

Asymptotic Behavior of Trajectory Functions and Size of Classical Orbits*

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The strong-coupling limits of several ladder-graph models are explored. A correlation is found between the power of the coupling constant appearing in the leading Regge trajectory $\alpha(t=0)$ and the size of classical orbits described by the coordinate-space Bethe-Salpeter wave function. Specifically: (i) For the ϕ^3 theory with exchanged mass $m \neq 0$, the orbit radius r_0 approaches a fixed value and $\alpha(t=0) \sim g^{1/2}$. (ii) For ϕ^3 with $m=0$, r_0 grows linearly with g and $\alpha(t=0) \sim g$. (iii) For ϕ^4 , $r_0 \rightarrow 0$ and the leading singularity is a fixed cut in t . Expansions about classical orbits are possible in the first two cases, and lead in lowest order to a harmonic-oscillator equation from which corrections to the classical result may be derived.

I. INTRODUCTION

Recently there has been renewed interest in the asymptotic behavior of trajectory functions in ϕ^3 ladder amplitudes. The behavior of the zero-energy ($t=0$) ladder amplitude with massless ($m=0$) exchange is known exactly through the work of Wick and Cutkosky¹ and Nakanishi.² However, no exact solution is known for the massive case ($m \neq 0$), or for the massless case with $t \neq 0$. In the limit of large coupling constant, it is possible to study the asymptotic behavior of the ladder amplitudes either by solving the Bethe-Salpeter (BS) integral equation approximately³⁻⁷ or by estimating the asymptotic behavior of individual terms in the sum of ladder amplitudes.^{8,9} However, in these approaches one does not obtain simple physical pictures or intuitive interpretations of the results.

In this paper, we shall study the asymptotic behavior of ladder amplitudes by analyzing the BS equation in coordinate space.¹⁰ It is easy to see that the zero-energy BS equation can be expressed as a fourth-order differential equation in coordinate space. Since the asymptotic energy (s) dependence of an amplitude is determined by the largest allowed angular momentum n in the t channel, we concentrate on finding n . In the strong-coupling limit $g \rightarrow \infty$, and for a large angular momentum n , the BS differential equation leads to a simple relation between n and the radius r .¹¹ This relation, $n=n(r)$, has a simple interpretation as the angular momentum for a classical orbit of radius r . The optimum orbit which gives rise to the maximum angular momentum, $n_0 = n(r_0) = \max_r n(r)$, can be obtained easily. By expanding the BS differential equation around the optimum orbit, we can work out systematically the inverse $-g$ corrections to $n=n_0$. The first-order correction to the classical result is worked out explicitly in the text by reducing the BS differential equation to a quantum harmonic oscillator. Physically, this correction

originates from the radial oscillation around the optimum orbit.

In ϕ^3 ladder models and in the strong-coupling limit, we find that the size of the optimum orbit for a massive exchange approaches a constant, while the size of the optimum orbit for a massless exchange increases linearly as the coupling constant. These results are not very surprising because the range of interaction due to a massive (mass m) exchange is always finite ($\sim 1/m$), and that due to a massless exchange is ∞ . Thus, what determines the size of the orbit for a massless exchange is the combination of the coupling strength and the mass (μ) of the constituents. It is interesting to see that the change of interaction range from finite ($m \neq 0$) to infinite ($m=0$) leads to a change of the power dependence on g in the Regge trajectory function α .

Further applications and physical interpretations of our approaches to other models are given in the discussion at the end of the paper.

II. ZERO-ENERGY BETHE-SALPETER EQUATION

In this section, we shall review briefly the method for deriving a differential equation for the zero-energy (i.e., $t=0$) BS amplitude. The homogeneous zero-energy BS equation is given by

$$(p^2 - \mu^2)^2 \phi(p) = \int \frac{d^4 q}{(2\pi)^4 i} K(p, q) \phi(q), \quad (2.1)$$

where $K(p, q)$ describes a two-particle irreducible kernel. For ϕ^3 and ϕ^4 ladder amplitudes (Fig. 1) we have

$$K(p, q) = - \frac{g^2}{(p-q)^2 - m^2 + i\epsilon} \quad (2.2a)$$

and

$$K(p, q) = -\frac{1}{2}ig^2 \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} (2\pi)^4 \delta^4(k_1 + k_2 + q - p) \frac{1}{(k_1^2 - m^2 + i\epsilon)(k_2^2 - m^2 + i\epsilon)}, \quad (2.2b)$$

respectively, where g is the coupling constant. The mass of the exchanged particles, m , and the mass of the two-particle intermediate states, μ , are in general different. In both the ϕ^3 and ϕ^4 models, $K(p, q)$ is a function of $p - q$ only.

