Spectral Forms for Four-Point Functions*

Richard J. Ivanetich
Department of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 12 February 1973)

Continuing the program initiated in a recent work on three-point functions, we present a detailed study of the establishment in source theory of spectral forms for four-point functions. Both single- and double-spectral forms are treated, and the particles are all allowed different masses (although some inequalities among the masses are employed). Most of the work is carried out for the lowest-order nontrivial contributions, but some considerations are also presented for higher-order contributions. The major new element here, relative to the three-point-function work, is the matter of mass extrapolation. That is, the double-spectral form as first obtained refers to external particles with certain off-shell momenta, since it is derived from a causal realization of the amplitude in which the momenta are thus specified. So one must then extrapolate to real external particles, and that in turn causes an extrapolation of the original spectral-mass domain. A somewhat different extrapolation occurs for the single-spectral form. The paper concludes by generally reviewing the source-theoretic procedures for establishing spectral forms, with some speculations regarding further developments, and by presenting a brief comparison with the conventional, analytic approaches.

I. INTRODUCTION

A principal motivation in the founding of source theory was to provide a theoretical framework, expressible in configuration and momentum space, that maintains a close association with the circumstances of high-energy experiment, in contrast to the remote connection provided in operator field theory. At the most rudimentary level, one thus studies[1,2,3] the passage, as in an accelerator beam, of noninteracting particles between causally related production and detection sources. Next, specific dynamical mechanisms are inserted for the general sources[2-5]; interaction among the particles is thereby realized, with the regions of interaction being causally separated. Upon consideration of a few examples of these causal processes,[2,4-6] it became obvious that such naturally provide the basis for the derivation of spectral forms for scattering amplitudes. Our interest is to more generally and systematically study this establishment of spectral forms from causal processes. In a previous work[7a] we investigated single-and double-spectral forms for three-point functions, and in the present one we turn to the consideration of four-point functions. A general goal for our program is to employ the physical ideas of source theory to elucidate the analytic structure of amplitudes more simply, and possibly more completely, than is done in the detailed complex-variable studies of S-matrix theory.

We shall in particular be concerned with the double-spectral form in $(p_\alpha + p_\beta)^2$ and $(p_\alpha + p_\delta)^2$, and with the single-spectral form in $(p_\alpha+p_\beta)^2$, where p_i, $i = \alpha, \beta, \gamma, \delta$ are the momenta of the particles to which the four-point function refers. The relevant causal processes, apart from some interchange among the external particles, are illustrated, respectively, in the causal diagrams of Figs. 1 and 2. (In such diagrams time is read vertically, the long thin lines refer to real particles, the short heavy ones to virtual particles, and K designates the source.) There are of course other four-source causal processes (two of which are shown in Fig. 3); they are related to spectral forms in sets of invariants which include some p_i^2, and it also would be of some interest to investigate these. Our work here is carried out for scalar particles with simple trilinear primitive interactions, as was the case in TPF,[7] but as discussed there, the conclusions are easily generalized. It is instructive, though, to explicitly illustrate some of the modifications necessitated by such matters as gauge invariance and identical particles, and as a simple example of such we have prepared a work on spectral forms for elastic photon-photon scattering.[7b]

Two main steps are required to proceed from the amplitude for the causal process illustrated in Fig. 1 to the double-spectral form applicable to the conventional scattering situation: space-time generalization and mass extrapolation. In the former the restrictions corresponding to the causal propagation of the internal particles are removed. In the latter the momenta of the external particles are brought on shell, and it turns out that this leads to an extrapolation of the causally established spectral-mass domain. Such an extrapolation also arises, in a somewhat different way, during the single-spectral studies, in addition to space-time generalization. The application of space-time generalization in both the single- and double-spectral studies is quite similar to that in TPF. And, as there, it can be carried out for

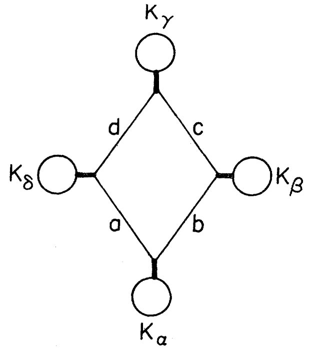

FIG. 1. The causal process leading to the double-spectral form.

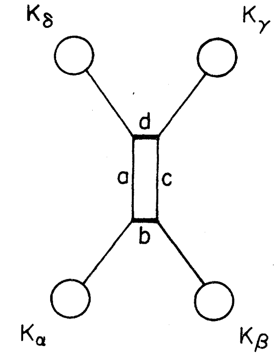

FIG. 2. The causal process leading to the single-spectral form.

causal processes of general order.

Although the matter of extrapolation was considered to some extent in TPF, its execution in the present instance is quite different. We defined in TPF two kinds of extrapolation, physical and mathematical. In the former one obtains the amplitude for the extrapolated values of the external momenta by directly considering new causal processes where the momenta have these values, whereas in the latter one must study the variation of the originally obtained amplitude as the momenta are varied.[8] Use was made in TPF of the former, but not the latter. In the present studies physical extrapolation is not directly available; rather, we follow an approach that is somewhat intermediate between the two. That is, we must study the variation of the original amplitude upon extrapolation, but this is not just a mathematical task: It is provided with some physical meaning and, in a loose sense, is related to causal processes with unphysical momenta.

Previously in source theory,[5,6] for a few simple examples, the extrapolation for the double-spectral form was carried out. That work was highly dependent on the specific form of the spectral weight function. Consequently, one was quite limited in the variety of mass values that could be assigned the particles in the lowest-order causal process (Fig. 1), and there was little hope of being able to consider higher-order processes. The extrapolation procedure developed in the present work is much less dependent on the specific form of the weight function. And we easily treat the situation in which all particles in the lowest-order process have different masses, although some inequalities involving the masses are imposed. Also, some progress is made in treating general-order processes, but further work remains there. Similar conclusions apply to the extrapolation required for the single-spectral form, a matter which has not been previously considered in source theory.

The double-spectral form is treated in Sec. II, and the single-spectral form in Sec. III. In each case, following the treatment of space-time generalization and then extrapolation, we next study the additional causal processes that are in general necessary for the complete lowest-order results. For example, for the single-spectral form in $(p_\alpha + p_\beta)^2$, with $(p_\beta + p_\gamma)^2$ considered as fixed, we must also consider the three additional causal processes obtained from that of Fig. 2 by the interchanges $\beta \leftrightarrow \gamma$, $\gamma \leftrightarrow \delta$, and $\beta \leftrightarrow \gamma, \beta \leftrightarrow \delta$. Lastly in each of these sections, some considerations on higher-order processes are presented. Finally, in Sec. IV, we generally review the source-theoretic methods for obtaining spectral forms and present some speculations toward further developments. Also provided there is a brief comparison with the conventional, analytic approaches.

II. DOUBLE-SPECTRAL FORM

A. Derivation of the Spectral Form

The vacuum-amplitude contribution to which the causal process of Fig. 1 corresponds is obtained by the effective-source technique, which was detailed in TPF. The result is

$$\int (dx)\cdots(dx''')\, \varphi_\alpha(x)\, \Delta_+^b(x-x')\, \varphi_\beta(x')\, \Delta_+^c(x'-x'') \times \varphi_\gamma(x'')\Delta_+^d(x''-x''')\, \varphi_\delta(x''')\, \Delta_+^a(x'''-x)\,. \quad (1)$$

Propagation functions are expressed as

$$\Delta_+(x-x') = \int \frac{(dp)}{(2\pi)^4}\, \frac{e^{ip(x-x')}}{p^2+m^2-i\epsilon}\,, \quad (2a)$$

and for the causal situation $x^0 > x'^0$ this reduces to

$$\Delta_+(x-x') = i\int d\omega_p e^{ip(x-x')} \quad (2b)$$

$$= i\int_{p^0>0} \frac{(dp)}{(2\pi)^3}\, \delta(p^2+m^2)\, e^{ip(x-x')}, \quad (2c)$$

where

$$d\omega_p = \frac{1}{2p^0}\frac{(d\vec{p})}{(2\pi)^3}, \quad p^0 = (\vec{p}^2 + m^2)^{1/2}. \tag{3}$$

Because the source specifications require $x''^0 > x'^0$, $x'''^0 > x^0$, the causal form is employed for the propagation functions in Eq. (1). The numerical-valued fields are defined in momentum space by

$$\varphi_j(x) = \int \frac{(dp_j)}{(2\pi)^4} e^{+ip_j x}\varphi_j(p_j), \tag{4}$$

$j = \alpha, \beta, \gamma, \delta$ (all external momenta thus being defined as incoming). According to the causal specifications, the $\varphi_j(p_j)$ can be nonvanishing only for the p_j satisfying $-p_\alpha^2 \geq (m_a + m_b)^2$, $p_\beta^2 \geq 0$, $-p_\gamma^2 \geq (m_c + m_d)^2$, $p_\delta^2 \geq 0$; more specifically, each $\varphi_j(p_j)$ is taken to be nonvanishing at just one fixed, but arbitrary, p_j^2 within this domain. Upon substitution of the propagation-function and field expressions, the vacuum amplitude (1) can be reworked to give

$$\int [(d\underline{p})\varphi(\underline{p})](2\pi)^4\delta(p_\alpha + p_\beta + p_\gamma + p_\delta)\bar{\rho}, \tag{5}$$

where

$$\begin{aligned}\bar{\rho} = \int (dp)\, &\delta((p - p_\alpha - p_\beta)^2 + m_a^2)\\ &\times \delta((p - p_\beta)^2 + m_b^2)\\ &\times \delta(p^2 + m_c^2)\,\delta((p + p_\gamma)^2 + m_d^2)\end{aligned} \tag{6}$$

and

$$[(d\underline{p})\varphi(\underline{p})] \equiv \prod_{j=\alpha,\beta,\gamma,\delta}\left[\frac{(dp_j)}{(2\pi)^4}\varphi_j(p_j)\right]. \tag{7}$$

It is useful to display some of the details of the evaluation of $\bar{\rho}$. Consider the frame in which $\vec{p}_\alpha + \vec{p}_\beta = 0$ and the orientations are chosen so that $\vec{p}_{\alpha,\beta}$ are along the x direction and $\vec{p}_{\gamma,\delta}$ are in the xy plane. We then have

$$\begin{aligned}\bar{\rho} = \int (dp)\, &\delta(-m_c^2 - (p_\alpha^0 + p_\beta^0)^2 + m_a^2\\ &\quad + 2(p_\alpha^0 + p_\beta^0)p^0)\\ &\times \delta(-m_c^2 + p_\beta^2 + m_b^2 + 2p_\beta^0 p^0 - 2p_{\beta x}p_x)\\ &\times \delta(-m_c^2 + p_\gamma^2 + m_d^2 - 2p_\gamma^0 p^0\\ &\quad + 2p_{\gamma x}p_x + 2p_{\gamma y}p_y)\\ &\times \delta(m_c^2 - p^{02} + p_x^2 + p_y^2 + p_z^2).\end{aligned} \tag{8}$$

