Chiral Perturbation Theory*

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(Received 11 June 1973)

We consider perturbation theory for SU(2) × SU(2) and SU(3) × SU(3) chiral symmetries realized by Nambu-Goldstone bosons. Exact expressions are derived for the derivatives with respect to the symmetry-breaking parameter ϵ of Green's functions, scattering amplitudes, and the matrix elements of operators, including the effects of renormalization and the external mass-shell constraints. These expressions are used to systematically classify all leading nonanalytic behavior in the expansion of these quantities around $\epsilon = 0$. We find (1) S-matrix elements go to finite limits as $\epsilon \to 0$. (2) They in general approach this limit in a nonanalytic $\epsilon \ln \epsilon$ manner. (3) At exceptional momentum points, corresponding to the low-energy theorems of current algebra, the leading nonanalytic corrections can be absorbed into the renormalization of the parameters (such as f_π) of the theory by the symmetry-breaking interaction. Hence leading-order corrections to low-energy theorems are expected to be analytic. (4) The errors in off-shell partial-conservation-of-axial-vector-current extrapolations are often of order $\epsilon \ln \epsilon$ and can be calculated exactly. (5) The matrix elements of two or more zero-energy operators can diverge as $\ln \epsilon$ or $1/\epsilon$ or worse in the chiral limit. (6) The leading corrections in SU(2) × SU(2) expansions are very small (a few percent). (7) Expansions around SU(3) × SU(3) are marginal. The corrections are often 30% and in one case are larger than the leading term. We calculate the leading renormalization of the meson decay constants and consider the $\pi\pi$ and πN amplitudes in some detail.

I. INTRODUCTION

The notion of an approximate hadron symmetry has proven to be very efficacious in unraveling the systematics of the hadron spectrum. According to this idea one writes for the Hamiltonian

$$H = H_0 + \epsilon H' ,$$

where H_0 is invariant under the symmetry group and H' breaks the symmetry. By making assumptions about the transformation properties of H', one can derive relations, like the Gell-Mann–Okubo mass formula, which are valid to leading order in ϵ. Essential to implementing the notion of an approximate symmetry are the assumptions of perturbation theory.

If one further assumes, as did Gell-Mann in the development of these ideas,[1] that the generators of the symmetry group can be identified with the charges of the currents measured in weak and electromagnetic interactions, then the results of current algebra[2] follow as a consequence of approximate symmetry. From this point of view the weak and electromagnetic interactions simply act as probes of the hadron symmetries and the structure of hadronic symmetry breaking.

As has already been emphasized by many people, the idea of an approximate hadron symmetry must be altered if the recently developed gauge theories of weak and electromagnetic interactions are correct. If the gauge theories are to be renormalizable in perturbation theory, they require that H possess an exact local (in general, non-Abelian) symmetry. Symmetry breaking is manifest in the states without the necessity of massless Goldstone bosons according to the Higgs mechanism.

There are at least two extreme positions one can take in the confrontation of gauge theories with the older ideas of current algebra. The gauge-theory revolutionist would maintain that little of the older ideas will remain and that nothing less than a unified world of hadrons and leptons will do. Perhaps even class distinctions between hadrons and leptons will be lost. The gauge-theory conservative, alternately, is dedicated to a program of reconciliation. The symmetries of the hadrons and the dynamics are a world in their own right and the weak interactions basically are a small perturbation, both conceptually and in fact, on the hadron dynamics. The weak interactions continue to function as a probe (as we conventionally have thought of them), and the hadron world may ignore the complications of the weak interactions.

This paper is predicated on the hope that some meaning will continue to be attached to the idea of an approximate hadron symmetry. Many of the ideas developed here can be taken over to a suitable, future gauge theory involving hadrons. It could turn out that the pion mass squared is proportional to the fine-structure constant,[4] so that isospin and chiral breaking are not independent. Then the methods presented here would have to be modified but the essential features will remain unaltered. If it turns out that $\mu_\pi^2 \sim \alpha$, then there exists the interesting possibility that matrix elements can behave like $\alpha \ln \alpha$ as $\alpha \to 0$.

Most of this paper will be devoted to a study of perturbations about chiral SU(2)×SU(2) realized by Nambu-Goldstone bosons (the ground-state pion). The role of the pion, as a ground-state meson, was originally emphasized by Nambu.[5] On this basis the success of the Goldberger-Treiman formula could be understood as a consequence of partial conservation of axial-vector current (PCAC). However, it was not clear how the approximate SU(2)×SU(2) symmetry of the Hamiltonian was compatible with the observed approximate SU(3) symmetry of the hadron spectrum.

The reconciliation of these symmetries was given by Glashow and Weinberg[6] and Gell-Mann, Oakes, and Renner.[7] They observed that the underlying symmetry might be SU(3)×SU(3), with an SU(3)-symmetric ground state. The Goldstone theorem required an octet of ground-state mesons which were identified with the π, K, and η. If the Hamiltonian symmetry was broken by an explicit term which preserved the SU(2)×SU(2) subgroup, then the pion would remain massless and the SU(3) degeneracy of supermultiplets would be broken in the usual way. The SU(2)×SU(2) symmetry could then be further explicitly broken to SU(2) or U(1).

Dashen, and Dashen and Weinstein[8] especially stressed the fact that the results of chirality and current algebra could be understood as a perturbation theory about the chiral limit. The success of current algebra on either the SU(2)×SU(2) or SU(3)×SU(3) level could be seen to rest on the effective convergence of a perturbation expansion in the symmetry breaking parameter.

Subsequently it was pointed out[9] that for a chiral symmetry realized by Goldstone bosons the S matrix and matrix elements of currents would not be analytic for small values of the perturbation parameter. This was because the Goldstone theorem entails the existence of massless bosons to realize the symmetry, and they give rise to long-range forces which destroy the analyticity of the perturbation expansion near the origin. It was pointed out in this series of papers that if one had a chiral symmetry, then, in general, one expects the S matrix to exist in the symmetry limit. The nonanalytic character of the perturbation expansion only influenced the usual assumption of how this limit is approached, not its existence. Hence the current-algebra low-energy theorems for S matrix elements are unaffected.

In general, if one does not have a chiral symmetry, then one does not expect the S matrix to exist in the massless-pion limit. Evidently more than the usual assumptions of S-matrix theory are required to establish the existence of this limit as discussed by Dashen and Gross.[10] Conversely, if one assumes the existence of the S matrix in the massless-pion limit, then this assumption contains much of the content of chirality. That the S matrix exists in the chiral limit is a consequence of the Adler zero and the p-wave coupling of pions. Amplitudes, rather than diverging like $1/\mu_\pi^2$ or $\ln\mu_\pi^2$ as $\mu_\pi^2\to 0$, approach finite limits.

In general one finds that the S matrix approaches the chiral limit nonanalytically to leading order. However, at the current-algebra points (and, in general, only at such points) for which we have exact theorems in the chiral limit, the corrections are evidently analytic to leading order.[11] For example, the corrections to the Adler-Weisberger relation, expressed as a low-energy theorem, are analytic to leading order in $\mu^2=$(pion mass).[2] This is true providing we express the theorem using the physical pion decay constant and g_A, not their values in the chiral-symmetric world. This will be elaborated in Sec. II E. In this paper we will discuss in detail these features of nonanalytic behavior and their influence on symmetry breaking. We find that when the corrections are nonanalytic and dictated by the chiral symmetry itself, then this leading order correction can be calculated exactly and perhaps be expected to dominate the corrections to the symmetric world.[12]

The corrections to the chiral limit that are analytic in the perturbation parameter are not easily estimated. Here one must make necessarily dynamical assumptions in order to obtain an estimate. For example the corrections to the Goldberger-Treiman relation are analytic to leading order.[9] It has been suggested by Michael,[13] and Pagels and Zepeda[9] that the observed 8% corrections are due to a heavy pion, the π'. The π' as a source of corrections to PCAC has been recently applied by Drell[14] in a study of the $\pi^0\to 2\gamma$ anomaly. Also, Bég and Zepeda[9] have pointed out that for nucleon- and pion-charge radii the nonleading terms must be important; in particular, the ρ resonance must play a role.

Without considering such infrared effects coming from loops Carruthers and Haymaker[15] in a series of interesting papers pointed to a second difficulty in the chiral perturbation theory present even in the tree approximation. In the Σ model if one solves the potential minimization problem for the vacuum values of the scalar fields $\langle\sigma\rangle_0=F(\epsilon)$, then $F(\epsilon)$ as a function of the explicit symmetry breaking parameter ϵ can have a very small radius of convergence, perhaps smaller than the physical value of ϵ. The reason for this is that the potential function has several minima for fixed values of the constants in the symmetric Hamiltonian and ϵ. $\langle\sigma\rangle_0$ is determined by the global minimum. However, as one changes ϵ the global minimum changes, in fact it can jump from one local minimum to another. Thus, one cannot do perturbation theory if the minimum is too shallow.[16]

While such an effect can take place it is difficult to apply to specific problems. The reason is that such behavior depends crucially on the structure of the symmetric Hamiltonian and the potential, and hence is very model-dependent. It would be of interest to try to extract some model-independent conclusions from this effect so we would know when to expect trouble.

This paper develops the general perturbation theory about a chiral symmetry. We restrict ourselves to the case where the symmetry breaking interactions are local. It is possible to extend these considerations to effectively nonlocal interactions due to currents as we do in the following article.[17]

Even for a local interaction our study involves a nontrivial extension of the usual perturbation theory because of the fact that masses are vanishing in the symmetry limit and because of the dependence of amplitudes on the external mass. This feature, while it makes for the interesting and physically relevant aspect of chiral symmetries also makes the perturbation theory more complicated. It is the purpose of this paper to examine these complications in detail. Throughout this work we are motivated more by the desire to develop a technique for doing practical calculations than considerations of axiomatic rigor. Hence we will assume the usual analytic structure of amplitudes in the kinematic variables.

We consider perturbation theory in a parameter ϵ; as ϵ vanishes so do the masses of the Goldstone bosons. In general, amplitudes depend on ϵ in four ways: (a) through the unrenormalized Green's functions, (b) through renormalization, (c) through Klein-Gordon operators $\Box+\mu^2$, and (d) through mass-shell constraints. All these delicate questions are examined here. Although we develop our formalism for SU(2)×SU(2) it can be extended to SU(3)×SU(3), where we have made applications to Dashen's sum rule[8] $\mu_{\pi^0}^2-\mu_{\pi^+}^2=\mu_{K^0}^2-\mu_{K^+}^2$.

The principal results of this paper are as follows:

(i) We give a general formalism and exact expressions [Eqs. (2.11), (2.44), (2.51), and (2.53)] for calculating the derivatives with respect to ϵ of on- and off-shell Green's functions, scattering amplitudes, and matrix elements of currents.

(ii) We have a set of simplified rules for extracting the leading nonanalytic behavior of these quantities in the chiral limit $\epsilon\to 0$. All leading nonanalytic behavior is systematically isolated and classified. These rules are given at the end of Sec. II D.

(iii) The S matrix, in general, exists in the chiral limit at nonexceptional momenta. There are no terms like $\ln\epsilon$ or $\epsilon^{-1/2}$ that diverge.

(iv) The S matrix, in general, *approaches* the chiral limit in a nonanalytic fashion like $\epsilon\ln\epsilon$ or even $\epsilon\ln^2\epsilon$ for nonexceptional momenta.

(v) For the exceptional momenta, corresponding precisely to current-algebra points for which one can prove low-energy theorems, we have found, from the examples studied, that the corrections to these theorems are *analytic* to leading order. What we find is that at the current-algebra point all the nonanalytic behavior of the correction can be absorbed into the renormalization of the parameters, like f_π, due to the symmetry breaking interactions.

(vi) Matrix elements of operators, not corresponding to S-matrix elements, and at exceptional momenta can diverge in the chiral limit like ϵ^{-1} or ϵ^{-2}.

(vii) A possible candidate for a *dimensionless* parameter to characterize SU(2)×SU(2) perturbations is $\mu^2/32\pi^2 f_\pi^2\simeq 0.006$, which is a ubiquitous factor in our calculations. For example, a result of theoretical interest is the renormalization of the pion-decay constant,

$$\frac{f_\pi(\mu^2)}{f_\pi(0)}=1-\frac{2\mu^2}{32\pi^2 f_\pi^2}\ln 4\mu^2+O(\mu^2),$$

which for $\ln 4\mu^2\simeq 2$ is a $2\frac{1}{2}\%$ renormalization.

(viii) While perturbation theory about SU(2)×SU(2) is excellent, perturbations about SU(3)×SU(3) symmetry are marginal. The attitude we take towards perturbation theory is to calculate the perturbation; if it is a small correction to the leading terms, then we would say it works, if it is comparable to the leading term, then it fails. While we know of no failure for SU(2)×SU(2), we do for SU(3)×SU(3).[17] We know of no general criterion for when perturbation theory fails for SU(3)×SU(3); all one can do is to calculate the perturbation, which in some cases is small and others not.

The plan of this article is as follows: In Sec. II we develop our general technique for perturbative expansions about an SU(2)×SU(2) chiral symmetry for pion amplitudes. The general formula for Green's functions and amplitudes is given. In particular we illustrate these techniques for the π-π scattering amplitude. The nonanalytic character of these expansions is explicitly examined. In Sec. III we discuss the extension of these methods to SU(3) ×SU(3). Finally, in Sec. IV, we discuss the properties of Green's functions including baryons as we approach the chiral limit. In the second paper[17] we will consider the dependence of the electromagnetic mass shifts of mesons on the chiral-symmetry-breaking parameter.

II. PERTURBATION OF PION AMPLITUDES AROUND SU(2)×SU(2)

In this chapter we will derive and illustrate a general formula for the derivatives of Green's functions, matrix elements of operators, and scattering amplitudes with respect to a symmetry-breaking parameter. We use this result to locate and calculate the leading nonanalytic terms in these quantities. The basic formulas are true for any symmetry, but in this section we will restrict ourselves to SU(2)×SU(2). For SU(2)×SU(2) the symmetry-breaking parameter is essentially μ^2, the square of the pion mass.

We will begin with a review of broken SU(2) ×SU(2). Then we will show how to compute the derivative of a Green's function with respect to μ^2, including the effects of renormalization. After illustrating these techniques by calculating the leading-order deviation of the pion decay constant from its SU(2)×SU(2) symmetric value, we will move on to a discussion of symmetry-breaking effects in full matrix elements and scattering amplitudes. These quantities are complicated in that not only the internal dynamics, but also the masses of the external particles depend on the symmetry breaking. Hence, we will first give a heuristic discussion of the dependence of amplitudes on the external masses. As a by-product of this discussion we indicate how to compute the leading corrections to off-mass-shell (PCAC) calculations. We then derive our formula for the total variation of an amplitude or matrix element with respect to μ^2 and use this to systematically classify all of the different sources of nonanalytic behavior. As an example, we will consider the behavior of the $\pi\pi$ amplitude at fixed s and t as a function of μ^2, $A(s,t;\mu^2)$. We show that, in general, $A(s,t;\mu^2)=A(s,t;0)+G(s,t)\mu^2\ln\mu^2+O(\mu^2)$. However, as s and $t\to 0$, Weinberg's expansion[18] is still valid; the $\mu^2\ln\mu^2$ term *at threshold* merely represents the renormalization of the pion-decay constant by the symmetry-breaking interaction.

We conclude this section by arguing that an appropriate *dimensionless* parameter to measure the strength of explicit, SU(2)×SU(2) breaking is $\mu^2/(32\pi^2 f_\pi^2)\approx 0.006$, indicating that SU(2) ×SU(2) is a very good symmetry of the Hamiltonian. Typically, this small number gets multiplied by factors of 2 or 4 times $\ln 4\mu^2 \simeq 2$, suggesting that chiral SU(2)×SU(2) breaking is a good symmetry to 5–10%.

A. Review of Broken SU(2)×SU(2)

We assume that the strong-interaction Hamiltonian H is approximately invariant under an SU(2)×SU(2) group generated by the vector and axial-vector charges F_i and ${}^5F_i$, $i=1,2,3$. We denote the corresponding vector and axial-vector currents by $V_i^\mu(x)$ and $A_i^\mu(x)$. The full Hamiltonian density is $H=H_0+\epsilon^0\sigma^0$, where H is SU(2) ×SU(2)-symmetric and σ^0 is an unrenormalized scalar field. In this paper we always assume that σ^0 and the unrenormalized pion field ϕ_i^0, $i=1,2,3$ transform according to the $(\frac{1}{2},\frac{1}{2})$ representation, defined by the equal-time commutation relations

$$\begin{aligned}
&[F^i,\phi_j^0]=i\epsilon_{ijk}\,\phi_k^0\\
&[F^i,\sigma^0]=0\\
&[{}^5F^i,\phi_j^0]=i\delta_{ij}\sigma^0\\
&[{}^5F^i,\sigma^0]=-i\phi_i^0\ .
\end{aligned}\tag{2.1}$$

The Celbsch-Gordan coefficients in our expansions around the $\epsilon^0=0$ limit will therefore be model-dependent; the general analysis of the nature and origin of the singularities is representation-independent, however.