After Wick rotations, we can Fourier-transform the BS equation (2.1) and obtain a fourth-order differential equation in four-dimensional Euclidean space,

$$(-\partial^2 + \mu^2)\psi(x) = g^2 V(x)\psi(x), \quad (2.3)$$

where

$$g^2 V(x) = \left[\int \frac{d^4p}{(2\pi)^4} K(p, 0) e^{-i p \cdot x} \right]_{\text{Euclidean}}. \quad (2.4)$$

For ϕ^3 and ϕ^4 ladder amplitudes, the x -space BS potentials are:

(i) ϕ^3 ladder, $m \neq 0$

$$V(x) = \frac{mK_1(mr)}{4\pi^2 r^2}, \quad (2.5a)$$

(ii) ϕ^3 ladder, $m = 0$

$$V(x) = \frac{1}{4\pi^2 r^2}, \quad (2.5b)$$

and (iii) ϕ^4 ladder,¹²

$$V(x) = \frac{1}{8\pi^2} \left[\frac{mK_1(mr)}{r} \right]^2, \quad (2.5c)$$

respectively, with $r = (x^2)^{1/2}$.

Since the potential $V(x)$ is a function of r only

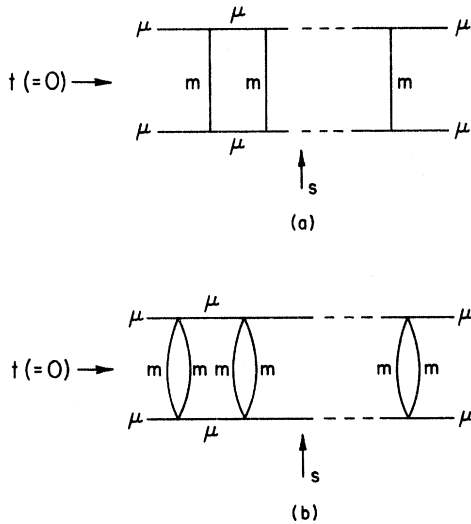


FIG. 1. Ladder amplitudes in the Bethe-Salpeter equation. (a) ϕ^3 case; (b) ϕ^4 case.

and thus Eq. (2.3) is $O(4)$ -symmetric, we can decompose the wave equation (2.3) according to the eigenvalues of 4-dimensional angular momentum n .¹⁰ For a given 4-dimensional angular momentum n , the radial equation is

$$\left[\frac{d^2}{dr^2} + \frac{3}{r} \frac{d}{dr} - \left(\frac{n^2 - 1}{r^2} + \mu^2 \right) \right]^2 \psi_n(r) = g^2 V(r) \psi_n(r). \quad (2.6)$$

Equation (2.6) is the desired differential equation obtained in Ref. 10.¹³

Given (2.6), we can determine the eigenvalues n for all possible zero-energy bound states. For each n , there exists a set of degenerate eigenstates of 3-dimensional angular momentum with $l = n - 1, n - 2, \dots$. In terms of Regge-pole language, each n corresponds to a Lorentz pole which can be decomposed into a series of Regge poles. In particular, the leading Regge trajectory function $\alpha(t=0)$ is related to the maximum eigenvalue n by

$$\alpha(t=0) = n - 1. \quad (2.7)$$

In the next section, we shall concentrate on the determination of maximum n , and consequently, the leading trajectory function $\alpha(0)$.

III. ϕ^3 LADDER MODEL

In the ϕ^3 or ϕ^4 ladder model and in the strong-coupling limit, the maximum eigenvalue n^2 also increases without limit. The asymptotic behavior of the leading trajectory function $\alpha(0) (= n - 1)$ can be studied readily by expanding the radial equation (2.6) and its eigenvalue n as inverse powers of g . To illustrate the method, we work out the ϕ^3 ladder model in detail.

A. Optimal Orbits

For a fixed g , the maximum n^2 is achieved as the radial kinetic term

$$-\left(\frac{d^2}{dr^2} + \frac{3}{r} \frac{d}{dr} \right)$$

in (2.6) becomes minimum.¹⁴ Thus, at large coupling limit and for maximum n^2 , we expect that the radial kinetic term can be ignored in comparison with the centrifugal term n^2/r^2 . However, we shall keep the μ^2 term in (2.6) because it has a different scaling property from the n^2/r^2 term and may become important at large r .