The δ functions, starting from the one on the left, determine the values of the components of p, so the integration is immediate. The result is

$$\begin{aligned}\bar{\rho} &= |2(p_\alpha^0 + p_\beta^0)\,2p_{\beta x}\,2p_{\gamma y}p_z|^{-1}\\ &= |8\epsilon_{\mu\nu\lambda\sigma}p_\alpha^\mu p_\beta^\nu p_\gamma^\lambda p^\sigma|^{-1},\end{aligned} \tag{9}$$

where the latter, covariant form reestablishes $\bar{\rho}$ in any frame. We further have ($p = p_c$)

$$\begin{aligned}(\epsilon_{\mu\nu\lambda\sigma}p_\alpha^\mu p_\beta^\nu p_\gamma^\lambda p^\sigma)^2 &= (\epsilon_{\mu\nu\lambda\sigma}p_a^\mu p_b^\nu p_c^\lambda p_d^\sigma)^2\\ &= -\Delta_4(p_a, p_b, p_c, p_d),\end{aligned} \tag{10}$$

the Gram determinant being generally defined by

$$\Delta_n(p_1, \ldots, p_n) = \det(p_i \cdot p_j), \quad i, j = 1, \ldots, n. \tag{11}$$

And the $p_i \cdot p_j$ for the case of present interest are evaluated as

$$\begin{aligned} p_a p_c &= -\tfrac{1}{2}(M^2 - m_a^2 - m_c^2),\\ p_b p_d &= -\tfrac{1}{2}(M'^2 - m_b^2 - m_d^2),\\ p_a p_b &= \tfrac{1}{2}(p_\alpha^2 + m_a^2 + m_b^2),\\ p_b p_c &= -\tfrac{1}{2}(p_\beta^2 + m_b^2 + m_c^2),\\ p_c p_d &= \tfrac{1}{2}(p_\gamma^2 + m_c^2 + m_d^2),\\ p_d p_a &= -\tfrac{1}{2}(p_\delta^2 + m_d^2 + m_a^2),\end{aligned} \tag{12}$$

where the definitions $M^2 = -(p_\alpha + p_\beta)^2$ and $M'^2 = -(p_\alpha + p_\delta)^2$ have been used. So, with the insertion of these evaluations understood, $\bar{\rho}$ is given by

$$\bar{\rho} = \tfrac{1}{8}[-\Delta_4(p_a, p_b, p_c, p_d)]^{-1/2}. \tag{13}$$

The existence of real particles in causal processes imposes certain kinematic constraints. That is, it must be that the simultaneous solution of the mass-shell and conservation-of-momentum conditions admit real (i.e., not complex) momentum components for these particles. For four particles—a, b, c, d—this reality condition is equivalently expressed by the following equations, as is common for physical-region statements[9]:

$$\begin{aligned}&\Delta_2(p_a, p_b) \leq 0,\\ &\Delta_3(p_a, p_b, p_c) \leq 0,\\ &\Delta_4(p_a, p_b, p_c, p_d) \leq 0.\end{aligned} \tag{14}$$

For our particular example we substitute into these equations the evaluations (12). This result, however, is not sufficient to uniquely specify the causal process of present interest since there are also two other four-source causal processes with four real internal particles, as illustrated in Fig. 3. But the present example is uniquely singled out by supplementing Eqs. (14) with the simple threshold statements

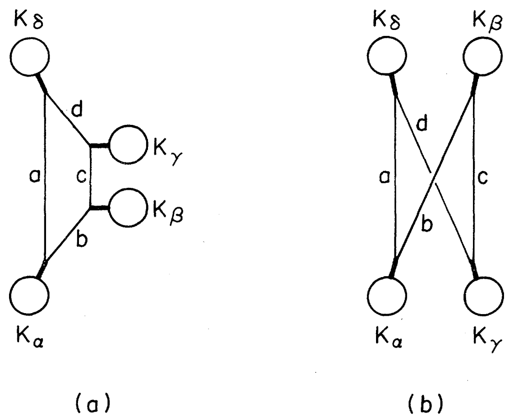

FIG. 3. The two additional four-source causal processes in which the four internal particles are real.

$$M \geqslant m_a + m_c,$$
$$M' \geqslant m_b + m_d. \tag{15}$$

The result (13) for $\bar{\rho}$ is of course true only if the arguments of all the δ functions in Eq. (8) can vanish; otherwise the result is zero. We expect that the former of these circumstances corresponds to the conditions for the existence of the causal process, and this is easy to explicitly show. Namely, the vanishing for the first three δ functions requires that the values of the p_i^2, $i = \alpha, \beta, \gamma, \delta$, be chosen so that the p_i, and hence $p_{0,x,y}$, are real. And for the remaining δ function it is required that the p_z^2 determined in terms of $p_{0,x,z}$ be positive. Or equivalently, by conservation of momentum, the internal momenta $p_{a,b,c,d}$ must have real components.

Returning to the vacuum amplitude proper, Eq. (5), we rework it to express the exhange of the two two-particle excitations ac and bd. Exchanged excitations, as noted in TPF, play a central role in the establishment of spectral forms. First the vacuum amplitude is multiplied by unity expressed as

$$1 = \int (dP)\,(dP')\,\delta(P - p_\alpha - p_\beta)\,\delta(P' - p_\alpha - p_\delta)$$
$$= \int (2\pi)^4\,\delta(P - p_\alpha - p_\beta)\,(2\pi)^4\,\delta(P' - p_\alpha - p_\delta)$$
$$\times \frac{dM^2}{2\pi}\,d\omega_P \frac{dM'^2}{2\pi}\,d\omega_{P'}, \tag{16}$$

in which use was made of the definitions (3) and $M^2 = -(p_\alpha + p_\beta)^2$, $M'^2 = -(p_\alpha + p_\delta)^2$. These last two δ functions and the one originally occurring in the vacuum amplitude are reexpressed according to

$$(2\pi)^{12}\,\delta(p_\alpha + p_\beta + p_\gamma + p_\delta)\,\delta(P - p_\alpha - p_\beta)\,\delta(P' - p_\alpha - p_\delta)$$
$$= \int (dx)(d\xi)(d\xi')\, e^{ip_\alpha(x-\xi-\xi')}\, e^{ip_\beta(x-\xi)}$$
$$\times e^{ip_\gamma x}\, e^{ip_\delta(x-\xi')}\, e^{iP\xi}\, e^{iP'\xi'}. \tag{17}$$

Then, upon return to configuration space, the vacuum amplitude may be stated as a double-spectral form[10]:

$$-\frac{1}{(2\pi)^2}\int (dx)(d\xi)(d\xi')\,\varphi_\gamma(x)\,\varphi_\beta(x-\xi)$$
$$\times \varphi_\delta(x-\xi')\,\varphi_\alpha(x-\xi-\xi')$$
$$\times \Delta_+(\xi, M^2)\,\Delta_+(\xi', M'^2)\,\bar{\rho}(M^2, M'^2)\,dM^2\,dM'^2. \tag{18}$$

Here, Eq. (4) was used for the fields, and the $\vec{P}$ and $\vec{P}'$ integrals supplied the propagation functions according to Eq. (2b) since, by the causal specifications, ξ and ξ' each refer to the displacement between causally related fields, i.e., $\xi^0, \xi'^0 > 0$. The range of the M^2, M'^2 integrations—the spectral-mass domain—is the variation admitted by the causal process. As discussed above, such is given by Eqs. (14) and (15), and in this domain the spectral weight function $\bar{\rho}$ is nonzero and given by Eq. (13). [An example of this domain, considered in Sec. II B1, is illustrated in Fig. 4(a).]

As discussed in TPF, structures such as Eq. (18) can be space-time generalized.[11] Namely, the description of the physical situation in which ξ^0 and ξ'^0 may have any value is provided by Eq. (18) with the conditions $\xi^0, \xi'^0 > 0$ removed. Furthermore, the temporal distinctions between the sources in this equation can be dropped. Henceforth, we shall always mean by Eq. (18) this space time–generalized result. The expression of this result in momentum space requires the use of the noncausal form of the propagation function, Eq. (2a), and one obtains

$$-\frac{1}{(2\pi)^2}\int [(d\underline{p})\,\varphi(\underline{p})](2\pi)^4\,\delta(p_\alpha + p_\beta + p_\gamma + p_\delta)$$
$$\times \frac{1}{(p_\alpha + p_\beta)^2 + M^2 - i\epsilon}\,\frac{1}{(p_\alpha + p_\delta)^2 + M'^2 - i\epsilon}$$
$$\times \bar{\rho}(M^2, M'^2)\,dM^2\,dM'^2. \tag{19}$$

Complementary to the generalization in ξ and ξ', $(p_\alpha + p_\beta)^2$ and $(p_\alpha + p_\delta)^2$ may now take on any values, timelike or spacelike.

B. Mass Extrapolation

In the casual process it is required that the displacements ξ and ξ' have positive time components and that the sources emit particles with certain off-shell momenta. The former of these restrictions was lifted from the vacuum amplitude by the process of space-time generalization. The removal of this restriction makes it no longer physically necessary to maintain the latter restriction, so we now consider if indeed it can be removed in the vacuum amplitude, and, in particular, if the sources can be brought to describe real particles. This procedure, known as mass extrapolation, is necessary to complete the establishment of the double-spectral form for the usual scattering situation of the four particles $\alpha, \beta, \gamma, \delta$, with any two forming the initial state.

1. *Previous Work*

Let us first briefly recall how mass extrapolation was treated previously in source theory for the four-point function.[6,12,13] The basics of that work are clearly illustrated by considering the simplest possible kinematic situation:

$$\begin{aligned} -p_a{}^2 = -p_b{}^2 = -p_c{}^2 = -p_d{}^2 = m^2\,, \\ -p_\alpha{}^2 = -p_\gamma{}^2 = p_\beta{}^2 = p_\delta{}^2 = \overline{m}^2\,, \end{aligned} \tag{20}$$

which is applicable only if one intends to extrapolate to real external particles of zero mass. For this case $\Delta_4(p_a, p_b, p_c, p_d)$ reduces to

$$\Delta_4 = \tfrac{1}{16} f_1 f_2\,, \tag{21}$$

with

$$f_1 = M^2 M'^2 - 4\overline{m}^2\,, \tag{22a}$$

$$f_2 = (M^2 - 4m^2)(M'^2 - 4m^2) - (4m^2)^2\,. \tag{22b}$$

Then the spectral-mass domain, generally determined by Eqs. (14) and (15), is easily specified. Namely, from the region admitted by $\Delta_4 \leq 0$ and the threshold statements, $\Delta_3 \leq 0$ picks out the part with $f_1 \leq 0$, $f_2 \geq 0$, while the Δ_2 condition guarantees that such is not empty. This result is illustrated in Fig. 4(a). Upon extrapolation to real external particles, $\overline{m} \to 0$, f_1 becomes a positive object, while f_2 is unaffected. The spectral-mass domain after mass extrapolation is thus taken to be given by $f_1 \geq 0$, $f_2 \geq 0$, although the former inequality can be dropped since it is now implied by the latter. This extrapolated spectral-mass domain is shown in Fig. 4(b). Also, after extrapolation $\overline{\rho} = \frac{1}{2}(-f_1 f_2)^{-1/2}$ is an imaginary object (with its over-all sign remaining to be determined, which is done below).

A critical feature of this analysis is the simple form into which Δ_4 factorizes. For a few other choices of masses such persists, but in general Δ_4, considered as a function of M^2 and M'^2, is neither simple nor factorizable. Also, the extrapolation of the external momenta brought about an extrapolation of the spectral-mass integration domain. And the manner in which this was carried out appeared natural for the simple example. But for a more complicated Δ_4, such a natural path is not apparent. It is possible, though, to generalize the above method to the situation where all particles are allowed different masses,[12] but that work requires a detailed examination of Δ_4 as a function of M^2 and M'^2, and the whole approach appears quite forced. So next we turn to two new methods of extrapolation that are better suited for treating the more general situations for the four-point function.

