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How Do Strings Radiate?*

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An invariant regulator field method is applied to the construction of vector-current vertices in a dual resonance model. If it is not required that the regulators be fully conformally invariant, but only that they have the invariance required for duality, Gaussian factors can be eliminated consistently from both the elastic form factor and the deep-inelastic structure functions. This method introduces new ghost states which can be pushed to arbitrarily large mass by judicious choice of the regulator parameters. However, there are still ghosts at low levels from the basic dynamical oscillator field, as in the Veneziano model with intercept $\alpha(0) \neq 1$. The structure function νW_2 does not scale in the model. The physical interpretation of this result is discussed in detail.

I. INTRODUCTION

In spite of the great progress made in recent years in understanding the structure of dual resonance models (DRM) that stem from the Veneziano formula and its generalizations for hadron-hadron scattering amplitudes, it has not yet proved possible to construct a comparably consistent dual theory of current-hadron interactions. The basic entity in terms of which the theory of the DRM can be most clearly formulated is the Nambu-Susskind "string."[1] There is no doubt that the classical problem of the radiation of a violin string on which charge has been distributed is well defined and can be solved. It is much more difficult to decide whether the analogous fully relativistic, quantum-mechanical problem is even well defined, let alone solvable in full generality. We will not do this in this paper, and the title is to be taken as a real question, not a rhetorical one.

Our aims in this paper are much less ambitious. Two principal strands of development[2,3] in the theory of "dual" currents will be interlaced in such a manner that at least formally the requirements of duality[4] are satisfied in a model for the vector current. The model will then be used to calculate the elastic form factor of the ground state of the string, and the behavior of the deep-inelastic structure function W_2 in the Bjorken limit. The results of these calculations have been published elsewhere.[5] In this paper, we elaborate the arguments leading to the results, and hopefully clarify some physical points. It must be pointed out that we accept at the outset shortcomings of a very serious nature in the formalism, such as the occurrence of both tachyons *and* negative-metric ghosts. Remarkably, in spite of this, the physical interpretation of our results is not implausible.

Some of the principal problems to be faced in constructing a dual theory of currents can be understood by first examining a single harmonic oscillator in three dimensions. The Hamiltonian is (in appropriate units)

$$H = \tfrac{1}{2}(\vec{p}^2 + \vec{x}^2) = \vec{a}^\dagger \cdot \vec{a} + \tfrac{3}{2},$$

with

$$\vec{a} = \tfrac{1}{2}(\vec{x} + i\vec{p}),$$

$$[a_i, a_j^\dagger] = \delta_{ij}.$$

The current in this theory has components (in momentum space)

$$j_0(\vec{k}) = e^{-i\vec{k}\cdot\vec{r}}, \qquad (1.1a)$$

$$\vec{j}(\vec{k}) = \{\tfrac{1}{2}\vec{p}, e^{-i\vec{k}\cdot\vec{r}}\}. \qquad (1.1b)$$

In terms of $\vec{a}$ and $\vec{a}^\dagger$ one has

$$e^{-i\vec{k}\cdot\vec{r}} = e^{-i\vec{k}\cdot(\vec{a}+\vec{a}^\dagger)/\sqrt{2}} \qquad (1.2a)$$

$$= e^{-i\vec{k}\cdot\vec{a}^\dagger/\sqrt{2}}\, e^{-i\vec{k}\cdot\vec{a}/\sqrt{2}}\, e^{-k^2/4}, \qquad (1.2b)$$

where use has been made of a well-known identity to pass to normal-ordered form.

One problem is that if, instead of a single mode, there are an infinite number of modes as in the string, there is a divergence in passing from (1.2a) to (1.2b). This problem with going to the continuum limit has been stressed by Nielsen and Susskind.[6] They argue that it is not reasonable for current probes to really see a continuum in the string, hence there should exist a maximum mode number corresponding to a minimum spacing of constituents of the string. Not unexpectedly, a careless implementation of this proposal leads to a breaking of duality. When the infinities are removed in such a manner as to preserve the duality properties of amplitudes involving currents, it is found that there are still an infinite number of modes in the problem. The details of this dual method of removing infinities, due to Drummond and to Rebbi, are discussed in Sec. II. It is found to be much like a field-theoretic regulator method.[7]

A second problem facing us in constructing dual current theories is how to generalize Eq. (1.1). It describes the current of a single particle in a harmonic well. The method of constructing the current at a space-time point (x_μ) due to the motions of the constituent particles on the string has been given by Nambu.[2] This formalism is also reviewed in Sec. II. One is still left with the problem of deciding how the total charge carried by the string is to be distributed. This distribution must be specified in such a manner that duality is not violated, yet gauge-invariance holds. A model for this distribution is given in Sec. III.

Section III also contains a discussion of a new set of fields which have the invariance properties required for duality. We have found it convenient to introduce these new fields because, even after a model for the vector current is introduced using the formalism of Nambu, and rendered finite by the methods of Drummond and Rebbi, the string is found to have an unpleasant Gaussian elastic form factor. This is the last remnant of the single-oscillator Gaussian evident in Eq. (1.2). The new fields which are introduced, and are applied as regulators, allow us to eliminate this Gaussian, leaving power-law falloff for the elastic form factor. This is demonstrated in Sec. IV. In this manner, it is seen that the string model is indeed capable of reproducing the qualitative behavior of the "electromagnetic" hadronic wave functions.

A third major obstacle in generalizing the single three-dimensional oscillator to the string model is that, for relativistic covariance, a fourth oscillator in a timelike direction is usually also introduced. Much of the recent progress in the DRM has consisted in demonstrating that, in the "manifestly covariant" formulation, there are enough subsidiary conditions to completely eliminate from the physical states all bad, negative-metric states created by the timelike oscillators.[8] Unfortunately, these conditions are only satisfied if the $(\text{mass})^2$ of the particles involved in the scattering process have the spectrum $-1, 0, 1, 2, \ldots, n, \ldots$. The problem of ensuring positivity for processes involving currents remains unsolved, although some attempts to preserve the Virasoro ghost-eliminating gauge conditions[9] in the construction of the currents have been made.[10] We reserve to the conclusion discussion of these attempts.

In this paper, no attempt has been made to ensure positivity of the norms of the states. On the contrary, our philosophy has been as follows. Since regulator fields are used, which by necessity introduce infinitely many ghosts that cannot be eliminated by Virasoro gauge conditions, it is unnecessary to require the regulators to be fully conformally invariant. It is sufficient that they be Möbius-invariant in order that the theory be dual.[11] This freedom is what permits us to use a set of regulator fields that eliminate the Gaussian behavior of the elastic form factors. The procedure is not as crazy as it might at first seem, because one is able to show that the net results, both for the elastic form factor and for W_2, are remarkably insensitive to the precise manner in which the regulators are chosen. Thus the bad effects of the regulators can be pushed to appear at arbitrarily high masses, while the good effects are present by mere virtue of the fact that regulators have been introduced. The hope is, then, that the regulators mimic the properties that a correctly cut-off, positive-definite theory might have, at least for certain matrix elements.[12] Some of these points are illustrated in Sec. VI.

Finally, it is appropriate in this Introduction to discuss why the structure function W_2 should be studied in the DRM at all. One can reasonably expect that a string picture will be totally antithetical to current explanations of the scaling of νW_2.[13] From a parton point of view,[14] it is predictable that the string model will fail to scale because the motions of the constituents of the string can never be considered to be free. From the light-cone considerations,[15] one might expect νW_2 to be nonscaling in the string model if one believes the "fishnet" argument that in DRM the constituent propagators are Gaussians, and hence have no light-cone singularity.

On the other hand, the work of Bloom and Gilman[4] clearly indicates that duality is a notion that is relevant to deep-inelastic processes. Furthermore, several authors have constructed pure resonance models which exhibit scaling behavior.[16] The question is whether the DRM in its present form realizes duality in a manner that is applicable to a discussion of these processes, or whether it really does fall prey to the diseases naive considerations lead us to expect.

Section V contains an analysis of W_2 in the model. We find that νW_2 does not scale in the Bjorken limit for the reasons suggested by the parton model concepts, namely, there is never a "time" scale so short that parton-parton forces can be neglected. Consequently, the "partons" of the string model have structure. There is a contribution to νW_2 from the string's partons which falls off; this is related to the "convective" form factor of the partons. But there is another contribution which grows, due to a "magnetic" parton form factor. Physically this is similar to a parton model in which the partons are quarks endowed with an anomalous magnetic moment. It is known that in such a case, W_2 scales rather than νW_2.

II. CONSTRUCTION OF THE VECTOR CURRENT

In this section we briefly review the construction for the vector current proposed by Nambu.[2] We will see how the sum over the infinite number of modes of the string produces a divergence, and introduce the Drummond-Rebbi procedure for eliminating the infinity.[3] The invariance properties required in DRM for the current to be dual are also discussed.

