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Tautologies and Optimization of N/D Equations

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It is usually accepted that the solutions of the N/D equations are seriously affected by the uncertainties of the input data as well as by the lack of information about short-range forces (distant regions of the left-hand cut of the scattering amplitude). However, so far little attention seems to have been paid to a systematic investigation of the mathematical ways in which the influence of this lack of knowledge could be minimized. To this end, it appeared desirable to also include as input data the real part of the amplitude which can be found in essentially the same way as the imaginary part by analytic continuation from the crossed reactions. Then, the basic idea is to exploit the fact that different theoretical methods which are logically equivalent and yield exactly the same results for the same set of "correct" data (we call these methods "tautological") nevertheless could behave in totally different ways when faced with "wrong" (error-affected) data. "Correct" in our case would mean left-hand-cut limiting values *absolutely* consistent both with analyticity and unitarity. Hence, the problem that arises is first to find all the possible tautological integral kernels and then to choose among them that one which is most insensitive to the lack of knowledge concerning the input data. Such a kernel is shown to be obtained by replacing one of the Cauchy kernels in the integral equations by a Poisson kernel weighted with a function determined by the errors. Optimal methods of construction, not based on integral equations, are also presented.

I. INTRODUCTION

As is well known, in relativistic scattering it is always possible to evaluate the partial waves (both their real and imaginary parts) on their left-hand cut by analytic continuation from the physical regions of the crossed reactions.[1] Then solving the relativistic N/D equations[2] essentially comes to performing a new analytic extrapolation from the left-hand cut Γ to the right-hand one γ, where the amplitude is moreover subjected to the unitarity condition

$$\operatorname{Im}\mathcal{A}_l(s)=\rho(s)\,|\mathcal{A}_l(s)|^2, \tag{1.1}$$

where $\rho(s)$ is a given function. So stated, the problem departs a bit from the conventional N/D one, as the latter starts solely from the imaginary part of the amplitude on Γ. Nevertheless, there is no *a priori* strong reason to throw away the information about the real part which can be obtained from the crossed reactions essentially in the same way as the imaginary part.[3] On the other hand, the use of the whole amplitude gives one much more flexibility in handling dispersion relations and leads one to trying to write integral equations optimized with respect to the input errors. This is an important question as in practice the knowledge of the amplitude is always limited to a small part Γ_1 of the left-hand cut Γ where, moreover, it is affected by errors. As the conventional N/D equations require the imaginary part over the whole Γ, the absence of information on $\Gamma_2=\Gamma\setminus\Gamma_1$ (the complement of Γ_1 with respect to the set Γ) as well as the errors on Γ_1 will cause their solution to depart in an uncontrollable way from the true amplitude. The purpose of this paper is just to optimize the way of writing N/D integral equations to make them as insensitive as we can to the errors of the input data.

Indeed there are many (infinitely many) ways of writing mathematically equivalent equations – i.e., logically deducible from each other and having exactly the same solutions – as long as the data we have to use are given with infinite accuracy. We will say that all these equations are tautologically equivalent. However, as long as the input data (or the calculation) are error-affected, the tautology may be broken, i.e., the results of formally equivalent equations are no longer identical. This can be seen on the simple example of dispersion relations:

If $f(z)$ is a holomorphic function in a domain D bounded by Γ, its values can be computed at any interior point either by writing a usual Cauchy integral

$$f(z)=\frac{1}{2\pi i}\oint_\Gamma \frac{f(t)}{t-z}\,dt \tag{1.2}$$

or by any of the weighted Cauchy integrals

$$f(z)=\frac{1}{2\pi i\, g(z)}\oint_\Gamma \frac{f(t)\,g(t)}{t-z}\,dt, \tag{1.3}$$

where $g(z)$ is a function holomorphic in D.

When $f(t)$ on the right-hand side of (1.2) and (1.3) actually takes on the boundary values of $f(z)$, the results of the integrals are of course identical. However, if instead of $f(t)$ we are forced to use approximate values (which in general are not the boundary values of functions holomorphic in D), or if owing to the lack of information on some part Γ_2 of Γ we restrict the range of integration only to the *open* contour $\Gamma_1 = \Gamma \setminus \Gamma_2$, the values of $f(z)$ computed by (1.2) and (1.3), respectively, may differ by arbitrarily large amounts. Therefore, it is meaningful to choose that $g(z)$ weight function which makes the calculation of $f(z)$ least sensitive to the uncertainties of its boundary values; for dispersion relations this has been solved.[4] A completely similar question arises in the N/D problem, in choosing those integral kernels which yield solutions least affected by the limitation of our knowledge concerning the left-hand cut.

At this point it may be worthwhile to notice that there may exist some subgroups of the initial tautology transformation group which leave the equations and their solutions invariant even if the data on the left-hand cut are not accurate. These subgroups are obviously meaningless for our optimization problem. For instance, changing the subtraction point in the dispersion relation for D, always amounts to multiplying D by a constant factor, irrespective of the uncertainties of the input data. This is a consequence of the linearity of the D equation. Indeed, if D obeys an equation of the type

$$D(z) = 1 + (\mathsf{K}D)(z) \tag{1.4}$$

with K a linear operator, one can obtain the subtracted equation by adding to (1.4) its value at a specified point z_0:

$$0 = D(z_0) - 1 - (\mathsf{K}D)(z_0). \tag{1.5}$$

Then D obeys

$$D(z) = D(z_0) + (\mathsf{K}D)(z) - (\mathsf{K}D)(z_0). \tag{1.6}$$

By dividing (1.6) by $D(z_0)$ one can see that $D'(z) = D(z)/D(z_0)$ is the solution of the subtracted equation

$$D'(z) = 1 + (\mathsf{K}_{z_0}D')(z), \tag{1.7}$$

where K_{z_0} is the subtracted kernel.

Since only the linearity of the equations was involved, the subtraction point is irrelevant to the optimization problem.

After some technical preliminaries in Sec. II, in Sec. III we shall look for that tautology τ_0 which realizes the minimum

$$\min_{\tau} \| \mathsf{K}_\tau - \mathfrak{K}_\tau \| \tag{1.8}$$

of the norm of the difference between the operators K_τ and $\mathfrak{K}_\tau$ of the exact and approximate equations for D,

$$D = 1 + \mathsf{K}_\tau D, \tag{1.9}$$

$$\mathfrak{D}_\tau = 1 + \mathfrak{K}_\tau \mathfrak{D}_\tau. \tag{1.10}$$

(Here 1 is the constant function equal to unity.) Notice that the solution of the exact equation (1.9) is independent of the choice of the tautological kernel K_τ (this is the whole point) whereas the solution of (1.10) is *not*.

Of course one would like to minimize directly the norm of the difference between the two resolvents

$$\begin{aligned}\mathfrak{R}_\tau - \mathsf{R}_\tau &\equiv (I - \mathfrak{K}_\tau)^{-1} - (I - \mathsf{K}_\tau)^{-1} \\ &= \mathfrak{R}_\tau(\mathfrak{K}_\tau - \mathsf{K}_\tau)\mathsf{R}_\tau,\end{aligned} \tag{1.11}$$

which is seen not to be identical to (1.8). However, in physics we are actually interested in the minimization not of the norm of the left-hand side of (1.11) but of that of its action on the function 1 [the unique inhomogeneous term of Eqs. (1.9) and (1.10)]. Noticing that

$$\mathsf{R}_\tau 1 = \mathsf{R}_0 1 = D \tag{1.12}$$

irrespectively of τ, solving (1.8) amounts to finding the minimum of (the bound of) the "relative error" ($\|D\|$ is constant with respect to τ):

$$\frac{\|(\mathfrak{R}_\tau - \mathsf{R}_\tau)1\|}{\|\mathfrak{R}_\tau\|} \leq \text{const}\|\mathfrak{K}_\tau - \mathsf{K}_\tau\|. \tag{1.13}$$

As will be seen in Sec. III, the result of the optimization is remarkably simple, although mathematically not straightforward. It will amount to replacing one of the Cauchy kernels appearing in the equations by a suitably weighted Poisson one.