We assume that as $\epsilon^0\to 0$ the symmetry is realized in the Nambu-Goldstone sense[5]: The vacuum $|0\rangle$ possesses only SU(2) symmetry. Hence, the spectrum of physical states is only SU(2) symmetric, σ^0 possesses a nonzero vacuum expectation value, and there are three Goldstone bosons (massless pions). For small but nonzero ϵ^0, the pions acquire a mass μ. That μ^2 is very small compared to other hadronic masses is the principal rationale for an approximate SU(2)×SU(2) Goldstone symmetry. Using our assumed Hamiltonian and commutation relations (2.1), we have

$$\begin{aligned}
\partial_\mu V_i^\mu(0)&=i[\epsilon^0\sigma^0(0),F^i]\\
&=0\\
\partial_\mu A_i^\mu(0)&=i[\epsilon^0\sigma^0(0),{}^5F^i]\\
&=-\epsilon^0\phi_i^0(0)\ .
\end{aligned}\tag{2.2}$$

We define a renormalized pion field $\phi_i=Z_\pi^{-1/2}\phi_i^0$; Z_π is chosen so that $\langle 0|\phi_i(0)|j\rangle=\delta_{ij}$, where $|j\rangle$ is a physical pion state[19] and $|0\rangle$ is the physical vacuum.

The pion decay constant f_π, defined by

$$\langle 0|A_i^\mu(0)|j\rangle=if_\pi K_j^\mu\delta_{ij}\ ,\tag{2.3}$$

is experimentally 95 MeV. Comparing Eq. (2.3) with (2.2) we see that

$$\epsilon^0 = \frac{-f_\pi \mu^2}{Z_\pi^{1/2}} , \tag{2.4}$$

Since there is no stable σ particle, we are free to renormalize σ^0 at our convenience. We define a renormalized σ field and a renormalized ϵ as $\sigma = Z_\pi^{-1/2}\sigma^0$ and $\epsilon = Z_\pi^{1/2}\epsilon^0 = -f_\pi \mu^2$. Then $\epsilon^0\sigma^0 = \epsilon\sigma$ and $(\sigma, \vec{\phi})$ still transforms under $(\frac{1}{2}, \frac{1}{2})$ representation. Of course, the renormalization constant Z_π itself depends on ϵ.

The renormalized pion propagator is

$$\begin{aligned} i\Delta_{ij}(K^2, \epsilon) &\equiv \int d^4x\, e^{iK\cdot x}\langle 0|T(\phi_i(x)\phi_j(0))|0\rangle \\ &= i\delta_{ij}\left[\frac{1}{K^2-\mu^2} + \Delta_c(K^2,\epsilon)\right] , \\ \Delta_c(K^2,\epsilon) &\equiv \int_{9\mu^2}^{\infty} \frac{ds}{K^2-s}\rho(s,\epsilon) , \end{aligned} \tag{2.5}$$

where T represents a time ordering. We have explicitly indicated the dependence of Δ_{ij}, the spectral function ρ, and the continuum integral Δ_c on ϵ. The unrenormalized propagator is $i\Delta^0_{ij}(K^2,\epsilon) \equiv i\Delta_{ij}(K^2,\epsilon)Z_\pi$.

Now, by considering

$$0 = \lim_{K_\mu \to 0} K_\mu \int d^4x\, e^{iK\cdot x}\langle 0|T(A^\mu_i(x)\phi_j(0))|0\rangle$$

and using the commutation relations (2.1), it is simple to show that

$$\langle\sigma(0)\rangle_0 = f_\pi(\epsilon)\left[1 + \mu^2 \int_{9\mu^2}^{\infty} \frac{ds}{s}\rho(s,\epsilon)\right] \tag{2.6}$$

where we have displayed the dependence of f_π on ϵ and where $\langle\,\rangle_0$ represents a vacuum expectation value. In the $\epsilon \to 0$ limit, ρ is well behaved $\{\rho(s,0) \underset{s\to 0}{\simeq} 8s/[3(8\pi)^4 f_\pi^4]$, Pagels and Zepeda[9]$\}$, so that the spectral integral is finite. Hence,

$$\langle\sigma\rangle_0(\epsilon = 0) = f_\pi(0) . \tag{2.7}$$

We can regard ϵ and $f_\pi(0)$ as the fundamental parameters of the theory: ϵ measures the explicit chiral-symmetry breaking and $f_\pi(0)$ the spontaneous (vacuum) symmetry breaking. The pion mass is related by $\epsilon = -f_\pi(\epsilon)\mu^2$. We shall see that $f_\pi(\epsilon) = f_\pi(0) + O(\epsilon \ln\epsilon)$, so that

$$\epsilon = -f_\pi(0)\mu^2 + O(\epsilon^2 \ln\epsilon) . \tag{2.8}$$

Equation (2.8) is just Goldstone's theorem: as $\epsilon \to 0$ either the vacuum is symmetric $[f_\pi(0) = 0]$ or there are Goldstone bosons $(\mu^2 = 0)$.

Of course, instead of taking ϵ and $f_\pi(0)$ as the independent parameters, we can regard μ^2 and $f_\pi(0)$ or μ^2 and $f_\pi(\epsilon)$ as independent.

B. Perturbation of Green's Functions

In this section we consider the dependence of a Green's function

$$G(x_1, \ldots, x_m; y_1, \ldots, y_k, \epsilon) \equiv \langle 0|T(\phi_1(x_1)\cdots\phi_m(x_m)A_1(y_1)\cdots A_k(y_k))|0\rangle$$

on the parameter ϵ. Here, the A's are any local operators, such as currents or other fields.

It is convenient to work with the unrenormalized functions $G^0 = Z_\pi^{m/2}G$ (of course, if the A operators are renormalized, G must be multiplied by their renormalization constants, also). In analogy with the standard interaction picture we may define a symmetry-breaking picture for G^0:

$$G^0 = \frac{\langle\hat{0}|T(\hat{\phi}^0_1\cdots\hat{\phi}^0_m\hat{A}_1\cdots\hat{A}_k\exp[-i\int d^4z\,\epsilon^0\hat{\sigma}^0(z)])|\hat{0}\rangle}{\langle\hat{0}|T(\exp[-i\int d^4z\,\epsilon^0\hat{\sigma}^0(z)])|\hat{0}\rangle}, \tag{2.9}$$

where $\hat{\phi}^0$, $\hat{\sigma}^0$, and $\hat{A}$ are unrenormalized operators, dressed with respect to the chiral-symmetric interactions, but bare in the symmetry-breaking intereactions. Similarly, $|\hat{0}\rangle$ is the symmetric vacuum. In (2.9) the entire ϵ^0 dependence is in the exponentials.

Taking $dG^0/d\epsilon^0$ and reexpressing the result in terms of fully dressed fields, we find

$$\begin{aligned} \frac{dG^0}{d\epsilon^0} &= -i\int d^4z[\langle 0|T(\phi^0_1\cdots\phi^0_m A_1\cdots A_k\sigma^0(z))|0\rangle \\ &\qquad -G^0\langle 0|\sigma^0(z)|0\rangle] \\ &\equiv -i\int d^4z\, G^0_\sigma(z); \end{aligned} \tag{2.10}$$

$G^0_\sigma(z)$ is connected with respect to the σ field and has the same type of connected structure in the other fields as the original Green's function.

Finally, the derivative of the renormalized function is

$$\frac{1}{Z_\pi^{1/2}}\frac{dG}{d\epsilon^0} = -i\int d^4z\, G_\sigma(z) - \frac{m}{Z_\pi}\frac{dZ_\pi^{1/2}}{d\epsilon^0}G, \tag{2.11}$$

with $G_\sigma = Z_\pi^{-(m+1)/2}G^0_\sigma$. If the A operators must be renormalized, the modification is obvious.

As a first application consider $\langle\sigma^0\rangle_0$, which by way of (2.6) equals $Z_\pi^{1/2}(\epsilon)f_\pi(\epsilon)[1 + O(\epsilon)]$. Then, using Eq. (2.10)

$$\frac{d}{d\epsilon^0}\langle\sigma^0(0)\rangle_0 = -i\int d^4z\langle 0|T(\sigma^0(z)\sigma^0(0))|0\rangle_C \tag{2.12}$$

This quantity *diverges in the chiral limit, reflecting the nonanalytic behavior in* ϵ. To see this, write a dispersion relation or spectral representation for (2.12):

$$\frac{d}{d\epsilon^0}\langle\sigma^0\rangle_0 = -(2\pi)^3\int_{4\mu^2}^{\infty}\frac{ds}{s}\times\sum_n \delta^4(P-P_n)|\langle n|\sigma^0(0)|0\rangle|^2, \qquad (2.13)$$

where $P=(\sqrt{s}, \vec{0})$. The two-pion contribution (Fig. 1) to (2.13) is

$$-\frac{1}{32\pi^2}\sum_{cd=1}^{3}\int_{4\mu^2}^{\infty}\frac{ds}{s}\left(\frac{s-4\mu^2}{s}\right)^{1/2}|\langle cd|\sigma^0(0)|0\rangle|^2, \qquad (2.14)$$

where we have utilized the s-wave nature of the matrix element. Now $\langle cd|\sigma^0(0)|0\rangle$ remains finite as s and $\epsilon\to 0$, so the integral diverges logarithmically at the lower limit as $\epsilon\to 0$. We see that

$$\frac{d}{d\epsilon^0}\langle\sigma^0\rangle_0 \xrightarrow[\epsilon\to 0]{} -\frac{1}{32\pi^2}\ln\frac{\Lambda}{4\mu^2}\sum_{c,d=1}^{3}|\langle cd|\sigma^0(0)|0\rangle|^2 + O(\text{constant}), \qquad (2.15)$$

where Λ is an arbitrary cutoff (such as m_ρ^2 or $4m_N^2$). Changing Λ merely changes the O(constant) contributions to (2.15).

The matrix element $\langle cd|\sigma^0(0)|0\rangle$ is to be evaluated at $s=0$ and $\epsilon=0$. According to our view that threshold values of amplitudes in the chiral limit can be calculated via PCAC, we have $\langle cd|\sigma^0(0)|0\rangle = -\delta_{cd}Z_\pi^{1/2}/f_\pi$; hence,

$$\frac{1}{Z_\pi}\frac{d}{d\epsilon^0}(Z_\pi^{1/2}f_\pi) \xrightarrow[\epsilon\to 0]{} \frac{-3\ln(\Lambda/4\mu^2)}{32\pi^2 f_\pi^2} + O(\text{constant}), \qquad (2.16)$$

or

$$\frac{Z_\pi^{1/2}(\epsilon)f_\pi(\epsilon)}{Z_\pi^{1/2}(0)f_\pi(0)} = 1 - \frac{3\epsilon\ln(\Lambda/4\mu^2)}{32\pi^2 f_\pi^3}$$
$$= 1 + \frac{3\mu^2\ln(\Lambda/4\mu^2)}{32\pi^2 f_\pi^2} + O(\epsilon). \qquad (2.17)$$

The pion-decay constant on the right-hand side can be evaluated either at $\epsilon=0$ or its physical value.

We will see in Sec. II C that

$$\frac{Z_\pi^{1/2}(\epsilon)}{Z_\pi^{1/2}(0)} = 1 + \frac{\mu^2}{32\pi^2 f_\pi^2}\ln\frac{\Lambda}{4\mu^2} + O(\epsilon), \qquad (2.18)$$

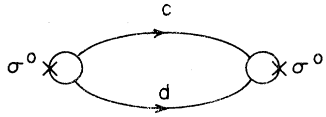

FIG. 1. Two-pion contribution to $-i\int d^4z\,\langle 0|T(\sigma^0(z)\times\sigma^0(0))|0\rangle$.

so that

$$\frac{f_\pi(\epsilon)}{f_\pi(0)} = 1 + \frac{2\mu^2}{32\pi^2 f_\pi^2}\ln\frac{\Lambda}{4\mu^2} + O(\epsilon). \qquad (2.19)$$

The physical value of $2\mu^2/32\pi^2 f_\pi^2$ is 0.012, and the logarithm is about 2 for $\Lambda=m_\rho^2$ and 3.8 for $\Lambda=4m_N^2$; hence, *the leading-order renormalization of f_π by the symmetry-breaking interaction is only 2–4%.*

As a second application, important for later work, consider the renormalized propagator (2.5). Then, since K^μ is an arbitrary four-vector independent of ϵ, we have

$$\frac{(K^2-\mu^2)}{Z_\pi^{1/2}}\frac{d}{d\epsilon^0}[(K^2-\mu^2)i\Delta_{ij}(K^2,\epsilon)]$$
$$= -\frac{1}{Z_\pi^{1/2}}\frac{d\mu^2}{d\epsilon^0}(K^2-\mu^2)i\Delta_{ij}(K^2,\epsilon) - i(K^2-\mu^2)^2\int d^4z\,d^4x\,e^{iK\cdot x}\times\langle 0|T(\phi_i(x)\phi_j(0)\sigma(z))|0\rangle - \frac{2}{Z_\pi}\frac{dZ_\pi^{1/2}}{d\epsilon^0}(K^2-\mu^2)^2 i\Delta_{ij}(K^2,\epsilon)$$
$$= i\delta_{ij}(K^2-\mu^2)\frac{1}{Z_\pi^{1/2}}\frac{d}{d\epsilon^0}[(K^2-\mu^2)\Delta_c(K^2,\epsilon)], \qquad (2.20)$$

where the second form comes from differentiating the spectral representation (2.5). Rearranging (2.20) we have

$$i^2\int d^4x\,d^4y\,e^{iK(x-y)}(K^2-\mu^2)^2\langle 0|T(\phi_i(x)\phi_j(y)\sigma(0))|0\rangle$$
$$= \frac{\delta_{ij}}{Z_\pi^{1/2}}\frac{d\mu^2}{d\epsilon^0} + \frac{2\delta_{ij}}{Z_\pi}\frac{dZ_\pi^{1/2}}{d\epsilon^0}(K^2-\mu^2) + \frac{\delta_{ij}}{Z_\pi^{3/2}}(K^2-\mu^2)^2\frac{d}{d\epsilon^0}[Z_\pi\Delta_c(K^2,\epsilon)]. \qquad (2.21)$$

Defining the off-shell amplitude $\langle i|\sigma(0)|j\rangle = \delta_{ij}F(K_i^2, K_j^2, t=(K_j-K_i)^2)$, we have the exact results

$$\frac{1}{Z_\pi^{1/2}}\frac{d\mu^2}{d\epsilon^0} = -\frac{1}{Z_\pi^{1/2}}\frac{d}{d\epsilon^0}\left(\frac{Z_\pi^{1/2}\epsilon^0}{f_\pi}\right) = F(\mu^2,\mu^2,0), \qquad (2.22)$$

and

$$\frac{2}{Z_\pi}\frac{dZ_\pi^{1/2}}{d\epsilon^0} = \frac{d}{dK^2}F(K^2,K^2,0)\Big|_{K^2=\mu^2} = 2\frac{d}{dK^2}F(K^2,\mu^2,0)\Big|_{K^2=\mu^2}. \qquad (2.23)$$

It is (2.23) that will allow us to calculate $Z_\pi^{1/2}(\epsilon)/Z_\pi^{1/2}(0)$.

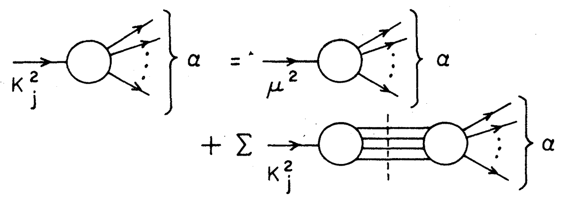

FIG. 2. The dispersion relation for $T(K_j^2, \epsilon)$.

We briefly mention two other well-known exact formulas for μ^2:

$$\mu^2 = \epsilon F(0, \mu^2, \mu^2), \tag{2.24}$$

from PCAC, and

$$2E^2\delta_{ij} = \langle i|H_0(0)|j\rangle|_{K_i{}^2 = K_j{}^2 = \mu^2;\, t=0} + \epsilon F(\mu^2, \mu^2, 0)\delta_{ij}, \tag{2.25}$$

the standard mass formula [$\langle i|H_0(0)|j\rangle$ vanishes in the chiral limit, but the physical states have symmetry breaking built into them]. Comparing (2.23) with (2.24) and (2.25), we see that

$$\frac{1}{Z_\pi}\frac{d}{d\epsilon^0}\left(\frac{Z_\pi{}^{1/2}}{f_\pi}\right)$$

is simply related to $F(\mu^2, \mu^2, 0) - F(0, \mu^2, \mu^2)$, or alternately to $\langle i|H_0(0)|j\rangle$. In fact, using the expressions (2.18) and (2.19) for $Z_\pi{}^{1/2}(\epsilon)$ and $f_\pi(\epsilon)$, we have, for pions at rest,

$$-\mu^2\delta_{ij} + \langle i|H_0(0)|j\rangle = \delta_{ij}\frac{\mu^4 \ln(\Lambda/4\mu^2)}{32\pi^2 f_\pi{}^2} + O(\epsilon^2).$$

$$\approx 0.02\mu^2\delta_{ij}, \tag{2.26}$$

where $-\mu^2\delta_{ij}$ is a kinematic factor.

C. Off-Mass-Shell Effects

A scattering amplitude or on-shell matrix element depends on ϵ in four ways: (i) through the unrenormalized Green's function, (ii) through the renormalization constants, (iii) through the Klein-Gordon operators $\Box + \mu^2$, and (iv) through the mass-shell constraints.