After ignoring the radial kinetic term, we obtain

the 4-dimensional angular momentum n for a classical circular orbit of radius r ,

$$n^2 = r^2 \{ g [V(r)]^{1/2} - \mu^2 \}, \quad (3.1)$$

where we have also ignored 1 relative to n^2 .¹⁵ Equation (3.1) defines $n = n(r)$ as a function of r . For ϕ^3 ladder amplitudes, $n(r)$ has a local maximum $n_0 \equiv n(r_0)$ at $r = r_0$. We shall refer to the orbit at $r = r_0$ as an optimum orbit.

For ϕ^3 ladder amplitudes with $m \neq 0$, we have

$$[n(r)]^2 = \left[\frac{g^2}{4\pi^2 m^2} (mr)^3 K_1(mr) \right]^{1/2} - \mu^2 r^2. \quad (3.2)$$

Using the fact that $x^3 K_1(x)$ has a maximum

$$\max_x [x^3 K_1(x)] = 1.1575 \quad \text{at } x = 2.3867, \quad (3.3)$$

we find that we can ignore the $\mu^2 r^2$ term and obtain at large g

$$\begin{aligned} n_0 &= \left[\frac{g^2}{4\pi^2 m^2} \times 1.1575 \right]^{1/4} + O(g^{-1/2}) \\ &\cong 1.4669 (g/4\pi m)^{1/2}, \end{aligned} \quad (3.4)$$

with

$$r_0 = 2.3867/m, \quad (3.5)$$

i.e., the radius of the optimum orbit for $m \neq 0$ approaches a constant, determined by the range of the interaction, as $g \rightarrow \infty$.

For ϕ^3 ladder amplitudes with $m = 0$, we have

$$\frac{1}{n_0^3} \left[\left(r \frac{d}{dr} \right)^2 - 2r \frac{d}{dr} - n^2 + 1 - \mu^2 r^2 \right] \left[\left(r \frac{d}{dr} \right)^2 + 2r \frac{d}{dr} - n^2 + 1 - \mu^2 r^2 \right] \psi_n(r) = \frac{1}{n_0^3} g^2 r^4 V(r) \psi_n(r). \quad (3.10)$$

Using the fact that $g^2 r^4 V(r)$ has a maximum $g^2 r_0^4 V(r_0) \equiv n_0^4$ at $r_0 = 2.3867/m$, we obtain

$$\begin{aligned} g^2 r^4 V(r) &= n_0^4 \left[1 - 2\omega^2 \left(\frac{r-r_0}{r_0} \right)^2 + O \left(\left(\frac{r-r_0}{r_0} \right)^3 \right) \right] \\ &= n_0^4 \left[1 - \frac{2\omega^2 y^2}{n_0} + O \left(\frac{y^3}{n_0^{3/2}} \right) \right], \end{aligned} \quad (3.11)$$

where

$$\begin{aligned} \omega^2 &= -\frac{1}{4} r_0^2 \frac{d^2}{dr^2} \left\{ \ln [g^2 r^4 V(r)] \right\} \Big|_{r=r_0} \\ &= 2 - \frac{1}{4} (m r_0)^2 = 0.57587. \end{aligned} \quad (3.12)$$

Substituting (3.9), (3.11), and (3.12) into (3.10), and making use of the relation $r(d/dr) = \sqrt{n_0} d/dy$, we have

$$\left[\frac{1}{n_0} \frac{d^4}{dy^4} - \frac{2n^2}{n_0^2} \frac{d^2}{dy^2} + \frac{n^4}{n_0^3} \right] \psi_n = [n_0 - 2\omega^2 y^2 + O(y^3/\sqrt{n_0})] \psi_n. \quad (3.13)$$

In (3.13), we have ignored terms which are manifestly $O(1/n_0)$ or smaller. Considering this equation without the $(1/n_0)d^4/dy^4$ and y^3 terms, we see

$$[n(r)]^2 = \frac{g r}{2\pi} - \mu^2 r^2. \quad (3.6)$$

The maximum value of $n(r)$ is located at

$$r_0 = \frac{g}{4\pi\mu^2}, \quad (3.7)$$

giving

$$n_0 \equiv n(r_0) = \frac{g}{4\pi\mu}. \quad (3.8)$$

Since r_0 increases linearly with g , it is clear that we can no longer ignore the $\mu^2 r^2$ term.