II. STATEMENT OF THE PROBLEM

For convenience, we first map the whole cut s plane onto the unit circle cut between 0 and 1 (see Fig. 1) so that the upper (lower) lip of the left-hand cut Γ comes onto the upper (lower) semicircle, and the right-hand cut γ onto the cut between (0, 1). If the left-hand cut Γ runs[5] from $-\infty$ to s_1, and the right-hand cut γ from s_2 to ∞, the mapping is given by

$$z = \frac{(s_1 - s)^{1/2} - i(s_2 - s_1)^{1/2}}{(s_1 - s)^{1/2} + i(s_2 - s_1)^{1/2}}, \tag{2.1a}$$

which, if $s_1 = 0$ and $s_2 = 4$, becomes

$$z = -\frac{2 - \sqrt{s}}{2 + \sqrt{s}}, \tag{2.1b}$$

where the square root is defined as having a cut along the negative semiaxis. As will be shown later [see discussion following Eq. (2.13)] there is

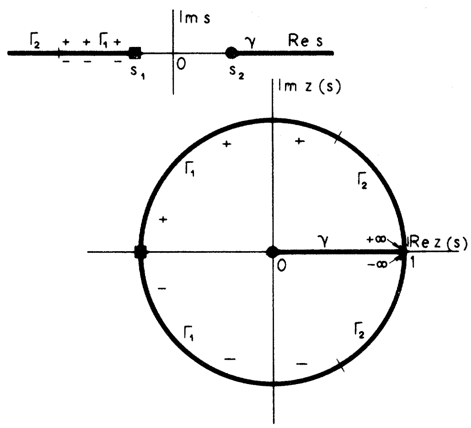

FIG. 1. The conformal mapping
$z=[(s_1-s)^{1/2}-i(s_2-s_1)^{1/2}]/[(s_1-s)^{1/2}+i(s_2-s_1)^{1/2}]$
leading from the cut energy plane to the cut unit circle.

no loss of generality with this special choice of the conformal mapping.

The elastic unitarity for the lth partial wave $\mathcal{A}_l$ of the whole scattering amplitude is

$$\operatorname{Im}\mathcal{A}_l(s)=\rho(s)\,|\mathcal{A}_l(s)|^2. \tag{2.2}$$

If $\mathcal{A}_l$ vanishes as s^{-1} when s goes to infinity ($z\to 1$), and if we take into account the k^{2l} threshold behavior, we define a "reduced partial wave" A_l,

$$A_l(z)\equiv \mathcal{A}_l(z)/z^l(1-z)^2, \tag{2.3}$$

for which the unitarity reads

$$\operatorname{Im}A_l(z)=\rho_l(z)\,|A_l(z)|^2, \tag{2.4a}$$

where

$$\rho_l(z)\equiv z^l(1-z)^2\rho(z)$$
$$=2z^l(1-z)^2\sqrt{z}/(1+z). \tag{2.4b}$$

The last equality of (2.4b) holds only in the special case $s_1=0$, $s_2=4$ when

$$\rho(s)=\sqrt{s-4}/\sqrt{s}$$
$$=2\sqrt{z}/(z+1).$$

We prove now that the amplitude can be written as N/D, with N analytic inside the *whole unit circle*, whereas D is analytic in the *whole z plane cut* along (0, 1) (see Fig. 2). To this end, assume that the true amplitude has no poles (their discussion is delayed until Sec. IV). Then, let all the zeros of A_l be included in a Blaschke factor $B(z)$. This can be done even if there is an accumulation of an infinity of zeros on Γ; however, if the zeros accumulate on the γ cut, the Blaschke product in the $z(s)$ variable diverges, and one has to resort to the variable $\zeta(s)$ defined in Sec. IV and the methods discussed there. Since $A_l(z)/B(z)$ has no zeros or poles in the cut unit circle, one can write a dispersion relation for its logarithm:

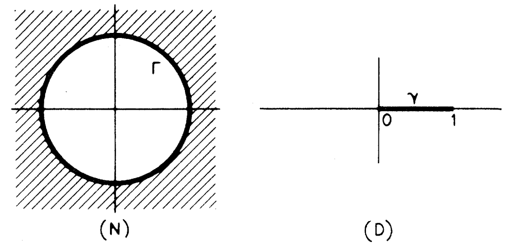

FIG. 2. Holomorphy domains for $N(z)$ and $D(z)$, respectively.

$$\ln[A_l(z)/B(z)]=n(z)-d(z) \tag{2.5a}$$

with

$$n(z)=\frac{1}{2\pi i}\oint_\Gamma \frac{\ln[A_l(z')/B(z')]}{z'-z}dz' \tag{2.5b}$$

and

$$d(z)=-\frac{1}{\pi}\int_0^1\frac{\Delta\ln[A_l(z')/B(z')]}{z'-z}dz'$$
$$\left(\Delta f(z)\equiv\frac{1}{2i}[f(z+i0)-f(z-i0)]\right). \tag{2.5c}$$

Notice that $d(z)$ is analytic in the whole z plane cut along (0, 1) and vanishes at infinity.

Therefore, defining

$$N(z)=B(z)e^{n(z)} \tag{2.6a}$$

and

$$D(z)=e^{d(z)}, \tag{2.6b}$$

$A_l(z)$ can be written as $N(z)/D(z)$, where $D(z)$ has no zeros [$A_l(z)$ has no poles] and is equal to unity at infinity. If furthermore $\operatorname{Re}\Delta\ln[A_l(1)/B(1)]$ in (2.5c) is smaller than π, one can write usual dispersion relations for $N(z)$ and $D(z)$. [If not, one has to introduce Castillejo-Dalitz-Dyson (CDD) poles, but this sort of questions will not be considered here; we refer in this context to the excellent analysis of Frye and Warnock.[2]] Hence,

$$N(z')=\frac{1}{2\pi i}\oint_\Gamma\frac{D(z'')A_l(z'')}{z''-z'}dz'' \tag{2.7a}$$

and

$$D(z)=1-\frac{1}{\pi}\int_0^1\frac{\rho_l(z')N(z')}{z'-z}dz'. \tag{2.7b}$$

These equations are true remembering that $D(z)$ is equal to unity at infinity and that

$$\mathrm{Im}D(z'\in\gamma) = -\rho_l(z')N(z') \tag{2.8}$$

as a consequence of the unitarity (2.4a).

Substituting (2.7a) into (2.7b) we get an integral equation of the Fredholm type for $D(z)$:

$$D(z) = 1 - \frac{1}{2\pi^2 i}\oint_\Gamma dz'' D(z'')A_l(z'') \times \int_0^1 dz' \frac{\rho_l(z')}{(z'-z)(z''-z')}. \tag{2.9}$$

This is a Fredholm integral equation of the second kind, which readily yields $D(z)$ on the circumference Γ. Further, one proceeds by analytic continuation, either using the right-hand side of the equation itself to obtain $D(z)$ at all interior points, or the dispersion relation for $N(z)$, (2.7a), to obtain first $N(z)$ everywhere, and then, $D(z)$ via Eq. (2.7b).

In practice, the data for $A_l(z'')$ in (2.9) are given on part of the left-hand cut of the s plane (i.e., a part Γ_1 of Γ in the z plane) under the form of a function $\mathcal{A}_l(e^{i\theta})$ which approximates the reduced true amplitude $A_l(e^{i\theta})$,

$$|A_l(e^{i\theta}) - \mathcal{A}_l(e^{i\theta})| < \epsilon(\theta), \quad z'' = e^{i\theta}\in\Gamma_1. \tag{2.10}$$

As in other papers[4,6] concerning stable extrapolations, we assume that some boundedness[7] condition has to be fulfilled by the reduced partial-wave amplitude $A_l(e^{i\theta})$ on the remaining part Γ_2 of the cut Γ. Thus, we assume

$$|A_l(e^{i\theta})| < M \quad \text{on} \quad \Gamma_2 = \Gamma\setminus\Gamma_1. \tag{2.11}$$

Both (2.10) and (2.11) can be written together defining

$$\mathcal{A}_l(e^{i\theta}) = 0 \ \text{on} \ \Gamma_2 \ \text{and} \ \epsilon(\theta) = \begin{cases}\epsilon(\theta) \text{ on } \Gamma_1\\ M \text{ on } \Gamma_2\end{cases},$$
$$|A_l(e^{i\theta}) - \mathcal{A}_l(e^{i\theta})|_\Gamma < \epsilon(\theta), \quad z = e^{i\theta}. \tag{2.12}$$

The conventional equation (2.9) is far from being the only one consistent with the requirements of analyticity and unitarity [Eq. (2.8)]. Indeed, following the Introduction, a tautological equation for $D(z)$ can be written using instead of (2.7a) an arbitrary reproducible kernel

$$N(z') = \frac{1}{2\pi i}\frac{1}{g_{z,z',\ldots}(z')}\oint_\Gamma dz'' \frac{N(z'')g_{z,z',\ldots}(z'')}{z''-z'}, \tag{2.13}$$

where $g_{z,z',\ldots}(z'')$ is an arbitrary holomorphic function of z'' and with any dependence on the remaining variables $z, z', \ldots$. Indeed, one can write Cauchy integrals for $N(z'')g(z'')$ as well as for $N(z'')$, as long as $N(z'')$ and $g(z'')$ are both holomorphic; on the other hand, the free dependence of $g(z'')$ on other variables means nothing else but the fact that in various situations one might use various weight functions $g(z'')$ (the only relevant parameters are nevertheless only z and z').

