Null-Plane Formalism and Polarization

A. Chakrabarti and C. Darzens
Centre de Physique Thêorique de l'Ecole Polytechnique, 17, rue Descartes, 75230 Paris, Cedex 05, France
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A systematic study is made of the properties of spin in the context of null-plane (or "infinite-momentum") formalism, where the spin projection is invariant under the action of a Galilean subgroup of the Poincaré group. We give explicitly its relations with the helicity and the canonical formalism. The crossing relations and the structure of kinematic singularities of the corresponding amplitudes are studied. It is shown that these amplitudes can be kinematically regularized in a fairly simple fashion. Reduction of direct products and partial-wave expansions are also discussed and some particular simple properties are pointed out.

I. INTRODUCTION

The significance of a two-dimensional Galilean subgroup of the Poincaré group and its relation to high-energy processes (the so-called "infinite momentum limit") have been studied by several authors. This has led to the "infinite momentum" or "null-plane formalism[1] for fields. Bacry and Chang[2] and Soper[2] have studied polarization by defining states which transform very simply under the Galilean subgroup. The same definition of states and the corresponding parametrization of the the momenta have been utilized also by other authors[3,4] for partial-wave analysis of a particular type and has been called the horispheric system or "O-system." Bacry and Chang study "kinematics at infinite momentum" through a transformation. Soper also emphasizes the limiting properties of "infinite-momentum helicity states." To avoid trivial confusion let us note explicitly that the following formalism is valid for any momentum. The results in Secs. II and III include the known results essential for our purposes.

For brevity we will also adopt the term "O states" for states defined through such spin projections. We will make a systematic study of their properties, including crossing relations and kinematical singularities of the corresponding scattering amplitudes. We will compare such properties with those of the well-known helicity and canonical states (which we will call henceforth "h states" and "c states," respectively). The so-called transversity states often used are related to the helicity states by a rotation $\frac{1}{2}\pi$ about an axis chosen conventionally to be the x axis. We will introduce states related by the same transformation to the O and c states and will henceforth use the terms "trans-O," "trans-c," and "trans-h" states, respectively ("trans-h" being the usual transversity states).

Let us now introduce some definitions and notations. To avoid confusion among different formalisms, we keep the usual significance of the indices $\mu = 0, 1, 2, 3$, with $g^{00} = -g^{ii} = 1$ $(i = 1, 2, 3)$ [i.e., we do not adopt the notations of Refs. 1(a) and 2(b)]. Let

$$n_\mu = (\tfrac{1}{2})^{1/2}(1, 0, 0, +1), \quad \bar{n}_\mu = (\tfrac{1}{2})^{1/2}(1, 0, 0, -1). \tag{1.1}$$

For any vector A_μ we define

$$\begin{aligned} A_n &= A\cdot n = (\tfrac{1}{2})^{1/2}(A^0 + A^3), \\ A_{\bar{n}} &= A\cdot \bar{n} = (\tfrac{1}{2})^{1/2}(A^0 - A^3), \end{aligned} \tag{1.2}$$

and the "transverse" components are $\underline{A} = (A^1, A^2)$. One can, of course, choose in a more general fashion any two vectors n, $\bar{n}$ such that $n^2 = 0 = \bar{n}^2$, $n\cdot\bar{n} = 1$. But the simple case (1.1) is adequate for our purpose.

In terms of the indices $(n, \bar{n}, 1, 2)$, we have

$$g_{n\bar{n}} = g_{\bar{n}n} = 1 = -g_{11} = -g_{22},$$

and

$$A\cdot B = A_n B_{\bar{n}} + A_{\bar{n}} B_n - \underline{A}\cdot\underline{B}. \tag{1.3}$$

(We reserve the notation $\vec{A}$ [$= (A^1, A^2, A^3)$] for the space part of A.)

$M_{\mu\nu}$ and P_μ are the generators of the Poincaré group and

$$\begin{aligned} &M_{ij} = \epsilon_{ijk} J_k, \quad M_{i0} = K_i, \\ &P_n = (\tfrac{1}{2})^{1/2}(P^0 + P^3), \quad P_{\bar{n}} = (\tfrac{1}{2})^{1/2}(P^0 - P^3), \\ &\underline{P} = (P^1, P^2), \\ &B_1 = (\tfrac{1}{2})^{1/2}(K_1 + J_2), \quad B_2 = (\tfrac{1}{2})^{1/2}(K_2 - J_1), \\ &S_1 = (\tfrac{1}{2})^{1/2}(K_1 - J_2), \quad S_2 = (\tfrac{1}{2})^{1/2}(K_2 + J_1), \end{aligned} \tag{1.4}$$

with the notation $\underline{B} = (B_1, B_2)$, $\underline{S} = (S_1, S_2)$.

$\underline{P}, J_3$ along with P_n, $\underline{B}$ or $P_{\bar{n}}$, $\underline{S}$ constitute a two-dimensional Galilean subgroup.[1] These are related through the parity operator $\mathcal{P}$, where

$$\mathcal{P}(P_n, \underline{B})\mathcal{P} = (P_{\bar{n}}, -\underline{S}) . \quad (1.5)$$

For any 4-vector operator A, we have

$$e^{i\omega K_3}Ae^{-i\omega K_3} = (e^{\omega}A_n, \underline{A}, e^{-\omega}A_{\bar{n}}),$$
$$e^{i\underline{v}\cdot\underline{B}}Ae^{-i\underline{v}\cdot\underline{B}} = (A_n, \underline{A}+\underline{v}A_n, \tilde{A}_{\bar{n}}),$$
$$e^{i\underline{u}\cdot\underline{S}}Ae^{-i\underline{u}\cdot\underline{S}} = (\tilde{A}_n, \underline{A}+\underline{u}A_{\bar{n}}, A_{\bar{n}}), \quad (1.6)$$

where

$$\tilde{A}_{\bar{n}} = (A_{\bar{n}} + \underline{v}\cdot\underline{A} + \tfrac{1}{2}A_n \underline{v}^2),$$
$$\tilde{A}_n = (A_n + \underline{u}\cdot A + \tfrac{1}{2}A_{\bar{n}}\underline{u}^2).$$

II. DEFINITION OF THE O STATE

We will start by considering *positive-energy massive particles*. The simple generalization to include negative-energy states will be indicated later and the limit to the zero-mass case presents no problem. Indeed, we obtain directly the well-known description of the zero-mass states in terms of the little group E_2.

Starting with the rest-frame state (mass $m > 0$) with the spin projection ν along the z axis,

$$|m, \vec{0}; \nu\rangle , \quad (2.1)$$

let us define

$$|p, \nu\rangle = e^{-iv\cdot B}e^{-i\omega K_3}|m, 0; \nu\rangle \quad (2.2)$$
$$= e^{-i\omega K_3}e^{-ie^{\omega}\underline{v}\cdot\underline{B}}|m, \vec{0}; \nu\rangle \quad (2.2')$$

and

$$|p, \nu\rangle = e^{-iv'\cdot S}e^{+i\omega' K_3}|m, \vec{0}; \nu\rangle , \quad (2.3)$$

where

$$e^{\omega} = \frac{\sqrt{2}\,p_n}{m}, \quad \underline{v} = \underline{p}/p_n ,$$
$$e^{\omega'} = \frac{\sqrt{2}\,p_{\bar{n}}}{m}, \quad \underline{v}' = p/p_{\bar{n}} . \quad (2.4)$$

Using (1.5) we obtain

$$\mathcal{P}|\vec{p}, \nu\rangle = \eta| -\vec{p}, \nu), \quad (2.5)$$

where η is the intrinsic parity.

Let us first study (2.2) in more detail. In the language of "tetrads,"[5,6] the three components of the spin operator are given by

$$-\frac{1}{m} W\cdot e_{(i)} \quad (i = 1, 2, 3),$$

where

$$W^{\mu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\lambda}P^{\nu}M^{\rho\lambda}$$
$$= -i[P^{\mu}, \vec{K}\cdot\vec{J}]$$
$$= (\vec{P}\cdot\vec{J}, P^0\vec{J} - \vec{P}\times\vec{K}), \quad (2.6)$$

and $e_{(i)}\cdot p = 0$, $e_{(i)}\cdot e_{(j)} = -\delta_{ij}$.

