Dispersion Inequalities for the Forward Compton-Scattering Differential Cross Section

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Dispersion inequalities involving integrals of the forward Compton-scattering differential cross section are written down starting from the low-energy theorems and the analyticity properties of the amplitudes. A previous result reported by Okubo is improved, the bounds obtained here being optimal with respect to the information taken as known.

I. INTRODUCTION

Using the analyticity properties of the form factors or scattering amplitudes in conjunction with a certain input information (which, for instance, can fix the values of these analytic functions at some points of the complex plane of the relevant kinematical variable), rigorous bounds have been recently obtained in the literature. Some of these bounds were found by means of the modulus representation for the holomorphic functions rather than by the usual technique of constructing the function in the whole complex plane starting from its imaginary part on the cut. The modulus representation appears particularly appealing in problems in which restrictions on the modulus of the needed function are given on the cut, and its usefulness relies on the possibility of easily applying the principle of maximum for holomorphic functions, which among many other things gives a way to majorize unreliable parts of the functions under consideration, related to their unknown zeros, once these zeros have been factorized in the form of Blaschke products. References concerning the derivation of such dispersion inequalities can be found in the papers of Okubo[1] and Radescu.[2]

In this article we shall make some simple applications of the methods mentioned above to the case of forward Compton scattering, exploiting the information coming from the low-energy theorems[3] valid for this process. Dispersion inequalities (representing optimal bounds under the stated conditions of the problem) are obtained for integrals of the forward Compton-scattering differential cross section. A previous result reported in Ref. 1 is presently improved.

II. DERIVATION OF THE RESULTS AND DISCUSSION

The forward Compton-scattering amplitude in the laboratory system, $f(\nu)$ (ν denotes the energy of the photon in the laboratory), written out between the Pauli spinors of the initial and final nucleons, has the form[4]

$$f(\nu)=\chi_f^*[f_1(\nu)\vec{\epsilon}_2^{\,*}\cdot\vec{\epsilon}_1+i\vec{\sigma}\cdot(\vec{\epsilon}_2^{\,*}\times\vec{\epsilon}_1)f_2(\nu)]\chi_i\,, \tag{1}$$

where $\vec{\epsilon}_1$ and $\vec{\epsilon}_2$ are the polarization vectors of the initial and final photons. The forward differential cross section is given by

$$\left.\frac{d\sigma}{d\Omega}\right|_{\theta=0^\circ}=\left.\frac{\nu^2}{\pi}\frac{d\sigma}{dt}\right|_{t=0} = |f_1(\nu)|^2+|f_2(\nu)|^2\,, \tag{2}$$

where t is minus the square of the four-momentum transfer. From the low-energy theorems[3] we have as $\nu\to 0$

$$f_1(\nu)\to f_1(0)=-\alpha/m\,, \tag{3a}$$

$$f_2(\nu)/\nu\to f_2'(0)=-\alpha(\mu_{\rm anom})^2/2m^2\,. \tag{3b}$$

In Eqs. (3a) and (3b) $\mu_{\rm anom}$ is the anomalous magnetic moment of the proton (mass m) and $\alpha=e^2/4\pi=\frac{1}{137}$ is the fine-structure constant. Noting that $f_1(\nu)$ is even and $f_2(\nu)$ is odd under crossing (i.e., $\nu\to-\nu$), we see that the dimensionless functions defined by

$$F_1(\nu^2)\equiv mf_1(\nu)\,, \tag{4a}$$

$$F_2(\nu^2)\equiv m^2f_2(\nu)/\nu \tag{4b}$$

are real analytic functions of ν^2 in the ν^2 complex plane with a cut along the real axis starting at

$$\nu_0^{\,2}=m_\pi^{\,2}(1+m_\pi/2m)^2\,, \tag{5}$$

the threshold for photoproducing single pions (mass m_π). With the definitions (4a), (4b), Eq. (2) becomes

$$|F_1(\nu'^2)|^2+\frac{\nu'^2}{m^2}|F_2(\nu'^2)|^2=\left.\frac{m^2\nu'^2}{\pi}\frac{d\sigma}{dt}\right|_{t=0}, \tag{2'}$$

where we wrote ν'^2 instead of ν^2 to keep in mind that Eq. (2′) is considered on the boundary region of the ν^2 complex plane, that is, on the cut from $\nu_0^{\,2}$ up to ∞.

