Analysis of Electromagnetic Mass Shift in Light-Cone Algebra

Fayyazuddin*
Rutherford High Energy Laboratory, Chilton, Didcot, Berkshire, England

Riazuddin
Institute of Physics, University of Islamabad, Islamabad, Pakistan†
and Institute for Theoretical Physics, University of Alberta, Edmonton, Canada
(Received 21 September 1973)

Within the framework of light-cone algebra, it is shown that a logarithmic divergence in the electromagnetic mass shift of a hadron arises precisely due to a mass term in the quark propagator. This divergence, therefore, can be removed by mass renormalization. The coefficient of this divergence is a scalar density which transforms as S_3 for the case of a nucleon. In our analysis, the gauge invariance to order $1/q^2$ is explicitly used.

A possible source of divergence in the electromagnetic self-energy of a nucleon is the deep-inelastic region for electron-nucleon scattering (the scaling region). This region in configuration space corresponds to the light cone. The light-cone algebra is relevant for discussing this region. It was shown recently[1] that within the framework of light-cone algebra, the electromagnetic mass shift comes out to be finite under the assumption that the contribution from the leading light-cone singularity of the current commutator to the absorptive part of electroproduction is manifestly gauge-invariant to order $1/q^2$. In abstracting light-cone algebra from the quark model, the contribution from the mass term in the quark propagator is neglected. The purpose of this paper is to show that the logarithmic divergence in the electromagnetic mass shift reappears and is precisely due to a mass term in the quark propagator. This divergence, therefore, can be removed by mass renormalization. Moreover, the coefficient of this divergence is a scalar density which transforms as S_3 for the case of a nucleon. This may be useful in understanding $\eta \to 3\pi$ decays and $\Delta I = 1$ mass differences.

In the free-quark model, the electromagnetic current is defined in the familiar way

$$J_\mu^{\text{em}} = i\bar{\psi}\gamma_\mu Q\psi , \tag{1}$$

where ψ is the column matrix for the free quark fields and Q is the charge matrix.

The algebra satisfied by the commutators of the electromagnetic currents can be easily worked out in the free-quark model and is given here:

$$[J_\mu^{\text{em}}(z), J_\nu^{\text{em}}(0)] = 2[s_{\mu\nu\rho\sigma} V_\sigma^{Q^2}(A; z, 0) + \epsilon_{\mu\nu\rho\sigma} A_\sigma^{Q^2}(S; z, 0)] \frac{\partial}{\partial z_\rho} \Delta(z) - 2i[\delta_{\mu\nu} J^{Q^2}(S; z, 0) + iT_{\mu\nu}^{Q^2}(A; z, 0)]\Delta(z) , \tag{2}$$

where $V_\sigma^{Q^2}(A; z, 0)$ and $A_\sigma^{Q^2}(S; z, 0)$ are, respectively, the usual antisymmetric (with respect to $z \leftrightarrow 0$) vector and symmetric axial-vector bilocal operators. $J^{Q^2}(S; z, 0)$ and $T_{\mu\nu}^{Q^2}(A; z, 0)$ are, respectively, symmetric and antisymmetric scalar and tensor bilocal operators and are defined as

$$J^{Q^2}(S, z, 0) = \tfrac{1}{2}[\bar{\psi}(z)MQ^2\psi(0) + \bar{\psi}(0)MQ^2\psi(z)] , \tag{3a}$$

$$T_{\mu\nu}^{Q^2}(A, z, 0) = \tfrac{1}{2}[\bar{\psi}(z)\sigma_{\mu\nu}MQ^2\psi(0) - (z \leftrightarrow 0)] . \tag{3b}$$

In Eq. (2) $s_{\mu\nu\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\nu\rho}\delta_{\mu\sigma} - \delta_{\mu\nu}\delta_{\rho\sigma}$. Note that M is the quark mass matrix; it is diagonal and so is Q^2. In deriving Eq. (2) we have made use of the anticommutation relation satisfied by free quark fields:

$$[\psi(x), \bar{\psi}(y)]_+ = -i\,S(x-y) , \tag{4a}$$

where

$$S(x-y) = (\gamma_\mu \partial_\mu - M)\Delta(x-y) . \tag{4b}$$

The algebra given in Eq. (2) satisfies electromagnetic current conservation. However, when one makes the approximation