[*Remark*. At this stage we can show that among this general set of dispersion relations we also encounter the one written for $N(z')$ in the original s plane. Indeed,

$$N(s') = \frac{1}{\pi}\int_{-\infty}^{s_1}\frac{\mathrm{Im}\,N(s'')}{s''-s'}ds'' \equiv \frac{1}{2\pi i}\int_\Gamma \frac{N(s'')}{s''-s'}ds'' = \frac{1}{2\pi i}\int_\Gamma \frac{N(z'')}{z''-z'}\left(\frac{z''-z'}{s''-s'}\frac{ds''}{dz''}\right)dz''.$$

But there exists a tautology, namely,

$$g_{z'}(z'') \equiv \frac{z''-z'}{s''-s'}\frac{ds''}{dz''}\frac{dz'}{ds'},$$

such that, since

$$\lim_{z''\to z'}\left(\frac{z''-z'}{s''-s'}\frac{ds''}{dz''}\right) = 1,$$

we have

$$\frac{z''-z'}{s''-s'}\frac{ds''}{dz''} \equiv \frac{g_{z'}(z'')}{g_{z'}(z')}.$$

In this way we have also implicitly proven that all conformal mappings of the Γ cut plane are tautological, and thus there is no loss of generality if one uses the special complex plane $z(s)$ of Fig. 1.]

No similar changes can be made in the dispersion relation for $D(z)$, since, in order not to spoil (2.7b), the $g\ldots(z')$ function has to be real along (0, 1) and analytic throughout the z plane, therefore a constant. [In principle it could have had a single pole, but this corresponds to changing the subtraction point and this has been shown in the Introduction to be an irrelevant tautology; see (1.4)–(1.7).]

For what follows, it is convenient to extract from $g\ldots(z'')$ an exterior weight function[8] $C(z'')$, defined as having modulus $\epsilon/\epsilon(\theta)$ on Γ and being without zeros in D,

$$C(z) = \exp\left(\frac{1}{2\pi}\oint\frac{e^{i\theta}+z}{e^{i\theta}-z}\ln[\epsilon/\epsilon(\theta)]d\theta\right), \tag{2.14}$$

so that

$$|A_l(z'')C(z'') - \mathcal{A}_l(z'')C(z'')|_{z''\in\Gamma} \le \epsilon, \tag{2.15}$$

where ϵ is, for instance, the mean value of the error $\epsilon(\theta)$.

In (2.13) we set

$$\frac{g_{z,z'}(z'')}{g_{z,z'}(z')} = \frac{C(z'')}{C(z')}[1 + (z'-z'')f_{zz'}(z'')], \tag{2.16}$$

where $f_{zz'}(z'')$ is a function holomorphic in z'' in the unit circle and with arbitrary dependence on z and z'.

After substituting (2.13) and (2.16), instead of (2.7a), in the dispersion relation for $D(z)$, we get in place of the conventional equation (2.9),

$$D(z)=1+\oint_\Gamma dz''k(z,z'')D(z''), \tag{2.17a}$$

where

$$\begin{aligned} k(z,z'')&=\frac{1}{2\pi i}A_l(z'')C(z'')\left(\frac{1}{\pi}\int_0^1 dz'\frac{\rho_l(z')}{C(z')(z'-z)(z'-z'')}+\frac{1}{\pi}\int_0^1 dz'\frac{\rho_l(z')}{C(z')(z'-z)}f_{zz'}(z'')\right)\\ &\equiv\frac{1}{2\pi i}\tilde{A}_l(z'')[G_z(z'')+F_z(z'')]. \end{aligned}\tag{2.17b}$$

Here

$$\tilde{A}_l(z'')=A_l(z'')C(z''), \tag{2.17c}$$

$$G_z(z'')=\frac{1}{\pi}\int_0^1 dz'\frac{\tilde{\rho}_l(z')}{(z'-z)(z'-z'')}, \tag{2.17d}$$

where

$$\tilde{\rho}_l(z')=\frac{\rho_l(z')}{C(z')}$$

and $F_z(z'')$ is an arbitrary function holomorphic in z'' and with any dependence on z. [This is due to the arbitrariness of $f_{zz'}(z'')$.]

In the next section we will look for those $F_z(z'')$ for which the difference between the exact equation (2.17a) and that obtained by replacing $\tilde{A}_l(z'')$ by $\tilde{\mathcal{A}}_l(z'')\equiv\mathcal{A}_l(z'')C(z'')$, is minimal.

III. OPTIMIZATION OF THE INTEGRAL EQUATION

As stated in the Introduction [Eqs. (1.8)–(1.13)] we shall use the tautologies produced by the function $F_z(z'')$ in the $D(z)$ equation (2.17) in order to minimize the norm of the difference $\mathsf{K}-\mathfrak{K}$, where $\mathfrak{K}$ is the integral operator of (2.17b) with $\tilde{A}_l(z'')$ replaced by the error-affected values $\tilde{\mathcal{A}}_l(z'')$.

A natural frame for this problem is the Banach space $C(\Gamma)$ of all complex functions f continuous on Γ with the norm:

$$\|f\|_C=\sup_{z''\in\Gamma}|f(z'')|. \tag{3.1}$$

Then it can be shown (see Appendix A) that the norm of the integral operator $\mathsf{K}:\ C(\Gamma)\to C(\Gamma)$, with continuous kernel $k(z,z'')$, is

$$\|\mathsf{K}\|_C=\sup_{z\in\Gamma}\oint_\Gamma|dz''|\,|k(z,z'')|. \tag{3.2}$$

As we do not know the true reduced amplitude, we cannot minimize the norm of the difference of the kernels, but we can minimize its upper bound with respect to all reduced amplitudes consistent with the error channel condition (2.15)

$$\begin{aligned} \sup_A\|\mathsf{K}_\tau-\mathfrak{K}_\tau\|_C&=\sup_A\sup_{z\in\Gamma}\frac{1}{2\pi}\oint_\Gamma|dz''|\,|\tilde{A}_l(z'')-\tilde{\mathcal{A}}_l(z'')|\,|G_z(z'')+F_z(z'')|\\ &\le\frac{\epsilon}{2\pi}\sup_{z\in\Gamma}\oint_\Gamma|dz''|\,|G_z(z'')+F_z(z'')|, \end{aligned}\tag{3.3a}$$

where $G_z(z'')$ and $F_z(z'')$ are given by (2.17d); the inequality comes from the condition[9] (2.15).

[*Remark*: At first glance one could believe that it is possible to obtain better bounds working with different tautologies for the exact and approximate kernels. Then (3.3a) turns into

$$\sup_A\|\mathsf{K}_{\tau_1}-\mathfrak{K}_{\tau_2}\|_C=\sup_A\sup_{z\in\Gamma}\oint_\Gamma|dz''|\,|[\tilde{A}(z'')-\tilde{\mathcal{A}}(z'')][G_z(z'')+F_z(z'')]+\tilde{\mathcal{A}}(z'')F'_z(z'')|, \tag{3.3b}$$

where F and F' are holomorphic functions defined by the tautologies τ_1 and τ_2. Now, if nothing is known about the relative phases of the error term $\tilde{A}(z'')-\tilde{\mathcal{A}}(z'')$ and the data function $\tilde{\mathcal{A}}(z'')$, in computing upper bounds one is forced to resort to the sum of the absolute values of the two terms in the right-hand side of (3.3b). The minimization with respect to F' is then immediate ($F'\equiv0$) and one gets back the right-hand side of the inequality (3.3a).]

Hence we are left with the problem of finding that function $F_z(z'')$, holomorphic in z'' in the unit disk (and continuous in z), which minimizes the upper bound with respect to z of the L^1 norm of $G_z(z'')+F_z(z'')$, i.e., finding

$$\inf_F\sup_{z\in\Gamma}\|G_z+F_z\|_{L^1}\equiv\inf_F\sup_{z\in\Gamma}\frac{1}{2\pi}\oint_\Gamma|dz''|\,|G_z(z'')+F_z(z'')|, \tag{3.4}$$

where according to (2.17d) $G_z(z'')$ is a given function holomorphic in the whole z'' plane except for a cut between zero and unity.