Defining for the rest frame

$$\hat{e}^{\mu}_{(\alpha)} = \delta_{\mu\alpha} , \quad e_{(\alpha)} = \Lambda_B\hat{e}_{(\alpha)} , \quad (\alpha = 0, 1, 2, 3),$$

with

$$U(\Lambda_B) = e^{-\underline{v}\cdot\underline{B}}e^{-i\omega K_3} \quad (2.7)$$

[as in (2.2)], one obtains:

$$e^{\mu}_{(0)} = p^{\mu}/m ,$$
$$e^{\mu}_{(1)} = \left(\frac{p^1}{\sqrt{2}\,p_n}, 1, 0, \frac{-p^1}{\sqrt{2}\,p_n}\right),$$
$$e^{\mu}_{(2)} = \left(\frac{p^2}{\sqrt{2}\,p_n}, 0, 1, \frac{-p^2}{\sqrt{2}\,p_n}\right), \quad (2.8)$$
$$e^{\mu}_{(3)} = \frac{p^{\mu}}{m} - \frac{m}{p_n}n^{\mu} .$$

Thus the component to be diagonalized is

$$-\frac{1}{m}W\cdot e_{(3)} = \frac{W_n}{P_n} , \quad (2.9)$$

which may be verified to be the same as the spin operator constructed by Soper.[2]

We see from (2.8), (2.9) the fundamental property *that for any momentum, though $e_{(3)}$ is a spacelike vector, what is effectively diagonalized (since $W\cdot p = 0$) is the projection of W on a fixed lightlike direction n .*

Instead of fixing our attention from the very beginning on momentum eigenstates, we can define, using the operators P^{μ}, the spin operators

$$\underline{\Sigma}_{(B)} = \frac{1}{M}\left(\underline{W} - \frac{W_n}{P_n}\underline{P}\right),$$
$$\Sigma_{3(B)} = \frac{W_n}{P_n} \quad [M \equiv (P^{\mu}P_{\mu})^{1/2}], \quad (2.10)$$

where

$$\Sigma_i|p, \nu\rangle = -\frac{W}{m}\cdot e_{(i)}|p, \nu\rangle .$$

But with such a definition of $\vec{\Sigma}$, one can consider directly its action on a more general class of states, which diagonalize P_n but not necessarily $\underline{P}$ and $P_{\bar{n}}$.

Using

$$[W_{\mu}, W_{\nu}] = i\epsilon_{\mu\nu\rho\sigma}P^{\rho}W^{\sigma} ,$$
$$[P_{\mu}, W_{\nu}] = 0 , \quad (2.11)$$

one obtains the usual spin algebra

$$[\Sigma_{(B)i}, \Sigma_{(B)j}] = i\epsilon_{ijk}\Sigma_{(B)k} . \quad (2.12)$$

The above definition may be compared with the canonical spin operator[7]

$$\vec{\Sigma}_{(c)} = \frac{1}{M}\left(\vec{W} - \frac{W^0}{P^0+M}\vec{P}\right). \tag{2.13}$$

These correspond to the tetrads

$$e_{(i)} = \Lambda_{(p)} \cdot \hat{e}_{(i)}, \tag{2.14}$$

where $\Lambda_{(p)}$ is the *pure Lorentz transformation* such that

$$\Lambda_{(p)}(1, \vec{0}) = \frac{p}{m} .$$

Using the results (1.6) for P and W, one obtains (for any ω and $\underline{v}, \underline{u}$)

$$e^{i\omega K_3}\vec{\Sigma}_{(B)}e^{-i\omega K_3} = \vec{\Sigma}_{(B)},$$
$$e^{i\underline{v}\cdot\underline{B}}\vec{\Sigma}_{(B)}e^{-i\underline{v}\cdot\underline{B}} = \vec{\Sigma}_{(B)}, \tag{2.15}$$

and

$$e^{i\underline{u}\cdot\underline{S}}\vec{\Sigma}_{(B)}e^{-i\underline{u}\cdot\underline{S}} = R_{(B)}\vec{\Sigma}_{(B)},$$

where the Wigner rotation $R_{(B)}$, written here formally in terms of operators, corresponds on 4-momentum basis to a rotation about the axis

$$\vec{a} = \left(u^2, -u^1, \frac{1}{m}(p^1u^2 - p^2u^1)\right)$$

through an angle χ where

$$\cos\chi = 1 - \frac{m^2\vec{a}^2}{2P_n\tilde{P}_n},$$
$$\sin\chi = \frac{m|\vec{a}|}{2P_n\tilde{P}_n}(2P_n + \underline{P}\cdot\underline{u}), \tag{2.16}$$

where $\tilde{P}_n = P_n + \underline{u}\circ\underline{P} + \frac{1}{2}P_{\bar{n}}\underline{u}^2$.

This formula is not particularly simple. But particular cases will be of crucial importance later on.

Under the action of J_3, $\vec{\Sigma}_{(B)}$ of course simply turns like a 3-vector round the z axis. As a consequence we have the following transformation laws for the momentum basis states of the representation $[m, s]$. Denoting the transformed momentum in each case by p' [see (1.6)],

$$e^{-i\varphi J_3}|p,\nu\rangle = e^{-i\varphi\nu}|p',\nu\rangle,$$
$$e^{-i\omega K_3}|p,\nu\rangle = |p',\nu\rangle,$$
$$e^{-i\underline{v}\cdot\underline{B}}|p,\nu\rangle = |p',\nu\rangle, \tag{2.17}$$
$$e^{-i\underline{u}\cdot\underline{S}}|p,\nu\rangle = |p',\nu'\rangle\mathfrak{D}^S_{\nu'\nu}(R_{(B)}(\underline{u},\vec{p})),$$

where $R_{(B)}(\underline{u},\vec{p})$ is given by (2.16). This is the only nontrivial Wigner rotation in this basis.

Starting with the states $|p,\nu)$, we obtain through similar considerations the spin operator

$$\underline{\Sigma}_{(S)} = \frac{1}{M}\left(\underline{W} - \frac{W_{\bar{n}}}{P_{\bar{n}}}\underline{P}\right),$$
$$\Sigma_{(S)3} = -\frac{W_{\bar{n}}}{P_{\bar{n}}}, \tag{2.18}$$

with

$$\mathcal{P}\vec{\Sigma}_{(B)}\mathcal{P} = \vec{\Sigma}_{(S)} . \tag{2.19}$$

Using this relation one obtains

$$e^{i\underline{v}\cdot\underline{B}}\vec{\Sigma}_{(S)}e^{-i\underline{v}\cdot\underline{B}} = R_{(S)}(\underline{v},\vec{p})\vec{\Sigma}_{(S)}, \tag{2.20}$$

where

$$R_{(S)}(\underline{v},\vec{p}) = R_{(B)}(-\underline{v}, -\vec{p}) . \tag{2.21}$$

As compared to (2.17), we have for $|p,\nu)$ similar results for J_3 and K_3, with

$$e^{-i\underline{u}\cdot\underline{S}}|p,\nu) = |p',\nu),$$
$$e^{-i\underline{v}\cdot\underline{B}}|p,\nu) = |p',\nu')\mathfrak{D}^S_{\nu'\nu}(R_{(S)}(\underline{v},\vec{p})) . \tag{2.22}$$

We will also need the actions of arbitrary pure rotations and pure Lorentz transformations. For this purpose it is sufficient to consider the action of J_2 on the states $|p,\nu\rangle$, which combined with the preceding results gives the rest.

It can be shown that

$$e^{-i\alpha J_2} = e^{i\zeta K_3}e^{(i/\sqrt{2})\sin\alpha B_1}e^{i\sqrt{2}\tan(\alpha/2)S_1} \tag{2.23}$$

[where $e^{-\zeta} = \cos^2(\alpha/2)$]

$$= e^{-i\zeta K_3}e^{(i/\sqrt{2})\sin\alpha S_1}e^{-i\sqrt{2}\tan(\alpha/2)B_1}. \tag{2.23'}$$

[For this purpose it is helpful to use a representation of SL(2, C) by 2×2 matrices. Also the apparently singular point $\alpha=\pi$ causes no problem; in subsequent related applications, unambiguous finite results are obtained on approaching π from either side.]

From (2.17),

$$e^{-i\alpha J_2}|p,\nu\rangle = |p',\nu'\rangle\mathfrak{D}^S_{\nu'\nu}(R(\alpha,p)), \tag{2.24}$$

where $R(\alpha,p)$ is given by (2.16) with

$$u^1 = -\sqrt{2}\tan\tfrac{1}{2}\alpha, \quad u^2 = 0 .$$

This angle we may denote by $\tilde{\alpha}_{\vec{p}}$.

Of particular interest for future use is the case where the momentum lies in the z-x plane:

$$p^3 = |p|\cos\theta, \quad p^1 = |p|\sin\theta, \quad p^2 = 0 .$$

Then (2.24) gives a rotation, *about the y axis itself,* through an angle $\tilde{\alpha}_{\vec{p}}$ such that

$$\cos\tilde{\alpha}_{\vec{p}} = 1 - \frac{2m^2\sin^2(\frac{1}{2}\alpha)}{\sqrt{2}\,p_n[p^0 + |p|\cos(\theta+\alpha)]},$$
$$\sin\tilde{\alpha}_{\vec{p}} = \frac{m\sin\alpha[p^0 + |p|\cos(\theta + (\frac{1}{2}\alpha)/\cos(\frac{1}{2}\alpha)]}{\sqrt{2}\,p_n[p^0 + |p|\cos(\theta+\alpha)]}; \tag{2.25}$$

the corresponding results for $|p,\nu)$ are obtained from (2.21) and (2.23′).

