as in Sec. III A by reference to the double-spectral form, it seems that as certain mass values are reached, the situation with four on-shell unphysical internal momenta can exist. Thus, the standard single-spectral form should be augmented by a contribution corresponding to this new "unphysical causal process." This seems to be related to the occurrence of anomalous thresholds in conventional studies.

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[32]See Eden *et al.* (Ref. 20) for references.

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[34]D. Iagolnitzer and H. P. Stapp, Commun. Math. Phys. 14, 15 (1969).

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[37]Even if one can treat the more general possibility, suggested near the begining of Sec. IV A, of more than one causal realization of a coupling occurring for a given source specification, localization and overlap considerations should still be necessary.

[38]For the purposes at hand, we should allow the possibility of more than one causal realization of a coupling for a given source specification.

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Spectral Representation for Elastic Photon-Photon Scattering*

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The derivation of spectral forms in source theory has recently been systematically developed for systems of scalar particles. Elastic photon-photon scattering in lowest nonvanishing order is here studied as a simple, but representative, example of the additional considerations necessitated by particles having internal quantum numbers. The amplitude is determined as a double-spectral form augmented by a single-spectral one, the latter being related to the imposition of gauge invariance. Also presented is some discussion on how, in source theory, one obtains at given order all the contributions to the amplitude.

I. INTRODUCTION

In two recent detailed works we have studied, in source theory, the establishment of single- and double-spectral forms for three-[1] and four-point functions.[2] The basic ideas in such developments center around space-time and energy-momentum considerations, independent of any internal quantum numbers that the particles may carry. To aid its systematic presentation, that work was thus carried out for scalar particles. In the present work we take up an example that, while rather simple, illustrates fairly well what further techniques are necessary when the more realistic situations of particles with internal quantum numbers are considered. In particular, elastic photon-photon scattering is studied to lowest nonvanishing order in spin-$\frac{1}{2}$ electrodynamics, with the photon polarizations chosen as equal and perpendicular to the scattering plane.[3]

A brief review of those aspects of the four-point-function work for scalar particles that are necessary here is presented in Sec. II. Enough detail is given there to make this paper understandable by itself, but to fully appreciate the matter, Ref. 2 should be consulted.

The main section of this paper is Sec. III, where the calculation for photon-photon scattering is carried out. The major new point there concerns gauge invariance. We show how it is maintained, with the consequence that the basic double-spectral structure is augmented with a single-spectral one. The final result has already been obtained by conventional analyticity techniques.[4] But we should emphasize, as Refs. 1 and 2 make clear, that the source-theoretic approach is independent of analyticity considerations, seemingly being simpler and more physical in its basis.

The calculational scheme for obtaining spectral forms starts from causal realizations of the amplitude, and then, after some reworking, proceeds to the final generally applicable scattering amplitude. The question thus arises as to just what set of these so-called causal processes one must consider in order to obtain the complete scattering amplitude to the given order. The matter is simple for three-point functions (at least when there are no anomalous thresholds), but for four-point functions the attitude that we have developed differs somewhat from that of Schwinger.[5] This point is discussed in Sec. IV, mainly within the context of photon-photon scattering, but also with some consideration of pair creation by two photons.

II. SCALAR PARTICLES

It is first necessary to review the establishment of the spectral forms for the scalar case. This is done here as briefly as possible; the details can be found in Ref. 2.

The vacuum-amplitude term to which the causal process of Fig. 1 corresponds is, as originally obtained, given by

$$\int (dx)\cdots(dx''')\varphi_\alpha(x)\Delta_+^b(x-x')\varphi_\beta(x')\Delta_+^c(x'-x'')$$
$$\times\varphi_\gamma(x'')\Delta_+^d(x''-x''')\varphi_\delta(x''')\Delta_+^a(x'''-x). \quad (1)$$

Here φ is the field, referring to the particle associated with the source K, and Δ_+ is the propagation function, referring to an internal particle. Two basic characteristics of the causal process are the existence of real internal particles and the pairwise combination of these particles to form propagating compound excitations (ac and bd) of variable mass (M and M'). The vacuum amplitude (1) may be rewritten to express these two characteristics. As detailed in Ref. 2, this rewriting makes use of the causal specifications of the process, and involves an intermediate expression of the vacuum amplitude in momentum space, followed by a return to configuration space. The result is

$$\frac{i}{(2\pi)^2}\int (dx)(d\xi)(d\xi')\varphi_\gamma(x)\varphi_\beta(x-\xi)$$
$$\times\varphi_\delta(x-\xi')\varphi_\alpha(x-\xi-\xi')$$
$$\times\Delta_+(\xi,M^2)\Delta_+(\xi',M'^2)i\bar{\rho}(M^2,M'^2)$$
$$\times dM^2dM'^2, \quad (2)$$

with

$$\bar{\rho}=\int(dp)\delta\big((p-p_\alpha-p_\beta)^2+m_a{}^2\big)$$
$$\times\delta\big((p-p_\beta)^2+m_b{}^2\big)$$
$$\times\delta(p^2+m_c{}^2)\delta\big((p+p_\gamma)^2+m_d{}^2\big), \quad (3)$$

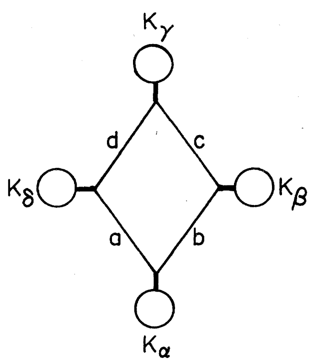

FIG. 1. Causal process leading to double-spectral form. The long thin lines refer to real particles, and the short heavy ones to virtual particles.

p_i being the momentum associated with external particle i. Thus, the structure of Eq. (2) refers to the propagation of the compound excitations, while the form of $\bar{\rho}$ bears on the reality of the internal particles.

