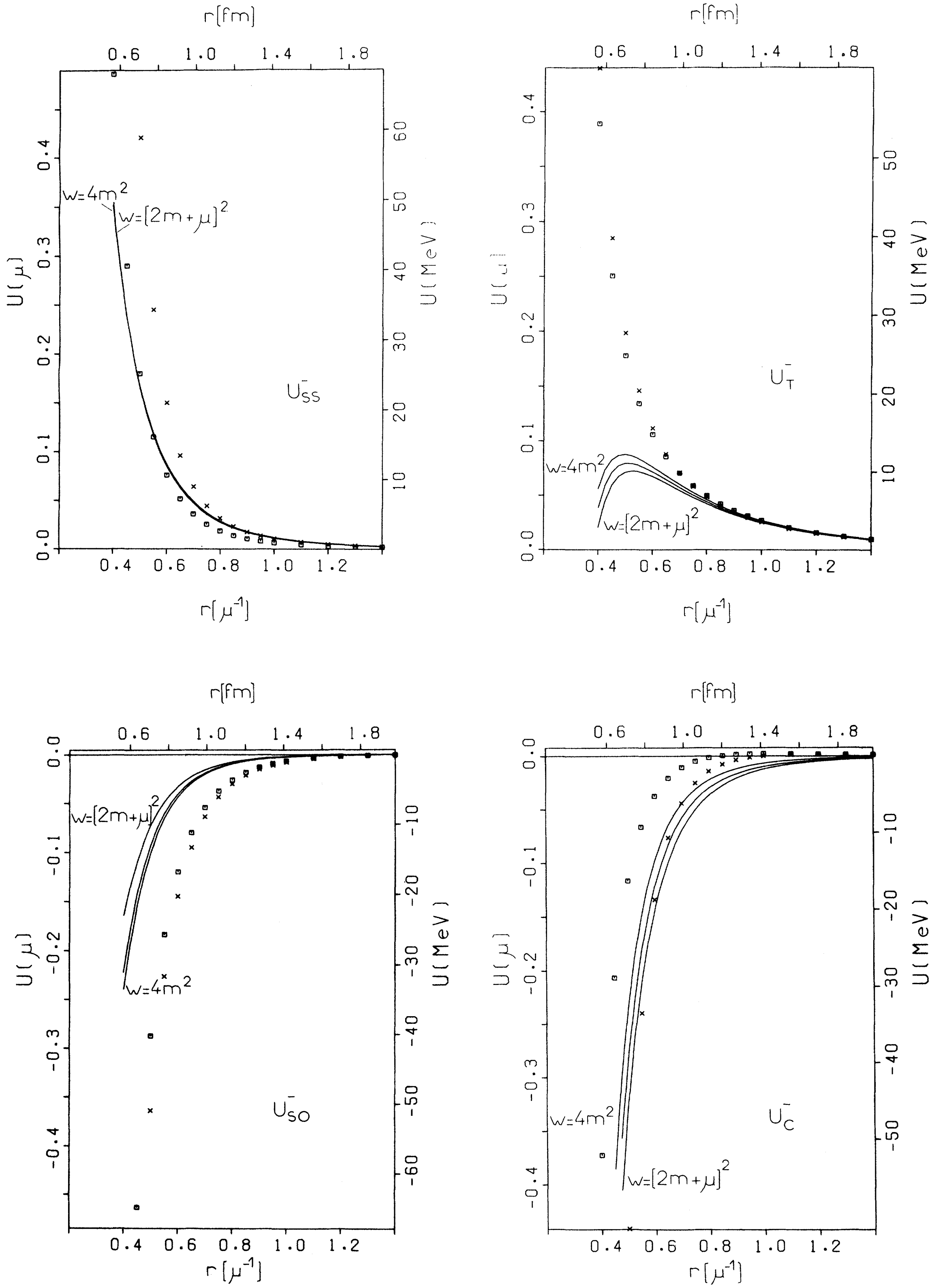

FIG. 8. The energy dependence of the isospin (−) potentials ($G+H$ in Fig. 6). The central curves correspond to $w=[4m^2+(2m+\mu)^2]/2$.

binations, our potentials evaluated at $w = 4m^2$ and obtained from the Glasgow A phase-shift solution and the nonresonant $\pi\pi$ S-wave model. We compare them with those calculated in II and with phenomenology. For $r > 0.6\mu^{-1}$ they are all in qualitative agreement. The agreement with II can be understood since for such large distances, apart from the well-established one-pion exchange, the potentials are dominated by both the "fourth-order contributions" and the S- and P-wave $\pi\pi$ exchanges