Let us now consider the explicit forms of the generators for the states $|p,\nu\rangle$, corresponding to the irreducible representation $[m,s]$. [The results for $|p, \nu)$ are obtained through parity transformation.]

The simplest forms are obtained if we consider $\underline{P}$, P_n as the independent components and

$$P_{\bar{n}} = \frac{(\underline{P})^2 + m^2}{2P_n} \quad (2.26)$$

when (with $\partial_i \equiv \partial/\partial P^i$, $i = 1, 2$)

$$\begin{aligned} &[\partial_i, P^i] = 1, [\partial_n, P_n] = 1, \\ &[\partial_i, P_{\bar{n}}] = \frac{P^i}{P_n}, \\ &[\partial_n, P_{\bar{n}}] = -\frac{P_{\bar{n}}}{P_n}. \end{aligned} \quad (2.27)$$

The corresponding scalar product may be taken to be

$$\langle p, \nu | p', \nu' \rangle = 2p_n \delta(p_n - p_n')\delta(\underline{p} - \underline{p}')\delta\nu\nu'. \quad (2.27')$$

Then it can be shown [by considering infinitesimal forms of (2.17)] that:

$$\begin{aligned} &J_3 = -i(P^1\partial_2 - P^2\partial_1) + \Sigma_3, \\ &K_3 = -iP_n\partial_n, \\ &B_i = -iP_n\partial_i, \\ &S_1 = -i(P_{\bar{n}}\partial_1 + P^1\partial_n) - \frac{P^2}{P_n}\Sigma_3 - \frac{m}{P_n}\Sigma_2, \\ &S_2 = -i(P_{\bar{n}}\partial_2 + P^2\partial_n) + \frac{P^1}{P_n}\Sigma_3 + \frac{m}{P_n}\Sigma_1. \end{aligned} \quad (2.28)$$

Here $\vec{\Sigma}$ are the usual $(2S+1)\times(2S+1)$ matrices, generators of the rotation group (e.g., $\vec{\Sigma} = \frac{1}{2}\vec{\tau}$ for $s = \frac{1}{2}$ in terms of the Pauli matrices). It may be verified that calculating W^μ from (2.28) and substituting in (2.10), we get simply

$$\vec{\Sigma}_{(B)} = \vec{\Sigma} \quad (2.29)$$

for this representation.[2(a)] [Instead of (2.27), if we consider, as usual, P^1, P^2, P^3 as independent components, the corresponding results, easily obtained, are somewhat less simple.]

In (2.28) m is to be replaced by ϵm, where $\epsilon = p^0/|p^0|$ if we consider both signs of energy. This is also true for other definitions involving m. This ensures that states with momenta p^μ and $-p^\mu$ transform in the same way, which is important in considering crossing relations.[3]

Also putting $m = 0$, we recover exactly the representation studied in Ref. 9 for the case of discrete spin. For the sake of completeness, we mention that for zero-mass continuous spin [Refs. 7(b), 10], by defining

$$\Sigma_i = W^i - \frac{W_n}{P_n}P^i + \delta_{3i}\frac{W_n}{P_n},$$

one generates the algebra of E_2. For discrete spin, this reduces to $\Sigma_{1,2} = 0$, $\Sigma_3 = W_n/P_n$.

III. RELATION WITH SPINOR REPRESENTATIONS

The relation of the above representations with the covariant spinor representations can be studied in exactly the same way as that between the canonical and spinor representations.[11]

For the important case of spin $\frac{1}{2}$, starting with the Dirac representation we note that the spinor matrix corresponding to the transformation (2.2) can be obtained (using the appropriate representation[11] of the γ matrices) as

$$Q = \begin{vmatrix} [p]_+ & 0 \\ 0 & [p]_- \end{vmatrix},$$

where

$$\begin{aligned} &[p]_+ = \left(\frac{\sqrt{2}\,p_n}{m}\right)^{-1/2} \begin{vmatrix} \sqrt{2}\,p_n/m & 0 \\ \sqrt{2}\,p_+/m & 1 \end{vmatrix}, \\ &[p]_- = \left(\frac{\sqrt{2}\,p_n}{m}\right)^{-1/2} \begin{vmatrix} 1 & -\sqrt{2}\,p_-/m \\ 0 & \sqrt{2}\,p_n/m \end{vmatrix}, \\ &[p_\pm = (\tfrac{1}{2})^{1/2}(p^1 \pm ip^2)]. \end{aligned} \quad (3.1)$$

Starting with the Dirac representation of $M^{\mu\nu}$, the transformed generators

$$Q^{-1}M^{\mu\nu}Q \quad (3.2)$$

correspond to (2.28) (for the same choice of independent variables), with

$$\vec{\Sigma} = \tfrac{1}{2}\begin{vmatrix} \vec{\tau} & 0 \\ 0 & \vec{\tau} \end{vmatrix}.$$

We have also for the transformed Dirac equation

$$Q^{-1}(\gamma^\mu p_\mu - m)Q = [\gamma^0(p^\mu p_\mu)^{1/2} - m], \quad (3.3)$$

exactly as for the canonical case.[11] This result follows directly from the covariance of the γ matrices.

The solutions of the Dirac equation diagonalizing W_n/P_n in the Dirac representation are given by[12]

$$u_\pm = \begin{vmatrix} [p]_+ & \chi_\pm \\ [p]_- & \chi_\pm \end{vmatrix}, \quad v_\pm = \begin{vmatrix} [p]_+ & C^{-1}\chi_\pm \\ [p]_- & C\chi_\pm \end{vmatrix}, \quad (3.4)$$

where

$$\chi_+ = \left|\begin{matrix} 1 \\ 0 \end{matrix}\right\rangle, \quad \chi_- = \left|\begin{matrix} 0 \\ 1 \end{matrix}\right\rangle,$$

$$C = i\tau_2 = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}.$$

Analogous transformations can be carried out for spinor representations for arbitrary spin (Soper[2] and Ref. 11).

IV. RELATION WITH HELICITY AND CANONICAL STATES

We have defined, for the representation $[m, s]$,

$$|p, \nu\rangle = e^{-i\underline{v}\cdot\underline{B}} e^{-i\omega K_3}|\vec{0}, \nu\rangle,$$

where $\underline{v} = \underline{p}/p_n$, $e^{\omega} = \sqrt{2}\, p_n/m$. The helicity states are defined as[13]

$$\begin{aligned} |p, \lambda\rangle &= R(\hat{p}) e^{-i\eta K_3}|\vec{0}, \lambda\rangle \\ &= e^{-i\varphi J_3} e^{-i\theta J_2} e^{i\varphi J_3} e^{-i\eta K_3}|\vec{0}, \lambda\rangle, \end{aligned} \tag{4.1}$$

where (θ, φ) are the angular coordinates of p and $\tanh\eta = |\vec{p}|/p^0$.

Hence (suppressing for our present purpose the δ functions corresponding to the 4-momenta),

$$\begin{aligned} \langle p, \lambda | p, \nu\rangle &= \langle \vec{0}, \lambda| e^{i\eta K_3} R^{-1}(\hat{p})|p, \nu\rangle \\ &= e^{i\varphi(\nu-\lambda)} d^s_{\lambda\nu}(-\tilde{\theta}), \end{aligned} \tag{4.2}$$

where putting $\alpha = -\theta$ in (2.25), denoting $(-\tilde{\theta})_{\vec{p}}$ by simply $-\tilde{\theta}$, we obtain

$$\cos\tilde{\theta} = \frac{p^0\cos\theta + |p|}{p^0 + |p|\cos\theta}, \quad \sin\tilde{\theta} = \frac{m\sin\theta}{p^0 + |p|\cos\theta}. \tag{4.3}$$

Some useful results are

$$\begin{aligned} \cos\tfrac{1}{2}\tilde{\theta} &= \left(\frac{p^0 + |p|}{p^0 + p^3}\right)^{1/2} \cos\tfrac{1}{2}\theta, \\ \sin\tfrac{1}{2}\tilde{\theta} &= \left(\frac{p^0 - |p|}{p^0 + p^3}\right)^{1/2} \sin\tfrac{1}{2}\theta \quad (p^3 = |p|\cos\theta), \end{aligned} \tag{4.3$'$}$$

or

$$\begin{aligned} \tan\tfrac{1}{2}\tilde{\theta} &= \left(\frac{p^0 - |p|}{p^0 + |p|}\right)^{1/2} \tan\tfrac{1}{2}\theta \\ &= \left(\frac{m}{p^0 + |p|}\right)\tan\tfrac{1}{2}\theta. \end{aligned} \tag{4.3$''$}$$

We will often use the case $\varphi = 0$ ($\vec{p}$ in zx plane). Let us also note that for $\theta = \frac{1}{2}\pi$ ($\vec{p}$ in xy plane),

$$\cos\tilde{\theta} = v, \quad \sin\tilde{\theta} = (1 - v^2)^{1/2}, \tag{4.4}$$

where $v = |\vec{p}|/p^0$, i.e., the rotation corresponds directly to the Lorentz contraction factor.