A. A Minimax Theorem

Assume (it can be proven[10]) that there exists a function $F^0_z(z'')$ analytic in z'' (and continuous in z) which for *each* z realizes

$$\inf_{F_z} \| G_z + F_z \|_{L^1} = \| G_z + F^0_z \|_{L^1} . \tag{3.5}$$

In that case it is true that

$$m^0 \equiv \inf_{F_z} \sup_{z \in \Gamma} \| G_z + F_z \|_{L^1} = \sup_{z \in \Gamma} \inf_{F_z} \| G_z + F_z \|_{L^1} . \tag{3.6}$$

Indeed, on one hand one always[11] has

$$\sup_{z \in \Gamma} \inf_{F_z} \| G_z + F_z \|_{L^1} \leq \inf_{F_z} \sup_{z \in \Gamma} \| G_z + F_z \|_{L^1} . \tag{3.7}$$

But since $F^0_z(z'')$ is continuous in z and analytic in z'', and therefore *is* an admissible function, we get, owing to (3.5),

$$\sup_{z \in \Gamma} \inf_{F_z} \| G_z + F_z \|_{L^1} \equiv \sup_{z \in \Gamma} \| G_z + F^0_z \|_{L^1} \geq \inf_{F_z} \sup_{z \in \Gamma} \| G_z + F_z \|_{L^1} , \tag{3.8}$$

this inequality being a consequence of the fact that the infimum of the right-hand side of (3.8) is meant to be taken over all functions $F_z(z'')$ depending not only on z'', but *also* on z. Hence the "minimax" property (3.6) is proven.

B. A Dual Problem

The search for the function $F^0_z(z'')$ analytic inside the unit circle which realizes the infimum of $\| G_z + F_z \|_{L^1}$ for every z, is a well-known problem in mathematics. Partial solutions can be found in the work of Riesz.[12] It can be readily[13] turned into its dual problem (see Fig. 3)

$$\inf_{F_z \in H^1} \| G_z(z'') + F_z(z'') \|_{L^1} = \sup_{\substack{h_z \in H^\infty \\ \|h_z\| \leq 1}} \left| \frac{1}{2\pi} \oint_\Gamma h_z(z'') G_z(z'') dz'' \right| . \tag{3.9a}$$

Here, first, $h_z \in L^\infty$, which is the space dual[14] to L^1, and also, because it has to be "orthogonal" to all $F_z \in H^1$ (the subspace of analytic functions belonging to L^1 on the boundary),

$$\frac{1}{2\pi i} \oint_\Gamma dz'' h_z(z'') F_z(z'') = 0 , \tag{3.9b}$$

$h_z(z'')$ has to be holomorphic in the unit disk, i.e., $h_z(z'') \in H^\infty$. The norm in L^∞ is

$$\| h_z \|_{L^\infty} = \operatorname*{ess\,sup}_{z'' \in \Gamma} | h_z(z'') |$$

so that the second condition in (3.9a) means

$$\operatorname*{ess\,sup}_{z'' \in \Gamma} | h_z(z'') | \leq 1 . \tag{3.9c}$$

It can actually be shown[15] that $| h_z(z'') | = 1$ almost everywhere on Γ. Then it is obvious that the optimal $h_z(z'') = h^0_z(z'')$ should be such that the phase of $z'' h^0_z(z'')[G_z(z'') + F^0_z(z'')]$ be constant along the unit circle. Indeed,

$$\begin{aligned} \frac{1}{2\pi} \oint_\Gamma | dz'' | \, | G_z(z'') + F^0_z(z'') | &= \frac{1}{2\pi i} \oint_\Gamma \frac{dz''}{z''} | h^0_z(z'') z'' [G_z(z'') + F^0_z(z'')] | \\ &\geq \left| \frac{1}{2\pi i} \oint_\Gamma dz'' h^0_z(z'') G_z(z'') \right| \end{aligned} \tag{3.10}$$

and equality holds in the last bit of (3.10) if and only if $z'' h^0_z(z'')[G_z(z'') + F^0_z(z'')]$ has a constant phase along Γ. On the other hand, we know from (3.9a) that equality does occur, so that $h^0_z(z'')$ has indeed the specified property.

This property of $h^0_z(z'')$ makes the construction of the function $F^0_z(z'')$ straightforward once $h^0_z(z'')$ is found:

$$F_z^0(z'') = \frac{1}{z''h_z^0(z'')}\frac{1}{2\pi i}\oint d\theta \frac{e^{i\theta}+z''}{e^{i\theta}-z''}\mathrm{Im}\left[e^{i\theta}h_z^0(e^{i\theta})G_z(e^{i\theta})\frac{|a|}{a}\right]\frac{a}{|a|}$$

$$\equiv \{a[a^{*-1}(z''G_z h_z^0)_-^*(1/z''^*)+1]-(z''G_z h_z^0)_+(z'')\}/z''h_z^0(z''), \quad (3.11)$$

where $(\cdots)_\pm(u)$ are the Fourier positive- and negative-frequency parts of $(\cdots)(u)$, where u stands either for z'' or for $1/z''^*$, and $a \equiv (z''G_z h_z^0)_+(0)$.

C. The Point $z=1$ and Determination of m^0

With these preliminaries, although we are not always[16] able to construct the function $F_z^0(z'')$, we are nevertheless in a position to find the bound

$$m^0 \equiv \sup_{z\in\Gamma}\inf_{F_z}\|G_z+F_z\|_{L^1} = \sup_{z\in\Gamma}\|G_z+F_z^0\|. \quad (3.12)$$

Indeed,

$$\begin{aligned}\sup_{z\in\Gamma}\inf_{F_z}\|G_z+F_z\|_{L^1} &= \sup_{z\in\Gamma}\sup_{\|h_z\|=1}\left|\frac{1}{2\pi i}\oint_\Gamma dz''h_z(z'')G_z(z'')\right| \\ &= \sup_{z\in\Gamma}\sup_{\|h_z\|=1}\left|\frac{1}{2\pi i}\oint_\Gamma dz''h_z(z'')\frac{1}{\pi}\int_0^1 dz'\frac{\tilde\rho_l(z')}{(z'-z)(z'-z'')}\right| \\ &= \sup_{z\in\Gamma}\sup_{\|h_z\|=1}\left|\frac{1}{\pi}\int_0^1 dz'\frac{\tilde\rho_l(z')h_z(z')}{z'-z}\right| \\ &\leqslant \sup_{z\in\Gamma}\frac{1}{\pi}\int_0^1 dz'\frac{\tilde\rho_l(z')}{|z'-z|} \\ &\leqslant \frac{1}{\pi}\int_0^1 dz'\frac{\tilde\rho_l(z')}{1-z'}. \end{aligned} \quad (3.13)$$

The first inequality in (3.13) holds because $\tilde\rho_l(z')$, defined by (2.17d), is always positive and $|h_z(z')|\leqslant 1$ for z' inside the unit disk, by the maximum modulus theorem and (3.9c). The second inequality is obvious on geometrical grounds: If $0\leqslant z'\leqslant 1$ and $z=e^{i\theta}$, then $|z-z'|\geqslant 1-z'$. Moreover, it can be immediately seen that for $z=1$, $h_{z=1}^0(z')\equiv 1$, because $h_{z=1}(z')=1$ effectively saturates all the inequalities[17] in (3.13). At the same time, the last inequality in (3.13) ensures that $\sup_{z\in\Gamma}\|G_z+F_z^0\|_{L^1}$ is *really attained for* $z=1$ *and equals*

$$m^0 = \frac{1}{\pi}\int_0^1 dz'\frac{\tilde\rho_l(z')}{1-z'}. \quad (3.14)$$

For $z=1$ and $h_{z=1}^0(z')\equiv 1$ we get from (3.11)

$$\begin{aligned} z''h_{z=1}^0(z'')[G_{z=1}(z'')+F_{z=1}^0(z'')] &= \frac{1}{\pi}\int_0^1 dz'\frac{\tilde\rho_l(z')z''}{(z'-1)(z'-z'')}+\frac{1}{\pi}\int_0^1 dz'\frac{\tilde\rho_l(z')\cdot 1/z''}{(z'-1)(z'-1/z'')}+\frac{1}{\pi}\int_0^1 dz'\frac{\tilde\rho_l(z')}{z'-1} \\ &\equiv \frac{1}{\pi}\int_0^1 dz'\frac{\tilde\rho_l(z')}{1-z'}\mathcal{P}(z'';z'). \end{aligned} \quad (3.15)$$