$$\Delta(z) \underset{z^2 \simeq 0}{\sim} -\tfrac{1}{2}\epsilon(z_0)\delta(z^2) \equiv D(z) \tag{4c}$$

and neglects the quark mass term, the light-cone algebra so obtained from Eq. (2) satisfies current conservation only to the leading order in the light-cone singularity. The next term to the leading light-cone singularity in Eq. (4c) is given by

$$d(z) \equiv -\frac{\mu^2}{2\pi} \tfrac{1}{2}\epsilon(z_0)\theta(z^2) , \tag{4d}$$

where μ is a constant with dimension of mass and may be identified with the mean quark mass. Note that the term $M\Delta(z)$ also gives a nonleading contribution to the algebra. We shall use algebra (2) near $z^2 \simeq 0$, but would include the terms next to the leading singularity mentioned above. Thus in

Eq. (2) we replace $(\partial/\partial z_\rho)\Delta(z)$ by $(\partial/\partial z_\rho)[D(z)+d(z)]$, and in the last term on the right-hand side of Eq. (2) we replace $\Delta(z)$ by $D(z)$. Thus we shall use light-cone algebra in the form

$$[J_\mu^{\rm em}(z), J_\nu^{\rm em}(0)] \underset{z^2\simeq 0}{\simeq} 2[s_{\mu\nu\rho\sigma}V_\sigma^{Q^2}(A;z,0)+\epsilon_{\mu\nu\rho\sigma}A_\sigma^{Q^2}(S;z,0)]\frac{\partial}{\partial z_\rho}[D(z)+d(z)]$$
$$-2i[\delta_{\mu\nu}J^{Q^2}(S;z,0)+iT_{\mu\nu}^{Q^2}(A;z,0)]D(z). \tag{5}$$

Note that in writing the light-cone algebra one usually makes the approximation $S(x-y) \simeq \gamma_\mu \partial_\mu \Delta(x-y)$. This approximation we have not made in writing Eq. (5). In Eq. (5), the second term on the right-hand side is due to the mass term in the quark propagator. It is precisely this term which gives rise to the logarithmic divergence in the electromagnetic mass shift to order α.

In order to use the algebra, let us define the spin-summed matrix elements of the bilocal operators between the one-nucleon states:

$$\tfrac{1}{2}(2\pi)^3\frac{p_0}{m}\langle p|\,V_\sigma^{Q^2}(A;z,0)|p\rangle = \tilde{G}^A(p\cdot z)\frac{P_\sigma}{m}+i\tilde{h}^A(p\cdot z)z_\sigma+\cdots, \tag{6a}$$

$$\tfrac{1}{2}(2\pi)^3\frac{p_0}{m}\langle p|\,J^{Q^2}(S;z,0)|p\rangle = \tilde{S}(p\cdot z)+\cdots, \tag{6b}$$

where z is lightlike and where $\cdots$ denotes terms which vanish at $z^2=0$. Note that the axial-vector bilocal operator does not contribute in the spin-summed matrix elements, and such is the case for the tensor bilocal operator if time-reversal invariance is assumed.

Let us now define Fourier transforms:

$$G^A(\xi)=\frac{1}{(2\pi)}\int d(p\cdot z)e^{i\xi(p\cdot z)}\tilde{G}^A(p\cdot z), \tag{7a}$$

$$\tilde{G}^A(p\cdot z)=\int d\xi\, e^{-i\xi(p\cdot z)}G^A(\xi), \tag{7b}$$

and similarly for $h^A(\xi)$ and $\tilde{h}^A(p\cdot z)$. The Fourier transform of $\tilde{S}(p\cdot z)$ is given by

$$S(\xi)=\frac{1}{2\pi}\int d(p\cdot z)e^{i\xi(p\cdot z)}\tilde{S}(p\cdot z). \tag{7c}$$

We note that

$$\int d^4z\, e^{-iq\cdot z}\tilde{G}^A(p\cdot z)\frac{\partial}{\partial z_\rho}D(z)=2\pi(q+\xi p)_\rho(\xi/q^2)G^A(\xi), \tag{8a}$$

$$\int d^4z\, e^{-iqz}\tilde{G}^A(p\cdot z)\frac{\partial}{\partial z_\rho}d(z) =2\pi\mu^2[(q+\xi p)_\rho(\xi/q^2)^2G^{A\prime}(\xi)+O(1/q^4)], \tag{8b}$$

$$i\int d^4z\, e^{-iq\cdot z}\tilde{h}^A(p\cdot z)z_\rho\frac{\partial}{\partial z_\rho}D(z) =-\frac{\partial}{\partial q_\rho}\left[2\pi(q+\xi p)_\rho\frac{\xi}{q^2}h^A(\xi)\right], \tag{8c}$$

$$\int d^4z\, e^{-iq\cdot z}\tilde{S}(p\cdot z)D(z)=-i(2\pi)(\xi/q^2)S(\xi). \tag{8d}$$