The canonical states are defined as (see Appendix A of Ref. 8 and Ref. 11)

$$\begin{aligned} |p, \sigma\rangle &= e^{-i\eta\hat{p}\cdot\vec{K}}|\vec{0}, \sigma\rangle \quad (\tanh\eta = |\vec{p}|/p^0, \hat{p} = \vec{p}/|\vec{p}|) \\ &= R(\hat{p}) e^{-i\eta K_3} R^{-1}(\hat{p})|\vec{0}, \sigma\rangle, \end{aligned} \tag{4.5}$$

Hence

$$\begin{aligned} \langle p, \sigma | p, \nu\rangle &= \langle \vec{0}, \sigma| R(\hat{p}) e^{i\eta K_3} R^{-1}(\hat{p})|p, \nu\rangle \\ &= e^{i\varphi(\nu-\sigma)} d^s_{\sigma\nu}(\theta - \tilde{\theta}). \end{aligned} \tag{4.6}$$

From (4.2) one obtains the property noted by Soper[2] that the O states tend to the h states as $|\vec{p}| \to \infty$. But this is not the only interesting limiting property. From (4.6) one can verify that the O states tend rapidly to the c states as $|\vec{p}| \to 0$. It is known that while h states possess certain *convenient* properties at high energy, the c states provide the most direct generalization of nonrelativistic formalism at low energy. *The O states thus gather together the convenient properties of the h and c states, in this respect.*

Let us now define the following states:

$$\text{trans-}h\text{:} \quad |p, \tilde{\lambda}\rangle = |p, \lambda\rangle \mathfrak{D}^S_{\lambda\tilde{\lambda}}(\tfrac{1}{2}\pi, \tfrac{1}{2}\pi, -\tfrac{1}{2}\pi), \tag{4.7}$$

$$\text{trans-}c\text{:} \quad |p, \tilde{\sigma}\rangle = |p, \sigma\rangle \mathfrak{D}^S_{\sigma\tilde{\sigma}}(\tfrac{1}{2}\pi, \tfrac{1}{2}\pi, -\tfrac{1}{2}\pi), \tag{4.8}$$

$$\text{trans-}O\text{:} \quad |p, \tilde{\nu}\rangle = |p, \nu\rangle \mathfrak{D}^S_{\nu\tilde{\nu}}(\tfrac{1}{2}\pi, \tfrac{1}{2}\pi, -\tfrac{1}{2}\pi) \tag{4.9}$$

(For the sake of uniformity we have denoted the usual transversity states as the trans-$h|p, \tilde{\lambda}\rangle$). These are often convenient for studying scattering amplitudes in the zx plane. For $\varphi = 0$ ($p^1 = |p|\sin\theta$, $p^2 = 0$, $p^3 = |p|\cos\theta$), we obtain from (4.2)–(4.9),

$$\begin{aligned} \langle p, \tilde{\lambda}| p, \tilde{\nu}\rangle &= e^{+i\tilde{\theta}\tilde{\nu}} \delta_{\tilde{\lambda}\tilde{\nu}}, \\ \langle p, \tilde{\sigma}| p, \tilde{\nu}\rangle &= e^{-i(\theta-\tilde{\theta})\tilde{\nu}} \delta_{\tilde{\sigma}\tilde{\nu}}. \end{aligned} \tag{4.10}$$

When considering scattering amplitudes in the zx plane we will adopt the convention that if the angular coordinates of $\vec{p}$ are $(0, \theta)$, those of $-\vec{p}$ are $(0, \theta + \pi)$ and not $(\pi, \pi, -\theta)$.

Let us note that for $p_2 = 0$,

$$\begin{aligned} e^{i\pi J_y}|p, \nu\rangle &= e^{i\pi J_y}(e^{-iv^1 B_1} e^{-i\omega K_3}) e^{-i\pi J_y} e^{i\pi J_y}|\vec{0}, \nu\rangle \\ &= e^{iv^1 S_1} e^{i\omega K_3}[(-1)^{S-\nu}|\vec{0}, -\nu\rangle] \\ &= (-1)^{S-\nu}|-\vec{p}, -\nu) \end{aligned}$$

or

$$|\vec{p}, \nu) = (-1)^{S+\nu} e^{i\pi J_y}| -\vec{p}, -\nu\rangle \tag{4.11}$$

$$= (-1)^{S+\nu}|\vec{p}, \nu'\rangle d^S_{\nu', -\nu}(-\tilde{\pi})_{-\vec{p}}, \tag{4.11$'$}$$

where [from (2.25)]

$$\cos(-\tilde{\pi})_{-\vec{p}} = 1 - \frac{2m^2}{p_0^{\,2} - |p|^2\cos^2\theta},$$

$$\sin(-\tilde{\pi})_{-\vec{p}} = \frac{2m|p|\sin\theta}{p_0^{\,2} - |p|^2\cos^2\theta}.$$

In terms of h states, one obtains

$$\langle p, \lambda | p, \nu) = d^S_{\lambda\nu}(-\tilde{\theta}), \tag{4.12}$$

where

$$\begin{aligned} \cos\tilde{\theta}' &= \frac{p^0\cos\theta - |p|}{p^0 - |p|\cos\theta}, \\ \sin\tilde{\theta}' &= \frac{m\sin\theta}{p^0 - |p|\cos\theta}. \end{aligned} \tag{4.13}$$

The results corresponding to (4.6) and those to (4.10) [for $|p, \tilde{\nu})$ defined as in (4.9)], are obtained by substituting $\tilde{\theta}'$ for $\tilde{\theta}$.

Corresponding to (4.3′) and (4.3″) we now have

$$\cos\tfrac{1}{2}\theta' = \left(\frac{p^0 - |p|}{p^0 - p^3}\right)^{1/2} \cos\tfrac{1}{2}\theta\,, \qquad \sin\tfrac{1}{2}\theta' = \left(\frac{p^0 + |p|}{p^0 - p^3}\right)^{1/2} \sin\tfrac{1}{2}\theta\,, \tag{4.13'}$$

and

$$\tan\tfrac{1}{2}\theta' = \left(\frac{p^0 + |p|}{p^0 - |p|}\right)^{1/2} \tan\tfrac{1}{2}\theta = \left(\frac{p^0 + |p|}{m}\right) \tan\tfrac{1}{2}\theta\,. \tag{4.13''}$$

V. CROSSING RELATIONS

The crossing relations for the O and trans-O amplitudes can be derived directly from the basic principles or deduced from the known relations for the h, c, or trans-h amplitudes.[8,11] In what follows, we will only point out certain basic results, relying heavily on the formalism of Ref. 8. It is simplest to start from the crossing relations between the c.m. s- and t-channel trans-h (or transversity) amplitudes in the zx plane (Figs. 1 and 2).

In Ref. 8 the conventions are so chosen as to make the crossing matrix really diagonal. Replacing the indices τ_i of Ref. 8 by $\tilde{\lambda}_i$ (according to our present conventions), we have

$$(\langle p_3\tilde{\lambda}_3, p_4\tilde{\lambda}_4 | T | p_1\tilde{\lambda}_1, p_2\tilde{\lambda}_2\rangle)_{\mathrm{cntd}} = (-1)^{\eta} e^{i\pi(\tilde{\lambda}_1+\tilde{\lambda}_4) - i(\chi_1\tilde{\lambda}_1 - \chi_2\tilde{\lambda}_2 - \chi_3\tilde{\lambda}_3 + \chi_4\tilde{\lambda}_4)} \langle P_3\tilde{\lambda}_3, P_1\tilde{\lambda}_1 | T | P_4\tilde{\lambda}_4, P_2\tilde{\lambda}_2\rangle \tag{5.1}$$

(here η is an over-all phase factor).

The angles χ_i are given in Ref. 8 and additional remarks about the phase factors and the continuation path are given in the Appendix.