Here $\mathcal{P}(z'';z')$ stands for the Poisson kernel for the unit circle

$$\mathcal{P}(z'';z')d\theta'' \equiv id[\mathcal{G}(z'';z')+i\mathcal{H}(z'';z')] \equiv \frac{\partial\mathcal{G}(z'';z')}{\partial n''_{\text{int}}}d\theta'', \quad (3.16\text{a})$$

where $\mathcal{G}(z'';z')$ is the Green's function of the unit circle with respect to z' and $\mathcal{H}(z'',z')$ its harmonic conjugate,

$$\mathcal{P}(z'',z') = \mathrm{Re}\frac{z''+z'}{z''-z'} \equiv \frac{1-r'^2}{1-2r'\cos(\theta''-\theta')+r'^2}. \quad (3.16\text{b})$$

D. The Poisson Kernel

For other points $z \neq 1$ $(z \in \Gamma)$ one might again proceed by finding directly the optimal function $F_z^0(z'')$. This is not an easy task,[18] but one can avoid this by noticing that the function $F_z(z'')$ which realizes (3.14)

$$m^0 \equiv \inf_{F_z} \sup_{z\in\Gamma} \| G_z + F_z \|_{L^1} \equiv \frac{1}{\pi}\int_0^1 dz' \frac{\tilde{\rho}_l(z')}{1-z'} \tag{3.17}$$

is not unique. Indeed we notice that the holomorphic function $F_z^P(z'')$ which is used to construct the Poisson kernel in (3.15), namely,

$$z'' F_z^P(z'') = \frac{1}{\pi}\int_0^1 dz' \frac{\tilde{\rho}_l(z') 1/z''}{(z'-z)(z'-1/z'')} + \frac{1}{\pi}\int_0^1 dz' \frac{\tilde{\rho}_l(z')}{z'-z}, \tag{3.18}$$

although for $z \neq 1$ it does not attain

$$\inf_{F_z} \| G_z + F_z \|_{L^1} \equiv \| G_z + F_z^0 \|_{L^1} \quad (F^P_{z\neq 1} \neq F^0_{z \neq 1}), \tag{3.19}$$

nevertheless satisfies

$$\| G_z + F_z^P \|_{L^1} = \frac{1}{2\pi}\oint_\Gamma |dz''| \left| \frac{1}{\pi}\int_0^1 dz' \frac{\tilde{\rho}_l(z')}{z'-z} \mathcal{P}(z''; z') \right|$$

$$\leq \frac{1}{2\pi}\oint_\Gamma |dz''| \frac{1}{\pi}\int_0^1 dz' \frac{\tilde{\rho}_l(z')}{|z'-z|} \mathcal{P}(z''; z') \equiv \frac{1}{\pi}\int_0^1 dz' \frac{\tilde{\rho}_l(z')}{|z'-z|} \leq \frac{1}{\pi}\int_0^1 dz' \frac{\tilde{\rho}_l(z')}{1-z'} \equiv m^0. \tag{3.20a}$$

In (3.20a) we have used $(1/2\pi)\oint d\theta'' \mathcal{P}(z'', z') \equiv 1$, and the last inequality is the same as the last inequality of (3.13).

Since for $z=1$, $F^P_{z=1}(z'') \equiv F^0_{z=1}(z'')$, the bound m^0 [Eq. (3.14)] is attained by $F_z^P(z'')$ as well, and so the proof of its optimality, together with $F_z^0(z'')$, is completed.

E. Short Review

Recollecting all the results obtained so far, the optimized kernel of the integral equation should then be written

$$k^{\text{optimal}}(z, z'') = \frac{i}{2\pi^2} \mathcal{A}_l(z'') C(z'') \times \int_0^1 dz' \frac{\rho_l(z')}{C(z')(z'-z)} \frac{\mathcal{P}(z'', z')}{z''}, \tag{3.21}$$

where $\mathcal{A}_l(z'')$represents [see (2.10)] the data function for the reduced amplitude $A_l(z'')$ defined in (2.3), whereas $\rho_l(z')$ and $C(z')$ are defined in (2.3), (2.4b), and (2.14), and $\mathcal{P}(z'', z')$ is the Poisson kernel (3.16). The optimized equation to be solved for $\mathfrak{D}(z)$ then reads

$$\mathfrak{D}(z) = 1 - \frac{1}{2\pi^2 i}\oint_\Gamma \frac{dz''}{z''} \mathfrak{D}(z'') \mathcal{A}_l(z'') C(z'') \times \int_0^1 dz' \frac{\rho_l(z')}{C(z')(z'-z)} \mathcal{P}(z''; z'). \tag{3.22}$$

It is probably illuminating to notice that the inequalities (3.20a) observed by $F_z^P(z'')$,

$$\frac{1}{2\pi}\oint |dz''| \, |G_z(z'') + F_z^P(z'')| \leq \frac{1}{\pi}\int_0^1 dz' \frac{\tilde{\rho}_l(z')}{1-z'}, \tag{3.20b}$$

together with the Schwarz inequality [and $\oint_\Gamma dz'' F_z(z'') = 0$]

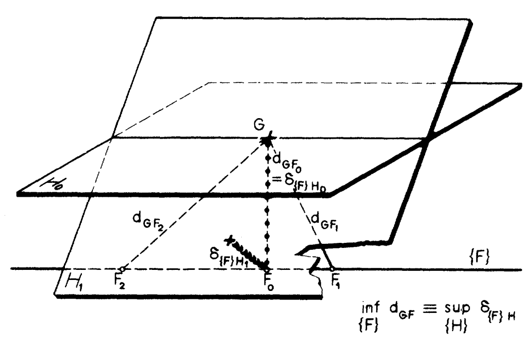

FIG. 3. A geometrical view of the duality principle: Points are to be interpreted as elements of a L^1 space, while planes mean level surfaces of linear functionals over L^1, i.e., elements of the dual space L^∞. Then, the smallest (L^1) distance α_{GF} between a given point G and points of a linear subset $\{F\}$ coincides with the greatest distance $\delta_{\{F\}H}$ between the linear subset $\{F\}$ and the planes whose normals are orthogonal to $\{F\}$, passing through G.

$$\sup_{z\in\Gamma}\frac{1}{2\pi}\oint_\Gamma |dz''|\,|G_z(z'')+F_z(z'')|$$
$$\geq \sup_{z\in\Gamma}\left|\frac{1}{2\pi i}\oint dz'' G_z(z'')\right|$$
$$=\sup_{z\in\Gamma}\left|\frac{1}{\pi}\int_0^1 dz'\frac{\tilde{\rho}_l(z')}{z'-z}\right|$$
$$=\frac{1}{\pi}\int_0^1 dz'\frac{\tilde{\rho}_l(z')}{1-z'}\equiv m^0, \quad (3.23)$$

are enough to prove the optimality of the Poisson kernel. However, by itself, Eq. (3.23) does not show that the bound m^0 is actually attained; for this, one has either to resort to a direct guess of the optimality of the Poisson kernel [i.e., to (3.20b)], or to the duality principle which, as we have seen, provides a constructive proof of it.

F. Analytic Continuation of N and D

We now proceed to the determination of the amplitude, i.e., of $N(z)$ and $D(z)$ in the whole cut unit disk; there arises the problem of finding optimal values $\mathfrak{N}(z)$, $\mathfrak{D}(z)$ for these two quantities at every point of the cut disk [optimal with respect to all admissible amplitudes in the error channel (2.12)]. In principle, an optimal $\mathfrak{D}(z)$ can be found by using the integral equation (2.17) as an integral representation for $\mathfrak{D}(z)$ in the cut unit disk in terms of the solution of (3.22) appearing on its left-hand side, and optimizing with respect to the available tautologies.

It is apparent that the latter again consist of the set of functions $F_z(z'')$, holomorphic in z'' [so that they do not spoil the integral representation when $\mathfrak{D}(z)$ is exact, i.e., $\mathfrak{D}(z)=D(z)$], and of any dependence on z. [This means that the optimal $\mathfrak{D}(z)$ need not be holomorphic in the cut disk, but that, as far as the tautology group is concerned, it departs least from the admissible exact $D(z)$.] As a matter of fact, if we use $F_z^P(z'')$, i.e., the one that led to Eq. (3.22), the $\mathfrak{D}(z)$ extrapolated by (3.22) is a function holomorphic in the whole cut *plane* [the solutions of (3.22) being its values on the unit circle]. Nevertheless, this extrapolated $\mathfrak{D}(z)$ is not optimal.