Here $G^{A\prime}(\xi)=(d/d\xi)G^A(\xi)$. We note that the two types of nonleading terms arise from the nonleading terms in the light-cone algebra of Eq. (5), namely $(\partial/\partial z_\rho)d(z)$ and $MD(z)$. The third type of nonleading term arises from the coefficient of z_ρ in the spin-summed matrix elements of the bilocal operator defined in Eq. (6a). In the previous paper we considered the nonleading term of the third type only. In this paper we include the nonleading terms of the first two types also. It is appropriate to impose the condition of manifest gauge invariance to order $1/q^2$ when all three types of nonleading contributions mentioned above are included.

Let us define the absorptive part of the electroproduction amplitude:

$$A_{\mu\nu}=\tfrac{1}{2}(2\pi)^3\frac{p_0}{m}\times\int d^4z\, e^{-iq\cdot z}\langle p|\,[J_\mu^{\rm em}(z),J_\nu^{\rm em}(0)]\,|p\rangle =\frac{2\pi}{m^2}W_2(q^2,\nu)P_\mu P_\nu-2\pi W_t(q^2,\nu)\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right), \tag{9a}$$

where

$$P_\mu P_\nu=p_\mu p_\nu+\frac{p_\mu q_\nu+p_\nu q_\mu}{2\xi}+\frac{q^2}{4\xi^2}\delta_{\mu\nu},$$
$$W_t(q^2,\nu)=\frac{\nu^2}{q^2}W_2(q^2,\nu)-W_1(q^2,\nu),$$
$$W_L(q^2,\nu)=W_2(q^2,\nu)+W_t(q^2,\nu),$$
$$\nu=-\frac{p\cdot q}{m},\quad \xi=\frac{q^2}{2m\nu}. \tag{9b}$$

The scaling region ($q^2\to\infty$, ξ finite) corresponds to the light cone in the configuration space. Thus using Eqs. (5)–(8), $A_{\mu\nu}$ in the deep-inelastic region is given by

$$A_{\mu\nu} \sim 2\pi\Big\{2\xi\Big[\frac{2\xi}{m}\Big(\frac{P_\mu P_\nu}{q^2}\Big)G^A(\xi) - \frac{\xi}{m}\frac{p^2}{q^2}\delta_{\mu\nu}G^A(\xi)\Big] + 2\xi\mu^2\Big[\frac{2\xi^2}{mq^2}\Big(\frac{P_\mu P_\nu}{q^2}\Big)G^{A\prime}(\xi)\Big]$$
$$-2\xi\Big[-\frac{2\delta_{\mu\nu}}{q^2}h^A(\xi) + \frac{8\xi^2}{q^2}\Big(\frac{P_\mu P_\nu}{q^2}\Big)h^A(\xi) - \frac{4\xi}{q^2}\Big(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\Big)h^{A\prime}(\xi) + \frac{8\xi^3}{q^2}\Big(\frac{P_\mu P_\nu}{q^2}\Big)h^{A\prime}(\xi) + \frac{2\xi}{q^2}\delta_{\mu\nu}h^{A\prime}(\xi)\Big]$$
$$-2\frac{\xi}{q^2}\delta_{\mu\nu}S(\xi) + O(1/q^4)\Big\}, \tag{10}$$

where $h^{A\prime}(\xi) = (d/d\xi)h^A(\xi)$.

