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Eikonal Estimates and Cancellations at High Energies*

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Using functional methods and the eikonal model, the leading s dependence of elastic scattering in a modified ϕ^3 theory is discussed. An approximate evaluation of ladder or tower graphs and certain nonplanar graphs reveals strong cancellations. The net contribution falls as a power of the energy rather than saturating the Froissart bound, as found in less complete treatments of multiperipheral models.

In view of the high current interest in measurements[1] of the total cross section for pp scattering, it may be worthwhile to describe a field-theoretic mechanism which can alter the Cheng-Wu[2] and Chang-Yan[3] prediction of $\sigma_T \sim \ln^2 s$ for large energies. This cancellation phenomenon which keeps the multiperipheral graphs from saturating the Froissart bound[4] has been demonstrated in great detail by using a completely different approach.[5] In this brief note we will use functional methods and the eikonal approximation to sum the leading $\ln s$ dependence of all tower graphs and nonplanar checkerboard graphs. A considerably less elegant method of estimation has been described elsewhere.[6]

For the purposes at hand, we will adopt a hybrid theory, one midway between the massive-photon quantum electrodynamics (QED) and the simple ϕ^3 models used in Refs. 2 and 3, respectively. A nucleon field ψ with sources η and $\overline{\eta}$, a neutral vector-meson (NVM) field W_μ with source k_μ, and a scalar pion field π with source j are introduced, and the interaction Lagrangian is written as

$$L' = -ig\overline{\psi}\gamma_\mu\psi W^\mu - \tfrac{1}{2}\lambda\pi W_\mu W^\mu . \tag{1}$$

All self-interactions will be neglected and only the eikonal-like graphs with NVMs being exchanged between a pair of scattering nucleons will be retained. However, all virtual-pion exchanges between the NVMs will be kept. As in simpler eikonal models,[7] it is assumed that the vector-meson exchanges eikonalize, carrying with them the composite substructures created by pion exchange in all possible ways, between all possible NVMs. In the large-energy limit of interest, we will only consider the helicity-nonflip amplitudes, which reduces the problem to essentially one involving scalar nucleons.

As in other similar treatments,[8] the generating functional appropriate to this theory is defined in terms of c-number sources j, k_μ, η, and $\overline{\eta}$:

$$Z[j, k_\mu; \eta, \overline{\eta}] = \left\langle \left(\exp\left[i\int (j\pi + k\cdot W + \overline{\eta}\psi + \overline{\psi}\eta)\right]\right)_+ \right\rangle .$$

The formal solution for Z is

$$\langle S\rangle Z = \exp\left\{ i\int \overline{\eta}G\left[-i\frac{\delta}{\delta k}\right]\eta + \mathrm{Tr}\ln\left(1 + g\gamma\cdot\frac{\delta}{\delta k}S_c\right)\right\}$$
$$\times\exp\left\{\frac{i}{2}\int k\cdot\overline{\Delta}_c\left[-i\frac{\delta}{\delta j}\right]\cdot k - \tfrac{1}{2}\mathrm{Tr}\ln\left(1 - i\lambda\frac{\delta}{\delta j}\Delta_c\right)\right\}\exp\left(\frac{i}{2}\int jD_c j\right), \tag{2}$$

where the propagators S_c, $g_{\mu\nu}\Delta_c$, and D_c are for the nucleon, NVM, and pion fields, respectively. In Eq. (2) the functions

$$G[A] = S_c(1 + ig\gamma\cdot AS_c)^{-1}$$

and

$$\overline{\Delta}_c[\pi] = \Delta_c(1 + \lambda\pi\Delta_c)^{-1}$$

denote relativistic propagators defined in terms of fictitious c-number "potentials" or sources $A_\mu(x)$ and

$\pi(x)$. The factor $\langle S\rangle$ represents the normalizing vacuum-to-vacuum amplitude.