Evaluation of the integral yields

$$\bar{\rho}=\tfrac{1}{8}(-\Delta)^{-1/2}, \quad (4)$$

where

$$\Delta\equiv\det(p_i\cdot p_j),\quad i,j=a,b,c,d, \quad (5)$$

a 4×4 Gram determinant. These internal momenta are eliminated in terms of the external momenta by conservation of momentum, and, corresponding to the compound excitations, one writes $-(p_\alpha+p_\beta)^2=M^2$ and $-(p_\alpha+p_\delta)^2=M'^2$. So, Δ is a function of M^2, M'^2, and $p_i{}^2$, $i=\alpha,\beta,\gamma,\delta$. Also, the kinematics of the causal process implies that Δ is negative. And it is this statement, along with an analogous one for the Gram determinant formed from p_a, p_b, p_c, plus $M\geq m_a+m_c$ and $M'\geq m_b+m_d$, that determines the M, M' integration domain in Eq. (2).

Two related steps are necessary in order to render the vacuum amplitude (2) applicable to the usual scattering situation of the four particles $\alpha,\beta,\gamma,\delta$ (with any two forming the incoming state). First, the stipulations referring to the causal propagation of the internal particles must be removed—space-time generalization—and this is simply achieved by directly applying the result (2) to such circumstances. Second—mass extrapolation—the external particles must be brought on shell, with the source localizations being changed to describe the scattering configurations. This amounts to changing the M, M' integration domain to all values satisfying $\Delta\geq0$ and $M\geq m_a+m_c$, $M'\geq m_b+m_d$, and in $\bar{\rho}$ we must write $(-\Delta)^{1/2}=+i\Delta^{1/2}$; here Δ is given by Eq. (5), but now with $-p^2\to m^2$ for each of the external particles. So, expressing it in momentum space, we have that the scattering vacuum amplitude is given by the double-spectral form[6]

$$\frac{i}{(2\pi)^2}\int\left[(d\underline{p})\varphi(\underline{p})\right](2\pi)^4\delta(p_\alpha+p_\beta+p_\gamma+p_\delta)$$
$$\times\frac{1}{(p_\alpha+p_\beta)^2+M^2-i\epsilon}\;\frac{1}{(p_\alpha+p_\delta)^2+M'^2-i\epsilon}$$
$$\times\rho(M^2,M'^2)dM^2dM'^2, \quad (6)$$

with the weight function

$$\rho=\tfrac{1}{8}\Delta^{-1/2}. \quad (7)$$

Also, we have written

$$\left[(d\underline{p})\varphi(\underline{p})\right]\equiv\prod_{i=\alpha,\beta,\gamma,\delta}\left[\frac{(dp_i)}{(2\pi)^4}\varphi_i(p_i)\right], \quad (8)$$

$\varphi_i(p)$ being the Fourier transform of $\varphi_i(x)$. In the result (6), as a consequence of space-time generalization, $(p_\alpha + p_\beta)^2$ and $(p_\alpha + p_\delta)^2$ are independent of M^2 and M'^2, and may take on any values, timelike or spacelike.

The causal process of Fig. 1 represents only one way in which, to the given order, the external particles may be coupled together. All told, there are three independent ways, as manifested in the causal processes of Fig. 2. (Of course, there may exist more than one causal process, or may exist none, for any given source arrangement of Fig. 2, depending on the particle interactions that are admitted.) The complete double-spectral form is thus Eq. (6) plus that result twice with the appropriate interchanges among the external particles (and with an appropriate relabeling if there is a change in the identity of the internal particles):

$$\text{Eq. (6)} + (\beta \leftrightarrow \gamma) + (\delta \leftrightarrow \gamma). \tag{9}$$

We shall also need the single-spectral form. The basic causal process here is that of Fig. 3, and the initial vacuum-amplitude expression has the structure of Eq. (1), but with the new source specifications applied. This vacuum amplitude is reworked, the compound excitation ac being brought to fore, and the final result is

$$\frac{i}{2\pi}\int [(d\underline{p})\varphi(\underline{p})](2\pi)^4\delta(p_\alpha+p_\beta+p_\gamma+p_\delta) \times \frac{1}{(p_\alpha+p_\beta)^2+M^2-i\epsilon}\chi(M^2)dM^2 . \tag{10}$$

Here we have

$$\chi = \int d\omega_{p_a} d\omega_{p_c} (2\pi)^4\delta(p_\alpha+p_\beta-p_a-p_c) \times [(p_\alpha-p_a)^2+m_b{}^2]^{-1} \times [(p_\gamma+p_c)^2+m_d{}^2]^{-1}, \tag{11}$$

where

$$d\omega_{p_i} = \frac{1}{2(\vec{p}_i{}^2+m_i{}^2)^{1/2}}\frac{(d\vec{p}_i)}{(2\pi)^3}. \tag{12}$$

The weight function χ also depends on $(p_\beta+p_\gamma)^2$ in addition to M^2, which is the causal $-(p_\alpha+p_\beta)^2$, and which ranges from $(m_a+m_c)^2$ to ∞.[7] The variable $(p_\beta+p_\gamma)^2$, in not being associated with a propagating compound excitation for the causal process considered, has the same identification before and after space-time generalization, in distinction to $(p_\alpha+p_\beta)^2$. Consequently, the result (10) is derived only for $(p_\beta+p_\gamma)^2$ having the range admitted by the causal process, that of a momentum transfer, whereas $(p_\alpha+p_\beta)^2$ may assume any value.

The set of independent causal processes that provides the complete single-spectral form is shown in Fig. 4. This complete vacuum amplitude is thus given as[8]

$$\text{Eq. (10)} + (\delta \leftrightarrow \gamma) + (\beta \leftrightarrow \gamma) + (\beta \leftrightarrow \gamma, \beta \leftrightarrow \delta). \tag{13}$$

Note in the third and fourth terms here that the momentum variable in the spectral denominator is

$$(p_\alpha+p_\gamma)^2 = -(p_\alpha+p_\beta)^2 - (p_\beta+p_\gamma)^2 - m_\alpha{}^2 - m_\beta{}^2 - m_\gamma{}^2 - m_\delta{}^2 . \tag{14}$$

Also, in the weight functions in these two terms one uses $-(p_\alpha+p_\gamma)^2 = M^2$, with the appropriate new threshold value, and the variable other than M^2 in all the weight functions is $(p_\beta+p_\gamma)^2$.