From Eq. (10) it is clear that we have gauge invariance to $O(1)$, and if we insist that the contribution to $O(1/q^2)$ should also be gauge-invariant, then the terms which behave like $1/q^2$ and are not individually gauge-invariant should combine to give a gauge-invariant contribution to $O(1/q^2)$. This means that (with $p^2 = -m^2$) we must have

$$m\xi G^A(\xi) + 2h^A(\xi) - 2\xi h^{A\prime}(\xi) = S(\xi). \tag{11}$$

From Eqs. (9) and (10), using Eq. (11), we have

$$\nu W_2(q^2,\nu) = 2\xi G^A(\xi) - \frac{8\xi^2}{q^2}m[h^A(\xi) + \xi h^{A\prime}(\xi)] + \mu^2\frac{2\xi^2}{q^2}G^{A\prime}(\xi) + O(1/q^4), \tag{12a}$$

$$W_t(q^2,\nu) = -\frac{8\xi^2}{q^2}h^{A\prime}(\xi) + O(1/q^4). \tag{12b}$$

Now with the usual definitions

$$\nu W_2(q^2,\nu) = F_2(\xi) + \frac{m^2}{q^2}H_2(\xi) + O(1/q^4), \tag{13a}$$

$$W_t(q^2,\nu) = \frac{1}{m}\Big[F_L(\xi) + \frac{m^2}{q^2}H_t(\xi)\Big] + O(1/q^4), \tag{13b}$$

$$W_L(q^2,\nu) = \frac{1}{m}\Big[F_L(\xi) + \frac{m^2}{q^2}H_L(\xi)\Big] + O(1/q^4), \tag{13c}$$

we have from Eq. (12)

$$F_2(\xi) = 2\xi G^A(\xi), \tag{14a}$$

$$F_L(\xi) = 0, \tag{14b}$$

$$H_2(\xi) = -\frac{8\xi^2}{m}[h^A(\xi) + \xi h^{A\prime}(\xi)] + \frac{\mu^2}{m^2}2\xi^2 G^{A\prime}(\xi), \tag{14c}$$

$$H_t(\xi) = -\frac{8\xi^2}{m}h^{A\prime}(\xi). \tag{14d}$$

Again using Eqs. (9), (13), and (14), we get

$$H_L(\xi) \equiv 2\xi F_2(\xi) + H_t(\xi) = 4\xi^2[G^A(\xi) - (2/m)h^{A\prime}(\xi)]. \tag{14e}$$

Now from the condition (11) we have

$$m\int_0^1 2\xi G^A(\xi)d\xi + 4\int_0^1[h^A(\xi) - \xi h^{A\prime}(\xi)]d\xi = 2\int_0^1 S(\xi)d\xi. \tag{15}$$

But it is easy to see that [integrating by parts and noting that at threshold $h^A(\xi=1)=0$]

$$\int_0^1 \xi h^{A\prime}(\xi)d\xi = -\int_0^1 h^A(\xi)d\xi. \tag{16}$$

Also we note that

$$2\int_0^1 S(\xi)d\xi = \tilde{S}(0) = \tfrac{1}{2}(2\pi)^3\frac{p_0}{m}\langle p|\bar\psi MQ^2\psi|p\rangle. \tag{17}$$

Hence from Eq. (15), using Eqs. (14e), (16), and (17), we get the sum rule

$$m\int_0^1\Big[\frac{H_L(\xi)}{\xi} - F_2(\xi)\Big]d\xi = \tilde{S}(0). \tag{18}$$

Note that the right-hand side of Eq. (18) is entirely due to the contribution from the mass term in the quark propagator to the light-cone algebra. In the usual light-cone algebra, this contribution is not taken into account and we recover our previous result.[1]

In general we expect the leading Regge behavior for $H_L(\xi)$ or $H_t(\xi)$ to be $\xi^{-\alpha}$ ($\alpha=1$) as $\xi\to 0$. Thus the integral $\int_0^1[H_L(\xi)/\xi]d\xi$ will diverge at $\xi\to 0$. It is clear that Regge behavior for $\xi h^{A\prime}(\xi)$ or $h^A(\xi)$ is $\xi^{-\alpha-1}$. In a special case $H_L(\xi)$ or $H_t(\xi)$ may not behave as $\xi^{-\alpha}$. This is the situation when the leading Regge contribution for W_1 and W_2 cancels in the linear combination (9b). In that case Eq. (18) is valid. For this case, the amplitude T_L (whose absorptive part is W_L) satisfies an unsubtracted dispersion relation. This is certainly a possibility and has extensively been considered in the literature.[2] The same assumption was made in our previous paper,[1] and here we confine ourselves to this case; the case where $H_L(\xi)$ or $H_t(\xi)$ behaves in general as $\xi^{-\alpha}$ as $\xi\to 0$ will be considered elsewhere together with the question of divergences in weak matrix elements.