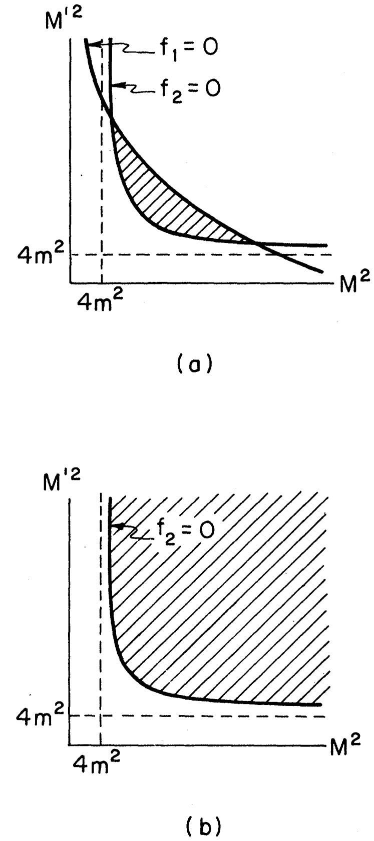

FIG. 4. The spectral-mass domain before extrapolation (a) and after extrapolation (b), for the simple choice of masses considered in Sec. II B 1.

2. *First Method*

The obvious thing to attempt first is a simplification of Δ_4, and also Δ_3, by change of variables, with this simplicity then being employed to construct a natural scheme of extrapolation. Toward this end, consider the frame in which $\vec{p}_\alpha + \vec{p}_\beta = 0$, the $\alpha\beta$ c.m. frame. Then, proceeding from Eq. (10), we can rewrite Δ_4 as

$$-\Delta_4=[M(\vec{p}_\beta\times\vec{p}_c)\cdot\vec{p}_\gamma]^2 = M^2\vec{p}_\beta^{\,2}\vec{p}_c^{\,2}\vec{p}_\gamma^{\,2}(1-z^2)(1-\sigma^2)\,, \tag{23}$$

and similarly

$$-\Delta_3=M^2\vec{p}_\beta^{\,2}\vec{p}_c^{\,2}(1-z^2)\,, \tag{24}$$

where z is the cosine of the angle between $\vec{p}_\beta$ and $\vec{p}_c$, and σ is the cosine of the angle between $\vec{p}_\gamma$ and the plane formed by $\vec{p}_\beta$ and $\vec{p}_c$.

In carrying out the extrapolation, we shall only consider, at least in the present paper, the situation where the factors $\vec{p}_\beta^{\,2}$, $\vec{p}_c^{\,2}$, and $\vec{p}_\gamma^{\,2}$ play an inert role, i.e., retain positive values after extrapolation. Furthermore, pending any possible indications to the contrary, we assume that the most minimal of the causal restrictions for the spectral masses, the threshold statements (15), persist after extrapolation. From the first threshold statement it immediately follows that $\vec{p}_c^{\,2}$, being evaluated in the $\alpha\beta$ c.m. frame, remains positive when the external particles are brought on shell. Similarly, the same conclusion is found for $\vec{p}_\beta^{\,2}$ and $\vec{p}_\gamma^{\,2}$ if the restrictions

$$m_\alpha+m_\beta,\; m_\gamma+m_\delta\le m_a+m_c \tag{25a}$$

are imposed. The evaluation of Δ_3 and Δ_4 and the analysis to follow could also have been carried out in the $\beta\gamma$ c.m. frame. So, for consistency we also require, analogous to Eq. (25a), that

$$m_\beta+m_\gamma,\; m_\alpha+m_\delta\le m_b+m_d\,. \tag{25b}$$

Mass restrictions also enter in another way. In order that the original causal process exist as specified (a point discussed extensively in TPF), it is required that particles α and γ be stable. Alternately, one could have employed the causal process in which K_β acted as the production source, so for consistency stability is also required of particles β and δ. Thus we impose the conditions

$$\begin{aligned} m_\alpha&\le m_a+m_b\,,\\ m_\beta&\le m_b+m_c\,,\\ m_\gamma&\le m_c+m_d\,,\\ m_\delta&\le m_d+m_a\,. \end{aligned} \tag{26}$$

Now, changing from the variables M^2, M'^2 to z, σ, we have from Eqs. (23) and (24) that before extrapolation the integration domain in the spectral form is specified by[14]

$$z:-1\to 1,\quad \sigma:-1\to 1\,. \tag{27}$$

This variation of z of course refers to the fact that z is a cosine evaluated in terms of the momenta at a physically realized vertex ($b\beta c$). But, when the external particles are brought on shell, this evaluation then becomes one referring to three on-shell particles at the vertex. Such is an unphysical situation, and thus[15] z takes on unphysical values, satisfying $z>1$. On the other hand, for the variable σ we find no physical argument relating to a change of domain. An obvious suggestion is thus presented for the spectral integration domain after extrapolation, with it being only natural to admit all values of z exceeding 1. We accordingly take this domain to be given by

$$z:1\to\infty,\quad \sigma:-1\to 1\,. \tag{28}$$

Next we return to the variables M^2 and M'^2. The spectral-mass integration domain after extrapolation is given by Eqs. (28) and (15), which, by use of Eq. (23), are equivalent to the statement that M^2 and M'^2 take on all values satisfying

$$\Delta_4\ge 0\,,\quad M\ge m_a+m_c\,,\quad M'\ge m_b+m_d\,. \tag{29}$$

[Δ_4 is given by Eq. (11), with the external momenta in Eq. (12) on shell.] This domain is similar to that illustrated in Fig. 4(b), with the asymptotes being $M^2=(m_a+m_c)^2$ and $M'^2=(m_b+m_d)^2$.

Since Δ_4 has changed sign, $\bar{\rho}$, Eq. (13), is now imaginary, but its over-all sign remains to be determined: $\bar{\rho}=\pm i\frac{1}{8}(\Delta_4)^{-1/2}$. This sign is determined by a simple comparison with a noncausal expression for the vacuum amplitude, as we now exhibit. The initially generated vacuum-amplitude term (1) may be space-time generalized as it stands. That is, it is meaningful[16] to remove the causal restrictions between the space-time coordinates, which introduces for the four propagation functions expressions of the form (2a). And furthermore, owing to the simple structure of this type of result, the extrapolation of the sources to describe real particles can immediately be carried out. The result—which is said to be a noncausal expression because its derivation did not make any substantial use of the specifications of the causal process—is given in momentum space by

$$\int[(d\underline{p})\,\varphi(\underline{p})](2\pi)^4\,\delta(p_\alpha+p_\beta+p_\gamma+p_\delta) \times\int(dp)\,\frac{1}{(p-p_\alpha-p_\beta)^2+m_a^{\,2}}\,\frac{1}{(p-p_\beta)^2+m_b^{\,2}} \times\frac{1}{p^2+m_c^{\,2}}\,\frac{1}{(p+p_\gamma)^2+m_d^{\,2}}\,, \tag{30}$$

where the $-i\epsilon$ factors have been suppressed. This result and the mass-extrapolated double-spectral form are two different expressions of the vacuum amplitude referring to the same source specification. Thus they must be equal, and their comparison will determine the over-all sign in the latter. Any special case may be considered to effect the comparison, and in particular we consider the mass choice $m_{\alpha,\beta,\gamma,\delta}=0$ in the limit of all external momenta approaching zero. In Eq. (30) one can then immediately transform to a Euclidean momentum $(p_0 \to ip_4)$, which illustrates that the p integral is $+i$ multiplied by a positive factor. Comparison with the double-spectral form then establishes[17]

$$\bar{\rho} = -i\tfrac{1}{8}(\Delta_4)^{-1/2} \equiv -i\rho\,. \tag{31}$$

So, in summary, the mass-extrapolated double-spectral form is given as

$$\frac{i}{(2\pi)^2}\int [(d\underline{p})\,\varphi(\underline{p})](2\pi)^4\,\delta(p_\alpha+p_\beta+p_\gamma+p_\delta) \times \frac{1}{(p_\alpha+p_\beta)^2+M^2-i\epsilon}\,\frac{1}{(p_\alpha+p_\delta)^2+M'^2-i\epsilon} \times \rho(M^2,M'^2)\,dM^2\,dM'^2 \tag{32}$$

or its configuration-space counterpart. Here the spectral weight function ρ is given by Eq. (31) and the spectral-mass integration domain is specified by Eq. (29). In this result all masses are allowed to be different, and each may take on any value, subject to Eqs. (25) and (26). This result is of course the same as that first obtained by Mandelstam[18]—and often referred to as the Mandelstam representation—and later by the conventional analytic methods, e.g., the Landau-Cutkosky approach.[19] However, less labor is required in our approach in that it is not necessary to study the analytic structure of the amplitude, which is a rather detailed task.[20] Also, the spectral form (32) is one with normal thresholds, and so Eqs. (25) and (26) may be regarded as predicting a region of masses where such spectral forms exist; this conclusion is consistent with the conventional predictions.

3. *Second Method*

There is one point in the above extrapolation procedure that stands out above the particular details: The spectral-mass domain after extrapolation is related to the unphysical situation specified by all particles, internal and external, being on shell. This suggests that we attempt a more direct approach to extrapolation; that is, work this unphysical situation into the consideration of the causal process, rather than waiting until the spectral form has been achieved and then extrapolating. Now we cannot directly mass extrapolate the vacuum amplitude for the causal process, since then some momentum components become imaginary, and such would invalidate the steps in the reworking of the vacuum amplitude into spectral form. But this reworking concentrates mainly on the external-particle structure of the vacuum amplitude—that apart from $\bar{\rho}$. Furthermore, $\bar{\rho}$ in the form directly obtained from the causal process, Eq. (6), expresses the central feature of the causal process, the reality of the internal particles. A more direct approach to the extrapolation is thus provided by the following procedure: Carry out the space-time generalization to spectral form as usual, but with $\bar{\rho}$ retained in the causal form (6). Then perform the mass extrapolation, and finally evaluate $\bar{\rho}$. In some ways this procedure turns out to be more appealing than the first method of extrapolation.

In this second method, the external momenta that appear in the evaluation of $\bar{\rho}$ are unphysical. On the other hand, those appearing in the external-particle structure of the extrapolated vacuum amplitude are physical on-shell momenta, since there the demands of the causal process were removed upon space-time generalization. The external momenta appear with these two different evaluations simply because, as made clear in the spectral-form derivation of Sec. II A, the external momenta occurring in $\bar{\rho}$ do not upon space-time generalization maintain their association with the fields, but rather serve to provide the spectral variables M and M'.

Now we turn to the specifics of this second method. Consider the space-time generalization, but not the mass extrapolation, to have been carried out, with $\bar{\rho}$ in causal form. And choose the particular Lorentz frame so that the explicit expression of $\bar{\rho}$ in terms of the momentum components is given by Eq. (8). Now bring the external momenta from the virtual to the on-shell values. To see the extrapolation that is thus induced in the vacuum amplitude, we need examine the components of the external momenta as they appear in $\bar{\rho}$. As in the first method, the two threshold statements (15) are assumed to apply after extrapolation. The external momenta are thus unphysical, which means that some of their components must be imaginary. With the mass-sum restrictions (25) imposed, it is easily seen[21] that such are the y components; that is,

$$\begin{aligned} p_\alpha &= (p_\alpha{}^0, p_{\alpha x}, 0, 0)\,,\\ p_\beta &= (p_\beta{}^0, -p_{\alpha x}, 0, 0)\,,\\ p_\gamma &= (p_\gamma{}^0, p_{\gamma x}, i\bar{p}_{\gamma y}, 0)\,,\\ p_\delta &= (p_\delta{}^0, -p_{\gamma x}, -i\bar{p}_{\gamma y}, 0)\,, \end{aligned} \tag{33}$$

where $\bar{p}_{\gamma y}$ is real. In the expression for $\bar{\rho}$, Eq. (8), the imaginary momentum component enters in only one place, the $p_{\gamma y}p_y$ term in the m_d δ function, which is also the only place p_y occurs. It is required that the vacuum amplitude, and hence $\bar{\rho}$, remain meaningful under extrapolation, and thus as a part of the extrapolation procedure we take

$$p_y \to -i\bar{p}_y, \tag{34}$$

where $\bar{p}_y$ is real. [$+i\bar{p}_y$ is of course also possible, thus giving an undetermined over-all sign, which is resolved in favor of Eq. (34) by the argument leading up to Eq. (31).]