We now construct[1,2] the function

$$S(\nu^2) = \exp\left[\frac{1}{2\pi}(\nu_0^2-\nu^2)^{1/2}\int_{\nu_0^2}^{\infty}(\nu'^2-\nu^2)^{-1}(\nu'^2-\nu_0^2)^{-1/2}\ln\left(\frac{m^2\nu'^2}{\pi}\left.\frac{d\sigma}{dt}\right|_{t=0}\right)d\nu'^2\right], \tag{6}$$

which, as can immediately be seen, is holomorphic, has no zeros in the cut ν^2 plane, and is of modulus

$$|S(\nu'^2)| = \left(\frac{m^2\nu'^2}{\pi}\left.\frac{d\sigma}{dt}\right|_{t=0}\right)^{1/2}$$

for $\nu'^2 \geq \nu_0^2$ along the cut. Similarly we introduce the function

$$T(\nu^2) = \exp\left(\frac{1}{\pi}(\nu_0^2-\nu^2)^{1/2}\int_{\nu_0^2}^{\infty}(\nu'^2-\nu^2)^{-1}(\nu'^2-\nu_0^2)^{-1/2}\ln(\nu'^2/m^2)d\nu'^2\right), \tag{7}$$

which is holomorphic, has no zeros in the cut ν^2 plane, and is of modulus

$$|T(\nu'^2)| = \frac{\nu'^2}{m^2}$$

for $\nu'^2 \geq \nu_0^2$ along the cut.

Defining again the new functions

$$\phi_1(\nu^2) \equiv \frac{F_1(\nu^2)}{S(\nu^2)} = \frac{mf_1(\nu)}{S(\nu^2)}, \tag{8a}$$

$$\phi_2(\nu^2) \equiv \frac{[T(\nu^2)]^{1/2}F_2(\nu^2)}{S(\nu^2)} = \frac{[T(\nu^2)]^{1/2}m^2f_2(\nu)}{\nu S(\nu^2)}, \tag{8b}$$

we see that $\phi_1(\nu^2)$, $\phi_2(\nu^2)$ are holomorphic in the complex ν^2 plane cut from ν_0^2 to ∞ along the real axis and satisfy the condition

$$|\phi_1(\nu'^2)|^2 + |\phi_2(\nu'^2)|^2 = 1 \tag{9}$$

on the cut ($\nu'^2 \geq \nu_0^2$). Moreover, from the low-energy theorems (3a) and (3b) we have

$$\phi_1(0) \equiv \alpha_1 = -\frac{\alpha}{S(0)}, \tag{10a}$$

$$\phi_2(0) \equiv \alpha_2 = -\frac{\alpha[T(0)]^{1/2}(\mu_{\text{anom}})^2}{2S(0)}. \tag{10b}$$

At this point it is useful to map the complex ν^2 cut plane onto the unit circle of a new complex plane z, through the conformal transformation

$$\nu^2 = 4\nu_0^2 z/(1+z)^2. \tag{11}$$

The upper and lower borders of the cut in the ν^2 plane map onto the line $|z|=1$, the points $z=1$, $z=0$, $z=-1$ corresponding, respectively, to the points $\nu^2=\nu_0^2$, $\nu^2=0$, $\nu^2=\infty$. We shall use the notation $\phi_{1,2}(z) \equiv \phi_{1,2}(\nu^2(z))$.