First recall the amplitudes of oscillation of the string in Minkowski space are described by the field

$$V_\mu^{(a)}(\theta,\tau)=x_\mu^{(0)}+p_\mu^{(0)}\tau+\sum_{n=1}^{\infty}\left(\frac{2}{n}\right)^{1/2}\cos n\theta \times(a_{n\mu}e^{in\tau}+a_{n\mu}^\dagger e^{-in\tau}).$$

Here θ is the length parameter along the string, and τ plays the role of a time variable. The string sweeps out a world sheet with parameters $\pi>\theta>0$, $\infty>\tau>-\infty$. The oscillator variables satisfy the commutation relations $[a_{n\mu},a_{m\nu}^\dagger]=-ig_{\mu\nu}\delta_{nm}$. If these modes of oscillation were absent, $V_\mu^{(a)}(\theta,\tau)$ would describe the position of a classical free particle that was at $x_\mu^{(0)}$ at $\tau=0$, and which has velocity $p_\mu^{(0)}$. Because of a conventional choice of units, $p_\mu^{(0)}$ is actually the center-of-mass momentum of the string.

This suggests that Eq. (1.1) be generalized to

$$j_\mu(x)=\tfrac{1}{2}\int\int d\theta d\tau\, J_\alpha(\theta,\tau)\left\{\frac{\partial V_\mu}{\partial\xi_\alpha},\,\delta^4(x-V)\right\},\tag{2.1}$$

where

$$J_\alpha\frac{\partial V_\mu}{\partial\xi_\alpha}=J_\theta\frac{\partial V_\mu}{\partial\theta}+J_\tau\frac{\partial V_\mu}{\partial\tau}.$$

The statement of current conservation, $\partial_\mu j^\mu(x)=0$, is satisfied if the internal current is conserved, $\partial_\alpha J^\alpha=0$; if the component of the internal current normal to the boundary vanishes at the boundary; if the integral in (2.1) is well defined; and if surface terms vanish.

Unfortunately, as it stands Eq. (2.1) is not well-defined.[17] We see this heuristically through the formal expression,

$$\delta^4(x-V^{(a)}(\theta,\tau))=\int d^4k\,\exp\{ik[x-V^{(a)}(\theta,\tau)]\}.$$

For, passing to normal-ordered form,

$$e^{-ikV^{(a)}}=:e^{-ikV^{(a)}}:\left[\exp\left(-k^2\sum_{n=1}^{\infty}\frac{\cos^2 n\theta}{n}\right)\right].\tag{2.2}$$

The sum in Eq. (2.2) is infinite, so the expression e^{-ikV} makes sense only for $k^2=0$.

The crucial observation that allows continuation of this line of development is that the infinity in the sum in Eq. (2.2) can be canceled if in addition to $V_\mu^{(a)}$ one uses a second field in Eq. (2.1):

$$\tilde V_\mu=V_\mu^{(a)}+i\sum_{n=1}^{\infty}\left(\frac{2}{n}\right)^{1/2}(b_{n\mu}e^{in\tau}+b_{n\mu}^\dagger e^{-in\tau})\sin n\theta.$$

The new b set of oscillators obeys the same commutation relations as the a set, and a and b sets are independent. The relative factor "i" between the sets of modes enables the infinities from the sums $(\sum n^{-1}\cos^2 n\theta)$ and $(\sum n^{-1}\sin^2 n\theta)$ to cancel. One then obtains

$$e^{-ik\tilde V}=:e^{-ik\tilde V}:(2\sin\theta)^{-k^2}.\tag{2.3}$$

The nontrivial factor $(2\sin\theta)^{-k^2}$ which appears in this expression is responsible for the properties of the elastic form factor to be discussed in Sec. IV. Although there is no purely "constant" infinite factor left in (2.3), we see that as θ approaches 0 or π, the expression is still divergent for $k^2>0$. This is not the same type of problem as was the over-all infinity, however, for these divergences show up as a series of poles for integer values of k^2. Thus the currents in this type of model are particle-dominated in the timelike region.

The properly "regulated" current (2.1) can be used to discuss photon-hadron interactions by use of the phenomenological Lagrange density

$$\mathcal{L}^{(\text{int})}=ej_\mu(x)A^\mu(x),$$

where $A_\mu(x)$ is a fixed external photon field. Then the single-current vertex in momentum space is

$$\Gamma_\mu(q)=\frac{e}{2}\int\int d\theta d\tau\, J_\alpha(\theta,\tau)\left\{\frac{\partial V_\mu(\theta,\tau)}{\partial\xi_\alpha},\,e^{iqV}\right\}.\tag{2.4}$$

We must now consider the requirements of duality, which place constraints on the possible forms of $J_\alpha(\theta,\tau)$, and on the candidates for regulators to be used in $V_\mu(\theta,\tau)$.

In the DRM, a minimal requirement for duality of an amplitude is the invariance of the operators used in the construction of the amplitude under Möbius transformations. In the operator formalism, this requirement is that

$$e^{i\vec\alpha\cdot\vec L}\,\Gamma_\mu(q)e^{-i\vec\alpha\cdot\vec L}=\Gamma_\mu(q),\tag{2.5}$$

where L_i are generators of SL(2, R).[18] To focus on the required transformation properties of the fields, it is useful to express our conserved internal current in the form

$$J_\alpha(\theta,\tau)=\epsilon_{\alpha\beta}\partial_\beta\rho(\theta,\tau),$$
$$\epsilon_{\alpha\beta}=-\epsilon_{\beta\alpha}. \tag{2.6}$$

In terms of the density $\rho(\theta,\tau)$, the current vertex Eq. (2.4) can be written

$$\Gamma_\mu(q)=\frac{e}{2}\int\int d\theta d\tau\left\{\frac{\partial(\rho,V_\mu)}{\partial(\theta,\tau)},e^{iqV}\right\}$$
$$=\frac{e}{2}\int\int_{\text{UHP}} dz d\bar{z}\left\{\frac{\partial(\rho,V_\mu)}{\partial(z,\bar{z})},e^{iqV}\right\}, \tag{2.7}$$

where $z=e^{\tau+i\theta}$, and UHP is the upper half complex z plane. The condition (2.5) is satisfied if

$$e^{i\vec{\alpha}\cdot\vec{L}}\left\{\frac{\partial(\rho(z),V_\mu(z))}{\partial(z,\bar{z})},e^{iqV(z)}\right\}e^{-i\vec{\alpha}\cdot\vec{L}}=\left\{\frac{\partial(\rho(z'),V_\mu(z'))}{\partial(z,\bar{z})},e^{iqV(z')}\right\}, \tag{2.8a}$$

where

$$z'=\frac{\alpha z+\beta}{\gamma z+\delta},\quad \alpha\beta-\gamma\delta=1. \tag{2.8b}$$

The real parameters $\alpha,\beta,\gamma,\delta$ are easily related to the parameters α_i in (2.5). The invariance is then satisfied by virtue of the manifest invariance of Eq. (2.4) under arbitrary coordinate transformations on θ and τ with nonsingular Jacobian.

We conclude this section with three observations on the gauge-invariant, dual vector current above:

(1) Inserting Eq. (2.6) into (2.1), one has formally

$$j_\mu(x)=\tfrac{1}{2}\int\int\{d\rho dV_\mu,\delta^4(x-V)\}. \tag{2.9}$$

This is the form of the current due to a collection of classical particles, with charge density ρ. The interaction $j_\mu A^\mu$ with $j_\mu(x)$ of the above form gives rise to the Lorentz equation of motion (without radiative reaction). Thus, the formalism proposed for the electromagnetic interaction of the string parallels the formalism of the free string in physical interpretation.

(2) However, if $J_\theta=\partial\rho/\partial\tau\neq0$, the current $j_\mu(x)$ contains a term involving $\partial V_\mu/\partial\theta$. This term has no analog in a minimally coupled classical theory, for which $j_\mu\propto\partial x_\mu/\partial\tau$. Thus, in general, our theory is not minimally coupled, and one may anticipate effects much like those due to an anomalous Pauli term. We will see such effects in W_2.

(3) At first sight, it appears as though Eqs. (2.8) cannot be satisfied for any nontrivial choice of ρ or V_μ. This stems from the conventional use of so-called multiplier representations of SU(1, 1), or of the isomorphic SL(2, R). We will see that if the duality invariance group is realized in a slightly different space from the one conventionally used, Eqs. (2.8) can be satisfied in a variety of ways. This is the subject of the next section.

III. INTRODUCTION OF NEW FIELDS

A. Invariant Equation

The Lagrange density of the free $V_\mu^{(a)}(z)$ field is

$$\mathcal{L}^{(a)}=\frac{1}{\pi}\left(\frac{\partial V_\mu^{(a)}}{\partial z}\right)\left(\frac{\partial V_\mu^{(a)}}{\partial\bar{z}}\right). \tag{3.1}$$

It follows that $V_\mu^{(a)}(z)$ satisfies the Laplace equation

$$\frac{\partial^2 V_\mu^{(a)}}{\partial z\partial\bar{z}}=0, \tag{3.2}$$

in the z domain. Equation (3.2) is well known to be conformally invariant.