From (4.10) and (5.1) we obtain (with $\theta_4 = \pi + \theta_s$, $\theta_2' = \pi + \theta_t$)

$$(e^{-i(\pi\tilde{\nu}_2 - \tilde{\theta}_s\tilde{\nu}_3 - \tilde{\theta}_4)\tilde{\nu}_4} \langle p_3\tilde{\nu}_3, p_4\tilde{\nu}_4 | T | p_1\tilde{\nu}_1, p_2\tilde{\nu}_2\rangle)_{\mathrm{cntd}} = (-1)^{\eta} e^{i\pi(\tilde{\nu}_1+\tilde{\nu}_4) - i(\chi_1\tilde{\nu}_1 - \chi_2\tilde{\nu}_2 - \chi_3\tilde{\nu}_3 + \chi_4\tilde{\nu}_4)} \times e^{-i(-\pi\tilde{\nu}_1 + \tilde{\theta}_t\tilde{\nu}_4 + \tilde{\theta}_2'\tilde{\nu}_2)} \langle p_3\tilde{\nu}_3, p_1\tilde{\nu}_1 | T | p_4\tilde{\nu}_4, p_2\tilde{\nu}_2\rangle\,. \tag{5.2}$$

For the O amplitudes, d matrices replace the exponentials, exactly as for the h amplitudes.

The kinematical significance of the angles in (5.2) may be seen as follows. Proceeding as in Ref. 8 [starting, for convenience, with case $P^2 = (p_3 - p_1)^2 = t > 0$], if we first apply the transformation

$$\Lambda(t) = e^{i\omega K_3} e^{i\underline{v}\cdot\underline{B}},$$

with

$$e^{\omega} = \sqrt{2}P_n/\sqrt{t}\,, \quad \underline{v} = \underline{P}/P_n \quad (P = p_{(3)} - p_{(1)}), \tag{5.3}$$

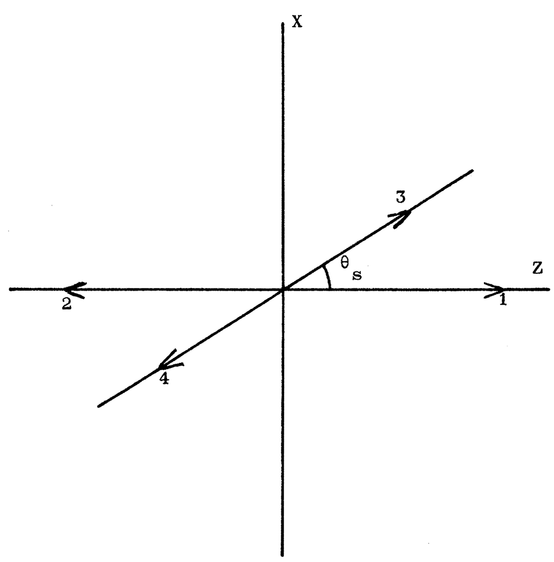

FIG. 1. s-channel center-of-mass frame. $\vec{p}_1 + \vec{p}_2 = 0 = \vec{p}_3 + \vec{p}_4$. (First configuration.)

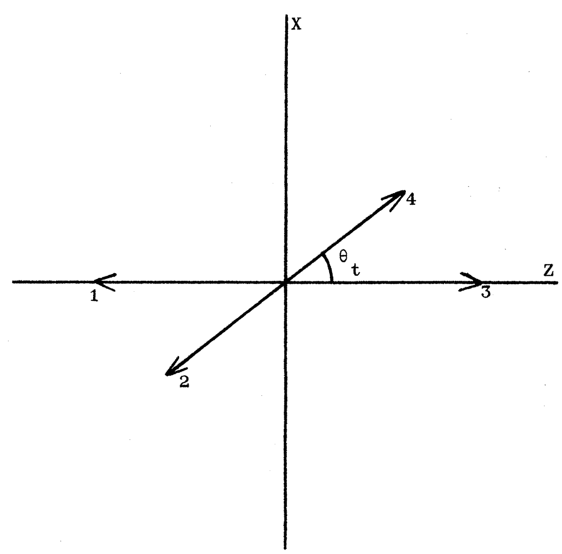

FIG. 2. t-channel center-of-mass frame. $\vec{p}_3 + \vec{p}_1 = \vec{p}_4 + \vec{p}_2$.

we obtain a zx plane amplitude with $p_i' = \Lambda_{(t)} \cdot p_i$ and $\nu_i' = \nu_i$. Then we have to apply a rotation around the y axis to attain the configuration of Fig. 2. This rotation alone gives the kinematical angles involved in (5.2).

Also, as compared to the result (2.27) of Ref. 8, we have for the O states,

$$[-p]_-^{-1} \times [-p]_+ = e^{\pm i(\pi/2)\tau_3} . \qquad (5.4)$$

Noting the foregoing two points, we may derive directly the crossing matrix for O and trans-O amplitudes by proceeding as in Ref. 8.

Let us finally note that we again obtain very simple crossing relations for the O and trans-O amplitudes for the "Breit" frame introduced in Ref. 8. These may again be written down at once from the corresponding results for the h and trans-h amplitudes.[8]

VI. KINEMATICAL SINGULARITIES

The structure of kinematical singularities of the O amplitudes has several interesting aspects. It can be derived in many ways, e.g., from the known results for the h and trans-h amplitudes, or by using the crossing relations, or by directly using methods analogous to those of CTMN[14] or Trueman.[15] Instead of going into full details we will take *the case of unequal masses* as an example and will summarize the results with a few comments, comparing them with the corresponding results for other well-known amplitudes. The results have been verified by deducing them from different points of view.

Let us define the zx-plane *s-channel c.m. amplitudes* (Fig. 1):

$$M_{\nu_3\nu_4,\,\nu_1\nu_2}(s,t) = \langle p_3\nu_3, p_4\nu_4 | T | p_1\nu_1, p_2\nu_2 \rangle , \qquad (6.1)$$

$$\mathfrak{M}_{\nu_3\nu_4,\,\nu_1\nu_2}(s,t) = (p_3\nu_3, p_4\nu_4 | T | p_1\nu_1, p_2\nu_2) , \qquad (6.2)$$

corresponding to the states $|p_i, \nu_i\rangle$ [(2.2)] and $|p_i, \nu_i)$ [(2.3)], respectively. The corresponding trans-O amplitudes will be denoted, respectively, by $M_{\bar\nu_3\bar\nu_4,\,\bar\nu_1\bar\nu_2}(s,t)$ and $\mathfrak{M}_{\bar\nu_3\bar\nu_4,\,\bar\nu_1\bar\nu_2}(s,t)$.

The parameters p_i^0, $|\vec{p}_i|$, $\cos\theta_s$, $\sin\theta_s$ are expressed in terms of the invariants s, t by very well-known expressions.[8,14] The particle "i" has mass m_i and spin S_i ($i = 1, 2, 3, 4$).

We will now examine the kinematical singularities in s for fixed t for the case of unequal masses. The following singularities are found for the amplitude (6.1). [The results for (6.2) are of course entirely similar. We will combine them at the end.]

(a) Physical region boundary. This boundary is given by $\Phi = 0$ where Φ is the well-known Kibble function.

As $\Phi \to 0$,

$$M_{\nu_3\nu_4,\,\nu_1\nu_2}(s,t) \to (\sin\theta)^{|\nu_1+\nu_2-\nu_3-\nu_4|} \hat{M}'_{\nu_3\nu_4,\,\nu_1\nu_2}(s,t)$$

or simply to

$$(\sqrt{\Phi})^{|\nu_1+\nu_2-\nu_3-\nu_4|} \hat{M}_{\nu_3\nu_4,\,\nu_1\nu_2}(s,t) , \qquad (6.3)$$

where $\hat{M}'$ and $\hat{M}$ are regular at $\Phi = 0$.

An exactly similar result holds for the c amplitudes where we have (in evident notation)

$$M_{\sigma_3\sigma_4,\,\sigma_1\sigma_2}(s,t) \to (\sqrt{\Phi})^{|\sigma_1+\sigma_2-\sigma_3-\sigma_4|} \times \hat{M}_{\sigma_3\sigma_4,\,\sigma_1\sigma_2}(s,t) . \qquad (6.4)$$

These may be compared with the result for the h amplitudes

$$M_{\lambda_3\lambda_4,\,\lambda_1\lambda_2}(s,t) \to (\sin\tfrac{1}{2}\theta)^{|\lambda_1-\lambda_2-\lambda_3+\lambda_4|} \times (\cos\tfrac{1}{2}\theta)^{|\lambda_1-\lambda_2+\lambda_3-\lambda_4|} \times \hat{M}_{\lambda_3\lambda_4,\,\lambda_1\lambda_2}(s,t) . \qquad (6.5)$$

Thus for the O and c amplitudes, the results are even simpler than for the h amplitudes.

For the trans-O amplitudes, the situation here is complicated just as for the trans-h amplitudes.