If the closed nucleon loops and the closed NVM loops are removed, then Z reduces to the simpler form

$$Z=\exp\left(i\int\bar{\eta}G\left[-i\frac{\delta}{\delta k}\right]\eta\right)\exp\left(\frac{i}{2}\int k\cdot\bar{\Delta}_c\left[-i\frac{\delta}{\delta j}\right]\cdot k\right)\exp\left(\frac{i}{2}\int jD_cj\right).\qquad(3)$$

Expressions for all physical processes of interest can be obtained by appropriate functional differentiation of Z. In particular, the configuration-space scattering amplitude for a pair of nucleons (assumed distinguishable to avoid the necessary symmetrization) is given by[9]

$$M(x_1y_1;x_2y_2)=i^2G_{\rm I}\left(y_1x_1\middle|-i\frac{\delta}{\delta k}\right)G_{\rm II}\left(y_2x_2\middle|-i\frac{\delta}{\delta k}\right)\exp\left(\frac{i}{2}\int k\cdot\bar{\Delta}_c\left[-i\frac{\delta}{\delta j}\right]\cdot k\right)\exp\left(\frac{i}{2}\int jD_cj\right)\Bigg|_{k=j=0}.\qquad(4)$$

A somewhat more convenient form follows if all groups in which a NVM is emitted and absorbed by the same nucleon are dropped:

$$M=i^2\exp\left(-i\int\frac{\delta}{\delta k_1}\bar{\Delta}_c\left[-i\frac{\delta}{\delta j}\right]\frac{\delta}{\delta k_2}\right)G_{\rm I}(y_1x_1|k_1)G_{\rm II}(y_2x_2|k_2)\exp\left(\frac{i}{2}\int jD_cj\right)\Bigg|_{j=k=0}.\qquad(5)$$

For $\lambda=0$, $\bar{\Delta}_c(\pi)=\Delta_c$ and the amputated, mass-shell Fourier transform of Eq. (5) yields the familiar NVM multiple-exchange eikonal model.

Using techniques exactly the same as those discussed in detail in Ref. 9, one performs the familiar Bloch-Nordsieck approximations to the nucleon $G_{\rm I,II}$ propagators, which then lead to the latters' representation in terms of parametric integrals over the exponential of linear functionals of $k_{1,2}$. The functional (translational) operations of Eq. (5) are then trivial, and exactly as in Ref. 9 one finds that the result may be expressed in eikonal form,

$$T(s,t)=i\frac{s}{2M^2}\int d^2b\,e^{iq\cdot b}(1-e^{i\chi(b,s)}),\qquad(6)$$

where the eikonal is now given by

$$e^{i\chi}=\exp\left(-\frac{i}{2}\int\frac{\delta}{\delta\pi}D_c\frac{\delta}{\delta\pi}\right)\exp\left[ig^2\int F_{\rm I}\cdot\bar{\Delta}_c(\pi)\cdot F_{\rm II}\right]\Bigg|_{\pi=0}\qquad(7)$$

and $p_1+p_2=p_1'+p_2'$, $q^2=(p_1-p_1')^2=-t>0$. The source currents F are given by

$$F^{\mu}_{\rm I,II}(x)=P^{\mu}_{1,2}\int_{-\infty}^{\infty}d\xi\,\delta(x-z_{1,2}+\xi P_{1,2}).\qquad(8)$$

The phase $i\chi$ is a function of the transverse center-of-mass coordinate difference $\vec{b}=(\vec{z}_1-\vec{z}_2)_\perp$. The eikonal phase may be expressed in terms of connected graphs only:

$$1+i\chi=\exp\left(-\frac{i}{2}\int\frac{\delta}{\delta\pi}D_c\frac{\delta}{\delta\pi}\right)\exp\left[ig^2\int F_{\rm I}\cdot\bar{\Delta}(\pi)\cdot F_{\rm II}\right]\Bigg|_{{\rm conn},\pi=0}.\qquad(9)$$

An expansion of χ in powers of g^2 produces the sum of all the connected t-channel amplitudes for n W's to scatter to n W's:

$$i\chi=\sum_i^{\infty}i\chi_n$$

where

$$i\chi_n=\frac{(ig^2)^n}{n!}\int F^{\mu_1}_{\rm I}\cdots F^{\mu_n}_{\rm I}M_{n,n}(u_1\cdots u_n;\,v_1\cdots v_n)F^{\mu_1}_{\rm II}\cdots F^{\mu_n}_{\rm II}.\qquad(10)$$

In the limit in which no pion is emitted and absorbed by the same NVM, the connected t-channel exchange amplitude is

$$M_{n,n}=\exp\left[\sum_{i\leq j=i}\left(-i\int\frac{\delta}{\delta\pi_i}D_c\frac{\delta}{\delta\pi_j}\right)\bar{\Delta}_c(u_1v_1|\pi_1)\cdots\bar{\Delta}_c(u_n,v_n)\right]\Bigg|_{{\rm conn},\pi=0}.\qquad(11)$$