In general, for an arbitrary $F_z(z'')$,

$$\sup_A |D(z)-\mathfrak{D}(z)|$$
$$=\sup_A\left|\frac{1}{2\pi i}\oint_\Gamma dz''[\tilde{A}_l(z'')D(z'')-\tilde{\mathcal{A}}_l(z'')\mathfrak{D}(z'')]\times[G_z(z'')+F_z(z'')]\right|$$
$$\leq \epsilon\,\text{const}\,\frac{1}{2\pi}\oint|dz''|\,|G_z(z'')+F_z(z'')|, \quad (3.24)$$

so that it is clear that the optimal extrapolated $\mathfrak{D}(z)$ is the one obtained by using that $F_z(z'')$ which renders the L^1 norm $\|G_z(z'')+F_z(z'')\|_{L^1}$ minimal, and so we fall back on the problem of Riesz.[12] However, as we have pointed out, only partial solutions (i.e., corresponding to some z only) are available at present.

In what follows we construct a solution which, although not optimal, should be good enough for practical purposes. To this end, we take $\mathcal{A}_l(z'')\mathfrak{D}(z'')$ with the $\mathfrak{D}(z'')$ solution of (3.22) as a good enough determination of $\mathfrak{N}(z'')$. We then use the tautologies (2.13) available for the dispersion relation (2.7a) in order to obtain an optimal extrapolation of this $\mathfrak{N}(z'')$. For this, we take exactly the same steps as described in Ref. 4. Following it closely, one comes to the conclusion that the required dispersion relation reads

$$\mathfrak{N}(z)=\frac{1}{2\pi i\,C(z)}\oint_\Gamma\frac{dz''}{z''}\mathcal{A}_l(z'')C(z'')\mathfrak{D}(z'')\mathcal{P}(z'';z). \quad (3.25)$$

Here, $C(z)$ is again a weight function, necessary to bring the problem to the canonical form $|A_l(z'')C(z'')-\mathcal{A}_l(z'')C(z'')|<\epsilon$ throughout the boundary of the unit circle, and is the same as that used in (3.21) for the optimal kernel $k^{\text{optimal}}(z,z'')$; $\mathcal{P}(z'',z)$ is the Poisson kernel (3.16). After having found $\mathfrak{N}(z)$, one uses it in the dispersion relation (2.7b) to find $\mathfrak{D}(z)$ everywhere. It is easily seen that due to the happy recurrence of the Poisson kernel in both (3.25) and (3.22), we find the same $\mathfrak{D}(z)$ as the one yielded by (3.22) used as an integral representation for $\mathfrak{D}(z)$ in terms of its values on Γ.

IV. DISCUSSION

We have shown that one can always find a tautology for which the norm of the difference between the real and approximative kernels is least. As was shown in Sec. III, the result turned out to be very simple, namely, one obtains this kernel by first introducing an exterior weight function [defined in (2.14)] in both $\mathcal{A}_l(z'')$ and $\rho_l(z')$ and substituting the Cauchy kernel of the dispersion relation for $\mathfrak{N}(z')$ with the Poisson one. The resulting optimal kernel is written in (3.21).

A. Range of Validity of the Results

There are two possible limitations of the above method.

(a) First, as it was pointed out in the Introduction, the fact that the kernels are close to each other does not always mean that the resolvents (the solutions) are close to each other. Indeed, it may happen that the data are such that the eigenvalues of the tautological kernel obtained with $F_{s}^{P}(z'')$ get close to unity, and in this case the norm of the resolvent $\mathfrak{R}$ of $\mathfrak{K}$ in (1.11) will be very large (and strongly dependent upon the chosen tautology).

However, at least in the case $\|\mathfrak{K}\|$ small (weak potential case), the exact and approximate solutions should approach each other, and the optimized kernel (3.21) should yield excellent results.

(b) Another question may arise when the input data $\mathcal{A}_l(z'')$ are wrong (at least from the point of view of analyticity) in the sense that there is no function $\mathcal{A}_l(z)$ holomorphic inside the cut unit disk and obeying the unitarity (2.4a) which passes through the given error channel (2.12). This situation arises in the dispersion relation case, too, and is treated in detail in Ref. 6.

When uncertainties arise concerning these two possible situations, one should compare the results with those obtained by the optimal unitary S-matrix method, to be sketched below.

B. Optimal Unitary S-Matrix Method

Following the methods of Sec. 3 of Ref. 19, one can directly find a function $S^0(s)$ satisfying the S-matrix partial-wave unitarity on the physical region γ,

$$|S^0(s)|_{\gamma_1}=1, \quad \gamma_1 \subset \gamma, \tag{4.1a}$$

$$|S^0(s)|_{\gamma_2}=\eta(s), \quad 0<\eta(s)<1, \quad \gamma_2=\gamma\setminus\gamma_1, \tag{4.1b}$$

and which is closest to a data function $\mathcal{S}=1+2i\rho\mathcal{A}(s)$ on the left-hand cut Γ,

$$|S^0(s)-\mathcal{S}^0(s)|/\epsilon(s)\to \text{minimum}\equiv\epsilon_{00}. \tag{4.2}$$

Here $\epsilon(s)$ is an error function connected with the progressive fading of the information along Γ, as in (2.12). If we use the canonical variable $\zeta(s)$, which maps the right/left-hand cut on the right/left-unit semicircle (see Fig. 4), and if we define the C-weight functions[19]

$$C_1(\zeta)=\exp\left(-\frac{1}{2\pi}\int_\Gamma \frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}\ln[\epsilon(\theta)]\,d\theta\right), \tag{4.3a}$$

$$C_2(\zeta)=\exp\left(-\frac{1}{2\pi}\int_{\gamma_2} \frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}\ln[\eta(\theta)]\,d\theta\right) \tag{4.3b}$$

such that

$$|C_1(\zeta)|_\Gamma=1/\epsilon(\theta), \quad |C_1(\zeta)|_\gamma=1, \tag{4.4a}$$

$$|C_2(\zeta)|_{\Gamma+\gamma_1}=1, \quad |C_2(\zeta)|_{\gamma_2}=1/\eta(\theta), \tag{4.4b}$$

we can bring the conditions to be obeyed by $S(\zeta)$ into a suitable standard form. Indeed, if

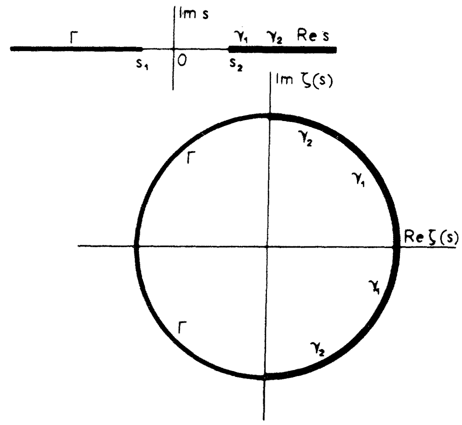

FIG. 4. The canonical mapping $\zeta(s)=[(1+u)^{1/2}-(1-u)^{1/2}]/[(1+u)^{1/2}+(1-u)^{1/2}]$, $u=(2s-s_1-s_2)/(s_2-s_1)$ transforming the left-hand cut Γ into the left unit semicircle and the right-hand cut γ into the right semicircle.

$$|S(\zeta)-\mathcal{S}(\zeta)|_\Gamma<\epsilon(\theta), \tag{4.5}$$

and if we define

$$S'(\zeta)\equiv S(\zeta)C_1(\zeta)C_2(\zeta),$$
$$\mathcal{S}'(e^{i\theta})\equiv\mathcal{S}(e^{i\theta})C_1(e^{i\theta})C_2(e^{i\theta}), \tag{4.6}$$

then (4.1) and (4.5) merge into

$$|S'(\zeta)-\mathcal{S}'(\zeta)|_\Gamma<1 \tag{4.7a}$$

and

$$|S'(\zeta)|_\gamma=1. \tag{4.7b}$$

Our aim is to discover among all functions $S'(\zeta)$ satisfying (4.7) (if they exist) that one, $S'^0(\zeta)$, which best approximates the data function $\mathcal{S}'(e^{i\theta})$ on the left-hand cut Γ. To this end, we shall take advantage of the fact that, given a (data) function $\tilde{h}(e^{i\theta})$ on the whole unit circle $|\zeta|=1$, one knows how to construct (Ref. 6 and Appendix B) that holomorphic function $f^0(\zeta)$ which is "minimal" in the sense that

$$|\tilde{h}(e^{i\theta})-\tilde{f}^0(e^{i\theta})|_{\Gamma+\gamma}=\epsilon_0[\tilde{h}] \tag{4.8}$$

so that there do not exist other holomorphic functions $\tilde{f}(\zeta)$ with

$$|\tilde{h}(e^{i\theta})-\tilde{f}(e^{i\theta})|_{\Gamma+\gamma}\leq\epsilon$$

throughout $\Gamma+\gamma$, with ϵ smaller than $\epsilon_0[\tilde{h}]$. The value of $\epsilon_0[\tilde{h}]$ is completely determined by the negative-frequency Fourier coefficients $c_{-1}, c_{-2}, \ldots$ of $\tilde{h}(e^{i\theta})$, being the norm of the matrix M,

$$M[\tilde{h}]=\begin{pmatrix} c_{-1} & c_{-2} & c_{-3} & \cdots \\ c_{-2} & c_{-3} & \cdots & \cdots \\ c_{-3} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \tag{4.9}$$

$\epsilon_0[\tilde{h}] = \|M\| \equiv$ square root of the largest eigenvalue of the Hermitian matrix $MM^\dagger$.