The extrapolated weight function is thus given as

$$\begin{aligned}\bar{\rho} = -i\int dp_0 dp_x d\bar{p}_y dp_z\, &\delta(-m_c^2-(p_\alpha{}^0+p_\beta{}^0)^2+m_a{}^2 \\ &+2(p_\alpha{}^0+p_\beta{}^0)\,p^0) \\ &\times\delta(-m_c{}^2-m_\beta{}^2+m_b{}^2+2p_\beta{}^0p^0-2p_{\beta x}p_x) \\ &\times\delta(-m_c{}^2-m_\gamma{}^2+m_d{}^2-2p_\gamma{}^0p^0+2p_{\gamma x}p_x \\ &+2\bar{p}_{\gamma y}\bar{p}_y) \\ &\times\delta(m_c{}^2-p^{02}+p_x{}^2-\bar{p}_y{}^2+p_z{}^2)\,.\end{aligned} \tag{35}$$

This integral, involving only real quantities, is evaluated as was Eq. (8) in Sec. IIA, the result being

$$\begin{aligned}\bar{\rho} &= -i\,|2\,(p_\alpha{}^0+p_\beta{}^0)\,2p_{\beta x}\,2\bar{p}_{\gamma y}p_z|^{-1} \\ &= -i\,|i\,2(p_\alpha{}^0+p_\beta{}^0)\,2p_{\beta x}\,2p_{\gamma y}p_z|^{-1}\,.\end{aligned} \tag{36}$$

And this is recast into covariant form as in Sec. II A, but with the i serving to provide $i^2=-1$:

$$\bar{\rho} = -i\tfrac{1}{8}\,(\Delta_4)^{-1/2}\,. \tag{37}$$

Here Δ_4 is given by Eq. (11) with the external momenta in Eq. (12) being on shell.

Concerning the spectral-mass integration domain in this second method, it is taken to be the domain for which the integral expression for the extrapolated weight function, Eq. (35), is nonvanishing (and $M \geq m_a + m_c$, $M' \geq m_b + m_d$). As the discussion below Eq. (15) shows, the unextrapolated domain can be viewed as similarly specified. But that domain is also directly associated with a physical-region statement; here, the association is, in some sense, with an unphysical region. It is manifest that the arguments of the first three δ functions in Eq. (35) can vanish (in distinction to the situation of Sec. II A), so the only restriction comes from the last δ function: $p_z{}^2 \geq 0$. The steps which lead from Eq. (36) to Eq. (37) also provide

$$p_z{}^2 = \Delta_4/M^2 p_{\beta x}{}^2\bar{p}_{\gamma y}{}^2\,, \tag{38}$$

and hence we have $\Delta_4 \geq 0$. Thus the mass-extrapolated double-spectral form (32) is again achieved.

C. Complete Lowest-Order Contribution

The causal process of Fig. 1 is not the only one providing a lowest-order contribution to the mass-extrapolated space-time–generalized vacuum amplitude. That is, contributions from other causal processes of the type of Fig. 1 but with sources interchanged must also be considered, since these contributions, upon space-time generalization and mass extrapolation, also apply to the same source specification as does the previous result—namely, to real-particle sources with general space-time disposition. The basic thing to determine is the full set of causal source arrangements necessary to obtain the complete vacuum amplitude. For a given source arrangement more than one lowest-order causal process may occur, or none may occur, depending on the primitive interactions that are allowed.

It is convenient to designate a given source arrangement by the order of the sources as read counterclockwise around the causal process, starting with the production source; so the causal process of Fig. 1 corresponds to the sequence $\alpha\beta\gamma\delta$. In forming the complete set of source arrangements, we do not admit any arrangement for which the source sequence is a cyclic permutation of one already in the set, since these two just provide two ways of calculating the same final vacuum-amplitude contribution. Also, we do not include in the set a sequence which is the reverse of one already in the set, since any coupling that can be established among the former configuration of sources may also occur for the latter. Thus, out of the 4! possible sequences, the complete set of source arrangements may be taken to be given by the three $\alpha\beta\gamma\delta$, $\alpha\gamma\beta\delta$, and $\alpha\beta\delta\gamma$. These are illustrated in Fig. 5.

The calculation for each of the new configurations is carried through exactly as in Secs. II A and II B for the original configuration. Analogous to Eq. (32), the complete lowest-order vacuum amplitude thus has the expression

$$\begin{aligned}\frac{i}{(2\pi)^2}\int[(d\underline{p})\,\varphi(\underline{p})]\,(2\pi)^4\,\delta(p_\alpha+p_\beta+p_\gamma+p_\delta)\,\Big[&\rho_{\rm I}\,\frac{1}{(p_\alpha+p_\beta)^2+M^2}\,\frac{1}{(p_\alpha+p_\delta)^2+M'^2}+\rho_{\rm II}\,\frac{1}{(p_\alpha+p_\gamma)^2+M^2}\,\frac{1}{(p_\alpha+p_\delta)^2+M'^2} \\ &+\rho_{\rm III}\,\frac{1}{(p_\alpha+p_\beta)^2+M^2}\,\frac{1}{(p_\alpha+p_\gamma)^2+M'^2}\Big]\,dM^2\,dM'^2\,,\end{aligned} \tag{39}$$

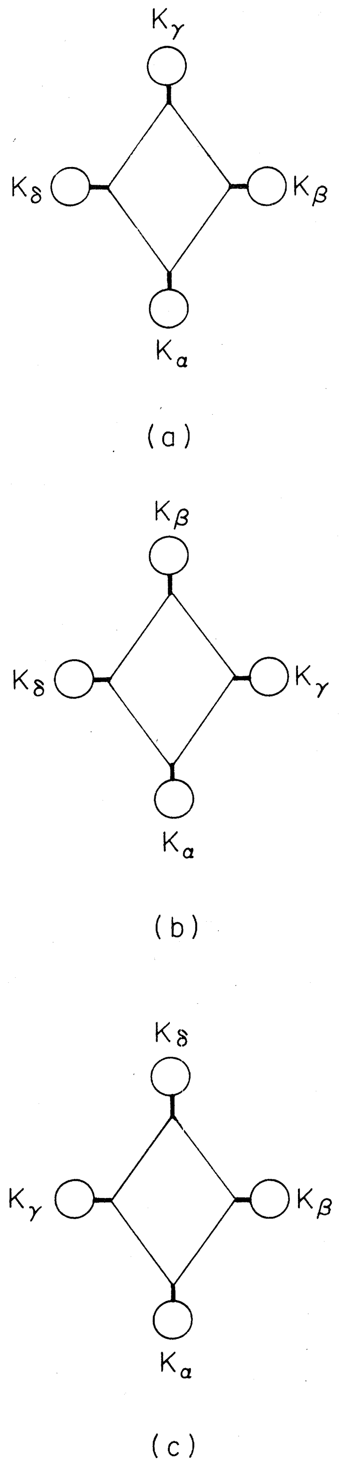

FIG. 5. The complete set of source arrangements necessary to obtain the complete lowest-order double-spectral form.

where I, II, and III refer to the contributions to which Figs. 5(a), 5(b), and 5(c), respectively, correspond. Also, the weight-function arguments and $-i\epsilon$ have been suppressed, and it is understood that the contributions from different causal processes may have different spectral-mass domains of the type (29) associated with them.[22]

D. Higher-Order Contributions

In TPF a discussion of the higher-order generalizations of the lowest-order processes was presented in which we argued the existence of the corresponding spectral forms. The same ideas can be applied here, prior to the point at which mass extrapolation must be considered, and this we now do.[23] Then some considerations toward the mass extrapolation will be presented.

The basic notion of the generalization, in the present instance, is the replacement of the individual particles that propagate between the sources by excitations composed of any number of interacting particles. These generalized causal processes can be schematically organized in the manner shown in Fig. 6(a), and analogously for the other two source arrangements. In that figure all shaded areas refer to regions of particle interaction. Those regions attached to the sources K_β and K_δ are localized ones (i.e., contain no propagating particles), while the other two are composed of an assemblage of such regions, causally related via the exchange of real particles. Furthermore, the region attached to K_α, and similarly that attached to K_γ, can be decomposed as shown in Fig. 6(b), where each shaded area refers to an assemblage of causally related localized interaction regions. Note that these generalizations of the lowest-order process do not include those processes in which particles pass directly between the regions associated with K_α and K_γ.

In order to specify the spectral-mass domains associated with these causal processes, it is useful to first establish some notation. The two systems of interacting particles into which the K_α interaction region has decomposed, as depicted in Fig. 6(b), and which propagate to the K_δ and K_β localized interaction regions, are referred to as, respectively, excitations a and b, and their momenta are designated by P_a and P_b. Continuing this analogy with the particles of the lowest-order process, we similarly define excitations c and d, with momenta P_c and P_d.[24] Among the various real multiparticle states that comprise, each at a different time, the excitation a in a given causal process, there is one state for which the sum of the individual particle masses is largest. Call this sum M_a, and likewise specify M_b, M_c, and M_d.

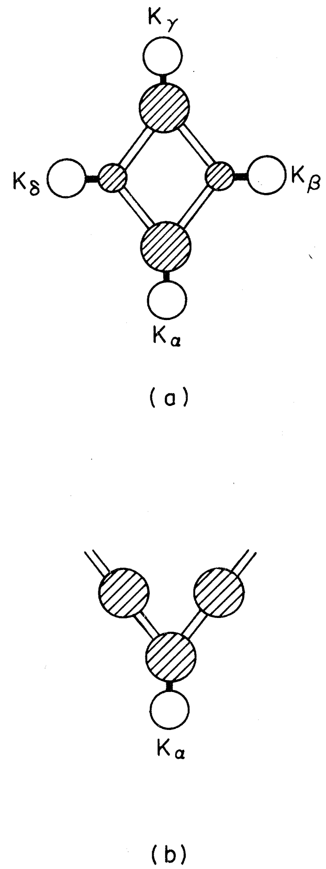

FIG. 6. (a) Illustrative example of the class of higher-order causal processes discussed in Sec. II D. (b) The further decomposition of the interaction region in (a) associated with K_α, with the emergent particles attached.

Also, for the system of interacting particles from which excitations a and b separate, and for that into which c and d coalesce—excitations ab and cd, respectively—we similarly define maximum mass sums, M_{ab} and M_{cd}.