We can separate out the last dependence as follows: Suppose we have an *invariant* off-mass-shell amplitude or matrix element $T(s_1, \ldots, s_r; K_1{}^2, \ldots, K_m{}^2; \epsilon)$, which depends on the independent kinematic invariants $s_1, \ldots, s_r$, the mass-squared values $K_1{}^2, \ldots, K_m{}^2$ of the m external particles, and ϵ. The off-shell extrapolation can be defined via PCAC or by the field theory (we will assume these are equivalent). Then the derivative with respect to ϵ^0 of the on-mass-shell amplitude is

$$\frac{1}{Z_\pi{}^{1/2}}\frac{d}{d\epsilon^0}T(s_1, \ldots, s_r; \mu^2, \ldots, \mu^2; \epsilon)$$

$$= \left[\frac{1}{Z_\pi{}^{1/2}}\frac{d}{d\epsilon^0} + \sum_{j=1}^{m}\frac{1}{Z_\pi{}^{1/2}}\frac{d\mu^2}{d\epsilon^0}\frac{d}{dK_j{}^2}\right] \times T(s_1, \ldots, s_r; K_j{}^2, \ldots, K_m{}^2; \epsilon), \tag{2.27}$$

evaluated at $K_1{}^2 = K_2{}^2 = \cdots = K_m{}^2 = \mu^2$. In taking the derivatives, $s_1, \ldots, s_r$ are held fixed but other dependent invariants vary with the K's. For example, in π-π scattering we can fix s and t; then $u = -s - t + \sum K_j{}^2$ and the $K_j{}^2$ derivatives of T include the dependence of T on u.

We define the quantity $T_j(s_1, \ldots, s_r; K_1{}^2, \ldots, K_j{}^2, \ldots, K_m{}^2; \epsilon)$ by

$$T(K_j{}^2, \epsilon) \equiv T(\mu^2, \epsilon) + (K_j{}^2 - \mu^2)T_j(K_j{}^2, \epsilon), \tag{2.28}$$

where we have suppressed the dependence on $s_1, \cdots, s_r$ and the other K^2 values. Now clearly $T(K_j{}^2, \epsilon)$ and $T_j(K_j{}^2, \epsilon)$ are analytic functions of $K_j{}^2$ for small values of $K_j{}^2$ (perhaps up to $9\mu^2$), unless T is a matrix element involving operators that can act on external pion legs producing poles in $K_j{}^2$. In this section we assume that any such pieces have been subtracted out (see the summary at the end of this section). Then we are interested in the following question: If we expand $T(K_j{}^2, \epsilon)$ as a power series in $K_j{}^2$, how singular in ϵ are the coefficients?

To answer this question we will write separate dispersion relations for $T(K_j{}^2, \epsilon)$ in terms of the invariant masses of each channel (the $K_j{}^2$ channel, each two-particle channel, etc.), and look for divergences at the threshold of each integral as $\epsilon \to 0$. Although any one dispersion relation will by itself represent T, the different dispersion integrals will single out for special attention different relevant classes of Feynman diagrams (in field-theoretic language) or Cutkosky diagrams (in S-matrix language). The *total singularity* is therefore the *sum* of the singularities in these different integrals. Of course we must always be careful to avoid double counting any given class of diagrams.

First we will consider the $K_j{}^2$ channel (Fig. 2):

$$T(K_j{}^2, \epsilon) = T(\mu^2, \epsilon) - (K_j{}^2 - \mu^2)(2\pi)^3\int_{9\mu^2}^{\infty}\frac{dq^2}{q^2 - K_j{}^2}\left\{\sum_n \delta^4(q - p_n)\langle 0|\phi_j(0)|n\rangle\langle n|\alpha\rangle\right\} + (u\text{-channel terms}). \tag{2.29}$$

We have singled out the pion pole. By u-channel terms we mean the contribution of cuts in the dependent channel invariants (such as u in π-π scattering) which vary with $K_j{}^2$; $|\alpha\rangle$ represents the rest of the original matrix element. In an expression like $\langle n|\alpha\rangle$ we always imply that *a factor* $i(2\pi)^4\delta^4(\Sigma)$ *has been removed.*

The possible need for subtractions in (2.29) is irrelevant for our purposes. By choosing large values for the subtraction points the behavior of the subtracted integrand at threshold will be the same as (2.29). Any singularities in the subtraction constants will be due to cuts in other channels and will be picked up separately later.

The integral in (2.29) begins with the three-pion cut, but the matrix elements in the integrand themselves contain pion poles (they are not single-particle irreducible in field-theory language). A more useful expression is obtained by replacing $\langle 0|\phi_j(0)|n\rangle$ and $\langle n|\alpha\rangle$ by the corresponding irreducible matrix elements and multiplying the whole right-hand side of (2.29) by $(K_j^2-\mu^2)\Delta(K_j^2,\epsilon)$. Then we can separate out $T_j(K_j^2,\epsilon)$ [see equation (2.28)] to obtain

$$T_j(K_j^2,\epsilon)=\Delta_c(K_j^2,\epsilon)T(\mu^2,\epsilon)-\left\{[1+(K_j^2-\mu^2)\Delta_c(K_j^2,\epsilon)](2\pi)^3\int_{9\mu^2}^{\infty}\frac{dq^2}{q^2-K_j^2}\sum_n\delta^4(q-p_n)\langle 0|\phi_j(0)|n\rangle_I\langle n|\alpha\rangle_I\right\}$$
$$+(u\text{-channel terms}), \tag{2.30}$$

where I means single-pion irreducible. The first term in (2.30) is a universal correction

$$\Delta_c(K_j^2,\epsilon)T(\mu^2,\epsilon)\underset{\substack{\epsilon\to 0\\ K_j^2\ \text{small}}}{\sim}\frac{-2T(\mu^2,\epsilon)K_j^2\ln[\Lambda/(9\mu^2-K_j^2)]}{3(32\pi^2f_\pi^2)^2}+O\,(\text{constant})+O(\epsilon\ln\epsilon)\,, \tag{2.31}$$

where we have used the result[9,20]

$$\rho(s,\epsilon)\underset{\substack{\epsilon\to 0\\ s\to 0}}{\sim}\frac{8s}{3(8\pi)^4f_\pi^4}\,. \tag{2.32}$$

The s is essentially three-pion phase space in the chiral limit (n-pion phase space is proportional to s^{n-2}).

Similarly, the three-pion contribution to the continuum integral will yield terms $K_j^2\ln[\Lambda/(9\mu^2-K_j^2)]$ if and only if $\langle 3\pi|\alpha\rangle_I\to$ constant.

Putting everything together we see that the class of diagrams illustrated in Fig. 2 (cuts in the K_j^2 channel) contribute to $T(K_j^2,\epsilon)$ as

$$T(K_j^2,\epsilon)=T(\mu^2,\epsilon)\left\{1+(K_j^2-\mu^2)\left[CK_j^2\ln\frac{\Lambda}{9\mu^2-K_j^2}+D+E\epsilon\ln\epsilon+\cdots\right]\right\}. \tag{2.33}$$

These corrections are fairly harmless in practice $[T(0,\epsilon)=T(\mu^2,\epsilon)+O(\epsilon)]$ but they illustrate our contention that although $T(K_j^2,\epsilon)$ is analytic in K_j^2 for small K_j^2, the coefficients of a power series expansion in K_j^2 are very singular in ϵ; hence, the successive terms in a power series expansion of $T(K_j^2,\epsilon)$ around $K_j^2=0$, evaluated at $K_j^2=\mu^2$, are not of higher and higher order in ϵ. This may be *part* of the explanation of the paradox[21] that the $K_S\to 2\pi$ rate is much higher than SU(3)×SU(3) arguments seem to indicate (see Sec. III).

Now consider the cuts in other channels. Channels with a two-pion cut can give far more singular contributions to $T_j(K_j^2,\epsilon)$ because two-body phase space is constant when $\mu^2=0$. Of course, these singularities are, in principle, included in (2.29); they do not show up explicitly, however, but are hidden within the matrix elements of the integrand, in the u-channel terms, or in possible subtraction terms.

Suppose we divide the particles and operators in T into two sets: β, which includes j plus other operators β', and α. We call the invariant-mass squared of the β channel s_B (which may be one of $s_1\cdots s_r$ or may be dependent). Then (see Fig. 3),

$$T(K_j^2,s_B;\epsilon)=(2\pi)\int_{s_0}^{\infty}\frac{ds}{s-s_B}\sum_n\delta^4(P-P_n)(-K_j^2+\mu^2)\times\langle\beta'|\phi_j(0)|n\rangle\langle n|\alpha\rangle\,. \tag{2.34}$$

We ignore cross-channel cuts because they too will be singled out as we run over all β'. If the two-pion state is allowed in (2.34) and if $\langle 2\pi|\alpha\rangle$ goes to a constant as s and ϵ go to zero, then for small s_B Eq. (2.34) will contain terms of the form $K_j^2\ln[\Lambda/(4\mu^2-s_B)]$, the coefficient of which we can calculate. Then, if s_B is an independent variable,

$$T(K_j^2,\epsilon)=T(\mu^2,\epsilon)+(K_j^2-\mu^2)\left[C\ln\frac{\Lambda}{4\mu^2-s_B}+O(\text{constant})\right]. \tag{2.35}$$

The singular part of $T_j(K_j^2)$ is independent of K_j^2.

If s_B is a dependent variable,

$$T_j(K_j^2,\epsilon)=\ln\frac{\Lambda}{4\mu^2-s_B(K_j^2)}\,. \tag{2.36}$$

This is the most singular behavior possible; dT_j/dK_j^2 may diverge as $1/\mu^2$.

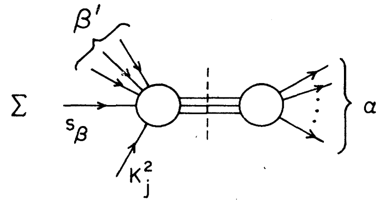

FIG. 3. The contribution of the β channel to $T(K_j^2, \epsilon)$.

If $\langle 2\pi|\alpha\rangle$ in (2.34) vanishes, there will still be logarithms in $dT(K_j^2, s_\beta, \epsilon)/ds_\beta$. That is, T may contain terms like $(K_j^2-\mu^2)s_\beta \ln[\Lambda/(4\mu^2-s_\beta)]$. These may be calculated by writing a dispersion relation for dT/ds_β.

As a very important example, suppose T is the matrix element of a scalar operator A between single-pion states:

$$T(K_i^2, K_j^2; t\equiv(K_i-K_j)^2; \epsilon)=\langle i|A(0)|j\rangle\,. \qquad (2.37)$$

Then the cuts in the K_i^2 and K_j^2 channels [Figs. 4(a), 4(b)] are harmless [(2.33)], yielding, for example,

$$\frac{d}{dK_i^2}\,T(K_i^2, \mu^2, 0; \epsilon)=\text{constant}+O(\epsilon \ln\epsilon). \qquad (2.38)$$

The two-pion contribution [Fig. 4(c)] to the t-channel integral is, however,

$$\frac{1}{32\pi^2}\sum_{cd=1}^{3}\int_{4\mu^2}^{\infty}\frac{ds}{s-t}\,(-K^2+\mu^2)\langle i|\phi_j(0)|cd\rangle\langle cd|A(0)|0\rangle, \qquad (2.39)$$

where we have already set the phase space equal to a constant and done the angular integral (A is a scalar). But at threshold in the chiral limit, the symmetrized amplitude for $ij\to cd$ is[18]

$$(-K_j^2+\mu^2)\langle i|\phi_j(0)|cd\rangle = \frac{1}{f_\pi^2}\left[s\delta_{ij}\,\delta_{cd}+\tfrac{1}{2}(\delta_{ic}\,\delta_{jd}+\delta_{id}\,\delta_{jc})(K_i^2+K_j^2-s)\right], \qquad (2.40)$$

from which we easily compute

$$\frac{d}{dK_i^2}\,T(K_i^2, \mu^2, t; \epsilon)=\frac{d}{dK_j^2}\,T(\mu^2, K_j^2, t; \epsilon) = \frac{1}{32\pi^2 f_\pi^2}\ln\frac{\Lambda}{4\mu^2-t}\,T(0,0,0;0) + O(\text{constant}). \qquad (2.41)$$

Hence, at $t=0$, the errors in the extrapolation of these matrix elements to $K^2=0$ are of order $\mu^2\ln\mu^2$, instead of just μ^2, and can be computed exactly.

We saw in (2.23) that with $\langle i|\sigma(0)|j\rangle \equiv \delta_{ij}F(K_i^2, K_j^2, t; \epsilon)$, there is an exact result

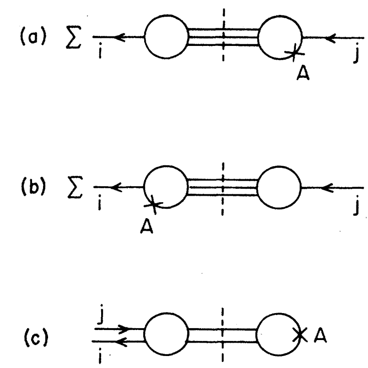

FIG. 4. Cuts in $\langle i|A(0)|j\rangle$: (a) cuts in K_i^2, (b) cuts in K_j^2, (c) cuts in t.

$$\frac{1}{Z_\pi}\frac{dZ_\pi^{1/2}}{d\epsilon^0}=\frac{d}{dK_i^2}F(K_i^2, \mu^2, 0; \epsilon)\bigg|_{K_i=\mu^2}\,.$$

But by PCAC, $F(0,0,0;0)=-1/f_\pi$, so from Eq. (2.41):

$$\frac{1}{Z_\pi}\frac{dZ_\pi^{1/2}}{d\epsilon^0}=\frac{-1}{32\pi^2 f_\pi^3}\ln\frac{\Lambda}{4\mu^2}+O(\text{constant}), \qquad (2.42)$$

which is the result we quoted in (2.18).

In summary:

(1) If the invariant matrix element T contains pion poles due to an operator acting on an external-pion leg, separate out these poles via a dispersion relation. The K_j^2 dependence of the pole residue is then studied in the same way as any other matrix element. If the invariant mass of the pole channel is a dependent variable, the K_j^2 dependence of the pole denominator is explicit.

(2) The derivative dT/dK_j^2 evaluated on the mass shell is $T_j(\mu^2;\epsilon)$. If T contains any channel $\beta\to\alpha$ (where β includes j) which has small or zero invariant mass s_β and which supports a two-pion cut, then T_j will have singularities of the form $\ln[\Lambda/(4\mu^2-s_\beta)]$ if and only if $T(\alpha\to 2\pi)$ and $dT(\beta\to 2\pi)/dK_j^2$ remain finite at threshold. These are given by (2.34) and (2.41) and lead to errors of order $\mu^2\ln\mu^2$ in off-shell extrapolations.

(3) If $T(\alpha\to 2\pi)$ vanishes, T_j may still have singularities $s_\beta\ln[\Lambda/(4\mu^2-s_\beta)]$. Strictly speaking, these terms are not singular at $\epsilon=0$ for fixed s_β. Three-pion cuts in the $\beta\to\alpha$ channel could also yield singularities of this form.

D. Perturbation of Amplitudes and Matrix Elements

We now have available all the machinery needed to study the dependence of both on- and off-shell matrix elements and scattering amplitudes on ϵ.

Suppose we have an invariant matrix element $T(s_1, \ldots, s_r; K_1^2, \ldots, K_m^2; \epsilon)$ defined in terms of the renormalized Green's function

$$G(x_1\cdots x_m;\, y_1\cdots y_K;\,\epsilon) = Z_\pi(\epsilon)^{-m/2}\langle 0|T(\phi_1^0\cdots\phi_m^0 A_1\cdots A_K)|0\rangle$$

(the A's are local operators) by

$$T(s_1\cdots s_r;\, K_1^2\cdots K_m^2;\,\epsilon)=\prod_{j=1}^{m}\left[i\int d^4x_j e^{-iK_jx_j}\times(\Box_j+\mu^2)\right]G, \tag{2.43}$$

where we have taken all momenta as incoming. We imply a Fourier transform of $y_1\cdots y_K$ in (2.43) so that the A operators carry momenta $r_1\cdots r_K$, which do not satisfy any mass-shell constraint; in addition, a factor $i(2\pi)^4\delta^4(\sum K_j+\sum r_l)$ is removed. We suppress these factors for notational simplicity. We illustrate T in Fig. 5.

The first case to consider is that in which the K_i^2 are fixed and off the mass shell. Then, using (2.11) we can differentiate T to obtain

$$\frac{1}{Z_\pi^{1/2}}\frac{dT}{d\epsilon^0}=-i\int d^4z\, T_\sigma + \sum_{j=1}^{m}\left(\frac{1}{Z_\pi^{1/2}}\frac{d\mu^2}{d\epsilon^0}\frac{1}{-K_j^2+\mu^2} - \frac{1}{Z_\pi}\frac{dZ_\pi^{1/2}}{d\epsilon^0}\right)T\,, \tag{2.44}$$

where $T_\sigma(z)$ is just (2.43), only with G replaced by $G_\sigma(z)\equiv\langle 0|T(\phi_1\cdots\phi_m A_1\cdots A_K\sigma(z))|0\rangle_c$. The pole terms in (2.44) come from differentiating the Klein-Gordon operators. If the A operator

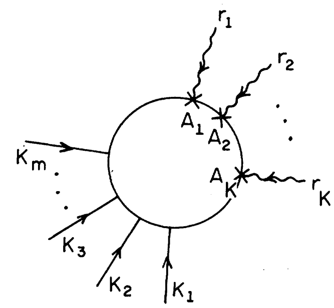

FIG. 5. The matrix element $T(s_1,\ldots,s_r;K_1^2,\ldots,K_m^2;\epsilon)$.

must be renormalized, derivatives of their Z's should be added to (2.44). Hence, the differentiation of T involves the insertion of zero-energy σ on every line and vertex in T, plus the addition of some additional counterterms associated with the external lines. When all the K_j^2 are zero, the counterterms are (using $\mu^2=-\epsilon_0 Z_\pi^{1/2}/f_\pi$)

$$\frac{-m}{f_\pi}\left(\frac{1}{\mu^2}+\frac{1}{Z_\pi^{1/2}}\frac{df_\pi}{d\epsilon^0}\right)T(s_1,\ldots,s_r;0,\ldots,0;\epsilon). \tag{2.45}$$

This has the effect of canceling the insertions on external lines and adding a term $-(1/f_\pi)df_\pi/d\epsilon^0$ for each external leg (these terms are expected because f_π defines how the off-shell extrapolation was done). We will give a simple application of (2.45) in Appendix A.