In addition, it has been established that, with $\mathcal{L}^{(a)}$, use of Nöther's theorem leads to the full set of Virasoro gauge operators,[19] which arise as generators of infinitesimal conformal transformations. In particular, the Gliozzi operators $\vec{L}_G$ generate infinitesimal Möbius transformations, and it is known that

$$e^{i\vec{\alpha}\cdot\vec{L}_G}V_\mu^{(a)}(z)e^{-i\vec{\alpha}\cdot\vec{L}_G}=V_\mu^{(a)}(z'), \tag{3.3}$$

where z' is given by (2.8b).

However, the Laplace equation is not the only equation invariant under Möbius transformations. The equation

$$\frac{\partial^2\phi}{\partial z\partial\bar{z}}=-\lambda(\lambda-1)\frac{\phi}{(z-\bar{z})^2} \tag{3.4}$$

is the most general homogeneous second-order differential equation left invariant under the transformation (2.8b).[20] First-order Dirac-like equations have also been given by Sakita,[21] and by Susskind, Casher, and Kogut.[22] Equation (3.4) is the Euler-Lagrange equation resulting from variation of

$$\mathcal{L}^{(\lambda)}=\frac{1}{\pi}\left[\left(\frac{\partial\phi}{\partial z}\right)\left(\frac{\partial\phi}{\partial\bar{z}}\right)-\frac{\lambda(\lambda-1)}{(z-\bar{z})^2}\phi^2\right]. \tag{3.5}$$

The action

$$I=\int d^2z\,\mathcal{L}^{(\lambda)}(z,\bar{z}) \tag{3.6}$$

is unchanged under real Möbius transformations on the integration domain z, so quantized solutions of (3.4) are likely candidates for operators that transform covariantly under SL(2, R).

In part B of this section, normalizable solutions of (3.4) will be displayed. Generators of the group in the quantized theory are constructed in part C, and special properties of the $\rho(z)$ field are treated in part D. The connection of these results to the representation theory of SL$(2, R)$ is discussed in Appendix A.

B. Solutions in the Strip Domain

In the Nambu-Susskind strip, the action (3.6) takes the following form after rotating $\tau \to i\tau$:

$$I = \frac{1}{4\pi}\int\int d\theta d\tau\left[\left(\frac{\partial\phi}{\partial\theta}\right)^2 - \left(\frac{\partial\phi}{\partial\tau}\right)^2 + \frac{\lambda(\lambda-1)}{\sin^2\theta}\phi^2\right]. \tag{3.7}$$

The equation of motion in the strip is

$$\left(\frac{\partial^2}{\partial\theta^2} - \frac{\partial^2}{\partial\tau^2} - \frac{\lambda(\lambda-1)}{\sin^2\theta}\right)\phi^{(\lambda)}(\tau,\theta) \equiv D^{(\lambda)}\phi^{(\lambda)} = 0. \tag{3.8}$$

Normalizable solutions have a very simple form; for 4-vector ϕ's,

$$\phi_\mu^{(\lambda)}(\theta,\tau) = \sum_{m=\lambda}^{\infty}\left(\frac{\pi}{m}\right)^{1/2}(b_{m\mu}^{(\lambda)}e^{im\tau} + b_{m\mu}^{(\lambda)\dagger}e^{-m\tau})\phi_m^{(\lambda)}(\theta), \tag{3.9a}$$

$$\phi_m^{(\lambda)}(\theta) = (2\sin\theta)^\lambda\Gamma(\lambda) \times\left[\frac{m(m-\lambda)!}{2\pi\Gamma(m+\lambda)}\right]^{1/2} C_{m-\lambda}^{\lambda}(\cos\theta), \tag{3.9b}$$

$$\int_0^\pi d\theta\, \phi_m^{(\lambda)}\phi_{m'}^{(\lambda)} = \delta_{mm'} \quad (\lambda > -\tfrac{1}{2}). \tag{3.9c}$$

Here $C_{m-\lambda}^{\lambda}$ are Gegenbauer polynomials as defined in Ref. 23. The boundary condition (b.c.)

$$\phi_\mu(\tau, 0) = \phi_\nu(\tau,\pi) = 0$$

is sufficient for normalizability if $\mathrm{Re}\lambda > 0$. If the b.c. is taken to be

$$(\phi_\mu \partial\phi_\nu/\partial\theta)_{\theta=0,\pi} = 0,$$

the case $\lambda = 0$ can also be included.

By use of the identity

$$\lim_{\lambda\to 0}\Gamma(\lambda)C_n^\lambda(\cos\theta) = (2/n)\cos n\theta, \quad n \neq 0$$

it follows that

$$\phi_n^{(0)}(\theta) = (2/\pi)^{1/2}\cos n\theta.$$

Similarly,

$$\phi_n^{(1)}(\theta) = (2/\pi)^{1/2}\sin n\theta.$$

Since for these values of λ (3.4) becomes the Laplace equation, any other solution of the Laplace equation can be added onto these, provided the boundary conditions are also satisfied.

The inverse of the differential operator $D^{(\lambda)}$ is the Green's function $G^{(\lambda)}$, which satisfies

$$D^{(\lambda)}(\theta,\tau)G^{(\lambda)}(\theta,\tau;\theta',\tau') = \kappa\delta(\theta-\theta')\delta(\tau-\tau'). \tag{3.10}$$

In physical terms, κ represents the strength of the source. This Green's function is constructed in Appendix B, where the boundary condition is also discussed. The result for $\lambda > 0$ is

$$G^{(\lambda)}(\theta,\tau;\theta',\tau') = Q_{\lambda-1}\left(\frac{\cos(\tau-\tau') - \cos\theta\cos\theta'}{\sin\theta\sin\theta'}\right) = Q_{\lambda-1}(w), \tag{3.11}$$

where $Q_\lambda(w)$ is the Legendre function of second kind, and G is normalized so as to agree with the special cases $\lambda = 0, 1$, which are easy to evaluate straightforwardly.

Upon reversion to the elliptic (τ,θ) domain, the argument of the Q_λ function in (3.11) may be transformed to the upper half-plane:

$$w = \frac{\cosh(\tau-\tau') - \cos\theta\cos\theta'}{\sin\theta\sin\theta'} = \frac{1}{4}\frac{|u-\bar{v}|^2 + |u-v|^2}{(\mathrm{Im}u)(\mathrm{Im}v)},$$

where $u = e^{\tau+i\theta}$, $v = e^{\tau'+i\theta'}$. This is invariant under real Möbius transformations on u and v, as required.

An important reason for using the strip domain is that canonical quantization of the fields $\phi_\mu^{(\lambda)}$ may be performed using τ as the time coordinate (see Sec. IV). From (3.7), the canonical momentum conjugate to $\phi_\mu^{(\lambda)}$ is

$$\pi_\mu^{(\lambda)} = -\frac{1}{2\pi}\frac{\partial\phi_\mu^{(\lambda)}}{2\tau}.$$

For Lorentz vector fields, the canonical commutation relation

$$[\pi_\mu^{(\lambda)}(\theta,\tau), \phi_\nu^{(\lambda)}(\theta',\tau)] = ig_{\mu\nu}\delta(\theta-\theta') \tag{3.12}$$

follows at once from the completeness of the functions $\phi_m^{(\lambda)}(\theta)$, provided the expansion coefficients satisfy the commutation relation

$$[b_{\mu m}^{(\lambda)}, b_{\nu m'}^{(\lambda')\dagger}] = -ig_{\mu\nu}\delta_{mm'}\delta_{\lambda\lambda'}, \tag{3.13}$$

and that all other commutators of the coefficients vanish.

For such quantized fields $\phi_\mu^{(\lambda)}(\theta,\tau)$, the τ-ordered two-point function is

$$\langle\phi_\mu^{(\lambda)}(\theta,\tau)\phi_\nu^{(\lambda')}(\theta',\tau')\rangle_0 = -g_{\mu\nu}\delta_{\lambda\lambda'}G^{(\lambda)}, \tag{3.14}$$

where $G^{(\lambda)}$ is given in (3.11). This is easily established using (3.13), the property of the vacuum $b_{\mu m}^{(\lambda)}|0\rangle = 0$, and examination of (B4).