The spectral-mass domain for a given one of the causal processes is specified, as in the lowest-order case, by the kinematic constraints due to the existence of the real particles in the process. For the real multiparticle states appearing in excitations ab and cd, the constraints are $-p_\alpha^2 \geq M_{ab}^2$ and $-p_\gamma^2 \geq M_{cd}^2$. These are taken simply as part of the source specifications, along with $-p_\alpha^2 \geq (M_a + M_b)^2$ and $-p_\gamma^2 \geq (M_c + M_d)^2$. For the excitations $i = a, b, c, d,$ the $-P_i^2$ may take on any values satisfying $-P_i^2 \geq M_i^2$ and $(-P_a^2)^{1/2} + (-P_b^2)^{1/2} \leq (-p_\alpha^2)^{1/2}$, $(-P_c^2)^{1/2} + (-P_d^2)^{1/2} \leq (-p_\gamma^2)^{1/2}$, which are also conditions independent of the spectral masses. The basic constraints are caused by the interrelation of all four of the excitations i. If the $-P_i^2$ are momentarily considered as fixed, then the matter is exactly as in the lowest-order case, and the constraints are given by Eqs. (14), (15), and (12), with the substitutions $m_i^2 \to -P_i^2$. These equations then specify a part of the spectral-mass domain. And the full domain is the union of all such parts obtained upon letting the $-P_i^2$ vary over their allowed range, as stated above.

Now consider the establishment, before mass extrapolation, of spectral forms from these processes. The basic point is that the steps in proceeding from the initially generated vacuum amplitude to the space-time generalized expression only involve a reworking of the external-particle structure of the vacuum amplitude, which is the same as in the lowest-order example. All reference to the internal particles is contained in the $\bar{\rho}$ associated with the generalized process and, in a much simpler manner, in the spectral-mass domain. The vacuum-amplitude contributions are thus expressible as double-spectral forms of the type (18) and (19), or their analogs from the other two source arrangements, although $\bar{\rho}$ remains only some general function.

Next is the matter of mass extrapolation. To begin, we again momentarily consider that the $-P_i^2$, $i = a, b, c, d,$ are held fixed. With such, the determination of the new spectral-mass domain that results upon mass extrapolation is carried out exactly as in the first method for the lowest-order example, because the relevant equations in the present case are those of the lowest-order case with $m_i^2 \to -P_i^2$. And so, the domain is given by Eq. (29) with this substitution. The union of all such domains which are obtained as the $-P_i^2$ vary over their range is the complete extrapolated spectral-mass domain corresponding to the given process. The range of the $-P_i^2$ after extrapolation is from M_i^2 to ∞, this latter value being chosen since after extrapolation the causal circumstances which enforced an upper bound in terms of the source momenta have been removed. Thus, the complete extrapolated spectral-mass domain is simply given by Eq. (29) with $m_i \to M_i$, since any domain for fixed $-P_i^2 > M_i^2$ is contained within that for $-P_i^2 = M_i^2$ (which also shows that the upper limit of the $-P_i^2$ is inconsequential in determining the complete domain). Also, in carrying out the steps of the extrapolation method, we must impose the analogs of the restrictions (25) and (26) for each fixed set of the $-P_i^2$. To enforce all these inequalities it is sufficient to demand that the restrictions (25) and (26) with $m_i \to M_i$ be satisfied.

So, the spectral-mass domain after mass extrapolation has been obtained. However, such is not all that is involved in mass extrapolation, and it is really not correct to just consider such by itself: In any extrapolation it is necessary that the vacuum amplitude remain meaningful. In particular, we must investigate if the spectral weight function remains well defined as the spectral-mass domain is extrapolated, or, in the language of the second method, as some momentum components become imaginary. But this is the point at which our present efforts stop. The fact that there is a structural similarity between the higher-order processes and the lowest-order one,[25] and that the extrapolation of the lowest-order weight function was quite simple, leads us to have some optimism concerning the task which remains.

Turning to a different matter, we briefly discuss the fact, passingly noted at the outset of this section, that not all possible higher-order causal processes with the given source specification are included in the set obtained by the generalization from the lowest-order process. In particular, not included in this class is the causal process shown in Fig. 7 (where the three-particle production and detection acts may each be realized via either a four-particle primitive interaction or two three-particle primitive interactions coupled by a virtual particle). From this process it should be possible to obtain, prior to mass extrapolation, a double-spectral form according to the methods presented in Sec. II A. But the determination of the spectral-mass domain requires a new set of Gram-determinant physical-region statements. Also, the internal momenta are no longer completely determined in terms of the invariants formed from the external momenta, as is the case in the lowest-order example (apart from the sign of p_z). For these reasons, and also because of the different form of the weight function, the structure upon which mass extrapolation should be performed would be quite different from the lowest-order one considered in Sec. II B. It would thus be quite interesting to test the general ideas of that section against this new, more involved example, and see what further insights (and complications !) we can uncover concerning mass extrapolation.

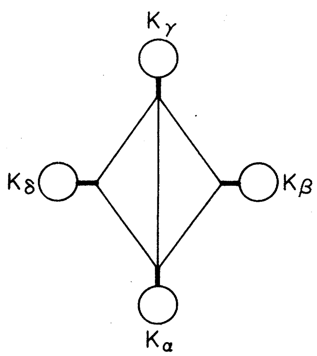

FIG. 7. A higher-order causal process, with the requisite source specification, not included in the class of Fig. 6(a).

III. SINGLE-SPECTRAL FORM

A. Derivation of the Spectral Form; Extrapolation

The relevant causal process is illustrated in Fig. 2. The vacuum-amplitude contribution to which this process corresponds is again found to be given by the expression (1), but the fields therein are now understood to refer to the new source specification. In order to guarantee the temporal separation of the production and detection scattering acts in the causal process, it is necessary to impose certain restrictions. This matter is discussed in detail in TPF, and the analysis there immediately applies to the present situation. Suitable restrictions are the inequalities

$$m_\alpha + m_\beta,\ m_\gamma + m_\delta \leq m_a + m_c \qquad (40a)$$

and

$$m_e \leq m_i + m_{i'}, \qquad (40b)$$

the latter being applied at each of the four vertices, where e refers to the external particle, and i and i' to the two internal ones. [These inequalities are of course the same as Eqs. (25a) and (26), and the arguments leading to them in the two instances are not unrelated.]

The reworking of the causal vacuum amplitude into single-spectral form proceeds according to the usual technique illustrated in TPF or in Sec. II A. Namely, upon use of the temporal stipulations of the causal process the vacuum amplitude is brought into momentum space, wherein the exchanged excitation (ac) is made manifest by the insertion of a unit factor, the vacuum amplitude then being returned to configuration space and space-time generalized. The result is

$$\frac{i}{2\pi}\int (dx)(dx')\,\varphi_\gamma(x)\,\varphi_\delta(x)\,\chi(M^2)\,\Delta_+(x-x',M^2)\times\varphi_\alpha(x')\,\varphi_\beta(x')\,dM^2, \qquad (41)$$

or, in momentum space,

$$\frac{i}{2\pi}\int [(d\underline{p})\,\varphi(\underline{p})]\,(2\pi)^4\,\delta(p_\alpha+p_\beta+p_\gamma+p_\delta)\times\frac{1}{(p_\alpha+p_\beta)^2+M^2-i\epsilon}\,\chi(M^2)\,dM^2, \qquad (42)$$

with the weight function being given by

$$\chi = \int d\omega_{p_a} d\omega_{p_c} (2\pi)^4 \delta(p_\alpha + p_\beta - p_a - p_c) \times[(p_\alpha - p_a)^2 + m_b^2]^{-1} [(p_\gamma + p_c)^2 + m_d^2]^{-1} . \quad (43)$$

Some remarks on the consequences of the space-time generalization should be made. This generalization removes the stipulation that the $\alpha\beta$ and $\gamma\delta$ interaction regions be causally related, i.e., that $(x-x')^0>0$, in the notation of Eq. (41). Complementary to the generalization of this displacement, the invariant $(p_\alpha + p_\beta)^2$ in Eq. (42) may take on any value, timelike or spacelike. But the other of the two independent invariants for a four-point function, $(p_\beta + p_\gamma)^2$, did not so participate in the space-time generalization, and thus maintains its original identity and domain of variation, that appropriate to a momentum transfer squared. [Accordingly, since all the momenta in the weight function refer to the causal situation, in the evaluation of the integral (43) one replaces $-(p_\alpha + p_\beta)^2$ by M^2 but leaves $(p_\beta + p_\gamma)^2$ as is.] Space-time generalization also allows the removal of the production-detection distinction of the sources, but only to the extent that $p_\beta + p_\gamma$ appears as a momentum transfer. Thus, the vacuum amplitude (42) applies to all possible (real-particle) source specifications in which $(p_\beta + p_\gamma)^2$ is so restricted, $(p_\alpha + p_\beta)^2$ having any value.

Still to be determined is the spectral-mass integration domain in the single-spectral form. To this end, as usual, we consider the conditions imposed by the existence of real particles in the causal process. But the entire causal process refers to six real particles, while the weight function only refers to the reality of two internal particles. This situation is different from those encountered in TPF and Sec. II A, where the demands of the causal process and the weight function were equivalent. So which set of demands do we employ to determine the spectral-mass domain? The former, referring to the usual physical-region conditions of scattering processes, requires for a given value of $(p_\beta + p_\gamma)^2$ that M in general is above and cannot reach the threshold value $m_a + m_c$, whereas the latter admits all values $M \geq m_a + m_c$ independent of $(p_\beta + p_\gamma)^2$. Now, the mass-extrapolated double-spectral form also provides an expression for the vacuum-amplitude contribution under consideration, so we may compare the single- and double-spectral forms to resolve between the two possibilities for the spectral-mass domain. And this comparison immediately shows the latter possibility to be the correct one.

The vacuum amplitude describing the causal process is *a priori* well defined since the causal process is guaranteed to exist. But the considerations just presented show that for the space-time–generalized single-spectral form, the spectral-mass domain must be extrapolated from that corresponding to the causal process by extending the lower limit down to $m_a + m_c$. Thus, we must explicitly investigate if the weight function remains well defined in this additional region. To this end one could study in detail the expression (43) or its explicit integrated form, but a simpler alternative is available. It is sufficient to show that for the allowed values of $(p_\beta + p_\gamma)^2$, particles b or d cannot be on-shell while particles a and c are, i.e., the denominators in Eq. (43) cannot vanish. This does not follow immediately from $(p_\beta + p_\gamma)^2$ being below the bd threshold since the momenta involved are unphysical. But it is a direct implication of the second method of mass extrapolation for the double-spectral form that no such kinematic configuration provides a contribution to the vacuum amplitude until $-(p_\beta + p_\gamma)^2$ exceeds the value $(m_b + m_d)^2$. So, this shows that χ remains well defined under the extrapolation of M, but it also allows us to extrapolate $(p_\beta + p_\gamma)^2$ slightly beyond the domain associated with a momentum transfer and claim the applicability of the single-spectral form for all

$$(p_\beta + p_\gamma)^2 \geq -(m_b + m_d)^2 . \quad (44)$$

Also, since the double-spectral form has been invoked here, and in the previous paragraph, the mass restrictions associated with it must now be imposed on the single-spectral form; thus Eqs. (40) are amended by Eq. (25b).

Because the single-spectral form only applies to source specifications included in those of the double-spectral form, and also because its derivation twice invoked the double-spectral form, one might consider the single-spectral form to be of little interest. However, it is useful as another example toward the development of general technique. And, in particular, in Sec. III B we find a certain feature which has not heretofore appeared in the source-theoretic study of spectral forms. In a different vein, for those examples of double-spectral forms involving contact terms, it is necessary to also calculate single-spectral forms in order to determine the contact terms, as will be illustrated in our fourthcoming work on elastic photon-photon scattering.[7b].