Now if T_L satisfies an unsubtracted dispersion relation and one has the Callan-Gross relation $F_L(\xi)=0$, then it is known[2] that the divergent part of the electromagnetic self-energy is given by

$$(\Delta^m)_{\text{divergent}} = -\frac{3\alpha}{4\pi}\left(\int_{q_m^2}^{\infty}\frac{dq^2}{q^2}\left\{\int_0^1\left[F_2(\xi)-\frac{H_L(\xi)}{\xi}\right]d\xi\right\}\right). \tag{19}$$

We note an important point: The choice $H_L(\xi)=0$ usually made in the literature[3] to get the right sign for the proton-neutron mass difference from the above deep-inelastic contribution is not compatible with the sum rule (18) as both $F_2(\xi)$ and $\tilde{S}(0)$ are positive definite.

If we make use of the sum rule (18) in (19) we get

$$(\Delta m)_{\text{divergent}} = \frac{3\alpha}{4\pi}\left(\int_{q_m^2}^{\infty}\frac{dq^2}{q^2}\right)\tilde{S}(0), \tag{20}$$

where as expressed in Eq. (17)

$$\tilde{S}(0)=\tfrac{1}{2}(2\pi)^3\frac{p_0}{m}\langle p|\bar{\psi}MQ^2\psi|p\rangle .$$

Thus we see that the logarithmic divergence in the electromagnetic mass shift arises from the mass term in the quark propagator, with its coefficient proportional to a scalar density, which for the case of nucleon transforms as S_3. This logarithmic divergence can be removed by renormalizing the quark mass matrix, viz., by introducing a counterterm which transforms as a scalar density S_3. Thus our analysis leads naturally to a term proportional to S_3, which is helpful for understanding $\eta\to 3\pi$ decay and $\Delta I=1$ electromagnetic mass differences. We also note from (17) and (20) that the coefficient of the logarithmic-divergent term is positive. We end with the following remarks:

(i) By using the Bjorken-Johnson-Low limit,[4] the divergent part of the electromagnetic mass shift is also given by the expression (20).

(ii) Since we have used gauge invariance explicitly in our analysis, therefore, although we have derived our results using free-quark model, our results are expected to hold in the quark-gluon model provided that the gluon interaction is introduced in the gauge-invariant way.

(iii) When $H_L(\xi)$ or $H_t(\xi)$ behaves as $\xi^{-\alpha}$, $\xi\to 0$, the sum rule (18) is modified to[5]

$$m\int_0^{\infty}\frac{\tilde{H}_L(\xi)}{\xi}d\xi-\int_0^1 F_2(\xi)d\xi=\tilde{S}(0), \tag{21}$$

where

$$\tilde{H}_L(\xi)=H_L(\xi)-H_L^R(\xi)$$

and where H_L^R denotes the Regge part.

(iv) If the quark mass is zero, the electromagnetic self-mass is finite and commutable. If the quark mass is nonzero, the divergence can be absorbed by renormalizing the quark mass.

One of us (R.) would like to thank Professor D. D. Betts and Professor A. N. Kamal for hospitality at the Institute for Theoretical Physics, University of Alberta. The other (F.) is grateful to Dr. Roger Phillips for hospitality at Rutherford High Energy Laboratory.

---

*On leave of absence from Institute of Physics, University of Islamabad, Pakistan. Address after 1 October 1973: Theory Division, CERN, Geneva, Switzerland.

†Permanent address.

[1]Fayyazuddin and Riazuddin, Phys. Rev. D 8, 510 (1973).

[2]See, for example, A. Zee, Phys. Rep. 3C, 129 (1972). References to earlier literature can be found in this article.

[3]R. Jackiw, R. Van Royen, and G. B. West, Phys. Rev. D 2, 2473 (1970); H. Pagels, *ibid.* 3, 610 (1971); 4, 1932(E) (1971); T. D. Lee, in *Proceedings of the Amsterdam International Conference on Elementary Particles, 1971,* edited by A. G. Tenner and M. Veltman (North-Holland, Amsterdam, 1972).

[4]J. D. Bjorken, Phys. Rev. 148, 1467 (1966); K. Johnson and F. E. Low, Prog. Theor. Phys. Suppl. 37–38, 74 (1966).

[5]D. J. Broadhurst, J. F. Gunion, and R. L. Jaffe, Phys. Rev. D 8, 566 (1973).