Let then $C(\epsilon;\zeta)$ be a weight function with

$$|C(\epsilon;\zeta)|_\Gamma = 1, \quad |C(\epsilon;\zeta)|_\gamma = \epsilon \tag{4.10a}$$

and with a still undefined ϵ, i.e.,

$$C(\epsilon;\zeta) = \exp\left[\left(\frac{1}{2} - \frac{i}{\pi}\ln\frac{1+i\zeta}{1-i\zeta}\right)\ln\epsilon\right]. \tag{4.10b}$$

Then, for the data function

$$\tilde{h}(e^{i\theta}) = \begin{cases} 0 \text{ on } \gamma & (-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi) \\ C(\epsilon; e^{i\theta})\mathcal{S}'(e^{i\theta}) \text{ on } \Gamma & (\frac{1}{2}\pi < \theta < \frac{3}{2}\pi) \end{cases},$$

the minimal function $\tilde{f}^0(\zeta)$ satisfies

$$|\tilde{f}^0(e^{i\theta})/C(\epsilon; e^{i\theta}) - \mathcal{S}'(e^{i\theta})|_\Gamma = \epsilon_0[\mathcal{S}';\epsilon], \tag{4.11a}$$

$$|\tilde{f}^0(e^{i\theta})/C(\epsilon; e^{i\theta})|_\gamma = \epsilon_0[\mathcal{S}';\epsilon]/\epsilon, \tag{4.11b}$$

$\epsilon_0[\mathcal{S}';\epsilon]$ being determined by (4.9) via the Fourier coefficients

$$c_{-n}[\mathcal{S};\epsilon] = \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2} \mathcal{S}(e^{i\theta}) \times\exp\left(-\frac{i}{\pi}\ln\epsilon\ln\frac{\cos\theta}{1+\sin\theta}\right)e^{in\theta}d\theta. \tag{4.12}$$

Therefore, it is apparent that if we pick out that very $\epsilon = \epsilon_{00}$ for which $\epsilon_0[\mathcal{S}';\epsilon]$ [corresponding to it by (4.9) and (4.12)] is just equal to it, i.e., the solution of the transcendental equation[19,20]

$$\epsilon_{00} = \epsilon_0[\mathcal{S}', \epsilon_{00}], \tag{4.13}$$

we manage [see (4.11b)] to satisfy the unitarity condition (4.7b) with the function

$$S'^0(\zeta) = \tilde{f}_{00}(\zeta)/C_0(\zeta). \tag{4.14}$$

Here $\tilde{f}_{00}(\zeta)$ is the minimal function $\tilde{f}^0(\zeta)$ corresponding to

$$\tilde{h}(\zeta) = C_0(\zeta)\mathcal{S}'(\zeta) \quad (\zeta = e^{i\theta}), \tag{4.15a}$$

where

$$C_0(\zeta) \equiv C(\epsilon_{00};\zeta). \tag{4.15b}$$

It is obvious that $S'^0(\zeta)$ given by (4.14) is the optimal holomorphic unitary function, which is closest to the data function $\mathcal{S}'(\zeta)$ on Γ. For the effective construction of $S'^0(\zeta)$, see Appendix B.

We would like to stress that, as far as unitarity was concerned, it was essential that the modulus of the minimal function happens (it is proved in Ref. 19) to be strictly constant along γ. As has been repeatedly stated, the optimal S-matrix partial wave constructed above is expected to be close to that obtained from the solution of the optimized N/D equation (3.22), at least when the norm of the Poisson-weighted kernel is small.

C. Dynamical Poles

If the data are wrong, i.e., $\mathcal{A}_l(e^{i\theta})$ are not the boundary values of a unitary function holomorphic in the z-plane cut unit disk (Fig. 1), but rather of a function exhibiting one pole, it is to be expected that the $\mathfrak{D}(z)$ constructed in Sec. III will have a zero (dynamical pole).

In contradistinction to the conventional (errorless) N/D equations, the appearance of the zero of $\mathfrak{D}(z)$ is dependent upon the relative magnitude of the influence of the pole upon the data on the left-hand cut Γ and of the width of the error channel. This is especially clear in the optimal unitary S-matrix approach (described in Sec. IV B). Indeed, if the error is so small that the analyticity of the boundary values is strongly affected by the pole, it will be very hard to find a function holomorphic in the unit disk and close (within the error) to these data. In other words, ϵ_{00} will be very high. Then,[20] to find the actual position of the pole, one multiplies $\mathcal{S}'(e^{i\theta})$ with a factor

$$B_{\zeta_0}(e^{i\theta}) = \frac{e^{i\theta}-\zeta_0}{1-\zeta_0^* e^{i\theta}},$$

and then solves the ϵ_{00} problem for the function $\mathcal{S}'(e^{i\theta})B_{\zeta_0}(e^{i\theta})$ for every (real) ζ_0 until $\epsilon_{00}(\zeta_0)$ reaches a minimum. This minimum corresponds to the exact cancellation of the pole of the amplitude by the zero of the factor $B_{\zeta_0}(\zeta)$. Computer calculations done[20] on model amplitudes show indeed a dramatically narrow dip of $\epsilon_{00}(\zeta_0)$ in the neighborhood of the pole.

ACKNOWLEDGMENTS

We would like to thank Professor C. Foiaş for his keen interest in the mathematics connected with these optimization methods. Without several of his proofs many points of the above assertions would have been a bit more shaky (although probably easier to read) and we acknowledge with gratitude the many hours he spent in discussing with us many functional analysis topics related to our problem. It is also a pleasure for us to acknowledge many useful discussions with our colleagues I. Caprini and G. Nenciu.

APPENDIX A: PROOF OF FORMULA (3.2)

We show that if K is an integral mapping from $C(\Gamma)$ to $C(\Gamma)$ with continuous kernel $k(z, z'')$, then its norm is

$$\|K\|_C = \sup_{z\in\Gamma} \oint_\Gamma |dz''|\,|k(z,z'')|. \tag{A1}$$

Indeed,

$$\|K\|_C \equiv \sup_{\substack{f\in C(\Gamma)\\ \|f\|_C=1}} \sup_{z\in\Gamma} \left| \oint_\Gamma dz'' k(z,z'') f(z'')\right| \leq \|f\|_C \sup_{z\in\Gamma}\oint_\Gamma |dz''|\,|k(z,z'')| = \sup_{z\in\Gamma}\oint |dz''|\,|k(z,z'')|. \tag{A2}$$

Conversely, there exists a function $f_0 \in C(\Gamma)$ for which the right-hand side of (A2) is attained. Since

$$\oint_\Gamma |dz''|\,|k(z,z'')|$$

is continuous, there exists a point $z = z_0$ where the supremum is attained; then let

$$f_0(z'') = \exp\{-i\,\text{phase}[k(z_0, z'')]\}. \tag{A3}$$

$f_0(z'')$ is continuous, since k is continuous and has norm unity. Thus

$$\|Kf_0\|_C \equiv \sup_{z\in\Gamma}\left|\oint_\Gamma dz'' k(z,z'') f_0(z'')\right| \geq \oint_\Gamma |dz''|\,|k(z_0,z'')| = \sup_{z\in\Gamma}\oint_\Gamma |dz''|\,|k(z,z'')|, \tag{A4}$$

the last equality of (A4) resulting from the definition of z_0. Combining (A4) with (A2), we get (A1).