Now we can state the problem in the following terms: Considering two holomorphic functions inside the unit circle ($|z|\leq 1$), $\phi_1(z)$, $\phi_2(z)$, which satisfy on the boundary of the holomorphy region ($|z'|=1$) the condition

$$|\phi_1(z')|^2 + |\phi_2(z')|^2 = 1, \quad z' = e^{i\theta} \tag{9'}$$

we must find the best bound on $|\phi_2(z=0)| \equiv |\alpha_2|$ when $|\phi_1(z=0)|$ is given and equal to $|\alpha_1|$.

The answer is

$$|\alpha_2| \leq (1-|\alpha_1|^2)^{1/2}. \tag{12}$$

To give a simple proof of the inequality (12), we write down the modulus representation for $\phi_i(z)$ $(i=1,2)$:

$$\phi_i(z) = B_i(z)\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln|\phi_i(z'=e^{i\theta})|\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right) \quad (i=1,2), \tag{13}$$

where $B_i(z)$ represent the Blaschke products containing the factorized zeros of $\phi_i(z)$ which satisfy

$$|B_i(z'=e^{i\theta})| = 1, \tag{14}$$

and hence

$$|B_i(z)| \leq 1 \quad \text{for} \quad |z| \leq 1. \tag{15}$$

[The so-called singular measure function has been disregarded in Eq. (13) on physical grounds.[1]] From relations (13) and (15) one has for $z=0$ the Jensen inequalities

$$|\phi_i(0)| \leq \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|\phi_i(\theta)|\,d\theta\right) \qquad (i=1,2), \tag{16}$$

where we have put $|\phi_i(z'=e^{i\theta})| \equiv |\phi_i(\theta)|$ to simplify the notation. The above inequalities become equalities in the absence of zeros.

The inequalities (16) obviously imply

$$|\phi_1(0)|^p |\phi_2(0)| \leq \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln[|\phi_1(\theta)|^p|\phi_2(\theta)|]\,d\theta\right), \tag{17}$$

for any $p \geq 0$. Noting that due to the condition (9′) we have the elementary inequality

$$|\phi_1(\theta)|^p|\phi_2(\theta)| \leq \left(\frac{p}{p+1}\right)^{p/2}(p+1)^{-1/2}, \tag{18}$$

one gets from (17)

$$|\phi_1(0)|^p|\phi_2(0)| \leq \left(\frac{p}{p+1}\right)^{p/2}(p+1)^{-1/2}. \tag{19}$$

Now choosing p such that

$$|\alpha_1| = \left(\frac{p}{p+1}\right)^{1/2}, \quad p = \frac{|\alpha_1|^2}{1-|\alpha_1|^2}, \tag{20}$$

one finds the desired result expressed by the relation (12). The value taken above for p realizes the extremum for $|\phi_2(0)|$ in the inequality (19) at given $|\phi_1(0)|$. The bound given by the inequality (12) is optimal in the context of the problem, because it is saturated by the particular holomorphic functions $\phi_1(z) = |\alpha_1|$, $\phi_2(z) = (1-|\alpha_1|^2)^{1/2}$. [The condition (9′) is satisfied and the inequality (12) becomes for these particular $\phi_{1,2}$ an equality.]

Coming back to the ν^2 plane, we find from relations (12), (10a), (10b), and (6) the result

$$(\mu_{\text{anom}})^2 \leq \frac{2(M-1)^{1/2}}{[T(0)]^{1/2}}, \tag{21}$$

where

$$M \equiv \frac{S^2(0)}{\alpha^2} = \exp\left[\frac{\nu_0}{\pi}\int_{\nu_0^2}^{\infty} \nu^{-2}(\nu^2-\nu_0^2)^{-1/2} \ln\left(\frac{m^2\nu^2}{\pi\alpha^2}\frac{d\sigma}{dt}\bigg|_{t=0}\right)d\nu^2\right], \tag{22}$$

and from Eq. (7)

$$[T(0)]^{1/2} = 2\frac{\nu_0}{m} = 2\frac{m_\pi}{m}\left(1+\frac{m_\pi}{2m}\right). \tag{23}$$