(b) $s = 0$. Like the h amplitudes, the O amplitudes are also regular at $s = 0$. The same result then evidently holds for the trans-O amplitudes.

The c amplitudes are not, in general, regular at $s = 0$.

(c) Thresholds and pseudothresholds. It is known that the h amplitudes have complicated behavior at the points

$$s = (m_3 \pm m_4)^2 , \quad s = (m_1 \pm m_2)^2$$

and for trans-h amplitudes, one obtains factorizable singularities.

The O amplitudes can be shown to be regular at $s = (m_3 \pm m_4)^2$ (for the configuration of Fig. 1). The c amplitudes are regular at $s = (m_3 + m_4)^4$ and behave as $(p_j^0 + m_j)^{-s}j$ ($j = 4, 3$ as $m_3 \gtrless m_4$) at $s = (m_3 - m_4)^2$. Thus we have the simplest behavior for the O and trans-O amplitudes.

Let us now consider the points (for Fig. 1) $s = (m_1 \pm m_2)^2$. One convenient method of exhibiting a factorized singularity is to pass to the configuration of Fig. 3. If M and M' denote the amplitudes corresponding to the configurations of Figs. 1 and 3, respectively, we have the trans-O amplitudes:

$$M_{\bar\nu_3\bar\nu_4,\,\bar\nu_1\bar\nu_2}(s,t) = M'_{\bar\nu_3\bar\nu_4,\,\bar\nu_1\bar\nu_2}(s,t) \times e^{i(\tilde\theta_1\bar\nu_1 + \tilde\theta_2\bar\nu_2 - \tilde\theta_3\bar\nu_3 - \tilde\theta_4\bar\nu_4)} , \qquad (6.6)$$

where

$$\sin\tilde\theta_j = \frac{m_j \sin\theta}{p_j^0 \pm |p_j|\cos\theta} , \quad \cos\tilde\theta_j = \frac{p_j^0\cos\theta \pm |p_j|}{p_j^0 \pm |p_j|\cos\theta} , \qquad (6.7)$$

where the "+" sign corresponds to $j = 1, 3$ *and "−"*

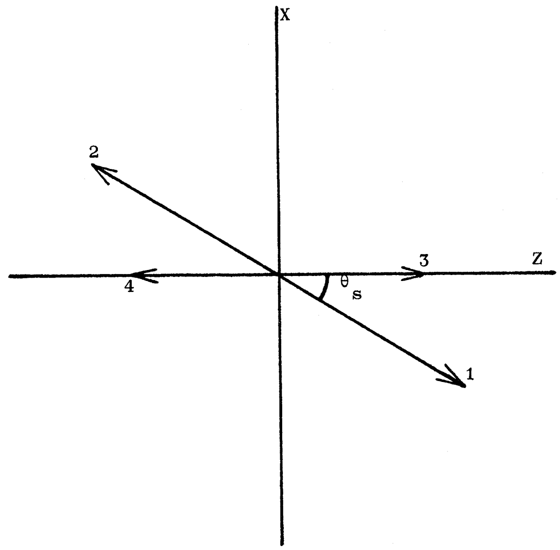

FIG. 3. s-channel center-of-mass frame. (Second configuration.)

sign to $j=2, 4$. Now in (6.6) M' is regular at $s=(m_1 \pm m_2)^2$. Hence M behaves as the coefficient in (6.6) near these points.

For the trans-c amplitude the corresponding relation is simpler, namely,

$$M_{\tilde{\sigma}_3\tilde{\sigma}_4,\, \tilde{\sigma}_1\tilde{\sigma}_2}(s,t) = M'_{\tilde{\sigma}_3\tilde{\sigma}_4,\, \tilde{\sigma}_1\tilde{\sigma}_2}(s,t) \times e^{i\theta(\tilde{\sigma}_1+\tilde{\sigma}_2-\tilde{\sigma}_3-\tilde{\sigma}_4)}. \qquad (6.6')$$

In fact this amplitude is essentially the same as the xy-plane c amplitudes.[8] From (6.6) it can be deduced that

$$s_{12}^{(S_1+S_2)} M_{\nu_3\nu_4,\, \nu_1\nu_2}(s,t), \qquad (6.8)$$

where

$$s_{ij} = \{[s-(m_i+m_j)^2][s-(m_i-m_j)^2]\}^{1/2},$$

is finite at $s=(m_1 \pm m_2)^2$.

A similar result holds for $\mathfrak{M}_{\nu_3\nu_4,\, \nu_1\nu_2}(s,t)$. Again, as regards branch-point behavior, it will be convenient to write the amplitudes using the parameters $\vec{p}_i$. We note that on turning once around the point $s_{12}=0$,

$$M_{\nu_3\nu_4,\, \nu_1\nu_2}(\vec{p}_i) \to M_{\nu_3\nu_4,\, \nu_1\nu_2}(-\vec{p}_i) \quad (i=1,2,3,4),$$

and similarly

$$\mathfrak{M}_{\nu_3\nu_4,\, \nu_1\nu_2}(\vec{p}_i) \to \mathfrak{M}_{\nu_3\nu_4,\, \nu_1\nu_2}(-\vec{p}_i). \qquad (6.9)$$

But for zx-plane amplitudes, using (4.11), we have

$$M_{\nu_3\nu_4,\, \nu_1\nu_2}(\vec{p}_i) = (-1)^{\Sigma_i(S_i+\nu_i)} \mathfrak{M}_{-\nu_3-\nu_4,\, -\nu_1-\nu_2}(-\vec{p}_i). \qquad (6.10)$$

For parity-conserving reactions, this is also [using (2.5)] equal to

$$\eta\, \mathfrak{M}_{\nu_3\nu_4,\, \nu_1\nu_2}(-\vec{p}_i), \qquad (6.11)$$

where $\eta = \prod_{i=1}^4 \eta_i$, η_i being the intrinsic parity of the particle "i".

Let us define

$$M^{(\pm)}_{\nu_3\nu_4,\, \nu_1\nu_2}(\vec{p}_i) = [M_{\nu_3\nu_4,\, \nu_1\nu_2}(\vec{p}_i) \pm (-1)^{\Sigma_i(S_i+\nu_i)} \times \mathfrak{M}_{-\nu_3-\nu_4,\, -\nu_1-\nu_2}(\vec{p}_i)]. \qquad (6.12)$$

For parity-conserving reactions the right-hand side of (6.12) is equal to

$$[M_{\nu_3\nu_4,\, \nu_1\nu_2}(\vec{p}_i) \pm \eta\, \mathfrak{M}_{\nu_3\nu_4,\, \nu_1\nu_2}(\vec{p}_i)]; \qquad (6.13)$$

then on turning once around the point $s_{12}=0$,

$$M^{(\pm)}_{\nu_3\nu_4,\, \nu_1\nu_2}(\vec{p}_i) \to \pm M^{(\pm)}_{\nu_3\nu_4,\, \nu_1\nu_2}(\vec{p}_i). \qquad (6.14)$$

(d) $p_{n(i)}=0$, $p_{\bar{n}(i)}=0$. As compared to the h and c amplitudes, the O amplitudes have a new type of kinematical singularity, due to the special role played by the parameters p_n and $p_{\bar{n}}$, respectively, in the definitions (2.2) and (2.3). But fortunately these are simple factorizable ones. Thus for the configuration of Fig. 1 singularities arise at the points $p_{n(i)}=0$, $p_{\bar{n}(i)}=0$ for $i=3, 4$. The amplitudes

$$p_{n(3)}^{S_3} p_{n(4)}^{S_4} M_{\nu_3\nu_4,\, \nu_1\nu_2}(s,t)$$

and (6.15)

$$p_{\bar{n}(3)}^{S_3} p_{\bar{n}(4)}^{S_4} \mathfrak{M}_{\nu_3\nu_4,\, \nu_1\nu_2}(s,t)$$

are regular at these points.

Let us note that for

$$p_{n(j)}=0 \text{ or } p_{\bar{n}(j)}=0 \quad (j=3,4), \qquad (6.16)$$

$$\Phi = -m_j^2 s_{12}^2. \qquad (6.16')$$

Recently Leader and Pennington[16] have found it quite useful to treat the amplitudes as a function of the invariants s and n^2 (instead of s and t), where

$$n^2 = -\Phi/s^2. \qquad (6.17)$$

From (6.16′) we note that if in modifying (6.17) we keep fixed

$$-\Phi/s_{12}^2, \qquad (6.18)$$

then (for the massive particles we are considering) the points (6.16) cannot be attained. For large s, $s_{12}^2 \to s^2$. This gives a kinematical motivation for the choice of the denominator in (6.18).