APPENDIX B: OPTIMAL UNITARY PARTIAL WAVES

We show here how one can construct an optimal S-wave amplitude without solving N/D equations but following the direct functional methods described in Sec. 3 of Ref. 19. We shall, namely, construct the weighted S-matrix partial wave

$$S'^0(\zeta) = C_1(\zeta) C_2(\zeta) S^0(\zeta)$$

[see Eq. (4.6)] which obeys the (weighted) unitarity

$$|S'^0(\zeta)| = 1 \quad \text{for } \zeta\in\gamma \quad (\text{i.e., } \zeta = e^{i\theta},\ -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi) \tag{B1}$$

(see Fig. 4) and which deviates least from the (weighted) "data function"

$$\mathcal{S}'(\zeta) = C_1(\zeta) C_2(\zeta) \mathcal{S}(\zeta)$$

on the left semicircle Γ,

$$|S'^0(\zeta) - \mathcal{S}'(\zeta)| = \epsilon_{00} \quad \text{for } \zeta\in\Gamma \quad (\text{i.e., } \zeta = e^{i\theta},\ \tfrac{1}{2}\pi < \theta < \tfrac{3}{2}\pi). \tag{B2}$$

Here ϵ_{00} is the smallest number ϵ for which a holomorphic function $S'(\zeta)$ satisfying both (B1) and $|S'(\zeta) - \mathcal{S}'(\zeta)|_\Gamma < \epsilon$ still exists, and is the solution ϵ_{00} of Eq. (4.13) [see also (4.9) and (4.12)]

$$\epsilon_{00} = \epsilon_0[\mathcal{S}'; \epsilon_{00}]. \tag{B3}$$

Conditions (B1) and (B2) are equivalent to

$$|\tilde{S}'^0(\zeta) - \tilde{\mathcal{S}}'(\zeta)| = \epsilon_{00} \tag{B4}$$

on the whole unit circle $\Gamma + \gamma$ with

$$\tilde{\mathcal{S}}'(e^{i\theta}) = \begin{cases} 0 & \text{for } -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi \\ C_0(e^{i\theta})\mathcal{S}'(e^{i\theta}) & \text{for } \tfrac{1}{2}\pi < \theta < \tfrac{3}{2}\pi, \end{cases}$$

where the new C-weight function $C_0(\zeta)$ is defined in terms of ϵ_{00} by (4.15b) and (4.10). We write

$$\begin{aligned} \tilde{\mathcal{S}}'(e^{i\theta}) &= \tilde{\mathcal{S}}'_+(e^{i\theta}) + \tilde{\mathcal{S}}'_-(e^{i\theta}), \\ \tilde{S}'(\zeta) &= \tilde{\mathcal{S}}'_+(\zeta) - \chi(\zeta), \end{aligned} \tag{B5}$$

where $\tilde{\mathcal{S}}'_\pm$ are the positive- and negative-frequency parts of the Fourier series for $\tilde{\mathcal{S}}'(e^{i\theta})$ on the unit circle. Since $\tilde{\mathcal{S}}'_+(e^{i\theta})$ can be continued analytically inside the unit disk, the problem comes to finding that holomorphic function $-\chi(\zeta)$ which best approximates the "nonanalytic" part $\tilde{\mathcal{S}}'_-(e^{i\theta})$ of $\tilde{\mathcal{S}}'(e^{i\theta})$ on $\Gamma + \gamma$:

$$|\chi(e^{i\theta}) + \tilde{\mathcal{S}}'_-(e^{i\theta})| = \epsilon_{00}. \tag{B6}$$

Restricting ourselves to a finite number, N, of negative Fourier coefficients (4.12),

$$\tilde{\mathcal{S}}'_-(e^{i\theta}) \approx \tilde{\mathcal{S}}'^{(N)}_-(e^{i\theta}) \equiv c_{-1}e^{-i\theta} + c_{-2}e^{-2i\theta} + \cdots + c_{-N}e^{-Ni\theta}, \tag{B7}$$

we come to the problem of finding[6] a holomorphic function

$$\psi_0(\zeta) = \zeta^N[\tilde{\mathcal{S}}'^{(N)}_-(\zeta) + \chi(\zeta)]/\epsilon_{00} \tag{B8}$$

of unit modulus on $|\zeta| = 1$ and with its first N Taylor coefficients fixed,

$$\begin{aligned} \psi_0(\zeta) &= \psi_{0,0} + \psi_{0,1}\zeta + \cdots + \psi_{0,N-1}\zeta^{N-1} + \frac{\zeta^N\chi(\zeta)}{\epsilon_{00}}, \\ \psi_{0,i} &\equiv c_{-(N-i)}/\epsilon_{00}, \\ |\psi_0(e^{i\theta})| &= 1. \end{aligned} \tag{B9}$$

The existence of this function is granted by Eq. (B3) and the function itself can be constructed in a recurrent way. Indeed, if the first $N-k$ Taylor coefficients $\psi_{k,i}$ of the unit-modular function $\psi_k(\zeta)$, $|\psi_k(e^{i\theta})| = 1$ are given, then

$$\psi_{k+1}(\zeta) = \frac{1}{\zeta}\,\frac{\psi_k(\zeta) - \psi_{k,0}}{1 - \psi^*_{k,0}\psi_k(\zeta)} \tag{B10}$$

is again a unimodular function,

$$|\psi_{k+1}(e^{i\theta})|=1,$$

but with only $N-(k+1)$ preassigned coefficients $\psi_{k+1,i}$ [which may be determined via Eq. (B10) from the $\psi_{k,j}$, and hence from $c_{-1}, c_{-2}, \ldots, c_{-N}$]. It can be shown that it follows from (B3) that the last function $\psi_N(\zeta)$ has to vanish identically; hence, coming back step by step, one can determine $\psi_0(\zeta)$ completely [and hence $\chi(\zeta)$ and also $S^0(\zeta)$] in terms of the constants $c_{-1}, \ldots, c_{-N}$ [Eq. (4.12)], in the form of a (finite, for N finite) Blaschke product. It is obvious that the amplitude constructed above is the best unitary amplitude one can produce from the left-hand cut data function, $\mathcal{S}(e^{i\theta})$.

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[1]This is an unstable problem which nevertheless can be solved in a mathematically neat way if stabilizing regularity conditions (e.g., Hölder continuity) are added. See R. E. Cutkosky, Ann. Phys. (N.Y.) 54, 110 (1969); S. Ciulli, Nuovo Cimento 61A, 787 (1969); S. Ciulli, *ibid.* 62A, 301 (1969).

[2]G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960); G. Frye and R. L. Warnock, *ibid.* 130, 478 (1963); R. L. Warnock, *ibid.* 131, 1320 (1963).

[3]Unfortunately many people are haunted by Lehmann ellipses and Legendre series extrapolations. There exists nevertheless an optimal way of handling these extrapolations from interior points to the cut (see papers quoted in Ref. 1).

[4]S. Ciulli and J. Fischer, Nucl. Phys. B24, 537 (1970); G. Nenciu, Lett. Nuovo Cimento 4, 96 (1970).

[5]In the unequal-mass case one prefers to start from the k^2 complex plane where the circle cut does not exist.

[6]S. Ciulli and G. Nenciu, J. Math. Phys. (to be published).

[7]The boundedness condition combined with a smoothness condition (e.g., Hölder-type continuity) are necessary to ensure the stability of the solutions of the various possible equations.

[8]S. Ciulli and G. Nenciu, in *Proceedings of the Fifth International Conference on Elementary Particles, Lund, 1969,* edited by G. von Dardel (Berlingska Boktryckeriet, Lund, Sweden, 1970); Commun. Math. Phys. 26, 236 (1972).

[9]We should point out that if the error is too small, there may exist no analytic amplitudes at all in the error channel. See Sec. IV.

[10]C. Foiaş (private communication).

[11]For any quantity $q(z,F)$, we have $q(z,F)\le \sup_z q(z,F)$ wherefrom $\inf_F q(z,F)\le \inf_F \sup_z q(z,F)$ irrespective of z; i.e., also $\sup_z \inf_F q(z,F)\le \inf_F \sup_z q(z,F)$.

[12]F. Riesz, Acta Math. Acad. Sci. Hung., 42, 145 (1920).

[13]D. Luenberger, *Optimization by Vector Space Methods* (Wiley, New York, 1969), Chap. 5.

[14]F. Riesz and B. Sz.-Nagy, *Leçons d' Analyse Fonctionnelle* (Akad. Kiadô, Budapest, 1952).

[15]C. Foiaş (private communication).

[16]As will be shown further in (3.15), the function $F_z^0(z'')$ can be readily produced for special points like, for instance, $z=1$; but, as will be seen, the solution of the problem being not unique, for practical purposes we will use another function $F_z^P(z'')$ related to the Poisson kernel.

[17]The last integral is convergent, since, by definition, $\tilde{\rho}_I(z')$ contains a factor $1-z'$.

[18]Although it can be shown that $h_z^0(z'')$ is for any z a finite Blaschke product [C. Foiaş (private communication)].

[19]S. Ciulli, Nuovo Cimento (to be published).

[20]I. Caprini, S. Ciulli, A. Pomponiu, and I. Sabba-Stefanescu, Phys. Rev. D 5, 1658 (1972).