B. Expansions Around the Optimum Orbits

We now take n_0 and the optimum orbit $r = r_0$ as a first-order approximation, and expand the wave equation (2.6) and its eigenvalue n around these classical values. We shall use the massive ($m \neq 0$) ϕ^3 ladder model as an explicit example to demonstrate that the optimum orbit gives a result valid to $O(1/n_0)$, and that more accuracy can be achieved if desired.

We introduce a variable y via

$$\begin{aligned} r &= r_0 e^{y/\sqrt{n_0}} \\ &\cong r_0 (1 + y/\sqrt{n_0}) \end{aligned} \quad (3.9)$$

to describe the deviation of r from the classical orbit $r = r_0$. The factor $1/\sqrt{n_0}$ is included to ensure that quantities of physical interest remain finite as $g^2 \rightarrow \infty$.

Multiplying (2.6) by r^4/n_0^3 , we have

that it reduces to the harmonic-oscillator problem. The expectation values of y and d/dy are hence of order 1, so that in fact the $(1/n_0)d^4/dy^4$ and $y^3/\sqrt{n_0}$

terms also may be dropped for large n_0 . Doing so, we find, for large g^2 , $n \simeq n_0$, $r \simeq r_0[1 + O(1/\sqrt{n_0})] \simeq r_0$, and

$$\left(-\frac{d^2}{dy^2} + \omega^2 y^2\right) \psi_n = \left(\frac{n_0}{2} - \frac{n^4}{2n_0^3}\right) \psi_n. \quad (3.14)$$

The energy eigenvalues of the harmonic oscillator equation (3.14) are well known:

$$E_{n_r} \equiv \frac{1}{2} \left(\frac{n_0}{2} - \frac{n^4}{2n_0^3}\right) = (n_r + \frac{1}{2})\omega, \quad (3.15)$$

with

$$n_r = 0, 1, 2, \dots, \quad \omega = 0.75886. \quad (3.16)$$

We can solve for n trivially from (3.15), getting

$$\begin{aligned} n &= n_0 - (n_r + \frac{1}{2})\omega + O\left(\frac{1}{n_0}\right) \\ &= 1.4669 \left(\frac{g}{4\pi m}\right)^{1/2} - (n_r + \frac{1}{2})\omega + O\left(\frac{1}{n_0}\right). \end{aligned} \quad (3.17)$$

One can see that the corrections to Eq. (3.17) due to the terms ignored in (3.13) are $O(1/n_0)$ as well. A perturbation expansion in $1/n_0$ can then be used systematically to calculate higher-order corrections.

It is straightforward to see that this method also can be applied to $t=0$, massless ($m=0$) ϕ^3 ladder amplitudes. We expand both the wave equation (2.6) and the radius r around the classical values $r_0 = g/4\pi\mu^2$ and $n_0 = g/4\pi\mu$. The equation again reduces, up to terms of $O(1/n_0)$, to a quantum harmonic oscillator. The resultant eigenvalues can be shown to be

$$n = \frac{g}{4\pi\mu} - (n_r + \frac{1}{2}) + O\left(\frac{1}{n_0}\right). \quad (3.18)$$

The exact solution to a $t=0$, massless ϕ^3 ladder is known,^{1,2} and is consistent with (3.18). The eigenvalue n of a massive ϕ^3 ladder was obtained recently by Cheng and Wu through solving an approximate integral equation.⁷ Their result is identical to (3.17).¹⁶

C. Semiclassical Features

The results obtained in Secs. III A and III B have features expected from the semiclassical approximation. It is known that for a system with large quantum numbers, we can ignore the quantum nature of the system and treat the problem classically. Under this classical approximation we have a c -number relation,

$$\left(p_r^2 + \frac{n^2}{r^2} + \mu^2\right)^2 = g^2 V(r), \quad (3.19)$$

where p_r is the radial momentum, and n^2/r^2 is the relativistic centrifugal potential. The classical bound-state solutions correspond to the positive square root of (3.19),