C. Generators in the Quantized Theory

From Eq. (3.7), identify

$$\mathcal{L}^{(\lambda)}=\frac{1}{4\pi}\left[\left(\frac{\partial\phi}{\partial\theta}\right)^2-\left(\frac{\partial\phi}{\partial\tau}\right)^2+\frac{\lambda(\lambda-1)}{\sin^2\theta}\phi^2\right], \tag{3.15}$$

and introduce

$$T^{(\lambda)}_{\alpha\beta}\equiv-g_{\alpha\beta}\mathcal{L}^{(\lambda)}+\frac{\delta\mathcal{L}^{(\lambda)}}{\delta(\partial\phi_\mu/\partial\xi_\alpha)}\frac{\partial\phi_\mu}{\partial\xi_\beta}, \tag{3.16}$$

where

$$\xi_{0,1}=\tau,\theta,$$
$$g_{00}=-g_{11}=1,$$
$$g_{01}=g_{10}=0.$$

Consider two independent components[24] of $T_{\alpha\beta}$,

$$\begin{aligned}T_{00}&\equiv\mathcal{H}\\&=-\frac{1}{4\pi}\left[\left(\frac{\partial\phi}{\partial\tau}\right)^2+\left(\frac{\partial\phi}{\partial\theta}\right)^2+\frac{\lambda(\lambda-1)\phi^2}{\sin^2\theta}\right]\\&\equiv-\frac{1}{4\pi}[\dot\phi^2+\phi'^2+\mu(\theta)\phi^2],\end{aligned} \tag{3.17}$$

$$T_{01}\equiv\mathcal{P}=-\frac{1}{4\pi}\{\dot\phi_\mu,\phi'^\mu\}. \tag{3.18}$$

Caution has been exercised in writing T_{01} because of the noncommutativity of $\dot\phi_\mu$ with ϕ'_μ.

According to an argument by Nambu,[25] if these densities are smeared with test functions, an interesting and nontrivial algebra ensues. It is just the algebra of the Virasoro operators. For brevity, we indicate only the operators with direct application to our work:

$$\begin{aligned}L_0&=-2\int_0^\pi d\theta\,\mathcal{H}\\&=\sum_{n=\lambda}^\infty nb_n^\dagger b_n,\end{aligned} \tag{3.19a}$$

$$\begin{aligned}L_+&=-\int_0^\pi d\theta(\cos\theta\,\mathcal{H}+i\sin\theta\,\mathcal{P})_{\tau=0}\\&=\sum_{n=\lambda}^\infty[(n+\lambda)(n-\lambda+1)]^{1/2}b_{n+1}^\dagger b_n,\end{aligned} \tag{3.19b}$$

$$\begin{aligned}L_-&=-\int_0^\pi d\theta(\cos\theta\,\mathcal{H}-i\sin\theta\,\mathcal{P})_{\tau=0}\\&=\sum_{n=\lambda}^\infty[(n+\lambda)(n-\lambda+1)]^{1/2}b_n^\dagger b_{n+1}.\end{aligned} \tag{3.19c}$$

The action of the operators L_i on the fields $\phi_\mu^{(\lambda)}$ may be computed using the fully integrated expressions and the commutator (3.13); the same results are obtained directly using the integral representations and the canonical commutator (3.12):

$$[L_0,\phi_\mu^{(\lambda)}(\theta,\tau)]=-i\partial_\tau\phi_\mu^{(\lambda)}(\theta,\tau), \tag{3.20a}$$

$$[L_+,\phi_\mu^{(\lambda)}(\theta,\tau)]=e^{i\tau}(-i\cos\theta\,\partial_\tau+\sin\theta\,\partial_\theta)\times\phi_\mu^{(\lambda)}(\theta,\tau), \tag{3.20b}$$

$$[L_-,\phi_\mu^{(\lambda)}(\theta,\tau)]=e^{-i\tau}(-i\cos\theta\,\partial_\tau-\sin\theta\,\partial_\theta)\times\phi_\mu^{(\lambda)}(\theta,\tau). \tag{3.20c}$$

Thus, as indicated in Appendix A, the L_i generate infinitesimal real Möbius transformations on $\phi_\mu^{(\lambda)}$ after a change of variables to the complex z-plane.

The SL(2, R) algebra

$$\begin{aligned}&[L_+,L_-]=-2L_0,\\&[L_0,L_\pm]=\pm L_\pm\end{aligned} \tag{3.21}$$

may also be checked, with the usual proviso that the signs in (3.21) apply for the spacelike oscillators.

Evidently L_0, the generator of τ displacements, is the Hamiltonian operator in the theory. Note that all fields are taken to have the same sign for the energy, regardless of whether fields of different λ enter in the construction of V_μ with relative "i" factors between them. An alternate approach would be to absorb the "i" into the $b_\mu, b_\mu^\dagger$, giving them a wrong-sign commutator. We prefer to keep the non-Hermiticity in the interaction.

D. The Quantized ρ Field

The constructions of the last two paragraphs may be carried over to the case of the ρ field introduced in (2.6) by deleting μ indices, and replacing $g_{\mu\nu}$ by (-1). One further point requires explanation, however.

The charge carried by the internal current through the Nambu-Susskind strip at any "time" is defined by

$$\begin{aligned}\epsilon&=\int_0^\pi d\theta\,J_\tau(\theta,\tau)\\&=\rho(\pi)-\rho(0).\end{aligned} \tag{3.22}$$

Since the functions (3.9) vanish at the boundaries for $\lambda\neq0$, they cannot contribute to the internal charge ϵ. On the other hand, cosine mode solutions, which occur for $\lambda=0$, do not vanish at the boundary, but give rise to an unsatisfactory τ-dependent ϵ.

Thus, in order to describe a charged particle, ρ must include a zero-mode part, in analogy to the terms $(x_\mu^{(0)}+p_\mu^{(0)}\tau)$ in $V_\mu^{(a)}$, which are the net center-of-mass coordinate and momentum of the particle. The analogy is very close, for it is necessary that under Möbius transformations, the zero-mode and excited ("dynamical") modes get mixed together in such a way that the entire ρ is form-invariant.

Consider as an example

$$\rho(\theta,\tau)=\frac{\epsilon_0\theta}{\pi}+\sum_{n=1}^{\infty}\left(\frac{2}{n}\right)^{1/2}(q_n e^{in\tau}+q_n^\dagger e^{-in\tau})\sin n\theta\,. \tag{3.23}$$

Aside from being the simplest ρ utilizing the $\lambda=1$ eigenfunctions, an added appeal of (3.23) has been pointed out by Tze,[26] who derived it from the Sakita-Gervais current[19] $J_k=:\bar\psi\sigma_\alpha\psi:$.

This model for ρ transforms according to (3.20) provided the generators L_i are modified to include ϵ_0 in a manner reminiscent of the Gliozzi operators:

$$\tilde L_0=-\left(\epsilon_0{}^2/\pi^2+\sum_{n=1}^{\infty}nq_n^\dagger q_n\right), \tag{3.24a}$$

$$\tilde L_+=-\left(\epsilon_0 q_1^\dagger/\pi+\sum_{n=1}^{\infty}[n(n+1)]^{1/2}q_{n+1}^\dagger q_n\right), \tag{3.24b}$$

$$\tilde L_-=-\left(\epsilon_0 q_1/\pi+\sum_{n=1}^{\infty}[n(n+1)]^{1/2}q_n^\dagger q_{n+1}\right). \tag{3.24c}$$

Here

$$[q_n, q_m^\dagger]=\delta_{nm},\quad n=1,2,\ldots\ ;$$

all other commutators, including those with ϵ_0, are zero.

Since only a single zero mode is required to close the algebra of the L_i, and it commutes with all operators, it is not itself an operator in the Fock space in which $q_n, q_n^\dagger$ act. It may, however, be treated as an isospin component, $\epsilon_0\to T_3$, which may always be diagonalized within an SU(2) representation, regardless of its multiplicity. Analogous comments apply for SU(3), in which case ϵ_0 is the operator $T_3+\frac{1}{2}Y$.

For the remainder of this paper, incoming and outgoing particle states will tacitly be understood to be eigenstates of the charge operator, but this will not be written explicitly.

Note that in (3.23) ρ has been normalized in such a manner that, by the above discussion,

$$\epsilon=\langle\rho(\pi)-\rho(0)\rangle_0=1\,.$$

The connection between the internal and external charges will be established in the next section; with this normalization, "e" is what a real photon ($q^2=0$) measures in the elastic process γ+particle $\to$ particle.

Since ρ is a dynamical degree of freedom, it contributes to the total Hamiltonian of the system. The zero-mode part can be used to shift the trajectory intercept, so that, in general, it will depend on the quantum numbers of the trajectory. There is always an arbitrariness in the intercept to start with, unless stronger gauge conditions impose a specific choice for the external masses. Since such a restriction is unsatisfactory for off-shell currents, in the remainder of the paper we always write "α_s," with $\alpha_s(0)$ arbitrary, and discuss possible effects of specific s-channel intercepts wherever appropriate. For the simple Compton amplitude we compute, the t-channel intercept is completely determined by the model from the s-channel input.

However, the situation regarding the energy of the excitations created by $q_n^\dagger$ is not entirely satisfactory. As already mentioned, these do not contribute to the internal charge flow. They may, therefore, be thought of as closed charged loops within a complicated Feynman diagram.