Let us now examine the on-mass-shell case. All we have to do is apply Eq. (2.27), which means add derivatives of T with respect to K_j^2 to (2.44) and evaluate at $K_j^2=\mu^2$:

$$\frac{1}{Z_\pi^{1/2}}\frac{d}{d\epsilon^0}T(s_1,\ldots,s_r;\mu^2,\ldots,\mu^2;\epsilon)=\left\{-i\int d^4z\,T_\sigma(s_1,\ldots,s_r;K_1^2,\ldots,K_m^2;\epsilon;z) + \sum_{j=1}^{m}\left[-\frac{1}{Z_\pi}\frac{dZ_\pi^{1/2}}{d\epsilon^0}+\frac{1}{Z_\pi^{1/2}}\frac{d\mu^2}{d\epsilon^0}\left(\frac{1}{-K_j+\mu^2}+\frac{d}{dK_j^2}\right)\right]\times T(s_1,\ldots,s_r;K_1^2,\ldots,K_j^2;\epsilon)\right\}_{K_1^2=\cdots=K_m^2=\mu^2}. \tag{2.46}$$

Now, $T(s_1,\ldots,s_r;\mu^2,\ldots,K_j^2,\ldots,\mu^2;\epsilon)=T(\mu^2,\epsilon)+(K_j^2-\mu^2)T_j(K_j^2,\epsilon)$ in the notation of Sec. III C. Hence,

$$\left(\frac{1}{-K_j^2+\mu^2}+\frac{d}{dK_j^2}\right)T(K_j^2,\epsilon)\xrightarrow[K_j^2\to\mu^2]{}\frac{T(\mu^2,\epsilon)}{-K_j^2+\mu^2}+O(K_j^2-\mu^2)\,. \tag{2.47}$$

The pole will be canceled by a pole coming from a σ insertion on the external leg.

In order to study the σ-insertion term, consider the more general quantity (Fig. 6)

$$-iT_\sigma(q)\equiv i\int d^4z\,e^{-iqz}T_\sigma(z)\,, \tag{2.48}$$

which is an invariant function of $s_1,\ldots,s_r$, $K_1^2,\ldots,K_m^2$, q^2, $(q+K_i)^2$, $(q+K_i+K_j)^2$, $\ldots$, and ϵ. Of course, the kinematic variables are not all independent. In complete analogy with our study of the singularities of $T_j(K_j^2,\epsilon)$ in Sec. II C,

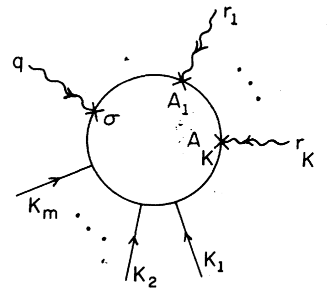

FIG. 6. The matrix element $T_\sigma(q)$.

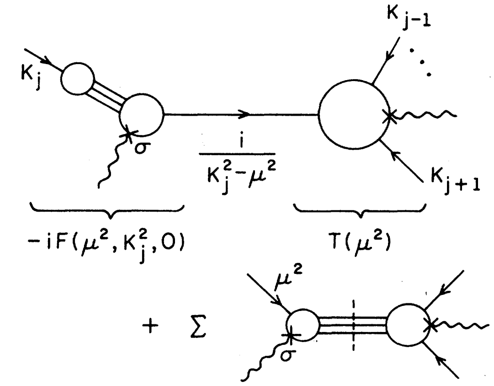

FIG. 7. A representation of $-iT_\sigma$ as a dispersion relation in the $(q+K_j)^2$ channel.

we will write dispersion relations for $-iT_\sigma(0)$ in each channel $[q^2,\ (q+K_i)^2$, etc.], putting the σ and A fields on equal footing with the external particles for the purpose of defining a channel. Although each dispersion integral will by itself represent $-iT_\sigma(0)$, the divergences coming from the lower limit of each integral as $\epsilon\to 0$ will be due to the independent classes of Feynman or Cutkosky diagrams associated with cuts in that channel. The various singularities must therefore be added to yield the singular behavior of $-iT_\sigma$ as $\epsilon\to 0$. We must, of course, be careful to not double count any type of diagram. As in Sec. III C, the possible existence of subtractions is totally irrelevant for our purposes.

We will first consider the $(q+K_j)^2$ channels (Fig. 7). The representation is

$$-iT_\sigma(0)=\frac{1}{K_j^{\,2}-\mu^2}F(\mu^2,K_j^{\,2},0)T(\mu^2)$$
$$-(2\pi)^3\int_{9\mu^2}^{\infty}\frac{ds}{s-\mu^2}$$
$$\times\sum_n \delta^4(P-P_n)\langle\alpha|n\rangle\langle n|\sigma(0)|j\rangle\,, \quad (2.49)$$

where $\delta_{ij}F(K_i^{\,2},K_j^{\,2},(K_i-K_j)^2)\equiv\langle i|\sigma(0)|j\rangle$. Notice that the residue of the pion pole is evaluated at $(q+K_j)^2=\mu^2$, while $K_j^{\,2}$ is left at its physical value. Now, we saw in (2.23) and (2.24) that

$$F(\mu^2,K_j^{\,2},0)=\frac{1}{Z_\pi^{\,1/2}}\frac{d\mu^2}{d\epsilon^0}+(K_j^{\,2}-\mu^2)\frac{1}{Z_\pi}\frac{dZ_\pi^{\,1/2}}{d\epsilon^0}$$
$$+O((K_j^{\,2}-\mu^2)^2)\,;$$

this means that the part of the pion-pole term in the $(q+K_j)^2$ channel that survives as $K_j^{\,2}\to\mu^2$ will exactly cancel the counterterms associated with j in (2.46) and (2.47). Let us explicitly remove these pole pieces from *each* $(q+K_j)^2$ channel in $-iT_\sigma(0)$ and call the remainder $-iT_\sigma^S(0)$ [the superscript S means S matrix; i.e., we have defined the poles in $(q+K_j)^2$ via a dispersion relation]. Then

$$-iT_\sigma\equiv -iT_\sigma^S+\sum_{j=1}^{m}\frac{1}{K_j^{\,2}-\mu^2}F(\mu^2,K_j^{\,2},0)T(\mu^2), \quad (2.50)$$

so that

$$\frac{1}{Z_\pi^{\,1/2}}\frac{d}{d\epsilon^0}T(s_1,\ldots,s_r;\mu^2,\ldots,\mu^2;\epsilon)$$
$$=-iT_\sigma^S(s_1,\ldots,s_r;\mu^2,\ldots,\mu^2;\epsilon;q=0)\,, \quad (2.51)$$

which means one must insert a zero-energy σ everywhere in the diagram, but omit those contributions shown in Fig. 7.

This is the most elegant formula possible for $dT/d\epsilon^0$, but it is not the most useful. This is because $-iT_\sigma^S(0)$ is not single-particle irreducible (in the field-theoretic sense) in the $(q+K_j)^2$ channels. The general class of diagrams containing pion poles is shown in Fig. 8. Therefore, let us define $-iT_\sigma^I(0)$ (I means irreducible) as $-iT_\sigma(0)$ minus these diagrams:

$$-iT_\sigma(0)=-iT_\sigma^I(0)-i\sum_{j=1}^{m}[F(K_j^{\,2},K_j^{\,2},0)\,i\Delta_{jj}(K_j^{\,2})\,T^I(s_1,\ldots,s_r;\mu^2,\ldots,K_j^{\,2},\ldots,\mu^2:\epsilon)]$$
$$=-iT_\sigma^I(0)+\sum_{j=1}^{m}\left\{\left[\frac{1}{K_j^{\,2}-\mu^2}\frac{1}{Z_\pi^{\,1/2}}\frac{d\mu^2}{d\epsilon^0}+\frac{2}{Z_\pi}\frac{dZ_\pi^{\,1/2}}{d\epsilon^0}\right]T(\mu^2)+\frac{1}{Z_\pi^{\,1/2}}\frac{d\mu^2}{d\epsilon^0}T_j(\mu^2)\right\}+O(K_j^{\,2}-\mu^2)\,. \quad (2.52)$$

We see that T_σ^I and T_σ^S differ only by terms which are finite at $K_j^{\,2}=\mu^2$. However, some of these extra terms are singular in ϵ; the singularities are related to cuts in the $(q+K_j)^2$ channels and would have been missed had we worked with $-iT_\sigma^S(0)$. In terms of $-iT_\sigma^I(0)$ we can write

$$\frac{1}{Z_\pi^{1/2}}\frac{d}{d\epsilon^0}T(s_1,\ldots,s_r;\mu^2,\ldots,\mu^2;\epsilon)=-iT_\sigma^I(s_1,\ldots,s_r;\mu^2,\ldots,\mu^2;\epsilon;q=0)$$
$$+mT(s_1,\ldots,s_r;\mu^2,\ldots,\mu^2;\epsilon)\frac{1}{Z_\pi}\frac{dZ_\pi^{1/2}}{d\epsilon^0}$$
$$+\frac{1}{Z_\pi^{1/2}}\frac{d\mu^2}{d\epsilon^0}\sum_{j=1}^{m}T_j(s_1,\ldots,s_r;\mu^2,\ldots,\mu^2;\epsilon)\ . \qquad (2.53)$$

The T_σ^I term represents the dependence of the internal dynamics on ϵ, while $\sum T_j$ is due to the external mass-shell constraints; the $dZ_\pi^{1/2}/d\epsilon^0$ counterterm can be thought of as due to the renormalization of the strong coupling through which the external pion hooks itself onto the diagram.

We can now continue our channel-by-channel search for $\ln\epsilon$ singularities in T_σ^I. A constant nuisance is the possibility of the A operators acting on external-pion legs of amplitudes (both the T_σ^I amplitude itself and the amplitudes that appear within the dispersion integrals). We will treat these diagrams separately. Therefore, it is always implied that these pole diagrams have been removed from all amplitudes (via dispersion relations).

The continuum cut in the $(q+K_j)^2$ channels of T_σ^I is harmless because of the vanishing of three-pion phase space.

Now turn to the q^2-channel representation for $-iT_\sigma^I(0)$, which is illustrated in Fig. 9. As should be clear by now, only the two-pion contribution to this cut can produce a $\ln\epsilon$ behavior. It is

$$-T_\sigma^I(0)=-\frac{1}{32\pi^2}\sum_{cd=1}^{3}\int_{4\mu^2}^{\infty}\frac{ds}{s}\left(\frac{s-4\mu^2}{s}\right)^{1/2}\times\langle\alpha|cd\rangle\langle cd|\sigma(0)|0\rangle$$
$$\underset{\epsilon\to 0}{\longrightarrow}\frac{1}{32\pi^2 f_\pi}\ln\frac{\Lambda}{4\mu^2}\sum_{cd=1}^{3}\delta_{cd}\langle\alpha|cd\rangle\,, \qquad (2.54)$$

where $\langle\alpha|cd\rangle$ is evaluated at $s=0$ and $\epsilon=0$. Of course, Cutkosky diagrams such as shown in Fig. 10 are omitted because we are dealing with $-iT_\sigma^I$. *There is no reason for this matrix element to vanish in general, so in most cases we expect*

$$T(s_1,\ldots,s_r,\mu^2,\ldots,\mu^2;\epsilon)=T(s_1,\ldots,s_r;0,\ldots,0;0)+F(s_1,\ldots,s_r)\epsilon\ln\epsilon+O(\epsilon)\,, \qquad (2.55)$$

the $\epsilon\ln\epsilon$ terms coming both from the counterterms and from the q^2 channel (and perhaps from other channels). In certain cases in which T contains a zero-energy operator, (2.54) will have to be modified. This is discussed before(2.59).

Now consider other channels. Let β' and σ define a channel of invariant mass squared s_B, as shown in Fig. 11. The representation is

$$-iT_\sigma^I(0)=-(2\pi)^3\int_{s_0}^{\infty}\frac{ds}{s-s_B}\times\sum_n\delta^4(P-P_n)\langle\beta'|\sigma(0)|n\rangle\langle n|\alpha\rangle. \qquad (2.56)$$

If the channel has odd G-parity, the integral (2.56) will contain both a pion pole and a continuum. The continuum cut, though not single-particle irreducible, is easily seen to remain finite as $\epsilon\to 0$. The cut can contribute $\ln\epsilon$ singularities to the slope dT_σ^I/ds_B, however.

The pion-pole contribution to (2.56) must be handled carefully to avoid double counting. It turns out that *the sum of the pion-pole pieces in both the* (β',σ) *and* (α,σ) *channels of* $-iT_\sigma^I$ (shown in Fig. 12) *plus the corresponding pion poles in the counterterms of Eq. (2.53)* (here, the pion poles are defined in the dispersion-relation sense) yield the contribution

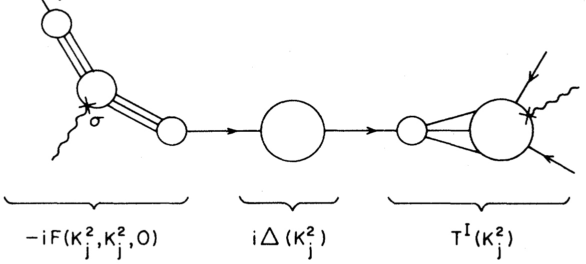

FIG. 8. The pion poles in the $(q+K_j)^2$ channel of $-iT_\sigma(0)$.

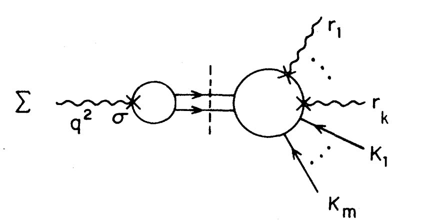

FIG. 9. The q^2-channel dispersion relation for $-iT_\sigma$.

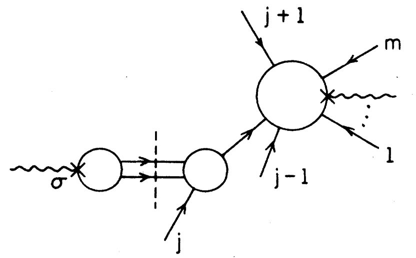

FIG. 10. Cutkosky diagrams omitted from $\langle \alpha | cd \rangle$ in Eq. (2.54).

$$\frac{1}{Z_\pi^{1/2}} \frac{d}{d\epsilon^0} \left(-\langle \beta' | \pi \rangle \frac{1}{s_\alpha - \mu^2} \langle \pi | \alpha \rangle \right) , \qquad (2.57)$$

to $(1/Z_\pi^{1/2}) dT/d\epsilon^0$. The T-matrix elements in (2.57) are on the mass shell and are differentiated just like any other amplitudes. These pion poles might, for example, appear in three-pion channels or be due to one of the A operators acting on an external-pion leg.

Now, consider even-G-parity channels. The two-pion cut will yield a logarithm if [see (2.56)] (i) both $\langle \beta' | \sigma(0) | 2\pi \rangle$ and $\langle 2\pi | \alpha \rangle$ remain finite as s and ϵ go to zero, and (ii) s_β is zero or at least small. If (i) and (ii) are satisfied, $-iT_\sigma^I$ contains

$$-\frac{1}{32\pi^2} \ln \frac{\Lambda}{4\mu^2 - s_\beta} \sum_{cd=1}^{3} \langle \beta' | \sigma(0) | cd \rangle_I \langle cd | \alpha \rangle , \qquad (2.58)$$

evaluated at $s=0$, $\epsilon=0$. Only for $s_\beta = 0$ is this truly a $\ln\epsilon$ singularity. We have put an I on $\langle \beta' | \sigma(0) | cd \rangle$ to indicate that the σ does not act on external legs: Insertions on the legs in β' are not included in $-iT_\sigma^I$ and singularities corresponding to insertions on c or d (Fig. 13) have already been included in the q^2-channel dispersion relation.

A number of comments must be made at this point: (i) the special case in which α is just one of the A operators is included in (2.58). (ii) The matrix element $\langle \beta' | \sigma(0) | cd \rangle_I$ is just $+iT_\sigma^I(\beta' \to cd)$, or minus the derivative of the $\beta' \to cd$ amplitude (omitting the counterterms). This derivative is likely to remain finite at threshold even if the $\beta' \to cd$ amplitude itself does not. If fact, $\langle \beta' | \sigma(0) | cd \rangle_I$ may itself be logarithmically divergent in ϵ, implying $\ln^2\epsilon$ terms

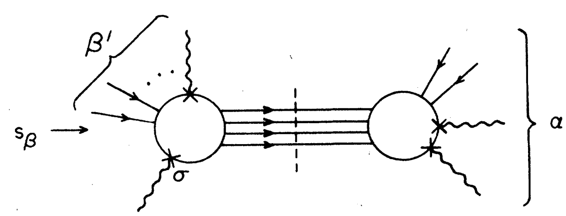

FIG. 11. The $(\beta', \sigma) \to (\alpha)$ channel.