III. PHOTON-PHOTON SCATTERING

Consider the causal process illustrated in Fig. 5. The corresponding vacuum amplitude is derived as was Eq. (1), with the appropriate electrodynamic quantities now being employed,[9] and is

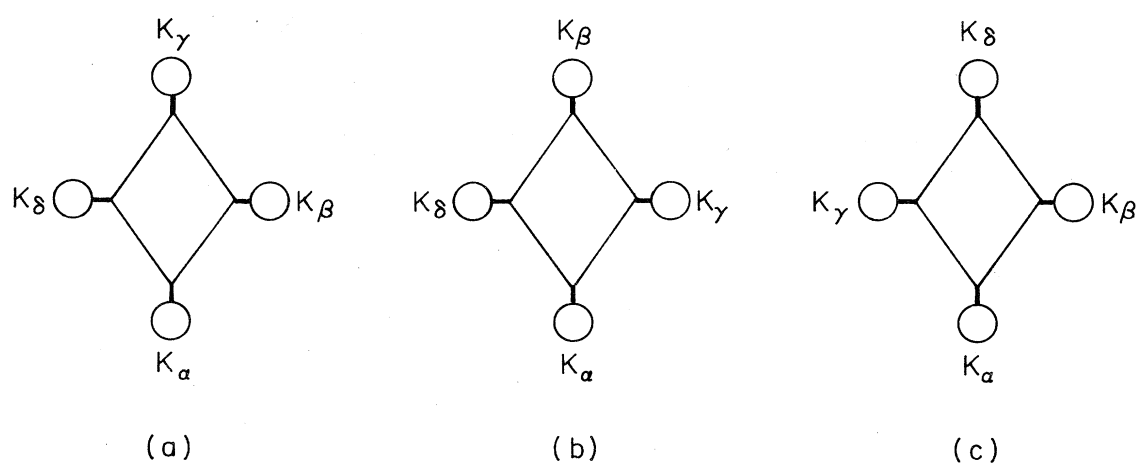

FIG. 2. The set of source arrangements necessary to obtain the complete lowest-order double-spectral form.

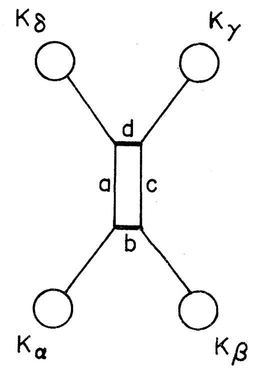

FIG. 3. Causal process leading to single-spectral form.

explicitly given by

$$
\begin{aligned}
-\int(dx)\cdots(dx''')\,\mathrm{tr}[&e\gamma^{\mu}A_{2\mu}(x)G_{+}(x-x')\\
&\times e\gamma^{\nu}A_{3\nu}(x')G_{+}(x'-x'')\\
&\times e\gamma^{\lambda}A_{1\lambda}(x'')G_{+}(x''-x''')\\
&\times e\gamma^{\sigma}A_{3'\sigma}(x''')G_{+}(x'''-x)]\,. \qquad (15)
\end{aligned}
$$

Here A is the photon field, associated with the source J, while G_+ is the electron (charge e) propagation function, and tr means trace over the γ-matrix structure.

First the vacuum amplitude (15) is rewritten to bring out the structure associated with the exchange of the compound excitations, i.e., to obtain the analog of Eq. (2). As noted, the steps leading from Eq. (1) to Eq. (2) employ the vacuum amplitude in momentum space. In the present instance we there write

$$A^{\mu}(k)=\epsilon^{\mu}A(k)\,, \qquad (16)$$

where $\epsilon^{\mu}(k)$ is the photon polarization vector. Then, similarly to the scalar case, and labeling the external momenta as there, we obtain

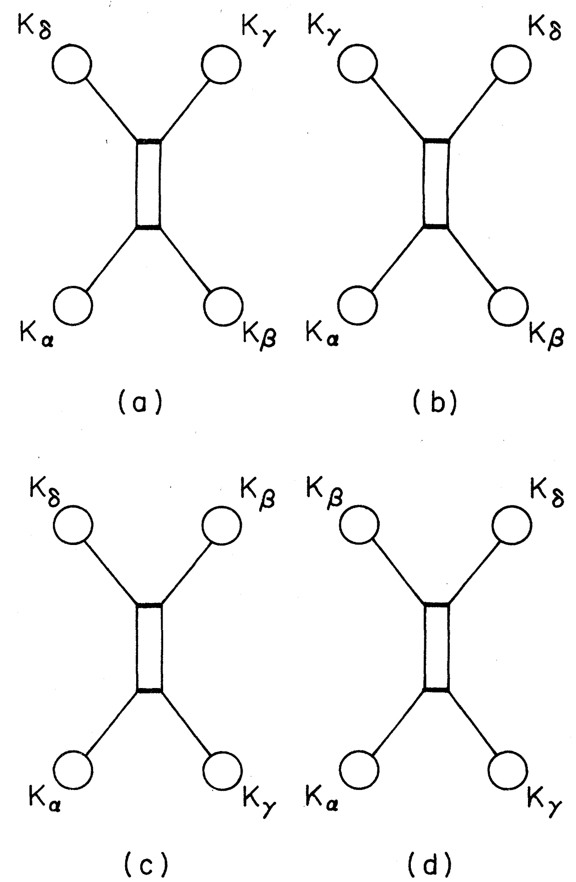

FIG. 4. The four types of causal processes necessary to obtain the complete lowest-order single-spectral form.

$$
\begin{aligned}
i\tfrac{1}{2}\int(dx)(d\xi)(d\xi')&A_{1}(x)A_{3}(x-\xi)\\
&\times A_{3'}(x-\xi')A_{2}(x-\xi-\xi')\\
&\times\Delta_{+}(\xi,M^{2})\Delta_{+}(\xi',M'^{2})i\bar{\rho}(M^{2},M'^{2})dM^{2}dM'^{2}\,, \qquad (17)
\end{aligned}
$$

with

$$
\begin{aligned}
\bar{\rho}=-8\alpha^{2}\int(dp)\,&\delta((p-k_{\alpha}-k_{\beta})^{2}+m^{2})\,\delta((p-k_{\beta})^{2}+m^{2})\,\delta(p^{2}+m^{2})\,\delta((p+k_{\gamma})^{2}+m^{2})\\
&\times\mathrm{tr}\{\epsilon\gamma[m-\gamma(p-k_{\alpha}-k_{\beta})]\,\epsilon\gamma[m-\gamma(p-k_{\beta})]\,\epsilon\gamma(m-\gamma p)\epsilon\gamma[m-\gamma(p+k_{\gamma})]\}\,, \qquad (18)
\end{aligned}
$$

α being the fine-structure constant, and m the electron mass. For simplicity in calculations below, the polarizations of the four photons have here been chosen as equal, and furthermore we shall take them as perpendicular to the scattering plane of the photons. (So, with the scattering plane and polarizations of the real photons in the final scattering situation considered as stated, the virtual photons in the causal process are taken to be in this plane and have the given polarizations.)