(e) Regularized O amplitudes. We start with the two amplitudes $M_{\nu_i}(\vec{p}_i)$ and $\mathfrak{M}_{\nu_i}(\vec{p}_i)$ [(6.1), (6.2)]. Putting together the foregoing results [and noting the behavior of the factors $p_{n(3,4)}, p_{\bar{n}(3,4)}$ in (6.15) for $s \to 0$ and $s_{12} \to 0$], we obtain the following two regularized combinations:

$$\Phi^{-|\nu_1+\nu_2-\nu_3-\nu_4|/2}\, s_{12}^{(S_1+S_2+S_3+S_4+\epsilon_\pm)} s^{(S_3+S_4)/2}$$
$$\times [\, p_{n(3)}{}^{S_3}\, p_{n(4)}{}^{S_4}\, M_{\nu_3\nu_4,\,\nu_1\nu_2}(\vec{p}_i)$$
$$\pm\, p_{\bar{n}(3)}{}^{S_3}\, p_{\bar{n}(4)}{}^{S_4}(-1)^{\Sigma_i(S_i+\nu_i)}\mathfrak{M}_{-\nu_3-\nu_4,\,-\nu_1-\nu_2}(\vec{p}_i)], \tag{6.19}$$

where,

$$\text{for } \sum_i S_i = 2n, \quad \epsilon_+ = 0, \quad \epsilon_- = 1,$$

and (6.20)

$$\text{for } \sum_i S_i = 2n+1, \quad \epsilon_+ = 1, \quad \epsilon_- = 0.$$

For parity-conserving reactions we may use (6.11) to substitute $\eta\mathfrak{M}_{\nu_i}(\vec{p}_i)$ in the second term of (6.19).

Thus finally we see that for the general mass case we have discussed, the O amplitudes can be regularized quite easily. In fact the results are simpler than for the h amplitudes and *do not fail for the case* $BF \to B'F'$ (as in the case for h amplitudes). We did not even assume parity conservation in (6.19).

The fact that we are using [in (6.19)] a combination of the amplitudes M_{ν_i} and $\mathfrak{M}_{\nu_i}$ causes no special inconvenience. Since compact and analogous transformation formulas [(2.17), (2.22)] are available for both types of states, such a combination can be handled as easily, say, as the combination of two h amplitudes. However, if we want to, for some particular reason, express one type of O amplitude as a linear combination of the other type, it can be done using (4.11′).

(f) Elastic πN *scattering.* When the masses are not all unequal, several particular cases of importance arise. Since the O amplitudes for all such cases can be treated easily by starting from known results for other amplitudes, instead of listing all the possibilities systematically, we will only briefly consider the important case of elastic πN scattering.

In Fig. 1 the particles 1 and 3 are now pions (mass μ) and particles 2 and 4 are nucleons (mass m). For the helicity amplitudes one has[14]

$$H_{++} = H_{--} = (\cos\tfrac{1}{2}\theta_s) F_1,$$
$$H_{+-} = -H_{-+} = \frac{1}{\sqrt{s}}(\sin\tfrac{1}{2}\theta_s) F_2, \tag{6.21}$$

where in terms of the well-known invariant amplitudes A and B,

$$F_1 = A + B\,\frac{s - m^2 - \mu^2}{2m},$$
$$F_2 = \frac{1}{2}\left[\frac{A}{m}(s + m^2 - \mu^2) + B(s + \mu^2 - m^2)\right]. \tag{6.22}$$

Denoting the amplitudes $M_{\nu_30,\,\nu_10}$ and $\mathfrak{M}_{\nu_30,\,\nu_10}$ by $O_{\nu_3\nu_1}$ and $O'_{\nu_3\nu_1}$, respectively, we have [from (4.2) and (4.12), with $\varphi = \pi + \theta_s$ and $\varphi' = \pi + \theta'_s$]

$$O_{\nu_3\nu_1} = H_{\lambda_3\lambda_1}\, d^S_{\nu_3\lambda_3}(\tilde{\varphi})(-1)^{S+\nu_1}\delta_{\nu_1,\,-\lambda_1},$$
$$O'_{\nu_3\nu_1} = H_{\lambda_3\lambda_1}\, d^S_{\nu_3\lambda_3}(\tilde{\varphi}')(-1)^{S+\nu_1}\delta_{\nu_1,\,-\lambda_1}. \tag{6.23}$$

Thus the singularities due to the half-angles [$\frac{1}{2}\theta_s$ in (6.21)] are eliminated while those due to $p_{n(3)}$ and $p_{\bar{n}(3)}$ are introduced.

VII. REDUCTION OF DIRECT PRODUCTS AND PARTIAL-WAVE ANALYSIS

In this section we will briefly indicate the relation of the present formalism to certain known techniques for reduction of direct products of states and partial-wave expansions of scattering amplitudes.

For massive particles two well-known methods of reduction employ l-s and helicity coupling, respectively. Moussa and Stora[6] have given results for these two couplings *valid for arbitrary tetrads*. Let us take the direct product $|p_1, \nu_1\rangle \times |p_2, \nu_2\rangle$. For simplicity we will consider c.m. frame, namely, $\vec{p}_1 + \vec{p}_2 = \vec{P} = 0$.

Let us start with the l-s coupling. Using the general result of Ref. 6 (Sec. II, a, 5) and adopting their notation, one finds that the calculation of the coupling coefficients

$$\langle [p_1], \nu_1; [p_2], \nu_2 \| [P], \nu, j, l, s, P^2 \rangle \tag{7.1}$$

for the states $|p, \nu\rangle$ [or similarly for $|p, \nu)$] amounts to first transforming them to the c-states [using (4.6)], and then using l-s coupling for the c-states. This corresponds to the fact that the l-s coupling arised naturally for the c-states and has the simplest expression for them.

For a similar reason the general result for helicity coupling[6] applied to $|p_1\nu_1, p_2\nu_2\rangle$ amounts to transforming $|p_i, \nu_i\rangle$ to $|p_i, \lambda_i\rangle$ [using (4.2)] and then using the standard result for the helicity states. The above two couplings may then of course be utilized as usual to expand the O amplitudes in the conventional l-s or helicity partial waves.

A particularly interesting application of O states is in crossed-channel partial-wave analysis for lightlike momentum transfer.[3] In this formalism, instead of the usual direct- (s-)channel partial-wave analysis of

$$\langle p_3\nu_3, p_4\nu_4 | T | p_1\nu_1, p_2\nu_2 \rangle,$$

one reduces the direct products

$$|p_3\nu_3,\ -p_1\nu_1\rangle$$

and

$$|-p_4\nu_4,\ p_2\nu_2\rangle$$

for the case

$$t = (p_3 - p_1)^2 = (p_2 - p_4)^2 = 0. \tag{7.2}$$

The result can be found in Ref. 3 and the partial waves correspond effectively to Fourier-Bessel transforms.

So far we have been considering partial-wave expansions starting from reduction of the direct products of states and the use of the language of little groups. More general types of expansions of amplitudes are nowadays often introduced. For example, a generalization of the above-mentioned crossed-channel expansion for $t \neq 0$ (but for spin-0 particles) is given in Ref. 4.

In fact if we do not insist on an interpretation through reduction of direct product of states and little groups, various types of partial-wave expansions can formally be introduced (for arbitrary values of the momenta) by using the orthogonality properties of different functions (such as Bessel functions which are useful for studying certain models). In these types of expansions, certain necessary integrability properties are to be postulated.

One way of introducing group-theoretical expansions is to parametrize the amplitude by certain transformations of the Lorentz group (which serve to bring the momenta to suitably chosen standard values) and then expand the amplitude as a function of those transformation parameters.[17] Without going into any detail, let us just note some agreeable properties of the O amplitudes concerning such parametrizations. These are consequences of the simple action of the generators $\underline{B}$, K_3, and J_3 as given in (2.17) and the fact that a 4-momentum p can be brought to any arbitrary value p' (with $p'^2 = p^2$) by a transformation of the type

$$U(\Lambda) = e^{-i\underline{v}\cdot\underline{B}} e^{-i\omega K_3}, \tag{7.3}$$

with suitable values of $\underline{v}$ and ω.

Thus, for example, to start with, one may write

$$\langle p_3\nu_3, p_4\nu_4 | T | p_1\nu_1, p_2\nu_2\rangle = \langle p_3'\nu_3, p_4'\nu_4 | U(\Lambda_{34}) T U(\Lambda_{12}^{-1}) | p_1'\nu_1, p_2'\nu_2\rangle, \tag{7.4}$$

where Λ_{12}, Λ_{34} are independent transformations of the type (7.3) such that

$$\Lambda_{12}(p_1, p_2) = (p_1', p_2'), \quad \Lambda_{34}(p_3, p_4) = (p_3', p_4'). \tag{7.5}$$

Hence,

$$\langle p_3\nu_3, p_4\nu_4 | T | p_1\nu_1, p_2\nu_2\rangle = \langle p_3'\nu_3, p_4'\nu_4 | T U(\Lambda) | p_1'\nu_1, p_2'\nu_2\rangle, \tag{7.6}$$

where $\Lambda = \Lambda_{34} \cdot \Lambda_{12}^{-1}$.