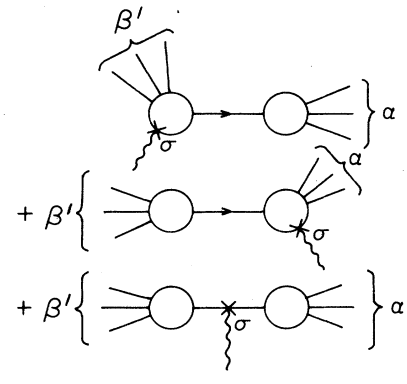

FIG. 12. Pion poles in $-iT_\sigma(0)$.

in $-iT_\sigma^I$. These would ultimately be due to diagrams such as Fig. 14. We will mainly be interested in the case that β' is itself a two-pion state, for which $\langle 2\pi | \sigma(0) | 2\pi \rangle_I$ goes to a known constant [the $\pi\pi$ amplitude is $O(\epsilon) + O(\epsilon^2 \ln\epsilon)$ at the relevant point $s=t=u=0$]. In more complicated processes for which β' possesses subchannels that are not at threshold, the $\ln^2\epsilon$ terms will in general be present and can be calculated by our methods. (iii) If the amplitude $\langle 2\pi | \alpha \rangle$ in (2.58) vanishes at threshold, there may still be singularities in $-idT_\sigma^I/ds_\beta$ (or in higher derivatives), as can be seen by writing a dispersion relation for this quantity. These give contributions $s_\beta \ln[\Lambda/(4\mu^2 - s_\beta)]$ to $-iT_\sigma^I$. These are not truly $\ln\mu^2$ singularities, and similar terms can come about from 3-pion cuts. They are still sometimes of interest, however, as we shall see in Sec. II E. (iv) If one or both of the amplitudes $\langle cd | \alpha \rangle$ or $\langle \beta' | \sigma(0) | cd \rangle_I$ contain pion poles (in a crossed channel), such as in Fig. 15, then (2.58) may have to be modified. This is because we tacitly assumed that only the S-wave part of the amplitudes were important [in the case of the q^2 channel (2.54) this is exactly true], and because the amplitudes may diverge at threshold. There will only be a problem if: (a) the pole approaches the physical region of $\langle cd | \alpha \rangle$ as s and ϵ go to zero; this only occurs if M_1 and/or M_2 are zero (M_1 and M_2 are the invariant masses defined in Fig. 15); and (b) one or both of the vertex amplitudes $\langle ce | \alpha_1 \rangle$ and $\langle de | \alpha_2 \rangle$ (Fig. 15) remains finite. *These conditions will never be fulfilled if α_1 and α_2 contain only pions.* At least

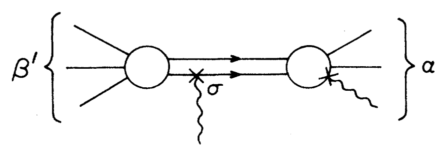

FIG. 13. Diagrams omitted from the evaluation of the (β', σ)-channel singularity.

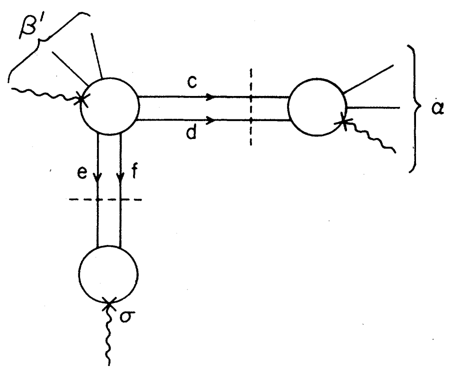

FIG. 14. Typical diagrams in the (β', σ) channel leading to $\ln^2 \epsilon$ singularities.

one of them must involve an A operator. If (a) and (b) are not both satisfied, then (2.58) is still valid. If they are satisfied, then (2.58) can still be applied for the nonpole part of $\langle cd|\alpha\rangle$.

For any given application one must calculate all of these triangle graphs (which modify the q^2-channel cuts, also) by hand, either using dispersion relations or as Feynman diagrams. In fact, if several A operators are present, there may be box graphs (Fig. 16) or even worse (we calculated such a graph in Ref. 9). The graph in Fig. 16 diverges as $(\mu^2)^{-2}$ in the chiral limit if at each corner there is a zero-energy scalar operator whose matrix element remains finite as $\epsilon \to 0$. If one of the corners involves a two-pion state (with zero invariant mass), the singularity would be $(\mu^2)^{-1}$ because the four-pion vertex vanishes in the limit (Weinberg's expansion).

We see that *the matrix elements of two or more operators carrying zero energy can have horrible singularities in the chiral limit.* In fact, if the matrix element itself diverges, one can work with it directly rather than with its derivatives. *These singularities never show up for scattering amplitudes, only for matrix elements of operators.* Of course, if these operators carry nonzero momentum, these singularities will not be present. This is why scattering amplitudes in the Σ model (in which the σ is treated as a particle of nonzero mass) remain finite as $\epsilon \to 0$.

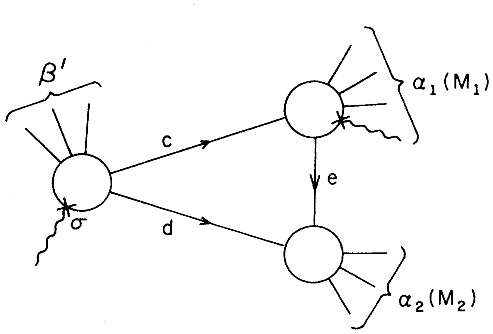

FIG. 15. A typical triangle graph that could invalidate (2.58).

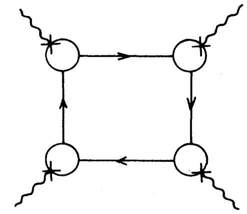

FIG. 16. A box graph that diverges as $(\mu^2)^{-2}$.

The most complicated matrix element we will be interested in is the matrix element $T_{ij}(K_i^2, K_j^2, (K_i-K_j)^2; \epsilon)$ of a scalar operator A between single-pion states i and j. In this case the triangle diagram in Fig. 17 will contribute a $\ln\epsilon$ singularity to $dT/d\epsilon^0$ if $(K_i-K_j)^2=0$. The result is

$$\frac{1}{Z_\pi^{1/2}} \frac{d}{d\epsilon^0} T_{ij}(\mu^2, \mu^2; 0; \epsilon) = -\frac{1}{8\pi^2 f_\pi^{\,3}} \ln\frac{\Lambda}{4\mu^2}\, T_{ij}(0, 0, 0; 0) + O(\text{constant})\,. \qquad (2.59)$$

Of course we must add singularities due to cuts in other channels to (2.59).

We summarize our unavoidably complicated discussion as follows: In order to calculate the singular part of $Z_\pi^{1/2}\; dT/d\epsilon^0$:

(i) Separate out any pion poles from T, including those due to operators acting on external-pion legs, via dispersion relations. Then differentiate the residue, which is itself a product of matrix elements, and look for singularities (renormalization of the "vertices"). The pole denominator should also be differentiated, but there the ϵ dependence is explicit [see (2.57)].

For the remainder of the amplitude:

(ii) Add a term $(1/Z_\pi)(dZ_\pi^{1/2}/d\epsilon^0)T = -T(32\pi^2 f_\pi^{\,3})^{-1}\ln(\Lambda/4\mu^2)$ for each external line (renormalization of external pion "coupling constants").

(iii) Add a term $(1/Z_\pi^{1/2})(d\mu^2/d\epsilon^0)dT/dK_j^2$, evaluated at $K_j^2=\mu^2$, for each external line (variation of the mass-shell constraints). The

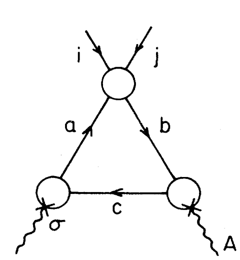

FIG. 17. The triangle graph relevant to $\langle i|A(0)|j\rangle$.

rules for the singularities in dT/dK_j^2 are summarized at the end of Sec. III C.

(iv) Add the logarithm due to a general internal pion loop [Fig. 9 and Eq. (2.54)]. If T contains any zero-energy operators, omit the triangle graphs (such as in Fig. 17) from the matrix element in (2.54).

(v) For every channel $\beta' \to \alpha$ in T which has zero invariant mass and allows a two-pion cut, add a term

$$+\frac{1}{32\pi^2}\ln\frac{\Lambda}{4\mu^2}\sum_{cd=1}^{3}\left[\left(\frac{1}{Z_\pi^{1/2}}\frac{d}{d\epsilon^0}\langle\beta'|cd\rangle\right)_I\langle cd|\alpha\rangle + \langle\beta'|cd\rangle\left(\frac{1}{Z_\pi^{1/2}}\frac{d}{d\epsilon^0}\langle cd|\alpha\rangle\right)_I\right]. \quad (2.60)$$

The amplitudes in (2.59) are on the mass shell and evaluated at $(P_c+P_d)^2=0$. By $(d\langle\,|\rangle/d\epsilon^0)_I$ we mean just keep the $-iT_\sigma^I$ insertions and ignore counterterms. Equation (2.60) represents the dependence on ϵ^0 of the amplitudes at the ends of the two-pion cut. In most *simple* cases derivatives in (2.60) will be finite, but in complicated amplitudes they may themselves diverge as $\ln(\Lambda/4\mu^2)$. If T contains zero-energy operators, omit triangle and box graphs from (2.60).

(vi) If the matrix elements $\langle cd|\alpha\rangle$ or $\langle\beta'|cd\rangle$ in (2.60) vanish at threshold, there may still be logarithms in dT_σ^I/ds_α. There may also be logarithms in dT_σ^I/ds_α due to the three-pion cut for channels with odd G parity. These may be calculated by writing dispersion relations in s_α for these quantities.

(viii) If T involves zero-energy operators, calculate all triangle and box diagrams, such as shown in Figs. 15, 16, and 17, by hand [the most common class of these singularities is calculated in (2.59)]. If the singularity is worse than logarithmic, the matrix element itself diverges as $\epsilon \to 0$. In this case, one should calculate the singular diagrams for T, rather than working with $dT/d\epsilon^0$.

We close this section with two comments:

(i) In the case that T is a scattering amplitude, we have seen no sign of $1/\mu^2$ or $1/\mu$ terms in $dT/d\epsilon^0$, which would be associated with $\ln\epsilon$ or $\epsilon^{1/2}$ behavior of T. That is, we see no evidence for T to diverge in the chiral limit. The leading corrections are $O(\epsilon\ln\epsilon)$. (ii) T generally contains terms like $\epsilon\ln\epsilon$, but only in very complicated amplitudes can there be terms $\epsilon\ln^2\epsilon$. Hence, in the simple cases the logarithms can not add up to yield a simple power of ϵ. Higher-order effects could conceivably add up to form a behavior like $\exp(A\epsilon\ln\epsilon)$, however.

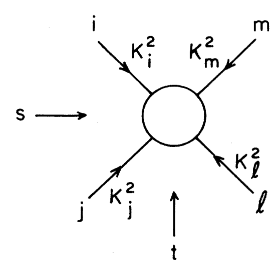

FIG. 18. The π-π scattering amplitude.

E. The $\pi\pi$ Scattering Amplitude

We will now illustrate the general considerations of Sec. II D with the specific example of $\pi\pi$ scattering. The amplitude, shown in Fig. 18, is

$$T(s,t;K_i^2,K_j^2,K_l^2,K_m^2;\epsilon)=(-K_i^2+\mu^2)\langle lm|\phi_i(0)|j\rangle, \quad (2.61)$$

and it is a function of $s=(K_i+K_j)^2$, $t=(K_i+K_m)^2$, and the K^2's. The variable $u=-s-t+\sum K_i^2$ is dependent. For small values of the invariants, we expect the Weinberg formula,[18]

$$T=\frac{1}{f_\pi^2}[\delta_{ij}\delta_{lm}(s-\mu^2)+\delta_{im}\delta_{jl}(t-\mu^2)+\delta_{il}\delta_{jm}(u-\mu^2)], \quad (2.62)$$

to be approximately correct. We show in Appendix A that at the off-shell point $s=t=u=K^2=0$ the corrections are of order $\epsilon^2\ln\epsilon$ and not $\epsilon\ln\epsilon$ (which would invalidate the expansion).

The derivative of the on-shell amplitude at fixed s and t can be calculated from our rules in Sec. II D. The amplitude has no pion poles; we will pick up the counterterms associated with rules (ii) and (iii) at the end.

According to (2.54) the logarithm associated with an internal pion loop is

$$\frac{1}{32\pi^2 f_\pi}\ln\frac{\Lambda}{4\mu^2}\sum_{cd=1}^{3}\delta_{cd}\langle ijlm|J_c(0)|d\rangle, \quad (2.63)$$

where J_c is the pion source and the amplitude is evaluated at $(K_c+K_d)^2=0$ and $\epsilon=0$. Other invariants are calculated with K_c and $K_d=0$. The six-pion amplitude includes the diagrams shown in Fig. 19(a), but not those in 19(b).

It is worth mentioning here that in the chiral limit amplitudes involving a single zero-energy pion vanish (the Adler condition), but for two zero-energy pions neither the full amplitudes nor the partial amplitudes considered here need vanish. Hence, the $\ln(\Lambda/4\mu^2)$ term will be present for arbitrary s and t.

When s and t are small we can calculate the amplitudes in Fig. 19(a) from a Weinberg-type expansion or effective Lagrangian,[22] yielding

$$\frac{4}{32\pi^2 f_\pi^{\,5}}\ln\frac{\Lambda}{4\mu^2}(\delta_{ij}\delta_{lm}s+\delta_{im}\delta_{jl}t+\delta_{il}\delta_{jm}u). \quad (2.64)$$

(The six-pion irreducible and the pole diagrams give contributions −6 and +10, respectively.)

There are no logarithms associated with the cuts in the s, t, or u channels when these invariants are zero, because the amplitudes such as $\langle cd|ij\rangle$ that multiply the log [Eq. (2.60)] vanish at threshold. There will be logs in, for example,

$$\frac{d}{ds}\frac{1}{Z_\pi^{\,1/2}}\frac{dT}{d\epsilon^0}\bigg|_{s=t=0}\,,$$

however. To calculate these, simply write dispersion relations for $-idT_\sigma^I(0)/ds$ in the (i,j,σ), (l,m,σ), (i,l,σ), and (j,m,σ) channels (with $t=0$ for simplicity). The singular part of the first integral, for example, is

$$-\frac{1}{32\pi^2}\int_{4\mu^2}^{\infty}\frac{ds'}{s'^2}\sum_{cd=1}^{3}\langle ij\,|\sigma(0)|cd\rangle_I\langle cd|J_I(0)|m\rangle\,. \quad (2.65)$$

The matrix element $\langle ij|\sigma(0)|cd\rangle_I$ is just

$$-\frac{1}{Z_\pi^{\,1/2}}\left(\frac{d}{d\epsilon^0}T_{ij\to cd}\right)_I$$
$$=-\frac{1}{f_\pi^{\,3}}(\delta_{ij}\delta_{cd}+\delta_{ic}\delta_{jd}+\delta_{id}\delta_{jc})+O(\epsilon\ln\epsilon), \quad (2.66)$$

the derivative of the Weinberg amplitude at $s'=0$, omitting renormalization terms and dT/dK^2 terms. The $\langle cd|J_I(0)|m\rangle$ amplitude, taken from (2.62), is proportional to s' when properly symmetrized in c and d. Thus, (2.65) diverges as $\ln(\Lambda/4\mu^2)$ at the threshold. The sum of the logarithms from the four cuts is

$$\frac{10}{32\pi^2 f_\pi^{\,5}}\ln\frac{\Lambda}{4\mu^2}(\delta_{ij}\delta_{lm}-\delta_{il}\delta_{jm})\,, \quad (2.67)$$

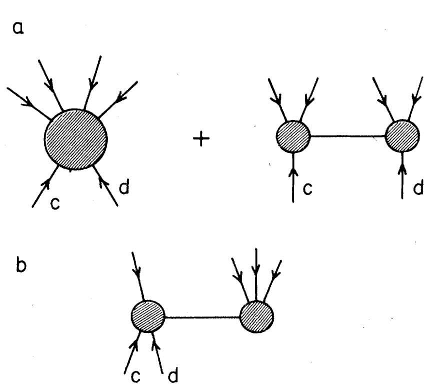

FIG. 19. (a) Diagrams included in the six-point function of (2.63). The shaded blobs are single-pion irreducible. (b) Diagrams excluded from (2.63).

from which we infer that the cut contribution to $-iT_\sigma^I(0)$ correct to first order in s and t is just $\frac{10}{4}$ of (2.64).

The renormalization contribution, for small s and t, is just minus the expression (2.64).

Finally, we must add

$$\frac{1}{Z_\pi^{\,1/2}}\frac{d\mu^2}{d\epsilon^0}\sum_{j=1}^{4}T_j(s,t;\mu^2,\ldots,\mu^2;\epsilon)\,. \quad (2.68)$$

Again, there are no logarithms in T_j at s or $t=0$ from the s, t, or u channel cuts, due to the softness of the $\pi\pi$ amplitude at threshold. They do show up in dT_j/ds, however, and in complete analogy to (2.65) they can be calculated by writing a fixed-t dispersion relation for dT/ds. The answer turns out to be $-\frac{3}{2}$ of (2.64).