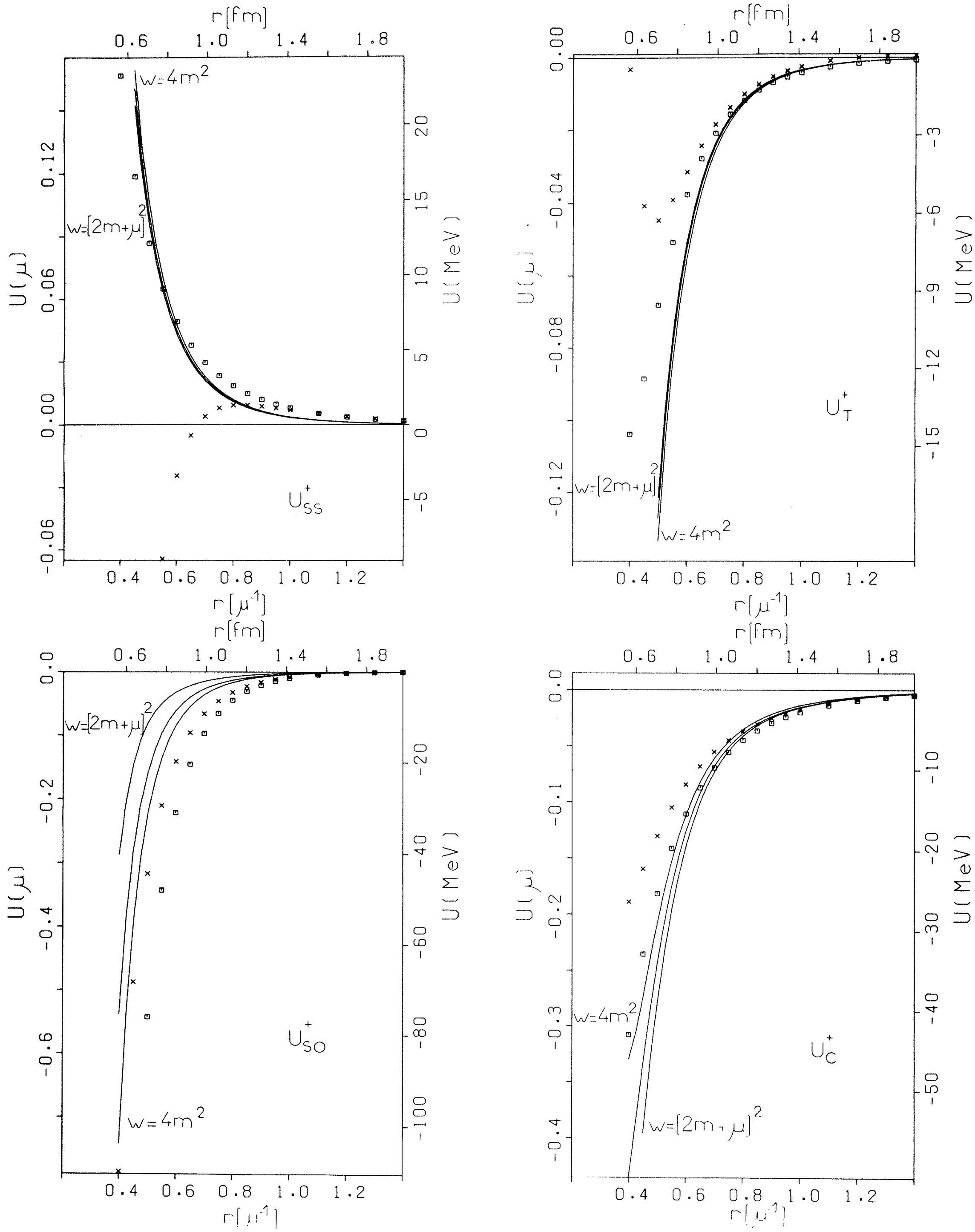

FIG. 9. The energy dependence of the isospin (+) potentials ($G + NR$ in Fig. 7). The central curves correspond to $w = [4m^2 + (2m+\mu)^2]/2$.

which are given in the two calculations by similar models. For $r < 0.6\mu^{-1}$, the agreement is not so good, especially for the spin-orbit and central potentials, because, as already discussed, the "double spectral contributions" become important there, and these are not the same in the two works.