The four δ functions in Eq. (18) completely determine p, apart from the sign of the component perpendicular to the scattering plane, the explicit determination being aided by working in, say, the $\alpha\beta$ center-of-mass frame. So, with this p being substituted into the explicit evaluation of the trace, the integration over p is immediate. This work is simplified by taking $-p_{\alpha}^{2}=-p_{\gamma}^{2}=p_{\beta}^{2}=p_{\delta}^{2}=\overline{M}^{2}$, which may be done since all these p_{i}^{2} will eventually be extrapolated to the on-shell value zero. The result is

$$\bar{\rho}=4\alpha^2\frac{\lambda\lambda'-\overline{M}^4/16m^4-2[1-\lambda\lambda'/(\lambda+\lambda')]^2}{[(-\lambda\lambda'+\overline{M}^4/4m^4)(\lambda\lambda'-\lambda-\lambda')]^{1/2}}\,, \tag{19}$$

where

$$\begin{aligned}-(k_\alpha+k_\beta)^2&=M^2=4m^2\lambda\,,\\ -(k_\alpha+k_\delta)^2&=M'^2=4m^2\lambda'\,.\end{aligned} \tag{20}$$

And the range over which M^2 and M'^2 vary is determined just as in the scalar case.

Next, as noted in Sec. II, in order to render the vacuum amplitude (17) applicable to the usual scattering situation, a space-time generalization and mass extrapolation of it must be carried out. Concerning the former, we first must take into account an important aspect of electrodynamics—gauge invariance. The original vacuum-amplitude expression, Eq. (15), is gauge-invariant,[10] and hence so is Eq. (17) because the two equations are mathematically equivalent. But when Eq. (17) is space-time generalized, a new result is obtained, and it is not guaranteed that this result will have maintained the gauge invariance, as it should. In order that Eq. (17) does maintain its gauge invariance upon space-time generalization, we first rewrite it in manifestly gauge-invariant form by expressing it in terms of the field-strength tensor.

To this end we consider the vacuum amplitude when expressed in momentum space during the transition from Eq. (15) to Eq. (17). The field-strength tensor is

$$\begin{aligned}F^{\mu\nu}(k)&=i[k^\mu A^\nu(k)-k^\nu A^\mu(k)]\\ &=i(k^\mu\epsilon^\nu-k^\nu\epsilon^\mu)A(k)\,.\end{aligned} \tag{21}$$

From it we can form scalars quartic in the fields such as

$$\begin{aligned}&[F_2(k_\alpha)F_3(k_\beta)][F_1(k_\gamma)F_{3'}(k_\delta)]\\ &\quad\equiv[F_2{}^{\mu\nu}(k_\alpha)F_{3\mu\nu}(k_\beta)][F_1{}^{\lambda\sigma}(k_\gamma)F_{3',\lambda\sigma}(k_\delta)]\\ &\quad=4(k_\alpha k_\beta)(k_\gamma k_\delta)A_2(k_\alpha)A_3(k_\beta)A_1(k_\gamma)A_{3'}(k_\delta)\,,\end{aligned} \tag{22}$$

where use was made of $\epsilon k_{\alpha,\beta,\gamma,\delta}=0$, as is appropriate for the above choice of polarizations. And from Eq. (20) this may be written as

$$(F_2F_3)(F_1F_{3'})=M^4A_2A_3A_1A_{3'}\,. \tag{23}$$

When the vacuum amplitude, thus expressed in terms of the field-strength tensor, is space-time generalized and then brought back to momentum space, one has in it, since Eq. (20) is no longer applicable,

$$(F_2F_3)(F_1F_{3'})/M^4=[-(k_\alpha+k_\beta)^2/M^2]^2A_2A_3A_1A_{3'}\,. \tag{24}$$

The above, however, is not the only way to express the vacuum amplitude in gauge-invariant form. There are three linearly-independent quartic structures, and they may be taken as Eq. (23) plus

$$(F_2F_{3'})(F_1F_3)=M'^4A_2A_3A_1A_{3'} \tag{25}$$

and

$$\begin{aligned}&(F_2F_1)(F_3F_{3'})-(F_2F_3)(F_1F_{3'})-(F_2F_{3'})(F_1F_3)\\ &\qquad=2(M^2M'^2-2\overline{M}^4)A_2A_3A_1A_{3'}\,.\end{aligned} \tag{26}$$

These three expressions for $A_2A_3A_1A_{3'}$ are of course equivalent in the causal situation, but after space-time generalization they are not. It is thus ambiguous just as to how the causal vacuum amplitude should be reexpressed in gauge-invariant form and space-time generalized. But the ambiguity is of a rather simple type. The difference between the space-time-generalized $A_2A_3A_1A_{3'}$ expressions provided by Eqs. (26) and (23) [similarly (25)] is $A_2A_3A_1A_{3'}$ multiplied by

$$\begin{aligned}&\left(\frac{-(k_\alpha+k_\beta)^2}{M^2}\right)\left(\frac{-(k_\alpha+k_\delta)^2}{M'^2}\right)-\left[\frac{-(k_\alpha+k_\beta)^2}{M^2}\right]^2\\ &\qquad=\frac{-(k_\alpha+k_\beta)^2}{M^2}\left\{\frac{1}{M^2}[(k_\alpha+k_\beta)^2+M^2]\right.\\ &\qquad\qquad\left.-\frac{1}{M'^2}[(k_\alpha+k_\delta)^2+M'^2]\right\}\,,\end{aligned} \tag{27}$$

where the $\overline{M}\to0$ of mass extrapolation has been taken. And the $(k_\alpha+k_\beta)^2+M^2$ and $(k_\alpha+k_\delta)^2+M'^2$ factors here cancel against those factors in the spectral denominator. Thus, the ambiguity is that to the known double-spectral form, expressed using Eq. (26), one must add unknown single-spectral forms. Terms like the latter, local in the dis-

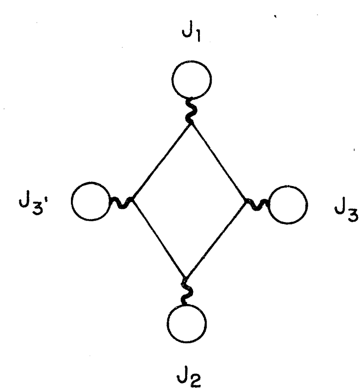

FIG. 5. Causal process leading to four-photon double-spectral form. The wavy lines refer to photons, and the straight ones to electrons.

placement of a compound excitation, are called contact terms. Shortly we will see in the present instance how they are determined.