The problem is that particles with different numbers of charged loops have different masses. If these states were exotic, this behavior would be gratifying. But any normalized excited state,

$$\sum_m \alpha_m q_m^\dagger|0\rangle\,,\qquad \sum_m|\alpha_m|^2=1\,,$$

has the same net internal quantum numbers as the unexcited state, as measured by the current at $q^2=0$.

The formalism using the field ρ is, therefore, only a partial solution to the problem of incorporating charges in the dual model. Deviations in "mean-square-charge" are allowed for, but these cannot be summed over to produce a "physical" particle state of a given mass. In this sense, our particles seem "bare." However, the (Thirring) low-energy theorem[27] provides an operational definition of the physical (charge)^2; we find in Appendix C that "e" in (2.4) is indeed the physical charge.

IV. ELASTIC FORM FACTOR

A. The Model

In this section, a specific model for the Lorentz vector field $V_\mu(\theta,\tau)$ will be constructed so that the elastic form factor of the ground-state particle has no Gaussian in q^2. [By "the" elastic form factor one means the convective $(p_\mu+p'_\mu)F(q^2)$, such that $F(0)$ measures the total charge.]

As discussed in the preceding section, the "particle" will be taken to be a state of unit charge, with no internal excitation of any kind. Thus, only the zero mode of ρ contributes. For the vector field V_μ, write

$$V_\mu=V_\mu^{(a)}+\sum_{\lambda\neq 0}\beta_\lambda\phi_\mu^{(\lambda)}\,, \tag{4.1}$$

where $V_\mu^{(a)}$ is the conventional oscillator field;

the values of β_λ and λ are yet to be determined.

The matrix element of the vector current is

$$M_\mu = \delta^4(p+q+p')\langle p|T_\mu(q)|p'\rangle$$
$$= (e/\pi)\delta^4(p+q+p')\delta(2p\cdot q+q^2)(p_\mu+p'_\mu)$$
$$\times \int_0^\pi d\theta \langle 0|e^{iq\bar{V}(\theta,0)}|0\rangle , \qquad (4.2)$$

$$\bar{V}_\mu \equiv V_\mu - x_\mu^{(0)} - p_\mu^{(0)}\tau .$$

The second equality is obtained from the first by performing the τ integration, which gives $\delta(2p\cdot q + q^2)$ from the zero-mode parts, and from use of (2.6) and (3.23) to account for J_τ.

The form factor is

$$F(q^2) = \int_0^\pi d\theta \langle 0|e^{iq\bar{V}(\theta,0)}|0\rangle ; \qquad (4.3)$$

since $\langle 0|:e^{iq\bar{V}}:|0\rangle = 1$, the sole q^2 dependence arises from passing to normal-ordered form. Let us express this step as

$$e^{iq\bar{V}(\theta,0)} = :e^{iq\bar{V}(\theta,0)}: \lim_{\theta\to\theta'} \exp[q^2\tilde{F}(\theta,\theta')] ,$$
$$\tilde{F}(\theta,\theta') = \sum_{n=1}^{\infty} \frac{\cos n\theta \cos n\theta'}{n} + \frac{1}{2}\sum_{\lambda\neq 0} \beta_\lambda{}^2 Q_{\lambda-1}(w(\tau=0)) . \qquad (4.4)$$

Since the argument of the Legendre function is approximately equal to unity, we obtain

$$\lim_{\theta\to\theta'} \tilde{F}(\theta,\theta') = \lim_{\theta\to\theta'} \left\{ \tfrac{1}{2}\Big(\sum_\lambda \beta_\lambda{}^2 - 1\Big)\ln\sin\theta - \tfrac{1}{4}\Big(\sum_\lambda \beta_\lambda{}^2+1\Big)\ln[1-\cos(\theta-\theta')] - \tfrac{1}{2}\Big[\ln 2 + \sum_\lambda \beta_\lambda{}^2\psi(\lambda) - \psi(1)\sum_\lambda \beta_\lambda{}^2\Big] + \tfrac{1}{4}\ln 2\Big(\sum_\lambda \beta_\lambda{}^2 - 1\Big)\right\} . \qquad (4.5)$$

Inspection of (4.5) provides the following set of constraints for a well-defined form factor with the desired properties:

(1) Cancellation of infinite term:

$$\sum_{\lambda\neq 0} \beta_\lambda{}^2 = -1 .$$

Thus, all β_λ cannot be real, and we will have ghosts.

(2) Cancellation of all constants (which give rise to Gaussian behavior):

$$\sum_{\lambda\neq 0} \beta_\lambda{}^2\psi(\lambda) = -\psi(1) - 2\ln 2 .$$

(3) Coefficient of surviving $\ln\sin\theta$ (responsible for off-shell behavior):

$$\tfrac{1}{2}\Big(\sum_{\lambda\neq 0}\beta_\lambda{}^2 - 1\Big) = -1 ,$$

by the first condition, and so is not independent.

Conditions (1) and (2) above may be satisfied simultaneously with a single extra field with $\lambda \cong 1.7$, or for an infinite number of λ values less than zero. For negative λ, the fields $\phi(\lambda)$ are not representations, however, so they will not be considered. In any case, an unpleasant feature of having a solution with only a single value of λ is that it is noninteger. Thus the poles due to this field appear at positions displaced relative to the positions of the a modes. Another possibility is to choose more than one auxiliary field. If this is done, integer values of λ can be used. Indeed, there are an infinite number of possibilities in that case.

For the elastic form factor, it makes no difference how the constraints are satisfied. Provided they *do* hold, one obtains

$$F(q^2) = \int_0^\pi d\theta (\sin\theta)^{-q^2} = B\left(\frac{1-q^2}{2}, \frac{1}{2}\right) . \qquad (4.6)$$

B. Discussion

We will now discuss the properties of this form factor, the modifications needed to produce more acceptable behavior, and comment on the physical interpretation of the results.

First, $F(q^2)$ has a series of poles for $q^2 = 1, 3, 5, \ldots, 2n+1$. These poles arise from the end points of the integration, near which the integrand behaves like $d\theta/\theta^{q^2}$. The position of the lowest pole can be modified if $\rho(\theta)$, Eq. (3.23), has a zero-mode dependence proportional to $\theta^{\alpha+1}$, rather than θ. This effects the modification $q^2 \to q^2 - \alpha$, so the (mass)^2 of the lowest pole moves higher or lower depending on whether α is positive or negative. *However, for the charge to be finite we must restrict* $\alpha > -1$. Susskind has argued that this may have an intimate relation to the Feynman ($x^\alpha dx$) "wee" parton distributions responsible for Regge behavior. Indeed, for $\alpha = -\frac{1}{2}$ one would have the lowest pole at the ρ-meson mass, and, identifying the α of the dual theory with the Feynman α, the Regge intercept would be $+\frac{1}{2}$. Unfortunately, we can say no more about this tantalizing possibility based on functions satisfying Eq. (2.4), since they cannot have a zero mode of the desired form. It will be interesting to see if the ρ [or equivalently $J_\alpha(\theta, \tau)$] functions in the "massive Dirac" two-dimensional theories[21,22] allow this behavior without violating the invariance needed for duality.

Consider next the asymptotic behavior. It is controlled by the central region of integration, as can be seen by the replacement

$$\theta = \psi + \tfrac{1}{2}\pi .$$

Then

$$\int_0^\pi d\theta\,(\sin\theta)^{-q^2} = \int_{-\pi/2}^{\pi/2} d\psi(\cos\psi)^{-q^2}$$

$$\propto \int_{(\psi\approx 0)} d\psi\, e^{q^2\psi^2/2} \quad \text{for large } q^2 .$$

Angles ψ up to $\theta(1/|q|)$ contribute, the net result being a power-law falloff $\sim|q|^{-1}$, as can also be seen from the beta function in Eq. (3.6).

Thus, the oscillator model can give rise to power-law behavior, provided there is a sizable concentration of the charge density in some "central" region.[28] Note that at each value of θ, one has the characteristic Gaussian behavior $e^{-f(\theta)q^2}$. It is by being able to add up the contributions from a neighborhood for which $f(\theta) = \ln\sin\theta \approx 0$ that one obtains the slower power-law decrease.

The precise power with which $F(q^2)$ falls off asymptotically can be modified if $\rho(\theta)$ is such that

$$J_\tau(\psi) \approx \psi^B \text{ for } \psi \approx 0 .$$

Then

$$F(q^2) \sim \int d\psi\, \psi^B e^{q^2\psi^2/2} \sim |q|^{-\beta-1} .$$

Not unexpectedly, $F(q^2)$ falls off faster if the charge density is depleted toward the important central region, by having $\beta > 0$. One must be careful, however, because the integration region is even in ψ, so odd integer β is forbidden. To obtain $|q|^{-4}$, e.g., we must choose $J_\tau(\psi) \sim |\psi|^3$. The need for such nonanalytic behavior will reappear when we study νW_2 in the Bjorken limit.