It is straightforward to see that

$$i\chi_1=-i\frac{g^2}{2\pi}K_0(\mu b)$$

for the present case of vector exchange. This term reproduces the results of the simplest eikonal model.[7] The function $i\chi_2$ alone generates the tower graphs which may be estimated for large s by a straightforward, if lengthy, graphical analysis. The leading $\ln s$ behavior in every λ^2 order arises from the pure ladder exchange of pions with ordered rapidities. If r such pions are exchanged between a pair of NVMs, the resulting nested rapidity integrals generate a contribtuion to $i\chi_2$ of the form

$$-\frac{g^4}{2(2\pi)^2}\frac{\ln^r(s/s_0)}{r!}\int\frac{d^2q_1d^2q_2e^{ib\cdot(q_1+q_2)}}{(q_1{}^2+M^2)(q_1{}^2+M^2)}[\alpha_2(q_1+q_2)]^r, \tag{12}$$

where

$$\alpha_2(q)=(2\pi)^{-2}\int d^2Q(Q^2+M^2)^{-1}[(Q-q)^2+M^2]^{-1}$$
$$=\frac{1}{4\pi}\int_0^1 dx[M^2+x(1-x)q^2]^{-1}. \tag{13}$$

Summing over all r except $r=0$, which is a disconnected graph corresponding to the second s-channel iterate of $i\chi_1$, yields the familiar result[2,3]

$$i\chi_2(b,s)=-2\left(\frac{g^2}{4\pi}\right)^2\int d^2q\,e^{iq\cdot b}\alpha_2(q)\times\left[\left(\frac{s}{s_0}\right)^{\lambda^2\alpha_2(q)/8\pi}-1\right]. \tag{14}$$

For large b, which is most sensitive to small q, the trajectory α_2 may be expanded as

$$\alpha_2(q^2)=\alpha_2(0)-q^2\alpha_2{}'(0),$$

and

$$i\chi_2\simeq-\frac{a}{\ln(s/s_0)}\left(\frac{s}{s_0}\right)^{\alpha_0-1}e^{-cb^2/\ln(s/s_0)}, \tag{15}$$

where

$$\alpha_0\equiv 1+\lambda^2\alpha_2(0)/8\pi,$$
$$a=2(g^2/\lambda)^2\alpha_2(0)/\alpha_2{}'(0),$$
$$c=2\pi/\lambda^2\alpha_2{}'(0).$$

If this result is used in Eq. (6), one finds that the Froissart bound is saturated.

Once it is understood that the source of the leading s dependence is the set of nested graphs, it is possible to devise a simple functional approach which reproduces the same result. One simply performs the replacement

$$ig^2\int F_{\rm I}\overline{\Delta}[\pi]F_{\rm II}\to-\frac{ig^2}{(2\pi)^2}\int\frac{d^2q\,e^{iq\cdot b}}{(q^2+M^2)}\exp\left[\lambda\int d^4u\,\bar\pi(u)\gamma(s,u_\pm)\sigma(u,q)\right], \tag{16}$$

where

$$\sigma(u,q)=[(u_\perp-q_\perp)^2+M^2]^{-1}, \tag{17}$$

and uses the prescription [which defines $\gamma(s,u_\pm)$]

$$\int du_+du_-\gamma^2(s,u_\pm)(\mu^2+u_\perp{}^2+u_+u_--i\epsilon)^{-1}\to i\pi\ln(s/s_0), \tag{18}$$

where $u_\pm=u_3\pm u_0$. These steps mirror the detailed graphical analysis. The ordering of the pion momenta produces a factor of $\ln^r(s/s_0)/r!$ whose coefficient is independent of the relative position along either NVM line. Precisely this dependence is produced by the replacement given by Eq. (16), with its exponential structure providing the factor $1/r!$ The correct $\ln(s/s_0)$ dependence and its coefficient follow from the replacement (17) and (18).