In particular we may choose $\Lambda_{12} = \Lambda_{34}$ such that

$$p_1' + p_2' = p_3' + p_4' = 0$$

to relate the c.m. amplitude in a *one-to-one fashion* to amplitudes for arbitrary values of $P = p_1 + p_2 = p_3 + p_4$.

More generally one can choose Λ_{12}, Λ_{34} so as to reduce, for example, $p_1 - p_2$ and $p_3 - p_4$ to suitably chosen standard values. For xy-plane amplitudes we can further use J_3 to rotate the whole configuration, introducing only a phase factor.

We can exploit this simplicity also in the reduction of direct products (many particle states) in calculating coefficients of the type

$$\langle p, \nu, \zeta | p_1, \nu_1, \ldots, p_n, \nu_n\rangle \tag{7.7}$$

(ζ denotes possible extra parameters which we need not specify for the moment, but which are supposed to have been expressed in an invariant way).

Let $U(\Lambda)$ be given by (7.3) and $\Lambda \cdot p = p'$ and $\Lambda p_i = p_i'$; then for arbitrary values of p_i,

$$\langle p, \nu, \zeta | p_1, \nu_1, \ldots, p_n\nu_n\rangle = \langle p', \nu, \zeta | p_1', \nu_1, \ldots, p_n'\nu_n\rangle. \tag{7.8}$$

Thus, for example, we can directly relate high-energy Clebsch-Gordan coefficients to the c.m. ones (where $\vec{p} = 0$) without introducing superpositions. This simplicity is not obtained for h or c states.

VIII. REMARKS

The Galilean subgroup of the Poincaré group plays a basic role in null-plane formalism, which is of interest in high-energy processes, and leads naturally to a definition of spin projection which we have studied at length. We have tried to point out many interesting properties of the O states and O amplitudes. We have concentrated on the study of massive particles since a smooth limit to zero mass leads directly to well-known description in terms of the little group E_2.

For helicity projection, one diagonalizes effectively the projection of W on a timelike vector (1, 0, 0, 0). For O projection, one diagonalizes the projection of W on a lightlike vector $(1, 0, 0, \pm 1)/\sqrt{2}$. Similarly one may also study the projection of W on a spacelike vector, say, (0, 0, 0, 1), for physical particles (and not only for $p^\mu p_\mu < 0$ where it is known). Now that not only do we have many interesting models for amplitudes, but also direct experimental determination of certain important cases are becoming possible,[18] a systematic comparison of the properties of different amplitudes (for both direct and crossed channels) may become of interest. Different amplitudes may provide a relatively simple description under different conditions. We hope to study these aspects more thoroughly elsewhere.

APPENDIX: PHASE FACTORS IN CROSSING RELATIONS

In Ref. 8 we did not give the explicit form of the overall phase factor η. Concerning this point it is interesting to compare the results of Ref. 8 with the corresponding ones of other authors, using the crossing relations for center-of-mass helicity amplitudes which have been studied by many.

The crossing relations for the helicity amplitudes corresponding to (5.1) is given in Ref. 8 as

$$\begin{aligned}(\langle p_3\lambda_3, p_4\lambda_4|T|p_1\lambda_1, p_2\lambda_2\rangle)_{\text{cntd}} &= e^{i\pi(\eta)}e^{i\pi(\lambda_1'-\lambda_4')}\langle p_3\lambda_3', p_1\lambda_1'|T|p_4\lambda_4', p_2\lambda_2'\rangle \\ &\times d^{S_1}_{\lambda_1'\lambda_1}(\chi_1)d^{S_2}_{\lambda_2'\lambda_2}(-\chi_2)d^{S_3}_{\lambda_3'\lambda_3}(\chi_3)d^{S_4}_{\lambda_4'\lambda_4}(-\chi_4)\,.\end{aligned} \tag{A1}$$

This corresponds exactly to (12) of Hara[19] upon making the correspondence

$$(A, B, C, D) \to (2, 4, 1, 3),$$

if we put (with $\epsilon = -1$ for $F+B \to B+F$ and $B+F \to F+B$ processes)

$$e^{i\pi(\eta)} = \epsilon e^{i\pi 2S_3}\,. \tag{A2}$$

The factor ϵ can be obtained explicitly in our formalism by choosing the negative sign in (2.27) of Ref. 8 in agreement with the continuation procedure of Hara.

Comparing with the result of CTMN,[14] noting that their χ_3 and χ_4 have opposite signs, we find apart from the condition

$$e^{i\pi\eta} = e^{i\pi[\sigma(p)+2S_2+2S_4]}, \tag{A3}$$

an extra helicity-dependent phase factor

$$e^{i\pi(\lambda_1'-\lambda_2'+\lambda_3'-\lambda_4')}\,. \tag{A4}$$

The origin of this factor is easily traced. It represents a rotation π about the n_3 axis and is a consequence of the fact that their n_2 axis (W_S in their notation) acquires a negative sign after continuation (this axis being common to all the particles). Our formalism implicitly defines the n_2 axis in the t channel with such an additional rotation, which is not only quite legitimate (amounting to a choice of phase conventions for the states), but has the additional advantage of making the transversity crossing relation really diagonal. In the CTMN[14] formalism the transversity indices have the crossing property

$$\tau_i \to -\tau_i \quad (i = 1, 2, 3, 4).$$

As compared to the original derivation of Trueman and Wick,[20] there are again several differences concerning phase factors (e.g., convention regarding "particle 2" and behavior of helicity states under crossing) [see Eqs. (3.13), (3.14) of Ref. 8.]

In fact, in our formalism (referring now to equations in Sec. II of Ref. 8) if we put Eq. (2.40) directly into (2.25) [using, like CTMN, the result of Bros, Epstein, and Glaser, so that their $\sigma(p) = \eta'$] we obtain finally,

$$\eta = \sigma(p) + 2(S_1 + S_4)\,. \tag{A5}$$

This is to be compared with (A2) and (A3). In our case (A5) the crossed particles play a symmetrical role (1 and 4) as do the uncrossed ones. It is to be noted that we have directly followed the relevant 2×2 matrices [(2.25)–(2.27) of Ref. 8] in the process of continuation, whereas CTMN follow the helicity frames, which necessitates a separate determination of η.

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Scaling and Chiral Ward Identities and the Axial-Vector Current Anomaly*

Sun-Sheng Shei
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850

A. Zee
The Rockefeller University, New York, New York 10021
(Received 26 March 1973)

We combine chiral and "scaling" Ward identities in an analysis of the axial-vector current anomaly. This leads to a proof of the Adler-Bardeen theorem. The "scaling" Ward identities we used may be properly called "response equations"; they are slight generalizations of the Callan-Symanzik equation. Our discussion is given for a class of theories which includes the σ model and massive vector-boson theory.

I. INTRODUCTION

Much has been learned from a systematic exploitation of chiral Ward identities. On the other hand, the scaling Ward identities of Callan[1] and Symanzik[2] have provided us with relatively little information. It is then natural to ask if these two types of Ward identities may be combined in any sensible way. In general the answer appears to be no, which is not to say that under specific circumstances the combination would not prove to be a potent one. Indeed, one of us has shown[3] that an example of such a set of circumstances may be found in the theory of axial-vector current anomalies.[4]

In field theory the scaling Ward identities merely translate what is essentially a trivial statement in the unrenormalized cutoff language into the renormalized language. But precisely because of this they are of great help in disentangling diagrams in renormalized perturbation theory. Unfortunately, their usefulness is severely limited by the fact that the so-called scalar insertion term that appears in them in general cannot be evaluated. These terms are thus often dropped in applications by going into the asymptotic Euclidean region and by appealing to Weinberg's theorem.[5] This tactic reduces the Callan-Symanzik equation to the renormalization group equation of Gell-Mann and Low,[6] with a costly loss of information content. One may ask if this loss may be avoided under certain circumstances. A limited answer was given in Ref. 3 and will be elaborated in this paper.

We outline our argument here, deferring the details to Sec. II. Consider the current correlation function

$$\begin{aligned} R_{D\mu\nu}(k,q) &\equiv i\int d^4x\, d^4y\, e^{i(kx+qy)} \\ &\quad\times\langle 0|T\partial A(0)V_\mu(x)V_\nu(y)|0\rangle \\ &\equiv \epsilon_{\mu\nu\lambda\sigma}k^\lambda q^\sigma f(k^2,q^2,(k+q)^2). \qquad (1.1)\end{aligned}$$