It is also worth mentioning that the present potentials $U_{SS}^{\pm}(r)$, $U_{T}^{\pm}(r)$, when compared with the results of the early calculations of paper I, are

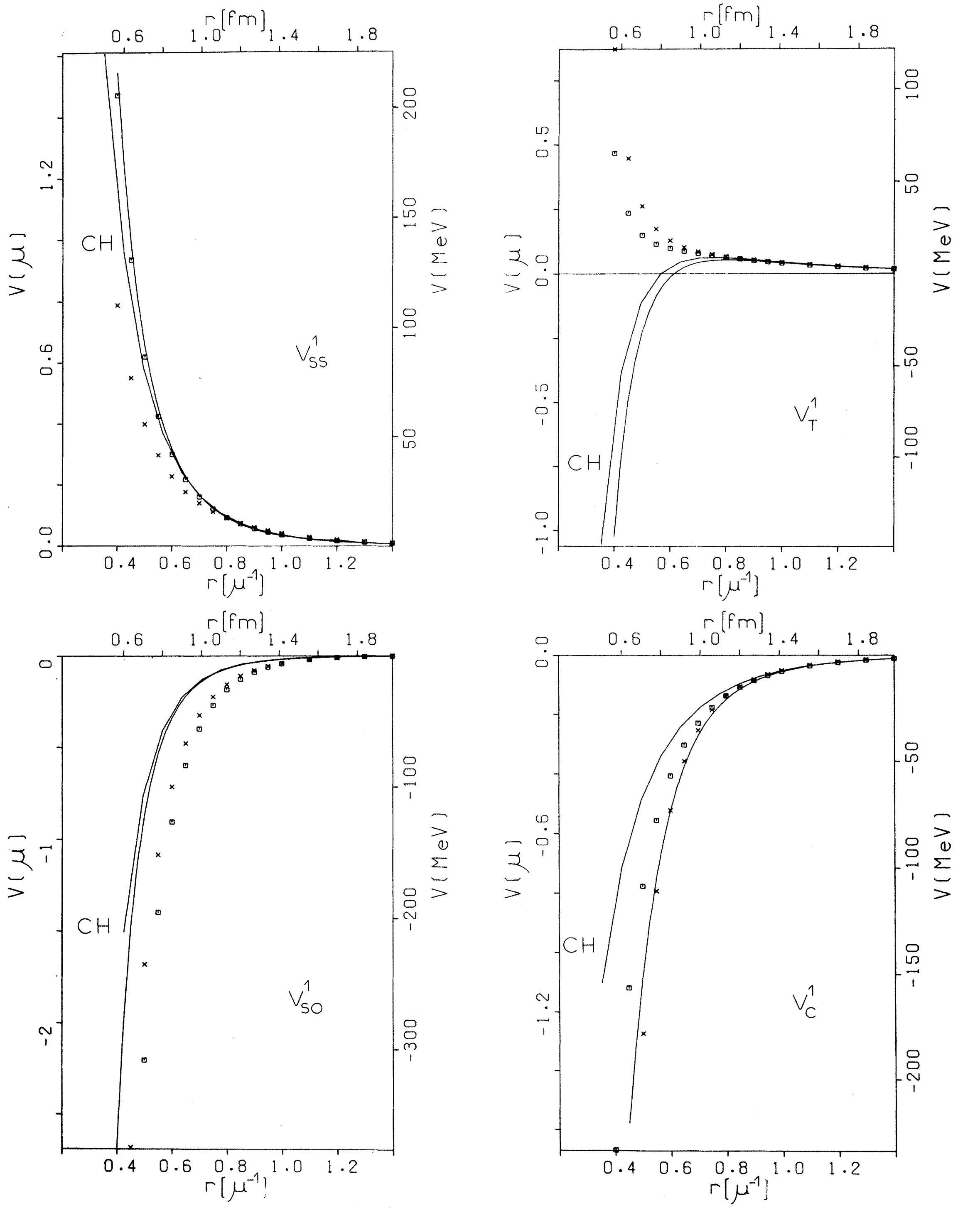

FIG. 10 *(Continued on following page)*.

almost unchanged.

One of the most striking features of this work is that the "fourth-order contribution" is large at large distances, and the "double spectral contribution" is important at smaller distances. This shows that the uncorrelated two-pion exchange is very important. It can be approximated neither by a fourth-order contribution nor by some low angular momentum $\pi\pi$ resonances.