B. Complete Lowest-Order Contribution

Analogous to the situation in Sec. II C for the double-spectral form, the single-spectral form obtained from the causal process of Fig. 2 is not the only lowest-order contribution to the vacuum

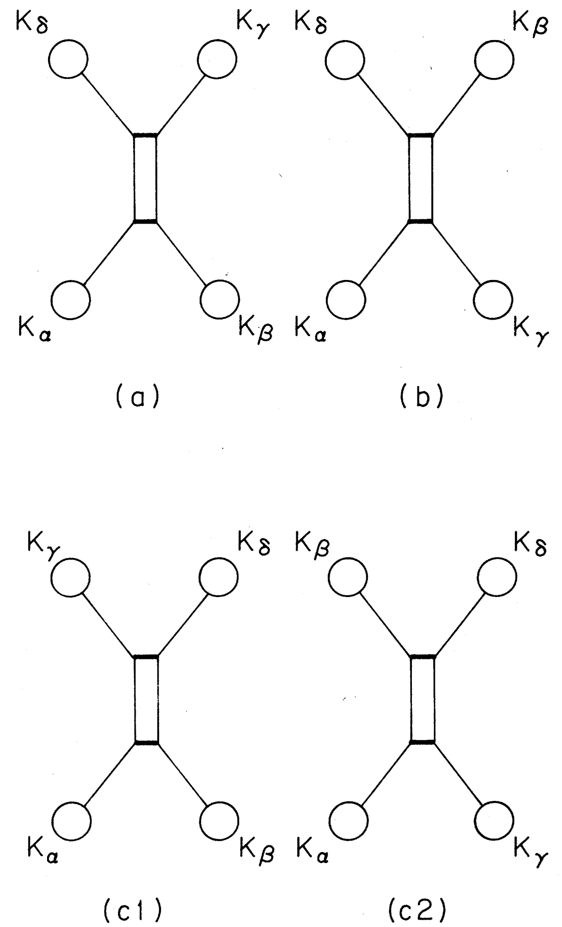

FIG. 8. The four types of causal processes necessary to obtain the complete lowest-order single-spectral form.

amplitude. Rather, we must also consider other causal processes of the structure of Fig. 2, but with sources interchanged, that provide space-time-generalized contributions referring to the same source specifications as does the above result. Since, in the original calculation, $(p_\beta+p_\gamma)^2$ remained fixed, we take it that only those causal processes should be included for which this variable again plays such a role. And this leads to the four types of causal processes shown in Fig. 8, the additional ones in which $p_\beta+p_\gamma$ also appears as a momentum transfer being excluded since they only provide alternate ways of calculating the contributions associated with the given ones. Also, note that all the causal processes of Fig. 8 may be realized between only two arrangements of sources, the original one and that differing from it by $\beta\leftrightarrow\gamma$.

The derivation of the spectral forms for the new contributions is carried through exactly as in the original case, except that $(p_\alpha+p_\gamma)^2$ is the spectral variable in two of the instances. The resulting single-spectral expression for the complete lowest-order vacuum amplitude is then

$$\frac{i}{2\pi}\int [(d\underline{p})\,\varphi(\underline{p})](2\pi)^4\,\delta(p_\alpha+p_\beta+p_\gamma+p_\delta)\times\left[\frac{\chi_{\rm I}+\chi_{{\rm III}_1}}{(p_\alpha+p_\beta)^2+M^2}+\frac{\chi_{\rm II}+\chi_{{\rm III}_2}}{(p_\alpha+p_\gamma)^2+M^2}\right]dM^2, \qquad (45)$$

where I, II, III_1, and III_2 refer to the contributions to which Figs. 8(a), 8(b), 8(c1), and 8(c2), respectively, correspond. It is understood that the contributions from different causal processes may refer to different spectral-mass thresholds, and in all cases the other variable on which the weight functions depend is $(p_\beta+p_\gamma)^2$. Also, $(p_\alpha+p_\gamma)^2$ may be eliminated in terms of $(p_\alpha+p_\beta)^2$ according to the relation

$$-(p_\alpha+p_\beta)^2-(p_\beta+p_\gamma)^2-(p_\alpha+p_\gamma)^2 = m_\alpha{}^2+m_\beta{}^2+m_\gamma{}^2+m_\delta{}^2. \qquad (46)$$

As the notation is meant to indicate, the contributions to the vacuum amplitude (45) designated by III_1 and III_2 are related. That is, once a set of primitive interactions is admitted so that a causal process of the type of Fig. 8(c1) exists, so does one of the type of Fig. 8(c2) [note Fig. 8(c1) "rotates" into Fig. 8(c2)], and vice versa, whereas in this sense the I and II contributions are independent from each other and from III_1 and III_2. A more specific statement of this matter is that two causal processes must be considered to provide the space-time–generalized vacuum-amplitude expression corresponding to one noncausal vacuum-amplitude contribution [the latter being defined in conjunction with Eq. (30)]. Such a feature has not appeared previously in the source-theoretic study of spectral forms.[26] That two causal processes corresponding to one noncausal structure must be considered is because the complete lowest-order vacuum amplitude is obtained from all causal processes (excluding those which just provide duplicate results) in which $p_\beta+p_\gamma$ appears as a momentum transfer. And that the two may occur within this restriction on $p_\beta+p_\gamma$ is because in them β and γ are not adjacent external particles.

This last property is also related to the fact, which we now discuss, that the range of $(p_\beta+p_\gamma)^2$ for which the spectral forms corresponding to these two processes are valid, is not the same as for the original process. For definiteness, first consider a causal process giving a III_1 contribution, and label its internal particles, in the manner corresponding to Fig. 2, by a', b', c', and d'. As in the original case the spectral mass domain is extrapolated down to threshold, $m_{a'}+m_{c'}$, and associated with this an argument analogous to that leading to Eq. (44) is carried out. Since the sources of the given causal process are related to those of the original one by the interchange $\gamma \leftrightarrow \delta$, the analog of Eq. (44) is

$$(p_\alpha+p_\gamma)^2 \geq -(m_{b'}+m_{d'})^2 . \tag{47}$$

To express this in terms of the variables of interest we use

$$M^2-(p_\beta+p_\gamma)^2-(p_\alpha+p_\gamma)^2 = m_\alpha{}^2+m_\beta{}^2+m_\gamma{}^2+m_\delta{}^2 \tag{48}$$

(the external momenta under consideration being those that appear in the evaluation of the weight function). Thus, the single-spectral form corresponding to the III_1 causal process is valid for all

$$(p_\beta+p_\gamma)^2 \leq (m_{a'}+m_{c'})^2+(m_{b'}+m_{d'})^2 - m_\alpha{}^2-m_\beta{}^2-m_\gamma{}^2-m_\delta{}^2 . \tag{49}$$

In other words, the extrapolation may be carried out, in the simple way we argued it, only for a limited portion of the values that a momentum transfer may assume, although in addition the vacuum amplitude is obtained for all timelike values of the variable. [That the right-hand side of Eq. (49) is a positive number follows from the mass restrictions required for the single-spectral forms.]

For the III_2 situation one obtains the analog of Eq. (49) in terms of the corresponding internal particles, and the II case is similarly related to the original restriction, Eq. (44). Thus, from Eqs. (44) and (49), we have that the complete lowest-order vacuum amplitude (45) is applicable for all $(p_\alpha+p_\beta)^2$ and those $(p_\beta+p_\gamma)^2$ satisfying

$$-(m_b+m_d)^2 \leq (p_\beta+p_\gamma)^2 \leq (m_{a'}+m_{c'})^2+(m_{b'}+m_{d'})^2 - m_\alpha{}^2-m_\beta{}^2-m_\gamma{}^2-m_\delta{}^2 , \tag{50}$$

where the internal-particle terms on the left-hand and right-hand sides now refer, respectively, to the largest and smallest such terms obtained from all relevant causal process.

Although in developing the single-spectral form we made a few comparisons with certain features of the double-spectral form in order to resolve some points, it remains a useful consistency check on our methods to show exact agreement between the complete single-spectral and double-spectral forms, the latter being taken only for the range (50). We shall not present the details, but this equivalence can be shown by explicit calculation.[27] In particular, the I terms in Eqs. (39) and (45) agree, likewise for the II terms, and the III term in Eq. (39) agrees with the $\mathrm{III}_1+\mathrm{III}_2$ term in Eq. (45).

C. Higher-Order Contributions

We now present, briefly, some considerations about the expression of four-point vacuum-amplitude contributions of arbitrary order as single-spectral forms. Following the discussion of the Sec. III B, we must consider all arrangements of sources in which $p_\beta+p_\gamma$ appears as a momentum transfer, and these are just the original one and that related to it by $\beta \leftrightarrow \gamma$. The causal processes for each of these arrangements are all formed by exchanging a compound excitation from the two incoming external particles to the two outgoing ones. And, as in the previous discussion of higher-order processes, this excitation is composed of real multiparticle states, causally related, and coupled by localized interaction regions. Some mass restrictions are required to ensure the causal stipulations associated with these processes; the treatment of this point need not be presented here since it is the same as for the single-spectral form in TPF.

The causal vacuum-amplitude expressions for these processes can be reworked into single-spectral form since the procedure for such only involves the external-particle structure of the vacuum amplitude. The complete space-time generalized vacuum amplitude for any given order is then that which is obtained from all causal processes of this order which can occur for the two source arrangements. But remaining is the extrapolation of the spectral mass domain. As a generalization of the lowest-order studies, we take the spectral mass for any given contribution to vary from M_0 to ∞, where M_0 is the largest of the mass sums formed over each of the multiparticle states in the compound excitation. Then, in that portion of the domain not realized in the causal process, we must investigate if the spectral weight function remains well defined, and this matter is left pending.

IV. DISCUSSION

A. Review and Extension

In this section we review the basic ideas presented in TPF and above for the establishment of spectral forms. Included are some remarks, often speculative, concerning the possible extension of these ideas to other processes, both general and specific.

The most fundamental idea, emphasized mainly in TPF, is how the identification of an exchanged excitation in a causal process leads to a spectral structure in terms of the displacement and variable mass of the excitation. In vacuum-amplitude expressions thus obtained, the spectral-mass domain is that specified by the causal process, and the displacements (or complementary momenta) have undergone space-time generalization. As is indicated by the examples in one and two variables that have been presented, this step of our methods appears to be similarly applicable to processes involving any number of exchanged excitations.

And this statement refers to processes of any order, owing to the basic considerations concerning the organization of such processes that were made in TPF and, to some extent, were also discussed in this paper. The central point is that such processes could be viewed as lowest-order ones—or more correctly, as lower-order ones, as indicated by the example at the very end of Sec. II D—in which one- or two-particle states have been replaced by excitations composed of any number of interacting particles. (These are not to be confused with what we refer to as exchanged excitations, which, briefly stated, are more specific in that their total momentum is equal to some sum of external-particle momenta.) Then, the identification of the exchanged excitations is immediate, given that of the parent process, while considerations such as the determination of a causal spectral-mass domain or the imposition of mass restrictions (see below) are just simple generalizations of that for the parent process, although one of course cannot claim explicit expressions for the spectral weight functions.

But two qualifications must be amended to the conclusion stated in the paragraph before the last. The first is simply that the displacements associated with the exchanged excitations must be independent—i.e., it must be possible to vary them independently of one another[28]—since the space-time generalization in them refers to each one separately. The second, which we now go on to discuss, relates to the basic matter, considered in TPF, of the enforcement of the temporal stipulations associated with a causal process.