Adding together the various contributions, we find that

$$T(s,t;\mu^2,\mu^2,\mu^2,\mu^2;\epsilon)$$
$$\xrightarrow[s,t\text{ small}]{\epsilon\to 0}\frac{1}{f_\pi^{\,2}(0)}[\delta_{ij}\delta_{lm}s+\delta_{im}\delta_{jl}t-\delta_{il}\delta_{jm}(s+t)]\times\left[1-\frac{4\mu^2}{32\pi^2 f_\pi^{\,2}}\ln\frac{\Lambda}{4\mu^2}+O(\mu^2)\right]\,, \quad (2.69)$$

where the zeroth-order term is from Weinberg's expansion. But by (2.19) we see that the logarithms are just due to the expansion of $f_\pi(\epsilon)$ around $\epsilon=0$. That is, we can absorb the singular terms into $1/f_\pi^{\,2}$ in (2.69) to give

$$T(s,t;\mu^2,\ldots,\mu^2;\epsilon)$$
$$\xrightarrow[s,t\text{ small}]{\epsilon\to 0}\frac{1}{f_\pi(\epsilon)^2}[\delta_{ij}\delta_{lm}s+\delta_{im}\delta_{jl}t-\delta_{il}\delta_{jm}(s+t)]+O(\epsilon).$$

This illustrates a result that we believe is valid for all amplitudes: *for arbitrary values of the invariants, there are nontrivial* $\mu^2\ln\mu^2$ *corrections to the chiral-limit values of the amplitudes. At the thresholds, however, these corrections can be absorbed into the constants (such as* f_π*) of the theory so as to reproduce the low-energy theorems of* PCAC *and current algebra, which are therefore still valid* (strictly speaking, a nontrivial $s\mu^2\ln\mu^2$ term in the $\pi\pi$ amplitude would not have been fatal, because the Weinberg expansion is not an exact theorem, but an expansion which is supposed to be valid to lowest order in s and ϵ).

It should be emphasized that we found no $\mu^2\ln\mu^2$ terms, but only terms like $s\mu^2\ln\mu^2$.

F. Conclusion and Approximate Formula

We have in Sec. II carefully classified all the sources of nonanalytic behavior in both on- and off-mass-shell scattering amplitudes and matrix elements. The singular terms come in from[23] (i) renormalization of the external "coupling constants," (ii) internal pion loops, (iii) two-pion cuts in various channels, and (iv) from the mass-shell constraints. As a by-product we saw that the off-mass-shell extrapolation of an amplitude from μ^2 to zero will often introduce errors of order $\mu^2 \ln\mu^2$.

We have argued that amplitudes typically have $\mu^2\ln\mu^2$ corrections, but (at least in an example) they do not violate the low-energy PCAC theorems because at threshold the corrections can be absorbed into f_π.

We have seen no indication of amplitudes diverging in the chiral limit and have argued that the logarithms cannot add up to a simple power. Matrix elements of zero-energy operators, on the other hand, can diverge quite severely in the chiral limit.

We wish to extract one particular result: for any quantity F that depends on ϵ, the expansion around $\epsilon=0$ is typically

$$\frac{F(\epsilon)}{F(0)} = 1 + \frac{C\mu^2}{32\pi^2 f_\pi^2} \ln\frac{\Lambda}{4\mu^2} + O(\mu^2)\ , \qquad (2.70)$$

where C is a Clebsch-Gordan coefficient of order unity. This is true for *each* source of nonanalyticity mentioned above. Therefore, we believe that $\mu^2/(32\pi^2 f_\pi^2) \approx 0.006$ is an appropriate dimensionless parameter to measure explicit SU(2) $\times$SU(2) breaking. For Λ one can use $m_\rho^{\,2}$, $4m_N^{\,2}$, or even use $\Lambda/4\mu^2 \approx 32\pi^2 f_\pi^2/\mu^2$, at his option $[\ln(32\pi^2 f_\pi^2/\mu^2) \approx 5]$. Hence, SU(2)$\times$SU(2) is a very good Goldstone symmetry to 5–10%.

Most of the formalism we have developed here can be trivially extended to SU(3)$\times$SU(3) and the inclusion of baryons. Before doing so we would like to give an approximate formula for the derivative of an on-shell amplitude to supplement the exact formulas (2.51) and (2.53). We would like a formula, correct up to order $\mu^2\ln\mu^2$, in which the insertion terms and counterterms involve off-shell massless external pions, so that we can use PCAC techniques. Now, $-iT_\sigma^I$ does not involve insertions on external-pion legs, so $-iT_\sigma^I(0)$ can be evaluated at $K^2=0$ for each pion, up to $O(\mu^2\ln\mu^2)$. This is not directly useful, however, because when we start applying PCAC and current-algebra techniques to $-iT_\sigma^I$, it is difficult to keep track of the external line insertion terms that are to be omitted. Hence we must add and subtract terms so that $-T_\sigma^I$ is replaced by the full $-iT_\sigma$ plus some extra counterterms. The bookkeeping is slightly complicated so we will just give the answer:

$$\begin{aligned}\frac{1}{Z_\pi^{\,1/2}}\frac{d}{d\epsilon^0}\, T(s_1,\ldots,s_r;\mu^2,\ldots,\mu^2;\epsilon) = -i\int d^4z\, T_\sigma(s_1,\ldots,s_r;0,\ldots,0;\epsilon)\\ +m\left(\frac{1}{\mu^2}\frac{1}{Z_\pi^{\,1/2}}\frac{d\mu^2}{d\epsilon^0} - \frac{1}{Z_\pi}\frac{dZ_\pi^{\,1/2}}{d\epsilon^0}\right) T(s_1,\ldots,s_r;0,\ldots,0;\epsilon)\\ + \frac{1}{Z_\pi^{\,1/2}}\frac{d\mu^2}{d\epsilon^0}\sum_{j=1}^{m} T_j(s_1,\ldots,s_r;0,\ldots,0;\epsilon) + O(\epsilon\ln\epsilon)\ ,\end{aligned} \qquad (2.71)$$

where T_σ is the full σ-insertion matrix element. We will use this formula in the next paper.[17]

III. EXTENSION TO SU(3)$\times$SU(3)

In this section we will extend the analysis of Sec. II to SU(3)$\times$SU(3). Most of the formalism already developed carries over with little modification, the only nontrivial complication being η-η' mixing. We will first review the formalism of broken SU(3)$\times$SU(3) and then show how to compute the nonanalytic parts of Green's functions and matrix elements. In particular, we will compute the leading corrections to the $\pi, K,$ and η decay constants from their symmetric value; from this we find an expression for f_K/f_π that agrees with experiment. Finally, we will argue that appropriate dimensionless parameters to measure explicit SU(3)$\times$SU(3) and SU(3) breaking are $\langle\mu^2\rangle/(32\pi^2 f_\pi^2) \approx 0.06$ and $\frac{2}{3}(\mu_K^{\,2}-\mu_\pi^{\,2})/(32\pi^2 f_\pi^2) \approx 0.05$, respectively, where $\langle\mu^2\rangle$ is the average pseudoscalar meson mass. However, for any given quantity the coefficients of these parameters may be as large as five or more, suggesting that perturbation theory around SU(3)$\times$SU(3) is at best marginal. Hence, before trusting any calculation that involves going to the SU(3)$\times$SU(3) limit, it is crucial to calculate the leading correction to see how it compares with the zero-order term. In many cases we find that the correction is small, but in the case of the electromagnetic mass shift of mesons we will find[17] that the leading correction is as big as the zero-order term.

A. SU(3)×SU(3) FORMALISM

Let the vector and axial-vector charges F_i and ${}^5F_i$, $i=1,\ldots,8$ be the generators of SU(3)×SU(3). We assume that the Hamiltonian density is $H=H_0+\epsilon_0^0u_0^0+\epsilon_8^0u_8^0$, where u_0^0 and u_8^0 are unrenormalized scalar fields that transform as a singlet and octet, respectively, under SU(3). In this paper we will adopt the $(\bar{3},3)+(3,\bar{3})$ model of Glashow and Weinberg[3] and Gell-Mann, Oakes, and Renner.[4] According to this scheme there exist nine unrenormalized scalar operators u_i^0, $i=0,\ldots,8$ and nine pseudoscalar operators ϕ_i^0, $i=0,\ldots,8$ that satisfy the equal-time $(\bar{3},3)+(3,\bar{3})$ algebra:

$$\begin{aligned}
[F_i, u_j^0] &= if_{ijk}u_k^0 ,\\
[F_i, \phi_j^0] &= if_{ijk}\phi_k^0 ,\\
[{}^5F_i, u_j^0] &= -id_{ijk}\phi_k^0 ,\\
[{}^5F_i, \phi_j^0] &= id_{ijk}u_k^0 ,
\end{aligned} \tag{3.1}$$

where i runs from 1 to 8 and j and k from 0 to 8 $[d_{0jk}=(\frac{2}{3})^{1/2}\delta_{jk}]$. As in the SU(2)×SU(2) case the basic analysis of nonanalytic behavior of amplitudes and matrix elements is independent of the transformation properties of u_0^0 and u_8^0. If these operators belonged to any other representation, the same calculations could be carried out; only the Clebsch-Gordan coefficients would change.

From the commutation relations (3.1) we can compute the divergences of the vector and axial-vector currents:

$$\begin{aligned}
\partial_\mu V_i^\mu &= \epsilon_8^0 f_{i8k}u_k^0\\
\partial_\mu A_i^\mu &= -[(\tfrac{2}{3})^{1/2}\epsilon_0^0+d^i\epsilon_8^0]\phi_i^0-(\tfrac{2}{3})^{1/2}\delta_{i8}\epsilon_8^0\phi_0^0 ,
\end{aligned} \tag{3.2}$$

where $d^i=d^{i88}$ is $1/\sqrt{3}$ for $i=1,2,3$, $-1/2\sqrt{3}$ for $i=4,\ldots,7$, and $-1/\sqrt{3}$ for $i=8$.

We will assume that the ϕ_i^0 fields, for $i=1,\ldots,7$, are associated with the π and K states and that linear combinations of ϕ_0^0 and ϕ_8^0 are associated with the η and η' (which we pretend is stable for simplicity). Then we can define renormalized fields $\phi_i=\phi_i^0/Z_i^{1/2}$, where

$$\langle 0|\phi_i^0|j\rangle=\delta_{ij}Z_i^{1/2} , \tag{3.3}$$

for $i,j=1,\ldots,7$. We will define $Z_0^{1/2}$ and $Z_8^{1/2}$ after a discussion of the mixing problem. Since we assume an exact SU(2) invariance, we have $Z_i=Z_\pi$, $i=1,2,3$ and $Z_i=Z_K$, $i=4,\ldots,7$. It will be convenient to work with the unrenormalized u_i^0 fields.

Before proceeding we must digress on the η-η' mixing problem. The physical $Y=I=0$ meson states are $|\eta\rangle$ and $|\eta'\rangle$. We will define singlet and octet states $|\bar{0}\rangle$ and $|8\rangle$ as

$$\begin{aligned}
|\bar{0}\rangle &= \cos\theta|\eta'\rangle-\sin\theta|\eta\rangle ,\\
|8\rangle &= \sin\theta|\eta'\rangle+\cos\theta|\eta\rangle ,
\end{aligned} \tag{3.4}$$

where θ is chosen so that $\langle 0|\phi_0^0|8\rangle=\langle 0|\phi_8^0|\bar{0}\rangle=0$. Then,

$$\begin{aligned}
\langle 0|\phi_0^0|\bar{0}\rangle &= Z_0^{1/2} ,\\
\langle 0|\phi_8^0|8\rangle &= Z_8^{1/2} .
\end{aligned} \tag{3.5}$$

When ϵ_8^0 goes to zero, $\sin\theta$ vanishes. In terms of the $|\bar{0}\rangle$ and $|8\rangle$ states the mass matrix is

$$\begin{aligned}
\mu^2{}_{00} &= \mu_{\eta'}{}^2\cos^2\theta+\mu_\eta{}^2\sin^2\theta ,\\
\mu^2{}_{88} &= \mu_{\eta'}{}^2\sin^2\theta+\mu_\eta{}^2\cos^2\theta ,\\
\mu^2{}_{08} &= (\mu_{\eta'}{}^2-\mu_\eta{}^2)\cos\theta\sin\theta .
\end{aligned} \tag{3.6}$$

Now we can define the meson decay constants f_i as

$$\begin{aligned}
\langle 0|\partial\cdot A_i|j\rangle &= \delta_{ij}\mu_i{}^2 f_i \quad (i,j=1,\ldots,7) ,\\
\langle 0|\partial\cdot A_8|\eta\rangle &= \mu_\eta{}^2 f_\eta ,\\
\langle 0|\partial\cdot A_8|\eta'\rangle &= \mu_{\eta'}{}^2 f_{\eta'} .
\end{aligned} \tag{3.7}$$

By SU(2), $f_i=f_\pi$, $i=1,2,3$ and $f_i=f_K$, $i=4,\ldots,7$. Comparing (3.7) with (3.2) we find that

$$\frac{f_i\mu_i{}^2}{Z_i{}^{1/2}}=-[(\tfrac{2}{3})^{1/2}\epsilon_0^0+d^i\epsilon_8^0], \quad i=1,\ldots,7 \tag{3.8a}$$

$$(\mu_\eta{}^2f_\eta\cos\theta+\mu_{\eta'}{}^2f_{\eta'}\sin\theta)/Z_8{}^{1/2} = -[(\tfrac{2}{3})^{1/2}\epsilon_0^0+d^8\epsilon_8^0], \tag{3.8b}$$

and

$$(\mu_\eta{}^2f_\eta\sin\theta-\mu_{\eta'}{}^2f_{\eta'}\cos\theta)/Z_0{}^{1/2}=(\tfrac{2}{3})^{1/2}\epsilon_8^0 . \tag{3.8c}$$

From these relations we discover that

$$\begin{aligned}
\epsilon_8^0 &= \frac{2}{\sqrt{3}}\left(\frac{f_K\mu_K{}^2}{Z_K{}^{1/2}}-\frac{f_\pi\mu_\pi{}^2}{Z_\pi{}^{1/2}}\right) ,\\
\epsilon_0^0 &= -\frac{1}{\sqrt{6}}\left(\frac{2f_K\mu_K{}^2}{Z_K{}^{1/2}}+\frac{f_\pi\mu_\pi{}^2}{Z_\pi{}^{1/2}}\right) .
\end{aligned} \tag{3.9}$$

In the limit $\epsilon_8^0=\epsilon_0^0=0$ we assume that the vacuum (and hence the spectrum of physical states) is exactly SU(3) symmetric, but that SU(3)×SU(3) is spontaneously broken. Hence, u_0^0 will have a nonzero vacuum expectation value, but $\langle u_i^0\rangle_0=0$ for $i=1,\ldots,8$. In this limit there will be eight Goldstone bosons (π,K,η), while the η' meson will have a finite mass. The long-range forces associated with the Goldstone bosons will lead to a nonanalytic approach to the symmetry point.

As we increase ϵ_0^0 from zero, but keep $\epsilon_8^0=0$, the eight Goldstone bosons acquire a common mass μ^2. Also,

$$
\begin{aligned}
Z_\pi(0,\epsilon_0^0) &= Z_K(0,\epsilon_0^0) \\
&= Z_8(0,\epsilon_0^0) \equiv Z_s(\epsilon_0^0) , \\
f_\pi(0,\epsilon_0^0) &= f_K(0,\epsilon_0^0) \\
&= f_\eta(0,\epsilon_0^0) \equiv f_s(\epsilon_0^0) .
\end{aligned} \tag{3.10}
$$

However, $Z_0 \neq Z_s$ and $f_{\eta'} \neq f_s$, not even for $\epsilon_0^0 = 0$. In fact, from (3.7) we see $f_{\eta'}(0,\epsilon_0^0)=0$. Also, we see that $\sin\theta$ vanishes as $\epsilon_8^0 \to 0$ from (3.6).

If we now increase ϵ_8^0 from zero, the octet masses, decay constants, and renormalization constants will split, $f_{\eta'}$ and $\sin\theta$ will be nonzero, and u_8^0 will pick up a nonzero vacuum expectation value.

As has been emphasized by Mathur and Okubo,[24] the values of ϵ_0^0 and ϵ_8^0 cannot be chosen arbitrarily but must fall into allowed domains. The physical value of $\epsilon_8^0/\epsilon_0^0$ is about $-0.88\sqrt{2}$. The boundary of this domain corresponds to $\epsilon_8^0/\epsilon_0^0 = -\sqrt{2}$, at which $\mu_\pi{}^2 = 0$. In the $(\overline{3},3)+(3,\overline{3})$ model this ratio $\epsilon_8^0/\epsilon_0^0 = -\sqrt{2}$ $(\mu_\pi{}^2=0)$ implies an exact $SU(2)\times SU(2)$ subsymmetry of the Hamiltonian.[4] This is the appealing feature of the scheme: the closeness of $\epsilon_8^0/\epsilon_0^0$ to $-\sqrt{2}$ naturally explains the small ratio $\mu_\pi{}^2/\mu_K{}^2$ and the approximate validity of $SU(2)\times SU(2)$, for the Hamiltonian, while the approximate SU(3) invariance of the vacuum accounts for the spectrum of physical states. However, Dashen[21] has pointed out that if u_0^0 and u_8^0 belong to any representation other than the $(\overline{3},3)+(3,\overline{3})$, the near vanishing of $\mu_\pi{}^2$ need not correspond to an underlying $SU(2)\times SU(2)$ symmetry.