Let us now turn to the problem of computing χ_n for $n\geq 3$. Unfortunately, an analytical evaluation of Eq. (11) is not possible. This problem has been discussed in Ref. 5, where upper and lower bounds were derived for each value of n. In this paper we will use the functional approach to derive results in a simple and transparent manner which are not exact but which lie between the rigorous limits. The consequences of a leading-log factor $(1/r!)\ln^r(s/s_0)$ (for the exchange of r pions between n NVM lines), multiplied by appropriate coefficients $\alpha_2(q_i+q_i)$ (assumed independent of position along any NVM line), combined with an obvious statistical factor (representing the number of ways of selecting pure ladder graphs of this form), may be reproduced by the equivalent functional replacements of Eqs. (16)–(18); one easily obtains

$$e^{i\chi}-1=\sum_{n=1}^{\infty}\frac{(ig^2)^n}{n!}\left[\prod_{l=1}^{n}(2\pi)^{-2}\int\frac{d^2q_le^{iq_l\cdot b}}{(q_l{}^2+M^2)}\right]\left(\frac{s}{s_0}\right)^Q, \tag{19}$$

where

$$Q=\frac{\lambda^2}{8\pi}\sum_{i<j=1}^{n}\alpha_2(q_i+q_j).$$

Two important features of this equation should be noted: the oscillating phase factor i^n which will provide cancellations between every other term, and the rapidly growing s dependence which is in the form of Regge behavior between *each possible pair* of exchanged NVM lines.

To illustrate our final result, we make the simplifying assumption that the q^2 dependence of α_2 can be neglected. This is a reasonable assumption sinch each q_l integral has a convergence factor of $(q_l^2+m^2)^{-1}$ and each $\alpha_2(q^2)$ has its q^2 dependence reduced by $\langle x(1-x)\rangle \le \frac{1}{4}$. This approximation does not change the qualitative behavior of the result. The total cross section becomes

$$\sigma_T = \frac{2\pi}{M^2}\,\mathrm{Re}\sum_{n=2}^{\infty} i^{n-2}C_n\left(\frac{g^2}{4\pi}\right)^n\left(\frac{s}{s_0}\right)^{\lambda^2\alpha_2(0)n(n-1)/16\pi}, \tag{20}$$

where

$$C_n = \frac{2}{n!}\int_0^\infty b\,db[2K_0(b)]^n \simeq 1\,. \tag{21}$$

Thus the total cross section can be written as

$$\sigma_T = \frac{2\pi}{M^2}\,\mathrm{Re}\sum_{n=2}^{\infty} C_n i^{n-2}(x/y)^n y^{n^2}\,, \tag{22}$$

where

$$x = g^2/4\pi\,,$$

$$y = \left(\frac{s}{s_0}\right)^{\lambda^2\alpha_2(0)/16\pi}$$

Since y is very large in the region of interest, where the previously made approximations make sense, this series is badly divergent. However, it can be defined as summed by using the formula

$$\sigma_T = \frac{2\pi}{M^2}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx\, e^{-z^2}F\left(\frac{x}{y}e^{2z\beta}\right), \tag{23}$$

where

$$F(X) \equiv \mathrm{Re}\sum_{n=2}^{\infty} C_n i^{n-2}X^n$$

$$\cong X^2(1+X^2)^{-1}$$

and $\beta^2 = \ln y$. A simple analysis of the z integral then shows that as $s\to\infty$,

$$\sigma_T \simeq \frac{\pi g}{2\sqrt{2}M^2}C_{-1/2}\left(\frac{s}{s_0}\right)^{-\lambda^2\alpha_2(0)/64\pi} \times\left[\ln\left(\frac{s}{s_0}\right)\right]^{-1/2}+\cdots, \tag{24}$$

where

$$C_{-1/2} = \frac{\sqrt{2}}{\pi}\int_0^\infty dx\, x^{-3/2}F(x)\,.$$

Therefore there is almost complete cancellation, and rather than behaving as the Froissart bound, $\simeq \ln^2 s$, the total cross section falls as a power of s. Further, it is easy to see that if C_n is not given by Eq. (21) but is given by a smooth function of n, the total cross section still falls as Eq. (24).

One may expect that certain of our results are quite independent of specific details and approximations used in this model. In particular, the strong cancellations exhibited between higher-order nonplanar graphs should be a general property of relativistic theories. For example, one should be able to use these cancellations to produce constant total cross sections rather than the typical Froissart-bound behavior. However, one must engineer the theory very carefully so that the forces between the NVMs saturate and do not produce the behavior exhibited above. In any case, multi-Regge-type models which do not contain the nonplanar graphs required by unitarity cannot be trusted at very large energies. One cannot rule out the possibility that such theories are accidentally accurate at intermediate energies.

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