With regard now to mass extrapolation, we first note that the Δ defined in the scalar case here reduces to

$$\Delta = 16m^8\lambda\lambda'(\lambda\lambda' - \lambda - \lambda'). \tag{28}$$

The argument for the change in the spectral-mass domain as the external particles are brought on shell is the same as that developed in the scalar work, it being independent of the additional quantum numbers of the present particles. Thus, the spectral-mass domain is given by $\Delta \geq 0$ with $M, M' \geq 2m$, or equivalently, by $\lambda\lambda' - \lambda - \lambda' \geq 0$ with $\lambda, \lambda' \geq 1$. The square root of negative argument thereby appearing in the weight function (19)—and this is the only place where there is any question about the $\overline{M} \to 0$ behavior in the vacuum amplitude—is exactly the one occurring in the scalar case, so we follow the prescription obtained there; namely, $[-\lambda\lambda'(\lambda\lambda' - \lambda - \lambda')]^{1/2} = +i[\lambda\lambda'(\lambda\lambda' - \lambda - \lambda')]^{1/2}$.

With all this, then, the four-photon vacuum amplitude has been developed to the point analogous to the scalar result (6). Next, the contributions due to additional causal processes must be considered. First, there is an additional one with the sources taken as in the original causal process, Fig. 5. In that causal process an electron and positron propagate from A_2 to A_1, one scattering off A_3, and the other off $A_{3'}$. What also must be included is the process in which the electron and positron are interchanged. This corresponds to interchanging A_3 and $A_{3'}$ in the initial vacuum amplitude (15), and it is easily argued that the resulting spectral form is the same as that obtained from the original causal process.

Second, the contributions from different source arrangements must be included, analogous to the situation associated with Fig. 2. In the present case these additional contributions must be included because, given the momenta of the external particles, they complete the different ways in which these momenta can be coupled together. The calculation proceeds exactly as for the original source arrangement with the appropriate simple interchanges, indicated in Eq. (9).

These considerations, then, bring us to the final expression for the four-photon vacuum amplitude. With the definitions

$$k = k_\alpha + k_\beta,\ \ k' = k_\alpha + k_\delta,\ \ k'' = k_\alpha + k_\gamma, \tag{29}$$

the result is given by ($-i\epsilon$'s suppressed)

$$i\int[(d\underline{k})A(\underline{k})](2\pi)^4\delta(k_\alpha + k_\beta + k_\gamma + k_\delta)$$
$$\times\left\{\int dM^2 dM'^2 \frac{\rho(M^2, M'^2)}{M^2M'^2}\left[\frac{k^2k'^2}{(k^2+M^2)(k'^2+M'^2)} + \frac{k^2k''^2}{(k^2+M^2)(k''^2+M'^2)} + \frac{k'^2k''^2}{(k'^2+M^2)(k''^2+M'^2)}\right]\right.$$
$$\left.+\int dM^2\frac{\sigma(M^2)}{(M^2)^2}\left[\frac{(k^2)^2}{k^2+M^2} + \frac{(k'^2)^2}{k'^2+M^2} + \frac{(k''^2)^2}{k''^2+M^2}\right]\right\}. \tag{30}$$

The double-spectral weight function is

$$\rho = 4\alpha^2\frac{\lambda\lambda' - 2[1 - \lambda\lambda'/(\lambda+\lambda')]^2}{[\lambda\lambda'(\lambda\lambda' - \lambda - \lambda')]^{1/2}} \tag{31}$$

and σ is the weight function, not yet determined, of the single-spectral forms introduced by gauge-invariance considerations. (By symmetry, the same weight function applies to all three of the single-spectral forms.)

To determine σ we calculate the complete vacuum amplitude as a single-spectral form and compare Eq. (30) with such. Thus, we consider the causal process of Fig. 6 and proceed to the analog of Eq. (10). To ensure gauge invariance we use, at the causal level, the substitution [cf. Eq. (23)]

$$[F_2(k_\alpha)F_{2'}(k_\beta)][F_{1'}(k_\gamma)F_1(k_\delta)]$$
$$= M^4A_2(k_\alpha)A_{2'}(k_\beta)A_{1'}(k_\gamma)A_1(k_\delta). \tag{32}$$

There is no ambiguity of substitution in the present case because there is only one spectral variable.

The additional contributions related to the latter three causal processes in Fig. 4 and specified by

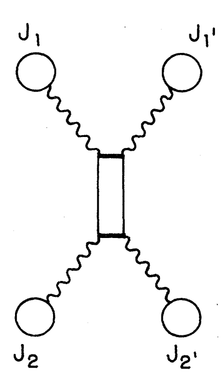

FIG. 6. Causal process leading to four-photon single-spectral form.