V. DEEP-INELASTIC SCATTERING

A. Results of this Section

We define the structure functions for deep-inelastic scattering in the standard form,

$$W_{\mu\nu} = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) W_1(s, q^2)$$

$$+ \frac{1}{M^2}\left(p_\mu - \frac{p\cdot q\, q_\mu}{q^2}\right)(\mu \to \nu) W_2(s, q^2) .$$

The differential cross section for the process is (see Fig. 1)

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{d^2}{4E^2 \sin^4(\frac{1}{2}\theta)} \times [W_2 \cos^2(\tfrac{1}{2}\theta) + 2W_1 \sin^2(\tfrac{1}{2}\theta)] .$$

Here θ is the lab-frame scattering angle; E and E' are, respectively, the electron initial and final energies.

The most remarked feature of the structure functions W_i, which are in principle functions of both invariants s and q^2, is that, for

$$\nu = p\cdot q/M = (s - q^2 - p^2)/2M \to \infty , \quad q^2 \to -\infty$$

with $\omega = 2\nu/-q^2$ = constant (Bjorken limit), the dependence is only on the ratio of the invariants,

$$W_1 \to F_1(\omega) ,$$

$$\nu W_2 \to F_2(\omega) .$$

The hypothesis that this should occur (Bjorken's scaling hypothesis) is supported by experiment for νW_2, even with surprisingly nonasymptotic s and q^2; it is widely believed that W_1 also scales, though all doubts have not yet been removed.[29]

Both structure functions can be calculated in the present model. However, it is well known that W_1 is very sensitive to the spin of the particle. From the vertex, $\Gamma_\mu(q)$, it can be argued that the photon field couples to elementary vectors, in the sense that every excitation created by $a_{\mu n}^\dagger$ and $b_{\mu n}^{\dagger(\lambda)}$ is independent of all the rest. (There is no coupling of modes.) Thus we expect W_1 to reflect this choice of vector excitations, and not to scale properly. We will not deal with it here.

On the other hand, it is possible to hope that the W_2 function calculated in the model could have the gross qualitative features of the true hadronic W_2. Parton-model calculations typically indicate that, provided elementary coupling of the partons to the

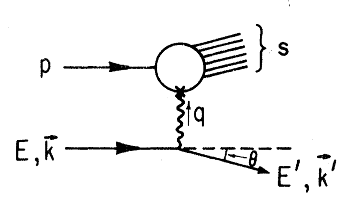

FIG. 1. Kinematics for inelastic lepton-hadron scattering. "X" is the action of the current vertex $\Gamma_\mu(q)$.

electromagnetic field can be enforced, *qualitative* scaling of νW_2 will result irrespective of other details of the constitution of the hadron state.

We find that in the Bjorken limit (with $Q^2 \equiv -q^2 > 0$), our model gives

$$\nu W_2 \approx \begin{cases} |Q|^{-1} \text{ from zero mode of } \rho(\theta,\tau), \\ |Q| \ln Q^2 \text{ from } \rho \text{ excitations.} \end{cases}$$

We will indicate how these results are derived in the second half of this section. In the remainder of this half, we will simply state heuristic reasons for these results, and attempt to make contact with "parton" rhetoric.

First, the $(1/\nu)$ factor in W_2 stems basically from an interaction "time" of $O(1/\nu)$. We stress this is a result of the model, the dominant contribution to the integral arising from this region; it is not an *ad hoc* assumption. In spite of this short life of the relevant intermediate states, we never couple to an element of the string in pointlike fashion. "Parton" rescatterings occur in this model on time scales much smaller than ν^{-1}, hence only dressed partons are observed. Nielsen and Susskind have stressed it is natural to expect this in any theory that achieves duality by being the limit of the "fishnet" Feynman graphs. We have a mathematical realization of this idea though we do not start from fishnets; it shows that even after eliminating the infinities due to the infinitely rising spectrum, one does not cut off the interaction. There is no "minimum wavelength" beneath which one sees the discrete parton lines.

From the zero-mode part of $\rho(\theta,\tau)$, we find the parton form factor is simply related to that of the hadron as a whole. We will say more about this later, at which point it will be more clear. In brief, as for the elastic form factor, it is a region of $O(|q|^{-1})$ separation between the points where the current acts that gives rise to the $|q|^{-1}$ falloff of νW_2. "Pointlike" coupling would correspond to having a δ function in this separation, rather than a small spread.

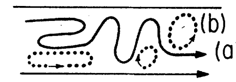

FIG. 2. Charge flow in Feynman diagram. (a) corresponds to possible single carrier of net charge, although for fractionally charged constituents, more than one such line is allowed; (b) corresponds to closed charged loops, which contribute to the mean-square-charge.

On physical grounds, one can anticipate that the "excitation" part of $\rho(\theta,\tau)$ will enhance W_2 relative to the zero-mode part. For, as discussed in Sec. III, $\rho_{\text{ex}}(\theta,\tau)$ contributes only to the mean-square-charge, a quantity that W_2 certainly measures. The idea is to bolster the Q^2 dependence by increasing the mean-square-charge the photon encounters (Fig. 2). Alternately, one can conceive of the zero-charge-flow excitations as being magnetic dipoles. It is known that in a parton model with quarks endowed with an "elementary" anomalous moment, it is W_2 that scales, and not νW_2. While we do not have quite this situation, the coupling, Eq. (2.7), is nonminimal if $\rho_{\text{ex}} \neq 0$. Thus the basic mechanism causing νW_2 to grow can be said to be comprehensible, although the law of growth cannot be taken seriously from this model.

Let us now proceed to the quantitative verification of these observations.

B. Calculation of the Structure Function W_2

We first write the contribution from Figs. 3 and 4 to the forward Compton amplitude:

$$T_{\mu\nu} = \tfrac{1}{4}\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\tau} d\tau' \int_0^{\pi} d\theta' \int_0^{\pi} d\theta \, \langle p | \{J_\alpha \partial_\alpha V_\mu, e^{iqV}\}_{\theta',\tau'} \{J_\alpha \partial_\alpha V_\nu, e^{-iqV}\}_{\theta,\tau} | -p \rangle . \tag{5.1}$$

The structure function W_2 can be picked out from this expression as the imaginary part of the coefficient of $p_\mu p_\nu$:

$$W_2 = \text{Im} \tfrac{1}{4} \int_0^{\infty} d\tau \, e^{\alpha_s \tau} \int \int d\theta d\theta' \, \langle (J_\tau e^{iq\overline{V}})_{(\theta,\tau)} (J_\tau e^{-iq\overline{V}})_{(\theta',0)} \rangle_0 . \tag{5.2}$$

(We have simplified the expression by removing the δ function associated with energy conservation, and using only the relative "time" variable, which we now call τ. We have made a Wick rotation, in accordance with the usual procedures in dual theory.)

The ordered product factorizes into two parts,

$$\mathcal{C}(w) \equiv \left\langle \frac{\partial \rho(\theta,\tau)}{\partial \theta} \frac{\partial \rho(\theta',0)}{\partial \theta'} \right\rangle_0$$

$$= \frac{1}{\pi^2} + \frac{\partial^2}{\partial\theta \partial\theta'} Q_0(w) , \tag{5.3a}$$

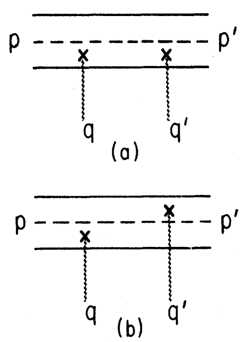

FIG. 3. Regions into which the direct Compton graph may be split for the angular integrations. (a) has $\frac{1}{2}\pi > \theta$, $\theta' > 0$; the region $\pi > \theta$, $\theta' > \frac{1}{2}\pi$ corresponds to reflection of the graph about $\frac{1}{2}\pi$, and gives a contribution equal to that shown. (b) has $\frac{1}{2}\pi > \theta > 0$, $\pi > \theta' > \frac{1}{2}\pi$. The contribution from reflecting about $\frac{1}{2}\pi$ is equal to that shown.

using Eq. (3.23) for $\rho(\theta, \tau)$, and

$$\langle e^{iq\overline{V}(\theta,\tau)} e^{-iq\overline{V}(\theta',0)}\rangle_0 = \exp\left\{(-q^2)\left[2N(\tau, \theta, \theta') + \sum_\lambda \beta_\lambda{}^2 Q_\lambda(w)\right]\right\}. \tag{5.3b}$$

Here,

$$N(\tau, \theta, \theta') = -\ln|1 - e^{-\tau} e^{-i(\theta+\theta')}|\,|1 - e^{-\tau} e^{-i(\theta-\theta')}|$$

is the Green's function of the usual $V_\mu^{(a)}$ DRM field.