In analogy with (2.6) we have the following Ward identities:

$$
(\tfrac{2}{3})^{1/2}\langle u_0^0\rangle_0 + d^i\langle u_8^0\rangle_0 = Z_i{}^{1/2} f_i\left(1+\mu_i{}^2\int \frac{ds}{s}\,\rho_i(s)\right), \qquad i=1,\ldots,7 \tag{3.11a}
$$

$$
(\tfrac{2}{3})^{1/2}\langle u_0^0\rangle_0 + d^8\langle u_8^0\rangle_0 = Z_8{}^{1/2} f_\eta \cos\theta\left(1+\mu_\eta{}^2\int \frac{ds}{s}\,\rho_\eta(s)\right) + Z_8{}^{1/2} f_{\eta'}\sin\theta\left(1+\mu_{\eta'}{}^2\int \frac{ds}{s}\,\rho_{\eta'}(s)\right), \tag{3.11b}
$$

and

$$
(\tfrac{2}{3})^{1/2}\langle u_8^0\rangle_0 = -Z_0{}^{1/2} f_\eta \sin\theta\left(1+\mu_\eta{}^2\int \frac{ds}{s}\,\rho_\eta(s)\right) + Z_0{}^{1/2} f_{\eta'}\cos\theta\left(1+\mu_{\eta'}{}^2\int \frac{ds}{s}\,\rho_{\eta'}(s)\right), \tag{3.11c}
$$

where the ρ's are the propagator spectral functions. We see that in the limit $\epsilon_0^0=\epsilon_8^0=0$, $\langle u_8^0\rangle_0=0$ and $(\tfrac{2}{3})^{1/2}\langle u_0^0\rangle_0 Z_s{}^{-1/2}(0) = f_s(0)$. Hence, we can regard ϵ_0^0, ϵ_8^0, and $f_s(0)$ as the physical parameters of the theory. We will generally state our final results in terms of the physical pion decay constant f_π and the quantities ϵ_8 and ϵ_0, which we define as

$$
\begin{aligned}
\epsilon_8 &= \frac{2}{\sqrt{3}} f_\pi(\mu_K{}^2-\mu_\pi{}^2) , \\
\epsilon_0 &= -\frac{1}{\sqrt{6}} f_\pi(2\mu_K{}^2+\mu_\pi{}^2).
\end{aligned} \tag{3.12}
$$

We will also refer to the average octet mass $\mu^2 = \tfrac{1}{3}(2\mu_K{}^2+\mu_\pi{}^2) = -\sqrt{6}\,\epsilon_0/3f_\pi \approx 0.17\ \text{GeV}^2$.

B. Perturbation of Green's Functions

The prescription for differentiating an unrenormalized Green's function is exactly the same as for $SU(2)\times SU(2)$:

$$
\begin{aligned}
&\frac{d}{d\epsilon_i^0}\langle 0|T(\phi_1^0\cdots\phi_m^0 A_1\cdots A_K)|0\rangle \\
&\qquad = -i\int d^4z\,\langle 0|T(\phi_1^0\cdots\phi_m^0 A_1\cdots A_K u_i^0(z))|0\rangle_c ,
\end{aligned} \tag{3.13}
$$

for $i=0$ or 8. Similarly, higher-order derivatives can be calculated by putting in more u^0 operators. For a renormalized Green's function one must add the appropriate derivatives of renormalization constants.

One word of warning is required: The renormalized η field is defined as

$$
\phi_\eta = -\frac{\sin\theta}{Z_0{}^{1/2}}\,\phi_0^0 + \frac{\cos\theta}{Z_8{}^{1/2}}\,\phi_8^0 \tag{3.14}
$$

[cf. (3.4) and (3.5)]. To apply (3.13) when an η operator is present one must express ϕ_η from (3.14) and put in derivatives of $\sin\theta$ and $\cos\theta$, as well as Z_0 and Z_8, by hand.

In all calculations our approach will be to set $\epsilon_8^0=0$ *after* taking the derivative and then look for singularities as $\mu^2\to 0$.

As a first application, consider $d\langle u_j^0\rangle_0/d\epsilon_i^0$, where i and j are 0 or 8. We have

$$
\frac{d}{d\epsilon_i^0}\langle u_j^0\rangle_0 = -i\int d^4z\,\langle 0|T(u_i^0(z)u_j^0(0))|0\rangle . \tag{3.15}
$$

As in Sec. II we write a dispersion relation for this quantity, saturate with the two octet-meson states (the singlet meson maintains a finite mass in the chiral limit), set $\epsilon_8^0=0$, and find that the dispersion integral diverges as $\ln\mu^2$ at threshold. The result is

$$\frac{d}{d\epsilon_i^0}\langle u_j^0\rangle_0 \underset{\substack{\epsilon_8^0=0\\ \epsilon_0^0\to 0}}{\longrightarrow} \frac{-\ln(\Lambda/4\mu^2)}{32\pi^2}\sum_{cd=1}^{8}\langle 0|u_i^0|cd\rangle\langle cd|u_j^0|0\rangle , \tag{3.16}$$

where Λ is an arbitrary cutoff. But in the chiral limit,

$$\langle cd|u_j^0|0\rangle = -d^{jcd}\frac{Z_s^{1/2}(0)}{f_s(0)}$$
$$= -d^{jcd}\frac{Z_s^{1/2}(0)}{f_\pi} + O(\epsilon\ln\epsilon) , \tag{3.17}$$

so that

$$\frac{d}{d\epsilon_i^0}\langle u_j^0\rangle_0 \to \frac{-Z_s\ln(\Lambda/4\mu^2)}{32\pi^2 f_\pi^2}\times\begin{cases}\frac{16}{3}, & i=j=0\\ \frac{5}{3}, & i=j=8\\ 0, & \text{otherwise}\end{cases}$$
$$+O(\text{constant}). \tag{3.18}$$

From (3.18) and the formulas (3.11) relating $\langle u_j^0\rangle_0$ to the decay constants (and anticipating our later results on the renormalization constants), we find that

$$\frac{f_i}{f_s} -1 = -(3\gamma_0 + \tfrac{3}{2}\sqrt{3}\,\gamma_8 d^i)\ln\frac{\Lambda}{4\mu^2} + O(\epsilon)$$
$$= (0.18 - 0.08\sqrt{3}d^i)\ln\frac{\Lambda}{4\mu^2} + O(\epsilon) , \tag{3.19}$$

where $\gamma_0 \equiv -\mu^2/(32\pi^2 f_\pi^2)\approx -0.059$ and $\gamma_8\equiv(\mu_K^2-\mu_\pi^2)/(48\pi^2 f_\pi^2)\approx 0.052$. In (3.19) i runs from 1 to 8. By f_8 we mean $f_\eta\cos\theta$, which to the order of (3.19) is just f_η.

For $\Lambda = 4m_N^2$, the logarithm in (3.19) is 1.65, indicating that the perturbation expansion of the decay constants around SU(3)×SU(3) is of marginal validity: the right-hand side of (3.19) is 0.17, 0.36, and 0.46 for $i=1, 4$, and 8. From (3.19) we can also calculate the leading contributions to f_K/f_π and f_η/f_π:

$$\frac{f_K}{f_\pi} - 1 = \frac{3(\mu_K^2-\mu_\pi^2)}{64\pi^2 f_\pi^2}\ln\frac{\Lambda}{4\mu^2}\approx 0.20,$$
$$\frac{f_\eta}{f_\pi} - 1 = \frac{4}{3}\left(\frac{f_K}{f_\pi}-1\right) . \tag{3.20}$$

The formula for f_K/f_π, which roughly agrees with the experimental value $f_K/f_\pi - 1 \simeq 0.26\pm 0.02$, was previously found from a different technique.[25]

In passing we mention that one can calculate the leading contributions to

$$\frac{d^2}{d\epsilon_i^0 d\epsilon_j^0}\langle u_k^0\rangle_0 = (-i)^2\int d^4x\, d^4y$$
$$\times\langle 0|T(u_i^0(x)u_j^0(y)u_k^0(0))|0\rangle . \tag{3.21}$$

The most singular terms go as $1/\mu^2$ and are due to the triangle graphs shown in Fig. 20. These terms yield new contributions of $5f_\pi\gamma_8^2/\sqrt{6}\,\gamma_0$ and $\sqrt{3}f_\pi\gamma_8^2/4\gamma_0$ to $\langle u_0^0\rangle_0$ and $\langle u_8^0\rangle_0$, respectively. Although these terms dominate the second derivatives, they are formally of order ϵ in the vacuum-expectations values because of the constraint $-\epsilon_8^0/\epsilon_0^0<\sqrt{2}$.

As in our discussion of SU(2)×SU(2) we can apply (3.13) to the unrenormalized meson propagators. The results are

$$\frac{d\mu_a^2}{d\epsilon_i^0} = \langle a|u_i^0|a\rangle , \tag{3.22}$$

where a is any physical on-shell meson (π, K, η, or η') and

$$\delta_{jm}\frac{1}{Z_j^{1/2}}\frac{dZ_j^{1/2}}{d\epsilon_i^0} = \frac{1}{2}\frac{d}{dK^2}F_{jm,i}(K^2,K^2,0)\Big|_{K^2=\mu_j^2}$$
$$= \frac{d}{dK^2}F_{jm;i}(K^2,\mu_j^2,0)\Big|_{K^2=\mu_j^2} ,$$
$$i=1,\ldots,7 \tag{3.23}$$

where $F_{jm;i}(K_j^2, K_m^2, (K_j-K_m)^2)$ is the off-shell amplitude $\langle j|u_i^0|m\rangle$. There are also several complicated expressions involving the derivatives of Z_0, Z_8, and $\sin\theta$. We will only quote the result we shall need later:

$$\frac{\sin^2\theta}{Z_0^{1/2}}\frac{dZ_0^{1/2}}{d\epsilon_i^0} + \frac{\cos^2\theta}{Z_8^{1/2}}\frac{dZ_8^{1/2}}{d\epsilon_i^0}$$
$$= \frac{1}{2}\frac{d}{dK^2}F_{\eta\eta;i}(K^2,K^2,0)\Big|_{K^2=\mu_\eta^2}$$
$$= \frac{d}{dK^2}F_{\eta\eta;i}(K^2,\mu_\eta^2,0)\Big|_{K^2=\mu_\eta^2} . \tag{3.24}$$

As in the SU(2)×SU(2) calculation we can compare (3.22) with the mass formula $2\mu_i^2 = \langle i|H_0|i\rangle + \langle i|\epsilon_0^0 u_0^0 + \epsilon_8^0 u_8^0|i\rangle$ to obtain

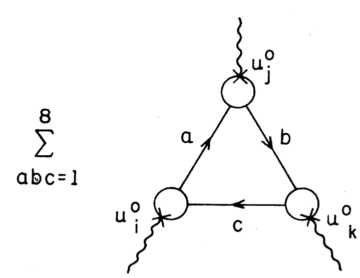

FIG. 20. The most singular contributions to Eq. (3.21).

$$-\mu_i^2+\langle i|H_0|i\rangle=\ln\frac{\Lambda}{4\mu^2}\left[\tfrac{2}{3}\gamma_0+\tfrac{4}{3}\sqrt{3}\; d^i\gamma_8\right]\mu_i^2+O(\epsilon^2)$$
$$=\ln\frac{\Lambda}{4\mu^2}\left[-0.04+0.12d^i\right]\mu_i^2+O(\epsilon^2)\,, \qquad (3.25)$$

for $i=1,\ldots,7$ (the calculation for $\langle\eta|H_0|\eta\rangle$ gets inextricably tangled with the mixing problem).

C. Perturbation of Amplitudes and Matrix Elements

Let $T(s_1,\ldots,s_r;K_1^2,\ldots,K_m^2;\epsilon_0,\epsilon_8)$ be an amplitude or matrix element involving m external π, K, or η mesons (we are not concerned with η' amplitudes). As in (2.28) we can single out the dependence on the mass of the jth meson by the definition

$$T(K_j^2;\epsilon_0;\epsilon_8)=T(\mu_j^2;\epsilon_0,\epsilon_8)+(K_j^2-\mu_j^2)T_j(K_j^2;\epsilon_0,\epsilon_8). \qquad (3.26)$$

As $\epsilon_0\to 0$, $T_j(K_j^2;\epsilon_0,0)$ will in general be nonanalytic in ϵ_0. The analysis of the sources of singularities is identical to the SU(2)×SU(2) analysis in Sec. II C. In particular, T_j may have $\ln\epsilon_0$ terms due to the two-meson cuts in any channel with zero invariant mass. For example, if A is a scalar operator and $T(K_i^2,K_j^2;t;\epsilon_0,\epsilon_8)\equiv\langle i|A(0)|j\rangle$ remains finite in the chiral limit, then Eq. (2.39) still holds, only with c and d ranging from 1 to 8 and with μ^2 being the average meson mass. The Weinberg formula can be generalized (in the $\epsilon_8=0$ limit) to SU(3)×SU(3) simply by replacing $\delta_{ab}\,\delta_{cd}$ by $T_{ab;cd}=\frac{2}{3}\delta_{ab}\,\delta_{cd}+\sum_{e=1}^{8}d_{abe}\,d_{ecd}$. Then,

$$\frac{d}{dK_i^2}T(K_i^2,\mu_j^2;t;\epsilon_0,0)\xrightarrow[\epsilon_0\to 0]{}\frac{1}{64\pi^2 f_\pi^2}\ln\frac{\Lambda}{4\mu^2}\times\sum_{cd=1}^{8}(T_{ic;jd}+T_{id;jc})\times\langle cd|A(0)|0\rangle\,. \qquad (3.27)$$

From (3.23), (3.24), and (3.27) the leading corrections to the octet renormalization constants can be calculated. They are

$$\frac{Z_i^{1/2}}{Z_s^{1/2}(0)}-1=\frac{-\ln(\Lambda/4\mu^2)}{192\pi^2 f_\pi^3}\left[14(\tfrac{2}{3})^{1/2}\epsilon_0+d^i\epsilon_8\right],\quad i=1,\ldots,8. \qquad (3.28)$$

Notice from (3.19) and (3.28) that the corrections to both f_i and $Z_i^{1/2}$ satisfy Gell-Mann–Okubo formulas, as they should.

Before proceeding, we note that for our purposes it is always sufficient to calculate $T_j(K_j^2;\epsilon_0,\epsilon_8)$ in the $\epsilon_8=0$ limit. In fact, one cannot even use PCAC to define an off-shell extrapolation of η amplitudes until one has set $\epsilon_8=0$.

The formulas for derivatives of a matrix element are virtually the same as for SU(2)×SU(2). Equation (2.44) for the derivative of an off-shell matrix element becomes

$$\frac{dT}{d\epsilon_i^0}=-iT_{u_i^0}(0)+\sum_{j=1}^{m}\left(\frac{d\mu_j^2}{d\epsilon_i^0}\,\frac{1}{-K_j^2+\mu_j^2}-\frac{1}{Z_j^{1/2}}\,\frac{dZ_j^{1/2}}{d\epsilon_i^0}\right)T\,, \qquad (3.29)$$

where $T_{u_i^0}(0)$ is the amplitude with a zero-energy unrenormalized u_i^0 field inserted. If T involves any external η fields, one must modify (3.29) slightly, as discussed after Eq. (3.14).

For on-shell amplitudes, the analog of (2.51) is

$$\frac{d}{d\epsilon_i^0}T(s_1,\ldots,s_r;\mu_1^2,\ldots,\mu_m^2;\epsilon_0,\epsilon_8)=-iT^{s}_{u_i^0}(s_1,\ldots,s_r;\mu_1^2,\ldots,\mu_m^2;\epsilon_0,\epsilon_8;q=0). \qquad (3.30)$$

Equation (2.52) is exact, even in the presence of external η particles.

Incidentally, if external η particles are present, both $T^{s}_{u_i^0}$ and $T^{I}_{u_i^0}$ still *include* η' poles in the (η,u_i^0) channels. These will never give rise to logarithmic singularities, however.

Equation (2.53) becomes

$$\frac{dT}{d\epsilon_i^0}=-iT^{I}_{u_i^0}(0)+\sum_{j=1}^{m}\left(\frac{d\mu_j^2}{d\epsilon_i^0}\,T_j+\frac{1}{Z_j^{1/2}}\,\frac{dZ_j^{1/2}}{d\epsilon_i^0}\,T\right), \qquad (3.31)$$

except that for η fields we replace the renormalization term by the quantity in (3.24) multiplied by T.

We can search for $\ln\epsilon_0$ singularities in (3.31) by writing dispersion relations for $T^{I}_{u_i^0}$ in each channel just as for SU(2)×SU(2). All of the analysis, conclusions, and rules are exactly as in Sec. II D. There will, in general, be both $\epsilon_0\ln\epsilon_0$ and $\epsilon_8\ln\epsilon_0$ singularities in T, but presumably these can be absorbed into the parameters of the theory at the current-algebra points.