The casting of the causal vacuum amplitude into spectral form, without space-time generalization, is nothing more than a mathematical rewriting. It is thus required that the stipulations on certain time variables which are central to this task are consequences of the original vacuum-amplitude expression, reflecting the fact that it vanishes outside the domain which they specify; the stipulations are not just conditions arbitrarily and externally imposed on the vacuum amplitude. So, for the given space-time and energy-momentum specification of the sources, it must be that the only way in which the given coupling between the sources can be realized is by the one specific causal process. However, often the source specification is not sufficient to guarantee such. Of use then are mass restrictions [e.g., Eqs. (40)] and overlap considerations (the latter being discussed only in TPF). But beyond these two things nothing further seems to be available, and consequently the structure of the causal processes suitable for calculation is severely limited. For example, the causal process shown in Fig. 9(a) cannot be employed in the scheme of calculation since, for the given source specification, it is not possible to exclude the existence of alternate realizations of the coupling, namely, the causal processes of Figs. 9(b) and 9(c). A general lesson then is that we cannot argue the types of spectral forms that would naively be associated with causal processes involving three or more real external particles interacting in the same localized region. On the other hand, available for calculation are realizations of these same couplings in terms of causal processes involving at any localized interaction region only one or two external particles, where in the latter instance, if the two particles are real, both are either incoming or outgoing.

These remarks, though, do suggest a possible direction for generalizing the calculational scheme, and we interject it here since it may bear looking into. Namely, in those instances where more than one causal realization can exist with the given source specification, one should calculate for each, in some way related to the previous, and then take the space-time–generalized vacuum amplitude for the given coupling to be the sum of such contributions.[29] Note, however, that different contributions will not generally refer to the same spectral variable or set of spectral variables. Also, on a somewhat different matter, one can generally expect that, at fixed order and for a given set of primitive interactions, more than one specification of the sources must be considered in order to get the complete space-time–generalized vacuum amplitude, as is simply illustrated in the example of Sec. III B.

So, the discussion up to this point has indicated that space-time–generalized spectral forms can be argued in some generality. But this, of course, is not the whole story: There is also the matter of extrapolation. That is, there is a change in the spectral-mass domain, and an attendant modification of the spectral weight function, caused by the bringing of virtual external particles on shell (or more generally, taking invariants not in spectral form from their original domain of variation), or caused simply by the structure of the weight function, as in the single-spectral example of Sec. III A. And this matter is the one point in which our methods are seriously incomplete.

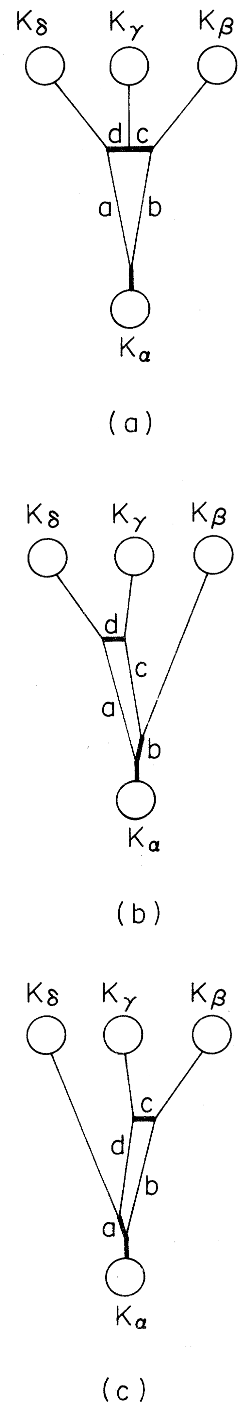

FIG. 9. Example of three causal realizations of one coupling for a given source specification.

Extrapolation was a major concern in the work presented earlier in this paper. We could have viewed the matter as just a mathematical problem, but, in the spirit of source theory, an attempt was made to provide a physical basis for it. In so doing we have gained some insight into the matter, but more is necessary. One general notion that might be pursued further is the point, made at the beginning of Sec. II B 3, that, loosely speaking, the extrapolated situation corresponds to causal processes with unphysical momenta. (The same remark holds for the extrapolation of the single-spectral form since there the weight function was also retained in causal form after the space-time generalization.) The more the extrapolation can be brought into the causal considerations, the more general its execution is likely to become.

More specifically, the study of certain new processes may shed some further light on extrapolation. In particular, the causal processes of Fig. 3, and also that of Fig. 1, reemployed, should lead to spectral forms in some of the p_i^2, $i=\alpha, \beta, \gamma, \delta$. So extrapolation will not be necessary in the corresponding external-particle masses (but will in other variables), and a comparison of these spectral forms with each other and the previously derived ones might serve to fix some of the elements of the extrapolation. Also, we should return to the consideration of extrapolation for higher-order processes, including that of Fig. 7, which was left hanging in Secs. II D and III C. On a related matter, we must also consider extrapolation away from mass restrictions such as Eqs. (25), (26), and (40), some work in this direction having already been done.[30]

There remains a broad and rather appealing speculation. We start by remarking that, in considering the various causal realizations of any given coupling contributing to an n-point function, it would seem desirable to especially study those realizations which would allow the contributions to be expressed with the largest number of invariants in spectral form. This is achieved by considering causal processes in which no more than one external particle couples to any localized interaction region, since in this way the number of independent exchanged excitations is maximized. A drawback to such might be that a more extensive extrapolation is required than in the case where two external particles are allowed at a localized interaction region, because in this latter instance the particles can be real. But if our experience with the four-point function says anything general, there is no such drawback; in fact, the extrapolation above for the double-spectral form proceeded more simply and more naturally than did that associated with the single-spectral form. So, considering the said class of causal processes, one might hope to get all independent invariants for the n-point function (not including the external masses) into spectral form. However, in the framework thus far presented this is not possible for $n>4$, since there the number of such invariants appears to exceed the number of exchanged excitations with independent displacements.

But, as a possible way of accomplishing this aim, we conjecture the following method: Consider a theory in which the number of time dimensions has been increased, establish the spectral-form in a manner analogous to the usual, and then extrapolate back to one time dimension (which means the imposition in the spectral form of nonlinear constraints[31] among the invariants formed from the external momenta). The details of course must be supplied, but the suggestion is that by increasing the number of time dimensions, the number of independent exchanged excitations is increased, and so the number of invariants that can be put in spectral form. Now, as the dimension of space-time is increased, the number of independent invariants also increases, but some simple considerations nonetheless suggest that it might be possible in the present scheme to get all invariants (including external masses) into spectral form. With this taken to be the case, an appealing picture emerges: We do not expect, as has been indicated by some conventional studies,[32] that any simple spectral forms exist for general n-point functions in the usual space-time. But rather, in the multiple-time situation these functions are expressible as spectral forms in all independent invariants, with the spectral-mass domain being specified by the causal process. The complicated situation occuring in the case of one time dimension then arises from this simple structure solely by the imposition in the spectral denominators of the nonlinear constraints which enter in the extrapolation back to one time dimension. It should be quite interesting to turn to specifics and see if there is any substance to this speculation.

Even if this scheme of extrapolation in the dimension of space-time does not live up to the sublime expectation just stated, there are still suggestions that it might be of some utility, as we now briefly mention. First off, a few paragraphs above we remarked that it might be helpful for carrying out extrapolations if the "causal processes with unphysical momenta" could be brought on a more equal footing with the usual causal processes. And this might be accomplished in the multiple-time situation since there the scalar product has additional terms with negative signature, which is what characterizes the unphysical momenta [e.g., $p_y \to i\bar{p}_y$, Eq. (33)]. Secondly, even if all independent invariants cannot be gotten into spectral form, it seems at least that more can than in the usual situation—including more external masses—and this fact alone might be of assistance. Lastly, the determination of the causal spectral-mass domain and the consideration of its extrapolation in the manner of the first method (Sec. II B 2) will be simpler than in the usual situation, since there the spectral variables will be related by the nonlinear constraints. (Increasing the number of space dimensions also provides the simplification.) And this may not be just a matter of algebra, but may also be, in a less encompassing manner, another expression of the point made at the end of the last paragraph concerning the complicated structure of spectral forms in the case of one time dimension.

B. Comparison with Conventional Approaches

In going through Secs. II and III, the reader may well have said to himself on occasion, "But this is obvious from conventional studies." Obviously. But let us again note, as we did in the Introduction, that we are attempting to create new methods, independent of the usual, analytic ones. If, when the opportunity seemed ripe, we were constantly to resort to the conventional arguments, we would most likely also inherit the conventional difficulties. And it is just such that we are attempting, in part at least, to overcome.

Nonetheless, some comparison between our approach and the conventional ones is called for. In particular, we should comment on those methods that employ something analogous to the causal process, the central element of our scheme. The first instance where such occurs is in the work of Coleman and Norton.[33] What they do is to show, within the context of perturbation theory, that whenever the external momenta, taken in the physical region (real components, not necessarily on shell), are such that there may occur between them a multiple-scattering process with real intermediate particles traveling over macroscopic distances, the scattering amplitude has a physical-region singularity, and vice versa. In the context of mass-shell S-matrix theory, the same conclusion (external particles on shell) was obtained by Stapp and co-workers,[34,35] and they developed it far more extensively, including the relation with the discontinuities across the singularities.

An important distinction is that the causal process, in leading to an expression for the amplitude itself, provides more information than does its analog in the work of these authors. Or, in conventional language, the former leads to a description of all physical-*sheet* singularities, whereas the latter is only concerned with physical-*region* singularities. This difference is a consequence of the source-theoretic premise, which has no counterpart in the other work, that all contributions to the amplitude may, upon space-time generalization and mass extrapolation, be generated solely from causal processes.

In connection with this, however, we should note that in another work Norton[36] showed, for the lowest-order three- and four-point-function contributions, that the multiple-scattering interpretation and Cutkosky's discontinuity formula could be used to obtain the amplitude itself. But the methods we used to obtain these results seem much more direct, and recall that for the three-point function we were also able to argue the existence of the spectral forms in general order. Also, in this work Norton makes the point that while the Landau-Cutkosky scheme provides an implicit determination of all singularities and their discontinuities, it does not provide any information on the important fact of which singularities are on the physical sheet. So Norton considers his work as providing a contribution toward the resolution of this deficiency, and, if one so wishes, our work can be viewed likewise.

The greater use made of the causal process, in the sense of the paragraph before the last, is reflected in certain restrictions that are placed upon it. As discussed in Sec. IV A, it is required, for a given source specification, that the only way in which the coupling may be realized is via one specific causal process. Consequently, one is concerned with the localization of the interaction regions occurring in the causal process (see TPF) and is led to guaranteeing the temporal stipulations associated with it, which often necessitates mass restrictions and overlap considerations.[37] Nothing analogous to any of these requirements appears in the work of the other authors.

A further distinction is that, although there is a one-to-one correspondence between causal processes and their analogs in the other work when the two are viewed as geometric figures, there is no such correspondence when the relation to their different calculational purposes is considered. This point is simply illustrated by the lowest-order contribution to the four-point function. Namely, in the analytic approach one is concerned with enumerating the singularities of the amplitude, and these include a normal threshold singularity, which has a multiple-scattering interpretation via a diagram like Fig. 2. But, according to the work of Sec. II for the double-spectral form, our calculation of the amplitude may be phrased solely in terms of the causal process of Fig. 1, an entirely different diagram.