The relation (21) can be viewed as a bound on the anomalous magnetic moment in terms of the input information represented by the knowledge of the forward Compton-scattering differential cross section in the specified energy region. We remark here that the Jensen inequality (16) applied directly to $f_1(\nu)$ gives $M \geq 1$ anyway, because

$$|f_1(\nu'^2)|^2 \leq \frac{\nu'^2}{\pi}\frac{d\sigma}{dt}\bigg|_{t=0}.$$

The bound (21) improves the result reported by Okubo,[1] which in our notation takes on the form

$$(\mu_{\text{anom}})^2 \leq \frac{M}{[T(0)]^{1/2}},$$

by a factor of $[2(M-1)^{1/2}/M] \leq 1$.

The inequality (21) has been obtained on the basis of the analyticity properties of the forward Compton-scattering amplitudes, starting only from the kinematical expression of the forward differential cross section Eq. (2) and the low-energy theorems (3a), (3b). If one enlarges the input information, other dispersion inequalities involving $d\sigma/dt|_{t=0}$ can be written down. For instance, let us take as known the spin-averaged total cross section for photon + nucleon → hadrons, $\sigma_T(\nu)$. Thus we have

$$\text{Im}f_1(\nu') = \frac{\nu'}{4\pi}\,\sigma_T(\nu'), \quad \nu' \geq \nu_0 \tag{24}$$

and the real part of $f_1(\nu)$ is also known on the cut by means of a dispersion relation with a subtraction at $\nu = 0$:

$$\text{Re}f_1(\nu) = -\frac{\alpha}{m} + \frac{\nu^2}{\pi}\,\text{P.V.}\int_{\nu_0^2}^{\infty} \frac{d\nu'^2}{\nu'^2 - \nu^2}\,\frac{\text{Im}f_1(\nu')}{\nu'^2}\ . \tag{25}$$

Then from Eqs. (2), (24), and (25), one can express $|f_2(\nu)|$ on the cut in terms of $d\sigma/dt|_{t=0}$, $\sigma_T(\nu)$, and a principal-value integral over $\sigma_T(\nu)$. A direct application of the Jensen inequality (16) to the function $F_2(\nu^2) \equiv m^2 f_2(\nu)/\nu$ gives, then, the following bound:

$$(\mu_{\text{anom}})^2 \leq 2\exp\left(\frac{\nu_0}{2\pi}\int_{\nu_0^2}^{\infty} \nu^{-2}(\nu^2 - \nu_0^2)^{-1/2} \ln\left\{\frac{m^4}{\pi\alpha^2}\left[\frac{d\sigma}{dt}\bigg|_{t=0} - \frac{\sigma_T^2}{16\pi} - \frac{\pi}{\nu^2}\left(-\frac{\alpha}{m} + \frac{\nu^2}{2\pi^2}\,\text{P.V.}\int_{\nu_0}^{\infty}\frac{d\nu'}{\nu'^2 - \nu^2}\,\sigma_T(\nu')\right)^2\right]\right\} d\nu^2\right). \tag{26}$$

Finally, we shall observe that while the validity of the once-subtracted dispersion relation (25) has been needed to get the dispersion inequality (26), no similar hypotheses regarding possible subtractions for f_1, f_2 appeared in the derivation of the bound (21). The modulus representation, unlike the usual dispersion relations, due to the appearance under the integral of the logarithm of the modulus of the function rather than of its imaginary part directly, requires much weaker assumptions concerning the asymptotic behavior of the physical amplitudes. In exchange, the explicit presence of the zeros in the modulus representation can limit its practical use insofar as one looks for equalities, but, even in the absence of the knowledge of zeros, one can still have rigorous and maybe interesting results in the form of inequalities.

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