Finally, the approximate formula (2.71) in which the derivatives are expressed in terms of off-shell amplitudes becomes

$$\frac{d}{d\epsilon_i^0} T(s_1, \ldots, s_r; \mu_1^2, \ldots, \mu_m^2; \epsilon_0, \epsilon_8) = -i \int d^4 z\, T_{u_i^0}(s_1, \ldots, s_r; 0, \ldots, 0; \epsilon_0, 0)$$
$$+ \sum_{j=1}^{m} \left(\frac{1}{\mu^2} \frac{d\mu_j^2}{d\epsilon_i^0} - \frac{1}{Z_j^{1/2}} \frac{dZ_j^{1/2}}{d\epsilon_i^0} \right) T(s_1, \ldots, s_r; 0, \ldots, 0; \epsilon_0, 0)$$
$$+ \sum_{j=1}^{m} \frac{d\mu_j^2}{d\epsilon_i^0} T_j(s_1, \ldots, s_r; 0, \ldots, 0; \epsilon_0, 0) + O(\epsilon \ln\epsilon_0) , \qquad (3.32)$$

where we have set $\epsilon_8 = 0$, as is appropriate to the accuracy of (3.32).

D. Conclusion to Extension to SU(3)×SU(3)

We have found that there will generally be nonanalytic terms both of the form $\epsilon_0 \ln\epsilon_0$ and $\epsilon_8 \ln\epsilon_0$ in perturbation expansions around SU(3)×SU(3). Furthermore, off-mass-shell extrapolations will generally induce errors of order $\mu^2 \ln\epsilon_0$.

We have seen that the dimensionless parameters $\gamma_0 = -\mu^2/(32\pi^2 f_\pi^2) = -0.059$ and $\gamma_8 = (\mu_K^2 - \mu_\pi^2)/(48\pi^2 f_\pi^2) = 0.052$ appear in all of the calculations. This is true not only for the logarithmic singularities but also for the ϵ_8^2/ϵ_0 behavior of triangle graphs (3.21) and box graphs.[9] Therefore, we propose that γ_0 and γ_8 are appropriate parameters to measure SU(3)×SU(3) and SU(3) breaking. But beware: The Clebsch-Gordan coefficients in front of γ_0 and γ_8 often range from one half to five or more. The validity of the perturbation expansion is marginal and must be checked for each application by calculating the ratio of the leading correction to the zero-order term. We will show[17] that the perturbation expansion breaks down completely for electromagnetic mass shifts of mesons; this explains the Dashen paradox.[8] A similar explanation might apply to the $K \to 2\pi$ problem.[21] However, the leading contribution to the renormalization of strangeness-changing vector form factors[9] is only 2%.

We close by mentioning that the η-η' mixing angle must be proportional to $\epsilon_8 \ln\epsilon_0$. This can be seen by a careful comparison of Eqs. (3.8c), (3.11c), and (3.18). This can be explained from the last of Eqs. (3.6). The matrix element $\langle \overline{0} | u_8^0 | 8 \rangle$ will diverge as $\ln\epsilon_0$ in the chiral limit due to the two-meson cut in the t channel [there is no reason for the ($\overline{0} + 8 \to$ two mesons) amplitude to vanish at threshold].

IV. INCLUSION OF BARYONS

In this section we will outline the techniques for perturbing a matrix element involving baryons around the SU(2)×SU(2) limit. We will find, for example, that the pion-nucleon amplitude approaches the chiral limit in a nonanalytic ($\mu^2 \ln\mu^2$) manner except at the exceptional momentum (current-algebra) point where the corrections are analytic to leading order. This is true provided that we express the amplitude at the current-algebra point in terms of the physical values of the parameters of the theory instead of using the chiral-symmetric values of the parameters.

Consider an amplitude or matrix element T involving two or more baryons. We will single out a final baryon for special attention. The modification for antibaryons or initial baryons will be obvious. Hence, $T = \langle ps | Q \rangle$, where p is the momentum and s the spin direction of the baryon, and Q is the rest of the matrix element. In order to incorporate the baryon mass-shell constraints we must consider an off-shell *invariant* amplitude

$$F(p^2; \epsilon) = \sum_s P_\lambda(p) u_\lambda(p, s) \langle ps | Q \rangle , \qquad (4.1)$$

where u_λ is a spinor and P_λ is some projection operator, which can depend on p and other momenta of the problem.[25] It is implied that a projection is done for each baryon. Let us also define the quantity $Q_\delta(p)$ as $\sum_s u_\delta(p, s) \langle ps | Q \rangle$. We can now derive the analogs of (2.51) and (2.53) for $dF(m^2; \epsilon)/d\epsilon^0$. The spinor algebra is very complicated and uninteresting so we will only give the results. The analog of (2.51) is

$$\frac{1}{Z_\pi^{1/2}} \frac{dF(m^2; \epsilon)}{d\epsilon^0} = -iF_\sigma^s(m^2; \epsilon; 0) + \frac{1}{Z_\pi^{1/2}} \frac{dm}{d\epsilon^0} \frac{d}{dp^2} \times [P_\lambda(p)(\not{p} + m)_{\lambda\delta} Q_\delta(\bar{p})] \Big|_{p^2 = m^2} . \qquad (4.2)$$

where F_σ^s is the original (projected) matrix element with a zero-energy σ inserted everywhere except on the external legs; the external leg insertions are removed via a dispersion relation. The vector $\bar{p}$ in (4.2) is on the mass shell ($\bar{p}^2 = m^2$) so the p^2 derivative in (4.2) only affects the projection operator.

Similarly, the analog of (2.53) is

$$\frac{1}{Z_\pi^{1/2}}\frac{dF(m^2;\epsilon)}{d\epsilon^0}=-iF_\sigma^I(m^2;\epsilon;0)+\frac{1}{Z_\pi^{1/2}Z_F^{1/2}}\frac{dZ_F^{1/2}}{d\epsilon^0}F(m^2;\epsilon)+\frac{1}{Z_\pi^{1/2}}\frac{dm}{d\epsilon^0}\frac{d}{dp^2}\times[P_\lambda(p)(\not p+m)_{\lambda\delta}Q_\delta(p)]\bigg|_{p^2=m^2}. \tag{4.3}$$

The baryon-σ channels in F_σ^I are single-baryon irreducible; Z_F is the baryon wave-function renormalization constant. The p^2 derivative in (4.3) acts on the entire matrix element. In (4.2) and (4.3) we have not explicitly written the counterterms and external-leg conventions for the external mesons. They are the same as before [Eqs. (2.51), (2.53), or (2.71)].

In deriving (4.20) and (4.3) we made use of the identity [cf. (2.21)]

$$\frac{dm}{d\epsilon^0}+\frac{1}{Z_F}\frac{dZ_F}{d\epsilon^0}(\not p-m)+\frac{(\not p-m)}{Z_F}\frac{d}{d\epsilon^0}[Z_FS_c(p)](\not p-m)=-\int d^4x\,d^4y\,e^{ip(x-y)}(\not p-m)\times\langle 0|T(\psi(x)\bar\psi(y)\sigma^0(0))|0\rangle(\not p-\mathrm{m}), \tag{4.4}$$

where $S_c(p)$ is the continuum part of the baryon propagator. From (4.4) we see that

$$\frac{1}{Z_\pi^{1/2}}\frac{dm^2}{d\epsilon^0}\delta_{ss'}=\langle ps|\sigma(0)|ps'\rangle, \tag{4.5}$$

with $p^2=m^2$. This remains finite in the chiral limit. The worst diagram $\sigma\to 2\pi\to B\bar B$ (B=baryon) is nonsingular because the $2\pi\to B\bar B$ amplitude vanishes at threshold. One could also obtain an expression for $dZ_F/d\epsilon^0$ from (4.4).

Finally, it can be shown with some difficulty that if $F(p^2,\epsilon)$ contains a baryon pole in some channel ($\alpha\to\beta'$), then the sum of baryon poles in the (α,σ) and (β',σ) channels of $-iF_\sigma^I$ (analogous to Fig. 12) plus the poles in the counterterms of (4.3) is

$$\frac{1}{Z_\pi^{1/2}}\frac{d}{d\epsilon^0}\left[-\sum_s\frac{\langle\alpha|ps\rangle\langle ps|\beta'\rangle}{s_\alpha-m^2}\right], \tag{4.6}$$

where p is the (on-shell) momentum of the pole. It is implied that appropriate projections have been performed on the *external* baryons. Equation (4.6) is exact and incorporates the mass-shell constraints on the pole residue.

One can in principle use (4.3) to determine the nonanalytic terms in $F(m^2,\epsilon)$ as $\epsilon\to 0$. First, separate out the baryon poles and differentiate the residue according to (4.6). Then look for $\ln\epsilon$ singularities in the non-pole-part of $-iF_\sigma^I$. *There will generally be singularities due to a general pion loop* (the q^2 channel cut, as in Fig. 9), *because the pions can hook onto the external-baryon legs*. There will also be logarithms due to any channel with zero invariant mass if there is a two-pion cut and the amplitudes at the ends of the cut do not vanish. There will *not* be logarithms due to pion-baryon cuts because the phase space vanishes at threshold. Finally, one must add the singular parts of the counterterms corresponding to the external mesons [Eq.(2.51), (2.53), or (5.71)]. We have not attempted an analysis of the possible singularities due to the baryon counterterms in (4.3).

Now consider a *suitably projected on-shell* amplitude $F(s,t,\epsilon)$ for $\pi_i(K_i)+B_j(p_j)\to\pi_m(K_m)+B_l(p_l)$, where i,j,l,m, are isospin indices, at fixed $s=(K_i+p_j)^2$ and $t=(K_i-K_m)^2$. For arbitrary fixed s and t, $-iF_\sigma^I(s,t;\epsilon)$ will possess $\ln\epsilon$ singularities due to a general pion loop (Fig. 9). The $dZ_\pi^{1/2}/d\epsilon^0$ meson counterterms [Eq. (2.53)] will also be singular. There may also be logarithms due to the residues of the s- and u-channel baryon poles. There will *not* be singularities due to the t-channel cut in $-iF_\sigma^I$ because both the $\pi\pi\to\pi\pi$ and the $\pi\pi\to B\bar B$ amplitudes vanish at threshold for $\epsilon=0$. Similarly, the meson counterterms $F_i(\mu^2)$ (i.e., dF/dK_i^2) and $F_m(\mu^2)$ will be nonsingular due to the vanishing of $\pi\pi\to B\bar B$ (Sec. II C). *In general*, therefore,

$$F(s,t;\epsilon)=F(s,t;0)+G(s,t)\epsilon\ln\epsilon+O(\epsilon). \tag{4.7}$$

Now consider the low-energy $s\to m^2$, $t\to 0$ limit of $F(s,t;K_i^2=\mu^2,K_m^2=\mu^2;\epsilon)$, where we have now indicated the pion masses explicitly. We can use (2.71) to express $dF(s,t,\mu^2,\mu^2;\epsilon)/d\epsilon^0$ in terms of matrix elements involving massless external pions up to $O(\epsilon\ln\epsilon)$. The baryons are still on-shell and treated according to (4.3). We can then apply PCAC to the pions in $-iF_\sigma$ and the counterterms, yielding

$$\frac{1}{Z_\pi^{1/2}}\frac{d}{d\epsilon^0}F(s,t,\mu^2,\mu^2;\epsilon)=\frac{1}{Z_\pi^{1/2}}\frac{d}{d\epsilon^0}\left[\frac{i}{f_\pi^2}\int d^4x\,e^{iK_mx}K_m^\mu K_i^\nu\langle l|T(A_\mu^m(x)A_\nu^i(0))|j\rangle+\frac{i}{f_\pi^2}\sum_n\epsilon_{min}K_i^\mu\langle l|V_\mu^n(0)|j\rangle-\frac{\epsilon}{f_\pi^2}\delta_{im}\langle l|\sigma(0)|j\rangle\right]+\frac{1}{Z_\pi^{1/2}}\frac{d\mu^2}{d\epsilon^0}[F_i(0)+F_m(0)]+O(\epsilon\ln\epsilon). \tag{4.8}$$

In (4.8), $K_i^2 = K_m^2 = 0$ and projections on the baryons are implied. The first three terms are simply the derivative of the PCAC expression for an (off-shell) zero-pion-mass amplitude. In the $K_i \to 0$, $K_m \to 0$ ($s \to m^2$, $t \to 0$) limit, only the s- and u-channel baryon poles will survive in the first term. This is true both for the derivative of $1/f_\pi^2$ and for the derivative of the matrix element, the pole part of which is given by (4.6). [The derivative of the matrix element has a pion loop which goes as $\ln\epsilon$ as $\epsilon \to 0$ for fixed K_i, K_m but vanishes as $(K_i, K_m) \to 0$ for fixed ϵ. The limits do not commute.] Hence, at the current-algebra point, all of the corrections to $F(s, t, 0, 0; 0)$ in the first three terms of (4.8) can be absorbed into the renormalization of the parameters of the theory (m, f_π, g_A, and the σ matrix element) by the symmetry breaking. The $F_i + F_m$ term is finite as $\epsilon \to 0$ [see discussion before (4.7)]. Hence, *at the current-algebra point,*

$$F(s, t, \mu^2, \mu^2; \epsilon) = F(s, t, 0, 0; 0) + O(\epsilon) + O(\epsilon^2 \ln\epsilon)\ , \tag{4.9}$$

provided that the current-algebra expression for $F(s, t, 0, 0; 0)$ is written in terms of $f_\pi(\epsilon)$, etc., instead of the chiral-symmetric values of these parameters. The corrections, analytic to leading order, *are due entirely* to the off-shell extrapolation of the external pions.

V. CONCLUSIONS

It has been our goal, in this article, to establish a well-defined procedure for extracting the leading nonanalytic behavior in chiral-symmetry breaking. S-matrix elements exist in the chiral limit, but in general approach such a limit in a nonanalytic fashion like $\epsilon \ln\epsilon$. At current-algebra points, for which we have low-energy theorems for S-matrix elements, the corrections are analytic $O(\epsilon)$ to leading order. For matrix elements of currents or other operators (at zero momentum) the chiral limit may not exist, reflecting a behavior like ϵ^{-1} or $\ln\epsilon$. In general we find perturbation theory about SU(2)×SU(2) to be quite good; perturbation theory about SU(3)×SU(3) is marginal.

There are further applications of these ideas which we have not made. For example one would like to know the expansion of $g_A(\epsilon)$ and $g_{\pi N}(\epsilon)$ similar to the result we found for $f_\pi(\epsilon)$.

We further suggest that one might apply these techniques to the $K_S \to 2\pi$ puzzle. Dashen[21] has pointed out that this decay is suppressed to $O(\epsilon^2)$ in chiral SU(3)×SU(3) breaking and estimates an anomalously low rate. However this argument depends on momentum expansions which we know to fail. Hence the leading order will be less than $O(\epsilon^2)$ and one might be able to estimate the rate. The $K_S \to 2\pi$ decay deserves further study. Similarly one might examine the $\eta \to 3\pi$ puzzle in the light of these remarks. We have examined the $\pi^0 \to 2\gamma$ decay for which one has a theorem for zero-mass pions. The extrapolation corrections to pions with finite mass are, in this instance, analytic to leading order. Hence, one expects the rate for $\pi^0 \to 2\gamma$ to be well-approximated by the low-energy theorem unless there is an enhancement of the type considered by Drell.[14]

APPREDIX A

We would like to give a simple illustration of formula (2.44) for the derivative of an *off-shell* matrix element. Consider the off-shell $\pi\pi$ amplitude $ij \to kl$ at the unphysical point at which all four-momenta vanish. Then, using PCAC and the commutators (2.1):

$$\begin{aligned} -i \int d^4z\, T_\sigma(z) &= -(\mu^2)^4 \int d^4x_1\, d^4x_2\, d^4x_3\, d^4z \\ &\qquad \times \langle 0|T(\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(0)\sigma(z))|0\rangle \\ &= \frac{1}{f_\pi^2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ &\qquad \times \left[-\frac{3}{f_\pi^2}\langle\sigma\rangle_0 + \frac{3i\mu^2}{f_\pi} \int d^4z \langle 0|T(\sigma(z)\sigma(0))|0\rangle \right. \\ &\qquad \left. + (\mu^2)^2 \int d^4z\, d^4y \langle 0|T(\sigma(z)\sigma(y)\sigma(0))|0\rangle \right]. \end{aligned} \tag{A1}$$

But $\langle\sigma\rangle_0 = f_\pi + O(\epsilon\ln\epsilon)$; the second term was shown in Sec. II B to be $O(\epsilon \ln\epsilon)$ (from Fig. 1), and the third term is $O(\epsilon)$, because the most singular diagram (the triangle diagram, analogous to Fig. 20) is proportional to $(\mu^2)^2/\mu^2$. Hence, $-i\int d^4z T_\sigma = -3(\delta\delta\cdots)/f_\pi^3 + O(\epsilon\ln\epsilon)$. Similarly, one can show that the counterterms are $4(\delta\delta\cdots)/f_\pi^3 + O(\epsilon\ln\epsilon)$, so that

$$\frac{dT}{d\mu^2} = -\frac{1}{f_\pi^2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + O(\epsilon\ln\epsilon)\ , \tag{A2}$$

in agreement with the Weinberg expansion.[18] There is no new physics in (A2), which could have been obtained directly from the PCAC calculation we did for the counterterms; we merely quote it to illustrate that the low-energy theorems of current algebra are valid. Corrections to Weinberg's expansion are of the form $\epsilon^2\ln\epsilon$, not $\epsilon\ln\epsilon$.

*Work supported in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-2232.

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