It is convenient to note that

$$w \equiv \frac{\cosh\tau - \cos\theta\cos\theta'}{\sin\theta\sin\theta'} = \frac{\cosh\tau - \cos\sigma - \cos\delta}{\cos\delta - \cos\sigma}, \tag{5.4}$$

with $\sigma = \theta + \theta'$, $\delta = \theta - \theta'$. Thus if $\tau \to 0$, $\delta \to 0$, $w = 1$. [Care is required if $\sigma \to 0$ also; we will return to this.] We first study this region, verifying it gives rise to the behavior discussed in earlier paragraphs. Afterward we will show it is really the dominant region.

In the range of (τ, θ, θ') for which $w \approx 1$ is valid, the functions $Q_\lambda(w)$ may be approximated as follows:

$$Q_\lambda(w) \underset{w\to 1}{\approx} -\tfrac{1}{2}\ln(w-1) + [\tfrac{1}{2}\ln 2 - \psi(\lambda+1) + \psi(1)].$$

Then, by virtue of the constraints following Eq. (4.5), required so the elastic form factor has no Gaussian term, one obtains simply

$$W_2 \approx \mathrm{Im}\, 2^{Q^2} \int d\tau\, e^{(\alpha_s + Q^2)\tau} \int\!\!\int d\theta d\theta' (w+1)^{-Q^2} \mathcal{C}(w). \tag{5.5}$$

Note it is independent of the manner in which the constraints are satisfied. Let us treat the constant part of $\mathcal{C}(w)$ first. We evaluate the assymptotic value of the imaginary part by the so-

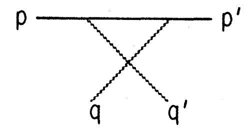

FIG. 4. Exchange Compton graph. It may also be split in the same way as the direct graph. A reference to Fig. 4(a), e.g., is the same as Fig. 3(a), but with q, q' interchanged.

called "statistical method."[30] For, note that in the approximation $\tau \approx 0$, $\delta \approx 0$ we may write

$$(w+1)^{-Q^2} \approx 2^{-Q^2} \exp(-Q^2)\left(\frac{\tau^2 + \delta^2}{4\sin^2(\frac{1}{2}\sigma)}\right). \tag{5.6}$$

Since

$$\alpha_s + Q^2 \propto \nu + (\text{const.}),$$

we take the limit $\nu \to -\infty$ (and in this sense average over s-channel poles), $Q^2 \to +\infty$, $|\nu|/Q^2$ fixed. Since, for nonvanishing $(\sin\frac{1}{2}\sigma)$, we can effectively extend the δ integration from $(-\infty)$ to $(+\infty)$, introduction of the new variables $\tau' = -|\nu|\tau$, $\delta' = \sqrt{Q^2}\,\delta$ gives us

$$W_2^{(a)} \sim \frac{1}{|\nu|\,|Q|} \int_0^\infty d\tau'\, e^{-\tau'} \times \int d\sigma \int_{-\infty}^{\infty} d\delta \exp[-(\delta'^2/4\sin^2(\tfrac{1}{2}\sigma))]. \tag{5.7}$$

There are three points to observe:

(1) As claimed, the "interaction time" of any significance is $O(\nu^{-1})$. This just gives the ν^{-1} falloff of W_2 (see Fig. 5).

(2) The relative "spacelike" variable between points where the current acts is δ. Only separations of order $\delta \sim |Q|^{-1}$ count. The Bjorken condition $|\nu|\,Q^2$ = fixed is essential in order that $\tau^2 \ll \delta^2$, and only the term linear in τ contributes.

(3) Even after making these approximations, there is considerable "phase space" embodied in the remaining integrations, so the numerical coefficient of the leading term is of $O(1)$. It is in-

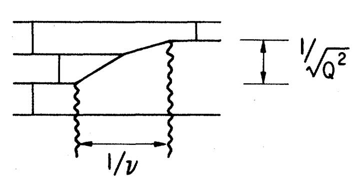

FIG. 5. Feynman graph interpretation of strip domain. If the graph is very dense, scattering of constituent lines may be likely between interactions with the current. The scale between interactions with the current is $(1/\nu)$ in τ direction, $(1/\sqrt{Q^2})$ in the θ direction.

structive in this regard to note that from the region $\theta \approx \theta' \approx \frac{1}{2}\pi$, one obtains a contribution to W_2 that goes like $|\nu|^{-1}$. The ν^{-1} arises from the τ integral as before, and $(Q^2)^{-1}$ is clearly the product of the form factors associated with the interactions of the two photons in the Compton scattering. By integrating over all possible regions $\theta \approx \theta'$, we pick up an extra factor $|Q|$. This is clearly a phase-space effect, changing the behavior from a 2-dimensional Gaussian in (δ, σ), to a single Gaussian in δ. But in parton terms, one would have to say the number of partons the photons see grows at least like $|Q|$ in the model.

Next, examine the contribution of the density excitations. We have

$$W_2^{(b)} = \frac{1}{2}\int d\tau\, e^{\nu\tau} \int\int d\theta d\theta' F(w)\, \frac{\partial^2}{\partial\theta\partial\theta'} \ln\left(\frac{w+1}{w-1}\right), \tag{5.8}$$

where $F(w)$ is the integral in Eq. (5.5). It is clear from this expression that we must be very careful in seeing whether we encounter a singularity as $w \to 1$. Note that $\langle J_\alpha(\tau) J_\beta(0)\rangle_0$ is just the propagator of the internal current; one might well expect such a Green's function to develop its singularity. However, the derivatives actually alter it sufficiently that there is no contribution to the Im part. (See, however, Appendix C.)

To prove this, integrate by parts once:

$$\tilde{W}_2 \equiv \frac{1}{2}\int d\tau\, e^{\nu\tau}\left\{\left[\int_0^\pi d\theta\left(\frac{\partial}{\partial\theta}\ln\frac{w+1}{w-1}\right)F(w)\right]_{\theta'=0,\pi} - \int\int d\theta d\theta'\left(\frac{\partial}{\partial\theta}\ln\frac{w+1}{w-1}\right)\frac{\partial F}{\partial\theta'}\right\}.$$

In the surface term, $F(0)=F(\pi)=0$, since $w\to\infty$ for $\theta'=0,\pi$; and $Q_\lambda(\infty)\to 0$. Also,

$$\frac{\partial}{\partial\theta}\ln\frac{w+1}{w-1} = \frac{\sin\sigma}{\cosh\tau-\cos\sigma} + \frac{\sin\delta}{\cos\delta-\cosh\tau}$$

shows this term is 0 for $\theta'=0,\pi$.

All of this is true except *at* the points $(\theta,\tau)=(0,0), (\pi,0)$, which are included in the remaining integrations in the surface term. In any neighborhood of these points, the surface term integrand is strictly zero. Thus the regions where the integrand is infinite have zero measure, and do not contribute to the value of the integral except in a δ-function sense. Since no further integrations on internal variables remain to be done on W_2, the surface term is safely zero.

Passing to the remaining integrals, note that $\partial F/\partial\theta'$ immediately brings down one power of Q^2. It is necessary to check that these integrals are convergent. After a bit of arithmetic,

$$\tilde{W}_2 = 2^{Q^2}Q^2\int d\tau\, e^{\nu\tau}\int\int d\sigma d\delta (w+1)^{-Q^2}\left[\frac{2\ln 2-1}{2\sin^2(\frac{1}{2}\sigma)}\left(\cos^2(\tfrac{1}{2}\sigma)(\tau^2+\delta^2)+\frac{4\delta^2}{\delta^2+\tau^2}\right)\right.$$
$$\left. - \frac{\ln 2}{4\sin^2(\frac{1}{2}\sigma)}\left(\frac{4\delta^2}{\delta^2+\tau^2}+2\cot^2(\tfrac{1}{2}\sigma)(\tau^2+\delta^2)\ln\frac{\tau^2+4\sin^2(\frac{1}{2}\sigma)}{\tau^2+\delta^2}\right)\right]. \tag{5.9}$$

(Terms linear in δ integrate to zero.) No piece of this expression is singular. [E.g.,

$$\frac{\delta^2}{\delta^2+\tau^2}\ln(\tau^2+\delta^2)$$

may be expressed in polar coordinates as $\cos^2\phi\ln r$. Since the area element contains $r\,dr$, there is no problem as $r\to 0$.] Two integrals are interesting:

$$I_1 = \int d\tau\, e^{\nu\tau}\int\int d\sigma d\delta\, \frac{\delta^2}{\delta^2+\tau^2}\left(1+\frac{\tau^2+\delta^2}{4\sin^2(\frac{1}{2}\sigma)}\right)^{-Q^2} \sim \frac{\text{const}}{(-\nu)\sqrt{Q^2}}, \tag{5.10a}$$

$$I_2 = \int d\tau\, e^{\nu\tau}\int\int d\sigma d\delta\, \frac{\delta^2}{\delta^2+\tau^2}\left(1+\frac{\tau^2+\delta^2}{4\sin^2(\frac{1}{2}\sigma)}\right)^{-Q^2}\ln\frac{\tau^2+4\sin^2(\frac{1}{2}\sigma)}{\tau^2+\delta^2} \sim -(\text{const})\,\frac{\ln Q^2}{\sqrt{Q^2}\,(-\nu)}. \tag{5.10b}$$

These estimates can be established by straightforward calculations which we need not repeat here.[31] Heuristically, we have again $\tau\to\frac{1}{2}$, $\delta\to|Q|^{-1}$. It is also easy to check that other terms in Eq. (5.9) fall off faster than this.