This uncorrelated two-pion exchange is calculat-

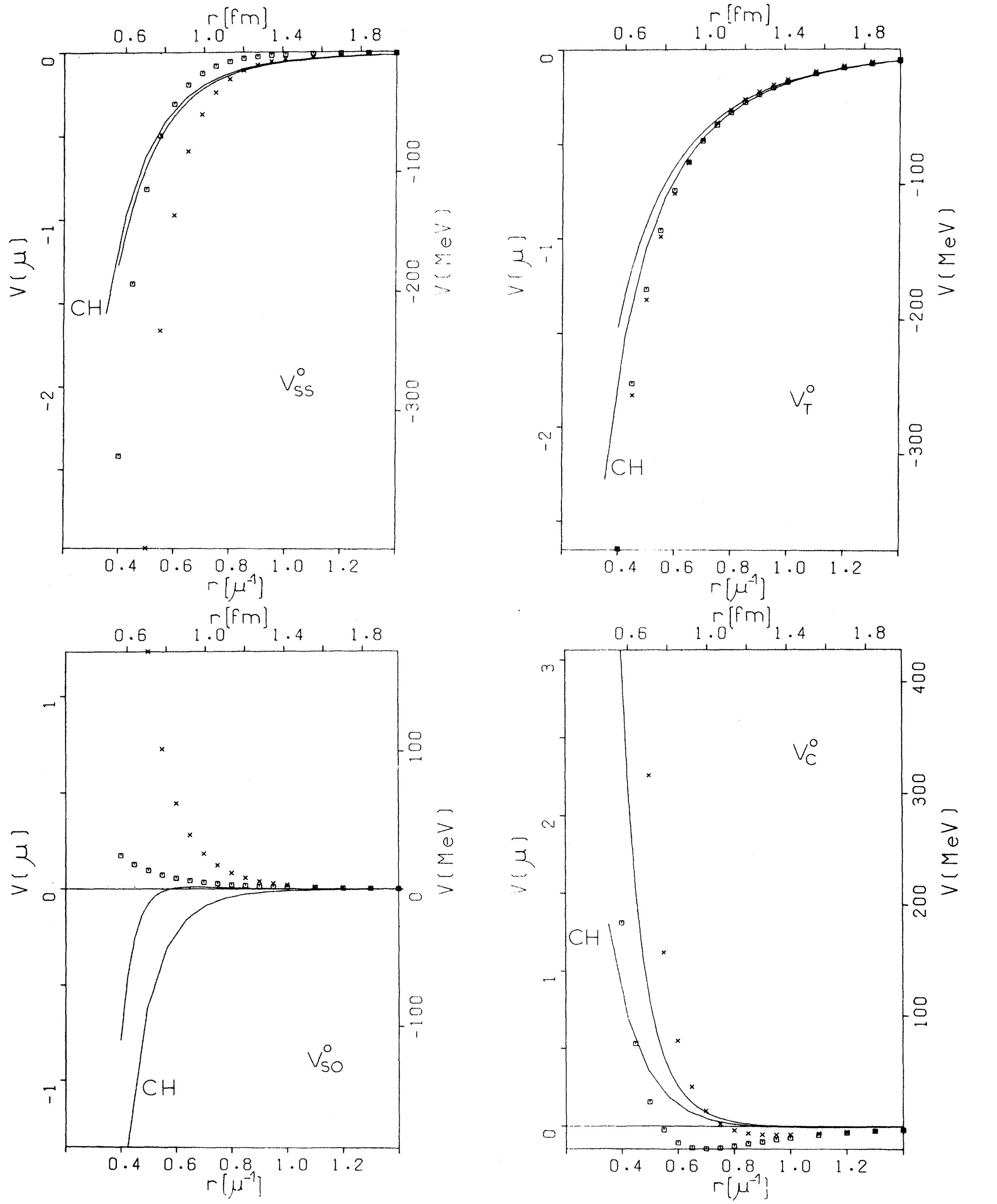

FIG. 10. Our potentials in the conventional isospin $I=1$ and $I=0$ states ($G+H$ and $G+NR$ in Figs. 6–7) are compared with phenomenological potentials and with the results of Chemtob, Durso, Riska (Ref. 4) referred to as (CH).

ed here in the framework of dispersion relations from the presently known properties of π mesons. Our aim was not to achieve a best fit of the nucleon-nucleon data. It was rather an attempt to relate independent pieces of information from different branches of meson physics. The fact that we have obtained, without adjusting any parameter, a good consistency between experimental data on the π-nucleon, $\pi\pi$, and nucleon-nucleon systems does give us confidence in the dispersion relation approach to the fundamental problem of nucleon-nucleon forces. Conversely, within the same framework, better nucleon-nucleon data could help in removing some ambiguities of the π nucleon or $\pi\pi$ interactions.

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