The determination of the structure of the whole amplitude is of course a more difficult task than the determination of the physical-region singularities. This is clearly indicated by the fact that the latter very nicely applies to contributions of general order for any n-point function, while the former has been completed only for three- and four-point functions, the major limiting factor being the necessity of extrapolation. In fact, as discussed in Sec. IV A, our program apart from extrapolation can be considered to be applicable for n-point functions of any order. And it is this aspect of our program that makes the most definite connection with the work of the other authors. In particular, if we take it that the causal processes[38] upon space-time generalization, but without any extrapolation, still provide contributions to the amplitude, although not the full amplitude, then we have the Coleman-Norton theorem in one direction: The existence of multiple-scattering process implies physical-region singularities. In this regard we also should make a comparison between the weight functions appearing in these space-time–generalized results and the discontinuity expressions of Coster and Stapp.[35] The form of these weight functions has not been examined in detail, but in general terms they refer to the propagation of the real internal particles in the causal process and their couplings via localized interaction regions, and such structures bear at least a cursory resemblance to the expressions given by Coster and Stapp.

ACKNOWLEDGMENT

The author gratefully acknowledges discussions with Professor Julian Schwinger.

*Work supported in part by the Air Force Office of Scientific Research under Contract No. F44620-70-C-0030.

[1] J. Schwinger, Phys. Rev. 152, 1219 (1966).

[2] J. Schwinger, *Particles and Sources* (Gordon and Breach, New York, 1969).

[3] J. Schwinger, *Particles, Sources, and Fields* (Addison-Wesley, Reading, Mass., 1970).

[4] J. Schwinger, Phys. Rev. 158, 1391 (1967).

[5] J. Schwinger, *Particles, Sources, and Fields* (Addison-Wesley, Reading, Mass., to be published), Vol. 2.

[6] R. J. Ivanetich, Ph.D. thesis, Harvard University, 1969 (unpublished).

[7a] R. J. Ivanetich, Phys. Rev. D 6, 2805 (1972). Succumbing to the use of acronyms, we shall hereafter refer to this paper as TPF. Many of the ideas employed in the present paper are developed in TPF, but we have attempted to make the present paper self-contained

to the extent that the major points of development should be clear without reference to TPF.

[7b]R. J. Ivanetich, following paper, Phys. Rev D 8, 4564 (1973).

[8]A third kind of extrapolation for an external momentum, which is immediate in execution and really a part of the space-time generalization, occurs in those instances where the square of the momentum is the spectral variable.

[9]R. A. Morrow, J. Math. Phys. 7, 844 (1966); cf. TPF.

[10]The substitution $x \to x + \frac{1}{2}(\xi + \xi')$ gives a more symmetric set of field arguments.

[11]In space-time generalization, contact terms may also occur, i.e., terms in which a propagation function is replaced by a δ function, but, as discussed in TPF, these do not occur for the simple system under study.

[12]R. J. Ivanetich, lectures, Harvard University, 1970 (unpublished).

[13]The treatment in Ref. 5 differs somewhat from that recorded here, but they share what is the central point—dependence on the simple form of Δ_4.

[14]Equation (27) is only what the Δ_n conditions imply immediately (Δ_2 being independent of the relevant variables), but it is easily argued that the threshold statements (15) impose no additional restriction. Namely, since Eqs. (27) are realized in their entirety over the full set of causal processes with four real internal particles (Figs. 1 and 3), and since these processes have disjoint spectral-mass domains, the occurrence of an additional restriction would imply a discontinuous variation of z or σ with M, which physically should not be the case.

[15]The argument is as follows. The momentum components at the vertex are completely specified by $\vec{p}_\beta^{\,2}$, $\vec{p}_c^{\,2}$, and z, and because the former two retain physical values, z must be unphysical. Complex values of z are excluded by

$$\begin{aligned} m_b^2 &= -(p_\beta - p_c)^2 \\ &= m_\beta^2 + m_c^2 - 2(\vec{p}_\beta^{\,2} + m_\beta^2)^{1/2}(\vec{p}_c^{\,2} + m_c^2)^{1/2} \\ &\quad + 2(\vec{p}_\beta^{\,2}\vec{p}_c^{\,2})^{1/2} z, \end{aligned}$$

since $\vec{p}_\beta^{\,2},\ \vec{p}_c^{\,2} \geq 0$. And evaluation for a simple choice of masses gives $z > 1$, z then generally being restricted to such values since as the masses vary continously, so must z.

[16]More on the noncausal expression of vacuum amplitudes is given in TPF.

[17]With $-i\epsilon$ added to each of the external p^2 variables in Δ_4, it is not difficult, for simple choices of the masses, to explicitly follow the variation of $(-\Delta_4)^{1/2}$ as the p^2 are brought on shell, thus determining the over-all sign of the weight function without use of the comparison. And by a simple continuity argument, one can claim that this sign applies for the general mass case.

[18]S. Mandelstam, Phys. Rev. 115, 1741 (1959).

[19]L. D. Landau, Nucl. Phys. 13, 181 (1959); R. E. Cutkosky, J. Math. Phys. 1, 429 (1960).

[20]J. Tarski, J. Math. Phys. 1, 149 (1960); R. J. Eden *et al.*, *The Analytic S-Matrix* (Cambridge University Press, Cambridge, 1966).

[21]The reality of all zero components and of $p_{\alpha x}$ follows from $\vec{p}_\beta^{\,2},\ \vec{p}_\gamma^{\,2} \geq 0$, while the same is implied for $p_{\gamma x}$ by

$$M'^2 = m_\alpha^2 + m_\delta^2 + 2p_\alpha^0 p_\delta^0 + 2p_{\alpha x}p_{\gamma x}.$$

Since $p_{\gamma y}$ is all that remains, it must be imaginary (complex values being ruled out by $m_\gamma^2 = p_\gamma^{02} - p_{\gamma x}^2 - p_{\gamma y}^2$).

[22]Since primitive interactions coupling two external particles (and one or more additional particles) have not been considered, there is a rather degenerate class of possible lowest-order contributions not included here; namely, those corresponding to causal processes in which external particles are coupled at the same point. Such contributions are basically three- and two-point functions and are treated by the methods available for such. Analogous remarks apply for the processes considered in Secs. II D and III. Also, we should say that in TPF we inadvertently failed to fully note that this degenerate possibility may occur for three-point functions.

[23]More detail on the relevant ideas is given in TPF. Especially, there is the important matter referred to as overlap, which is not discussed here.

[24]The P_i here play the same role as those in TPF, but the present manner of definition is perhaps clearer.

[25]The similarity could be enhanced if, instead of explicitly having to consider excitations a, b, c, d involving interaction, we could treat the causal processes simply as being composed of only four interaction regions, one associated with each source, and four noninteracting multiparticle states which couple these regions. That such can be done for certain single-spectral forms—i.e., the conventional unitarity statement for the spectral weight function—has been shown in the context of source theory by K. A. Milton, Ph.D. thesis, Harvard University, 1971 (unpublished); and unpublished.

[26]A simple exception, which is of a different nature, is provided by those situations where an additional causal process must be considered to determine a contact term.

[27]The relevant integrals are evaluated, in a different context, in Ref. 18.

[28]For example, the displacements associated with the exchanged excitations ac, bd, and ab in the causal process of Fig. 1 are not independent.

[29]This scheme is suggested not only for those cases in which one tries to couple three or more real external particles at one localized interaction region, but also for those cases involving two incoming or two outgoing real external particles in which the mass restrictions that guarantee the localization have not been imposed. On a different matter, a sum similar to the suggested one does already occur in the resolution of the overlap matter, but that suggested in the text is in addition to it, the distinguishing feature being that the external-particle configuration is the same for all the causal processes in the overlap sum.

[30]Such an extrapolation has been carried out in some cases involving simple choices of masses; see Ref. 5, Ref. 6, and K. A. Milton, Phys. Rev. D 4, 3579 (1971). In a more general vein, there is an idea under consideration which we now briefly discuss in the context of the single-spectral form for the four-point function; it is not unrelated to the suggestion, made near the beginning of Sec. IV A, concerning the situation where more than one causal realization of a coupling can occur for a given source specification. Namely, as the extrapolation is carried out we must consider if the weight function remains well defined, and, with the matter treated

as in Sec. III A by reference to the double-spectral form, it seems that as certain mass values are reached, the situation with four on-shell unphysical internal momenta can exist. Thus, the standard single-spectral form should be augmented by a contribution corresponding to this new "unphysical causal process." This seems to be related to the occurrence of anomalous thresholds in conventional studies.

[31]V. E. Asribekov, Nucl. Phys. 34, 461 (1962).

[32]See Eden *et al.* (Ref. 20) for references.

[33]S. Coleman and R. E. Norton, Nuovo Cimento 38, 438 (1965).

[34]D. Iagolnitzer and H. P. Stapp, Commun. Math. Phys. 14, 15 (1969).

[35]J. Coster and H. P. Stapp, J. Math. Phys. 11, 2743 (1970).

[36]R. E. Norton, Phys. Rev. 135, B1381 (1964).

[37]Even if one can treat the more general possibility, suggested near the begining of Sec. IV A, of more than one causal realization of a coupling occurring for a given source specification, localization and overlap considerations should still be necessary.

[38]For the purposes at hand, we should allow the possibility of more than one causal realization of a coupling for a given source specification.

PHYSICAL REVIEW D VOLUME 8, NUMBER 12 15 DECEMBER 1973

Spectral Representation for Elastic Photon-Photon Scattering*

Richard J. Ivanetich

Department of Physics, Harvard University, Cambridge, Massachusetts 02138

(Received 31 July 1973)

The derivation of spectral forms in source theory has recently been systematically developed for systems of scalar particles. Elastic photon-photon scattering in lowest nonvanishing order is here studied as a simple, but representative, example of the additional considerations necessitated by particles having internal quantum numbers. The amplitude is determined as a double-spectral form augmented by a single-spectral one, the latter being related to the imposition of gauge invariance. Also presented is some discussion on how, in source theory, one obtains at given order all the contributions to the amplitude.

I. INTRODUCTION

In two recent detailed works we have studied, in source theory, the establishment of single- and double-spectral forms for three-[1] and four-point functions.[2] The basic ideas in such developments center around space-time and energy-momentum considerations, independent of any internal quantum numbers that the particles may carry. To aid its systematic presentation, that work was thus carried out for scalar particles. In the present work we take up an example that, while rather simple, illustrates fairly well what further techniques are necessary when the more realistic situations of particles with internal quantum numbers are considered. In particular, elastic photon-photon scattering is studied to lowest nonvanishing order in spin-$\frac{1}{2}$ electrodynamics, with the photon polarizations chosen as equal and perpendicular to the scattering plane.[3]

A brief review of those aspects of the four-point-function work for scalar particles that are necessary here is presented in Sec. II. Enough detail is given there to make this paper understandable by itself, but to fully appreciate the matter, Ref. 2 should be consulted.

The main section of this paper is Sec. III, where the calculation for photon-photon scattering is carried out. The major new point there concerns gauge invariance. We show how it is maintained, with the consequence that the basic double-spectral structure is augmented with a single-spectral one. The final result has already been obtained by conventional analyticity techniques.[4] But we should emphasize, as Refs. 1 and 2 make clear, that the source-theoretic approach is independent of analyticity considerations, seemingly being simpler and more physical in its basis.

The calculational scheme for obtaining spectral forms starts from causal realizations of the amplitude, and then, after some reworking, proceeds to the final generally applicable scattering amplitude. The question thus arises as to just what set of these so-called causal processes one must consider in order to obtain the complete scattering amplitude to the given order. The matter is simple for three-point functions (at least when there are no anomalous thresholds), but for four-point functions the attitude that we have developed differs somewhat from that of Schwinger.[5] This point is discussed in Sec. IV, mainly within the context of photon-photon scattering, but also with some consideration of pair creation by two photons.