the interchanges of Eq. (13) must also be included. Because of the symmetry in the present example, the weight functions associated with the analogs of Figs. 4(a) and 4(c) are equal, as are those of Figs. 4(b) and 4(d). That is, $M^2=-(p_\alpha+p_\beta)^2$ or $-(p_\alpha+p_\gamma)^2$, respectively, enters identically in the two causal processes, and so does $(p_\beta+p_\gamma)^2$, the only other variable on which the weight function depends, it being unchanged by the interchange between the two processes. Furthermore, in addition to the four causal processes considered, we must also include those which differ from each of these by interchanging the exchanged electron and positron (or, if you prefer, by interchanging both the incoming photons and the outgoing photons). It is easily shown that each of these contributions is equal to its original counterpart. Thus, the complete single-spectral form is given as

$$i\int[(d\underline{k})A(\underline{k})](2\pi)^4\delta(k_\alpha+k_\beta+k_\gamma+k_\delta) \times\int dM^2\frac{\chi(M^2)}{(M^2)^2}\left[\frac{(k^2)^2}{k^2+M^2}+\frac{(k''^2)^2}{k''^2+M^2}\right], \quad (33)$$

with

$$\chi=-16\pi\alpha^2\Big\{\int d\omega_{p_a}d\omega_{p_c}(2\pi)^4\delta(k_\alpha+k_\beta-p_a-p_c) \times[(k_\alpha-p_a)^2+m^2]^{-1}[(k_\gamma+p_c)^2+m^2]^{-1} \times\mathrm{tr}\{\epsilon\gamma[m-\gamma(-p_a)]\epsilon\gamma[m-\gamma(k_\alpha-p_a)] \times\epsilon\gamma(m-\gamma p_c)\epsilon\gamma[m-\gamma(k_\gamma+p_c)]\} +(k_\gamma\to k_\delta)\Big\}, \quad (34)$$

M running from $2m$ to ∞. Expression of the integral (34) in the $\alpha\beta$ center-of-mass frame aids its explicit evaluation, with the result then being recast in terms of Lorentz scalars.

For our present task of determining σ, it is not necessary in the vacuum amplitude (33) to keep $(p_\beta+p_\gamma)^2$ as an arbitrary momentum transfer, since σ depends only on M^2. (That is, the contribution involving σ, being local in one of the displacements, is like a two-point function and so depends on only one invariant, as distinct from χ.) So we consider just forward scattering, $p_\beta=-p_\gamma$. The evaluation of the weight function χ that must be carried out is thereby simplified, and the result is

$$\chi_0\equiv\chi((p_\beta+p_\gamma)^2=0) =4\alpha^2[-(1+\tfrac{3}{2}\lambda^{-1})(1-\lambda^{-1})^{1/2} +2(1+\lambda^{-1}-\tfrac{3}{4}\lambda^{-2})\cosh^{-1}(\lambda^{1/2})]. \quad (35)$$

Now, the vacuum amplitude (30), with $k'^2\equiv(p_\beta+p_\gamma)^2$ taken as a momentum transfer,[8] and the vacuum amplitude (33) describe the same physical situation. Thus they must be equal, and, in particular, must be so for $k'^2=0$ (whence $k''^2=-k^2$). From this equality, upon taking $k^2\geq 4m^2$ and expressing $(k^2+M^2-i\epsilon)^{-1}$ as the sum of principal-part and δ-function terms, we extract the relation

$$-\int dM'^2\frac{\rho(M^2,M'^2)}{M^2M'^2}\frac{1}{M^2+M'^2}+\frac{\sigma(M^2)}{(M^2)^2}=\frac{\chi_0(M^2)}{(M^2)^2}. \quad (36)$$

The M' integration, whose range is determined by $\lambda\lambda'-\lambda-\lambda'\geq 0$, can be carried out upon substitution of λ^{-1} and λ'^{-1}, and the result gives

$$\sigma=32\alpha^2[-(\tfrac{1}{4}+\lambda^{-1})(1-\lambda^{-1})^{1/2} +(\tfrac{1}{2}+\lambda^{-1})\cosh^{-1}(\lambda^{1/2})]. \quad (37)$$

The derivation of the representation (30) is thus completed.[11] From that vacuum amplitude one can extract an expression for the scattering amplitude, and hence for the differential cross section, by following the general procedures for such given in Schwinger's text.[12] We are not here concerned with cross section calculations, but for the sake of reference we record the cross section formula for our conventions. Namely, with the term in the { } in Eq. (30) called $\mathfrak{M}$, and with the photons of momenta k_α and k_β taken as the two incoming ones, the differential cross section is given as

$$\frac{d\sigma}{d\Omega}=\frac{1}{64\pi^2}\frac{1}{-(k_\alpha+k_\beta)^2}|\mathfrak{M}|^2. \quad (38)$$

This formula refers to the $\alpha\beta$ center-of-mass frame, with $d\Omega$ being the element of solid angle for the relative spatial momentum of the final state.

IV. DISCUSSION

The complete vacuum amplitude, either as the single-spectral form (33) or the double-spectral form (30), is composed of more than one contribution. In this section we compare our method and Schwinger's method[5,10] for specifying these complete sets of contributions.

Central to Schwinger's scheme is the concept of the unity of the source. This means that the space-time–generalized vacuum amplitude should be expressible not only in terms of the field (or source) "pieces" referring to the individual external particles, but also, and solely, in terms of the total field, which is the sum of all these individual pieces. That is, the vacuum amplitude, in being a structure of general space-time validity, is taken to be dependent on only the acausal total field, rather than on just the causally-related field terms of the individual particles.

For the examples of this paper, unity of the source is applied as follows. Consider, in configuration space, the vacuum-amplitude contributions that have resulted from the causal processes of one given source arrangement—e.g., the double-spectral form (17) (rendered gauge-invariant, and multiplied by 2 to take into account the causal process with electron and positron interchanged). Then, after space-time generalization and mass extrapolation, this structure is expressed in terms of the total field, which means that in it one makes the replacement

$$A_1 A_3 A_{3'} A_2 \to \tfrac{1}{8} AAAA. \tag{39}$$

Here

$$A = A_1 + A_2 + A_3 + A_{3'} \tag{40}$$

is the total field, and the factor of $\frac{1}{8}$ is to reduce to unity the number of times the left-hand side (as written or in equivalent permutation) is included in the right-hand side in Eq. (39). What has been introduced by the replacement (39) is the additional vacuum-amplitude contributions with the field products $A_1A_2A_{3'}A_3$ and $A_1A_3A_2A_{3'}$, and these along with the $A_1A_3A_{3'}A_2$ term are taken to provide the complete vacuum amplitude to the given order. (Of course, there are also products in which a given label occurs more than once, but the associated vacuum-amplitude terms do not contribute for the conventional scattering circumstances, where the sources of the four external particles are separated in space-time.) In a similar way, proceeding from one source arrangement (e.g., Fig. 6), one generates the complete single-spectral form.