$$p_r^2 + \frac{n^2}{r^2} + \mu^2 = g[V(r)]^{1/2}. \quad (3.20)$$

The largest angular momentum n_0 is obtained classically by first setting $p_r = 0$, and then maximizing the remaining relation $n = n(r)$. The correction to $n = n_0$ due to small oscillations in r around r_0 can be understood readily. For $m \neq 0$, we expand (3.20) around $r = r_0$ to leading order in $1/n_0$ and obtain

$$\frac{p_r^2}{2M} + \frac{1}{2}M\omega^2 q^2 (\equiv E_r) = \frac{n_0^2 - n^2}{2n_0}, \quad (3.21)$$

where ω is given by (3.12), $q \equiv r - r_0$, and $M \equiv n_0/r_0^2$. The left-hand side of (3.21), which is denoted by E_r , is the Hamiltonian of a harmonic oscillator with frequency ω . Inverting (3.21), we have

$$n = n_0 - E_r + O(1/n_0). \quad (3.22)$$

Equation (3.22) coincides with (3.17) if one identifies the classical E_r with its quantum-mechanical analog $E_{n_r} = (n_r + \frac{1}{2})\omega$. A similar semiclassical picture applies to the massless case as well.

IV. DISCUSSION

The method outlined in Sec. III applies as long as the classical angular momentum $n(r)$ has a local maximum at $r = r_0 \neq 0$. However, there is an interesting class of theories in which $n(r)$ has a maximum at $r = 0$. The ϕ^4 ladder amplitude is a typical example; it appears that this property is shared by many other renormalizable theories. In the ϕ^4 ladder model, $[n(r)]^2$ is found to be

$$\begin{aligned} [n(r)]^2 - 1 &= g[V(r)]^{1/2} \\ &= \frac{g}{\pi\sqrt{8}} mr K_1(mr), \end{aligned} \quad (4.1)$$

which is a monotonically decreasing function of r . Its maximum value (at $r = 0$) is

$$n_0^2 - 1 = g/\pi\sqrt{8}. \quad (4.2)$$

Thus, the leading eigenvalue of n is dominated by the small- r behavior of the wave equation (2.6). For $n > n_0$, the combined centrifugal and potential terms in Eq. (2.6) produce a net repulsive potential for all r , while for $n < n_0$ the net potential is attractive and admits a continuum of bound states near $r = 0$.^{10,17} Hence the spectrum of n^2 is also continuous with branch points lying at

$$n = \pm(1 + g/\pi\sqrt{8})^{1/2}. \quad (4.3)$$

The emergence of cuts in this class of theories is

known, and has been discussed by several authors.^{5,10,18} Since our conclusion depends only on the small- r , or equivalently the large- p , behavior of the wave equation, one expects that the positions of the branch cuts are t -independent. This is in fact the case.^{10,18}

We wish to conclude this paper by summarizing the nature of classical orbits in various cases. In the massive ϕ^3 ladder model, the size of the orbit approaches a constant as $g \rightarrow \infty$. One finds in this case that the trajectory function $\alpha = n - 1$ is proportional to the square root of g . In the massless ϕ^3 ladder model, the size of the orbit increases linearly with g and the trajectory function is pro-

portional to g . Finally, in the ϕ^4 ladder model, the size of the orbit shrinks to zero at large g and one obtains a fixed cut rather than a leading pole. Since our results depend only on the gross features of the potentials, the above three examples may well represent three general categories of bound states and the corresponding asymptotic behavior of their trajectory functions.

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¹¹A similar approach was taken qualitatively by Tiktopoulos and Treiman, Ref. 3, but for the Schrödinger equation.

¹²An infinite subtraction constant contributing only to S-wave scattering in the t channel has been discarded.

¹³We have corrected some trivial numerical errors in the corresponding expression of Domokos and Suranyi, Ref. 10.

¹⁴The operator $-[d^2/dr^2 + (3/r)d/dr + 3/4r^2]$ is one whose eigenvalues are bounded from below by zero. This is the reason that an expansion about the radius for which $-[d^2/dr^2 + (3/r)d/dr]$ becomes minimum is possible, as the effect of the last term may be absorbed into n^2/r^2 and changes n by $O(1/n)$.

¹⁵Again, this changes n only by $O(1/n)$.

¹⁶Equation (3.4) was obtained earlier as an upper bound in Ref. 4, where the striking difference between $m = 0$ and $m \neq 0$ cases in the strong-coupling limit was pointed out.

¹⁷The physical reason for this is that an effective $1/r^2$ potential provides no scale for energy eigenvalues or bound-state sizes.

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