It is quite natural to generate the complete vacuum amplitude by application of unity of the source when treating systems with identical particles, but otherwise, this procedure is not available. And it was in these latter circumstances that the work of Ref. 2 was carried out, with the procedure adopted in the present paper for obtaining the complete vacuum amplitude being simply a reduction of that developed in Ref. 2. That is, we have fitted the present situation into the context of that previous work by employing in the role of the distinguishing particle properties the different momenta of the external particles.

When identical particles are not present, the most natural attitude for obtaining the complete double-spectral form is simply to admit the contributions of all different[13] causal processes of the given type because, upon space-time generalization and mass extrapolation, they all refer to the same scattering circumstances. For the complete single-spectral form this statement is somewhat amended. Namely, since the momentum transfer $[(p_\beta + p_\gamma)^2]$ of the original causal process is held fixed as such throughout the entire calculation, as opposed to the spectral-invariant $[(p_\alpha + p_\beta)^2]$, we take it that only those causal processes in which $(p_\beta + p_\gamma)^2$ thus appears should be considered. In this way, then, we obtain the complete vacuum-amplitude structures developed in Ref. 2 and applied here. Since no contact terms are expected in the scalar case, a check on the results in Ref. 2 is that the complete double-spectral form, with $(p_\beta + p_\gamma)^2$ taken as a momentum transfer,[8] should be equal to the complete single-spectral form; this indeed has been verified.

So, with all this prelude, let us now compare our vacuum-amplitude results with those of Schwinger. To this end we express the latter, given three paragraphs above in configuration space, in the momentum-space forms of Eqs. (30) and (33) (where the individual-particle fields have been reinstated). It is then easily seen that the two ways of obtaining the complete double-spectral form give the same result, while for the single-spectral form Schwinger's result differs from ours [Eq. (33)] by the addition to the sum of spectral propagators in that equation of the term $(k'^2)^2/(k'^2 + M^2)$. We should immediately note, though, that for their purpose in this paper and in Schwinger's work[5]—the determination of a contact term—the two single-spectral forms are identical because they are applied only for forward scattering, $k'^2 = 0$. However, if one employs the unity-of-the-source scheme as above for nonforward single-spectral forms—an approach which Schwinger neither encourages nor cautions against—then a real difference does develop. And note that the extra term in k'^2 is one that we would be rather suspicious about: k'^2 has been taken as a fixed momentum transfer, but in this term it comes out appearing as a general spectral variable.

Pair creation is another example where our scheme for obtaining complete single-spectral forms may be compared with that of Schwinger. In particular, he has carried out[5] the calculations in the instance of forward scattering with spin-0 charged particles. He considers the two causal processes of Fig. 7(a),[14] and upon source unification no additional contributions result. This present constant-angle situation is not one of constant momentum transfer, and it is the latter that our developments have been framed in terms of. But it seems quite reasonable to apply the basic reasoning there also to situations of fixed angle. So, according to the discussion above, we would consider *all* causal processes in which the angle can assume the given value. Thus, with forward scattering taken to refer to the particles associated with the sources J and K, we would consider in addition to the causal processes of Fig. 7(a), those of Fig. 7(b).[15] However, Schwinger employs his single-spectral form only to determine a contact term, analogous to the comparison (36), and the extra terms associated with Fig. 7(b) would not enter there, their spectral denominators being nonvanishing in the momentum range of the comparison. (Note, though, that single-spectral forms are of interest for purposes other than determining contact terms.[16]) But there are indications in Schwinger's work of the necessity of a contribution such as that provided by the extra terms. Namely, in the forward-scattering reduction of the pair-creation double-spectral form [see Eq. (4–13.152) in Ref. 5], there occurs a piece which is not matched in his single-spectral form and which has the same spectral-denominator structure as our extra terms. It would be of interest to explicitly carry out the calculations for the extra terms and see if they are exactly what is needed.

Unity of the source is a basic element of source theory, finding much wider application than the few examples mentioned here. And we should be quick to point out that the two disagreements with its application that we have discussed above do not bring its whole use into question. Rather, the explanation seems much simpler. Namely, this unification, in referring to the equivalence of different causal dispositions of the source, is a symmetric operation and so would seem to presume a symmetric vacuum-amplitude structure. But the single-spectral form is an asymmetric structure, treating quite differently the variables $(p_\alpha + p_\beta)^2$ and $(p_\beta + p_\gamma)^2$. So it simply appears that such structures are not suitable for application of the unification. On the other hand, the other methods of calculation developed in source theory—double-spectral forms and the so-called noncausal methods[17]—treat all variables on an equal footing. Such would thus appear to be the natural places to apply the unification. Also, there are the applications of unity of the source for single-particle-exchange processes,[12] and these too are appropriately symmetric situations.

ACKNOWLEDGMENT

As noted,[3] part of this work was done some time ago, when the author was a student under Professor Julian Schwinger. Professor Schwinger's instruction then, as well as afterwards, is gratefully appreciated.

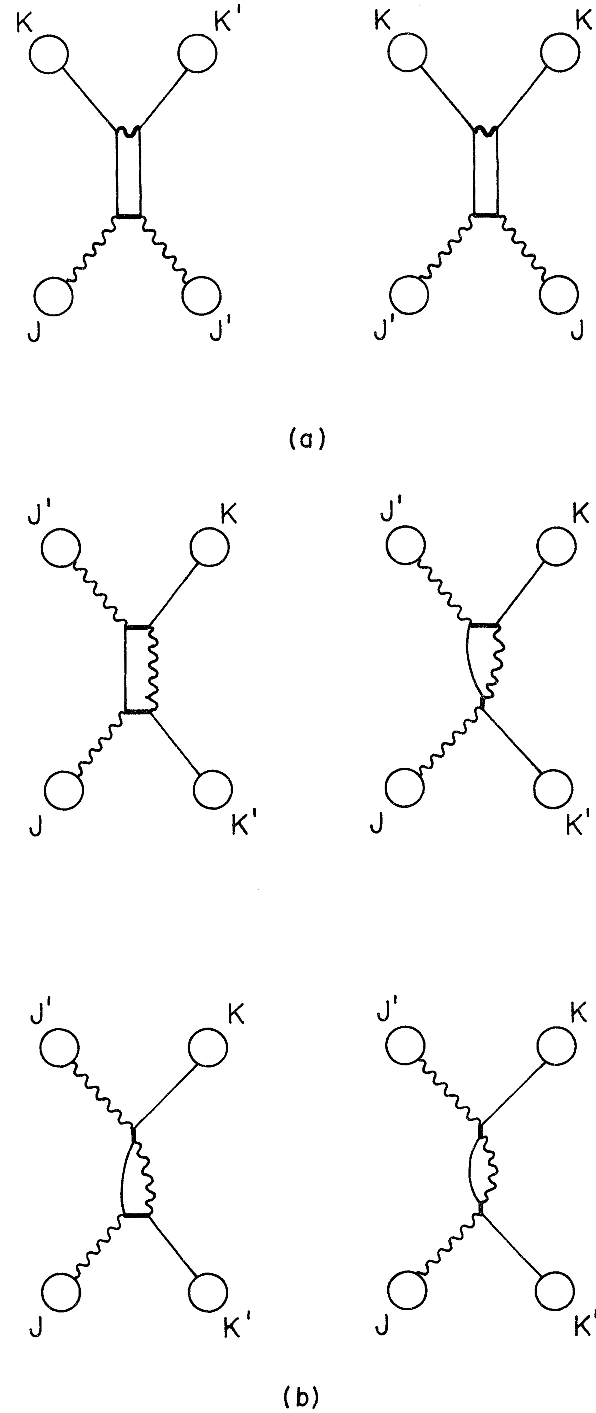

FIG. 7. (a) The two causal processes considered in Schwinger's calculation of the pair-creation single-spectral form. (b) The additional processes that would be brought in according to the approach of Ref. 2.

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[1]R. J. Ivanetich, Phys. Rev. D 6, 2805 (1972).

[2]R. J. Ivanetich, preceding paper, Phys. Rev. D 8, 4545 (1973).

[3]The calculations for a good part of this paper, namely, most of Sec. III, were done a few years ago, but it was felt that the exposition of spectral forms in source theory would be better served by presenting this work after the studies of Refs. 1 and 2 had been completed and published. At that same time Schwinger independently carried out the same calculations, and they will appear in the second volume of his text on source theory (Ref. 5). Also to appear there is his later calculation of the spectral representation for Compton scattering, another illustrative and somewhat more involved example of the additional techniques necessitated by the presence of internal quantum numbers. These calculations are for spin-0 charged particles; the work for spin-$\frac{1}{2}$ Compton scattering was subsequently carried out by K. A Milton, W.-y. Tsai, and L. L. DeRaad, Jr., Phys. Rev. D 6, 1411 (1972); 6, 1428 (1972). Several further examples, all involving either two- or three-point functions, are included in the works discussed in Ref. 9.

[4]B. DeTollis, Nuovo Cimento 32, 757 (1964). See also B. DeTollis, *ibid.* 35, 1182 (1965), and B. DeTollis and G. Violini, *ibid.* 41A, 12 (1966).

[5]J. Schwinger, *Particles, Sources, and Fields* (Addison-Wesley, Reading, Mass., to be published), Vol. 2.

[6]Certain restrictions on the masses are required in the derivation of this spectral form (see Ref. 2), and likewise for the single-spectral form given below. The example of this paper satisfies these restrictions.

[7]An extrapolation argument is required in obtaining this full range, except in the instance of forward scattering; see Ref. 2.

[8]The derivation of this complete result admits for $(p_\beta+p_\gamma)^2$ only a portion of the full range allowed a momentum transfer; see Ref. 2. When, below, the single- and double-spectral forms are compared, it is for this limited range, which includes $(p_\beta+p_\gamma)^2=0$.

[9]Although just a general knowledge of electrodynamics should suffice for reading the present paper, let us note the basic source-theory references on the subject. The matter was first treated by Schwinger in Ref. 10 [wherein one can find Eq. (15) in noncausal form]. Its systematic development is given in the two volumes of his text, Refs. 5 and 12. Also, some aspects are briefly, but lucidly, presented in J. Schwinger, *Particles and Sources* (Gordon and Breach, New York, 1969).

[10]J. Schwinger, Phys. Rev. 158, 1391 (1967).

[11]This representation has also been calculated for the case of spin-0 charged particles. The result is of the given form with the weight functions

$$\rho=4\alpha^2[1-\lambda\lambda'/(\lambda+\lambda')]^2\,[\lambda\lambda'(\lambda\lambda'-\lambda-\lambda')]^{-1/2}$$

and

$$\sigma=16\alpha^2[(\tfrac{1}{4}+\lambda^{-1})(1-\lambda^{-1})^{1/2}-\lambda^{-1}\cosh^{-1}(\lambda^{1/2})].$$

[12]J. Schwinger, *Particles, Sources, and Fields* (Addison-Wesley, Reading, Mass., 1970).

[13]By different we mean two causal processes which simply are not two alternative ways of calculating the same space-time–generalized mass-extrapolated result. For example, the causal process of Fig. 1 and the one "rotated from it by 90°" are not different.

[14]For simplicity, the further causal processes involving the point coupling of two photons and two spin-0 particles have not been included in Fig. 7; the modification of the discussion in the text to take into account these processes is simple and straightforward. The lowest-order processes are also omitted in Fig. 7.

[15]There is nothing special about the occurrence of the latter three causal processes in Fig. 7(b), which are of different structure from those considered previously; such could also have occurred in the scalar case, but for simplicity we assumed that the interactions which would lead to such did not exist. The basic point is that once an arrangement of sources is chosen—in the present instance an arrangement associated with a single-spectral form—it is physically meaningful only to admit all causal processes that may then occur (to given order).

Also, there are contributions from single-particle-exchange processes with vertex modifications. But, since these involve the form factors evaluated with all particles on shell, they are absent in the present situation of spin-0 charged particles because then there is only a charge form factor, for which the said evaluation is unity. That is, in the present situation the single-particle-exchange contribution is completely described by the lowest-order result.

[16]In particular, single-spectral forms would seem to present a simpler way than double-spectral forms for treating three-body decays and situations with anomalous thresholds. And, of course, they provide the first nontrivial example of determining a complete set of causal processes, which is a matter that must be well understood in proceeding to a study of spectral forms for higher-point functions.

[17]See Ref. 5; for a brief example see Eq. (30) and accompanying text in Ref. 2.