For completeness, we wish to note two distinct regions in the (τ,θ,θ') integration space not covered in the above discussion. If, in addition to $\delta\approx 0$, σ is also small, θ and θ' must separately be small. We must then worry about the fact σ and δ are not independent integrals in this neighborhood because of the wedged nature of the domain. (See Fig. 6.) For convenience, introduce polar coordinates, $\theta=r\cos\phi$, $\theta'=r\sin\phi$. Neglecting τ^2, we find $w\approx\csc 2\phi$. If θ and θ' approach zero together, i.e., $\phi\approx 45°$, one finds the behavior is qualitatively similar to Eq. (5.7) or (5.10), but with a very small coefficient corresponding to the negligible available phase space. If, on the other hand, ϕ is near 0 or $\frac{1}{4}\pi$, w can be quite large. The essential point then is that the careful cancellation of 2^{-Q^2} that was maintained for $w \approx 1$ breaks down, and the contributions are exponentially damped. This is generally true for all the regions of the integration domain where $w \gg 1$. The integral approaches

$$\int d\tau\, e^{\nu\tau} \int\int d\theta d\theta' (2w)^{-Q^2}.$$

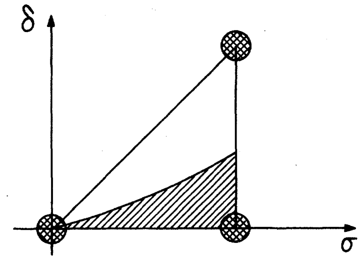

FIG. 6. Unshaded region is nondominant in calculation of W_2. The corners are dealt with separately. Hatched region has upper boundary $\cos\delta = \frac{1}{3}(2 + \cos\sigma)$.

VI. POLE AND ASYMPTOTIC STRUCTURE OF TWO-CURRENT AMPLITUDE

We have seen in the previous sections that the elastic form factor and W_2 do not depend on the way in which the regulator fields $\phi_\mu^{(\lambda)}$ are chosen. In this section we will briefly examine the properties of T_2, whose imaginary part for forward Compton scattering is W_2, in order to illustrate what effects the regulators *do* have.

The contribution of Fig. 3 to this amplitude is

$$T_2^{(s)} = \int_0^1 dx\, x^{-\alpha_s - 1} \int\int d\theta d\theta' (\sin\theta)^{-q^2} (\sin\theta')^{-q'^2} \exp\left\{ q\cdot q' \left[2N(x, \theta, \theta') + \sum_\lambda \beta_\lambda{}^2 Q^{(\lambda)}(w) \right] \right\}, \tag{6.1}$$

where $N(x, \theta, \theta')$ is the Green function of $V_\mu^{(a)}$, given in Sec. V, and w is defined in Eq. (3.11). The usual variable $x = e^{-\tau}$ has been introduced. Figure 4 gives a contribution $T_2(s \leftrightarrow u)$, where

$$s = (p+q)^2, \quad t = (q+q')^2, \quad u = (p+q')^2.$$

It will be useful for later reference to introduce the notation

$$\begin{aligned} T_2 &\equiv \int_0^1 dx\, x^{-\alpha_s - 1} F(x) \\ &= \int_0^1 dx\, x^{-\alpha_s - 1} \int\int d\theta d\theta' F(x, \theta, \theta'). \end{aligned} \tag{6.2}$$

We will not enter into much detail to establish some of the properties of T_2 we list below,[31] since the methods used are fairly conventional and well known to workers in this field.

A. Current Line Poles

The amplitude T_2 has an infinite number of poles for q^2 and q'^2 equal to positive integers, starting at $n = 1$. These arise from the end points in the θ and θ' in such a way that poles appear at even as well as odd integers. This is independent of the values of λ.

B. Poles and Residues

The residues in the s (t) channels are polynomials in t (s), corresponding to a maximum spin $J = \alpha_s$ (α_t), and all daughters, but with no ancestors. We will prove this for the s-channel case, taking for definiteness

$$\begin{aligned} &\lambda_1 = 1, \quad \beta_1{}^2 = 2\ln 2 - 1, \\ &\lambda_2 = 2, \quad \beta_2{}^2 = -2\ln 2. \end{aligned} \tag{6.3}$$

Using the generating function for Gegenbauer polynomials, we find

$$\begin{aligned} F(x, \theta, \theta') \approx \sum_{n,m,k} & x^{k+n} C_n^{-a q\cdot q'}(\cos(\theta+\theta')) \\ & \times C_m^{-b q\cdot q'}(\cos(\theta-\theta')) \frac{1}{k!} \left(-2b q\cdot q' \sum_{l=1}^\infty \frac{(2x \sin\theta \sin\theta')^{2l}}{2l+1} \right)^k. \end{aligned} \tag{6.4}$$

The last term of this expression may be rewritten for our purposes as

$$\left(\sum_l \frac{(2x\sin\theta\sin\theta')^{2l}}{2l+1}\right)^k = \sum_{p=0}^{\infty} \Lambda_{k,p}(2x\sin\theta\sin\theta')^{2(k+p)}, \tag{6.5}$$

where $\Lambda_{k,p}$ are combinative coefficients with the property $\Lambda_{0,p}=\delta_{0,p}$.

The Nth power of x in the expansion of $F(x,\theta,\theta')$ will have the form, in an obvious notation,

$$\sum_{q=0}^{N} C_{N-q}(\sigma)\Delta_q(\delta), \tag{6.6}$$

where

$$\Delta_q \equiv C_q(\delta) + \sum_{j=1}^{\left\{\substack{q/2,\, q \text{ even} \\ (q-1)/2,\, q \text{ odd}}\right\}} (1-\delta_{q0})C_{q-2j}(\delta)\sum_{l=0}^{j-1}\tilde{\Lambda}_{j-l,l},$$

$$\tilde{\Lambda}_{k,p} = \frac{\Lambda_{k,p}}{k!}(2\sin\theta\sin\theta')^{2(k+p)}(-2bq\cdot q')^k.$$

Now,

$$C_n^{\lambda}(\cos\theta) = g(\theta)[\lambda^n + f_1(\theta)\lambda^{n-1} + \cdots + f_n(\theta)].$$

Thus the leading power of $(q\cdot q')$ in $C_n^{-qq'}$ is $(-qq')^n$. Similarly, the leading power of $(q\cdot q')$ in $\tilde{\Lambda}_{j-l,l}$ comes from $l=0$. Thus in the term $q=k$ of the coefficient of x^N in $F(x,\theta,\theta')$, we have

$$C_{N-k}^{-aq\cdot q'}\Delta_k \sim (-aq\cdot q')^{n-k}\left[(-bq\cdot q')^k + \sum_{j=1}(-bq\cdot q')^{k-2j}(-2bq\cdot q')^j\right]. \tag{6.7}$$

This shows the highest spin at a pole $\alpha_s = N$ is N. It is followed by a complete sequence of daughters, but there are no ancestors.

One can easily see that this result does not crucially depend upon our choice of λ. All that is required is that in the s-pole region, $w \sim x^{-1} \to \infty$, so one can apply the asymptotic expansion in powers of x,

$$Q_\alpha(w) \sim \frac{\sqrt{\pi}}{(2w)^{\alpha+1}}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{3}{2})} \times F\left(\frac{2+\alpha}{2}, \frac{1+\alpha}{2}, |\alpha+\tfrac{3}{2}| \frac{1}{w^2}\right),$$

where F is a hypergeometric function.

The proof to show the t-channel poles have no ancestors is equally straightforward, so we will not present it here.

C. Ghosts in the Spectrum

Excitations created by the regulator fields appear in the spectrum, and because of the "i" factor appearing in their coupling, they give rise to "ghosts," that is, states of negative norm. Now, these can be pushed arbitrarily high in the spectrum by allowing the values of λ we choose to be arbitrarily large. However, we will maintain the values $\lambda_1 = 1$, $\lambda_2 = 2$ chosen for the previous example in order to demonstrate how they explicitly appear at the first few levels. We will also see that even the usual "a" modes give rise to ghosts, since the ghost-elimination theorems only apply when the common mass of the external lines is $M^2 = -1$.

Consider first $\alpha_s = N = 1$. In this first nontrivial case, the residue at the pole is

$$R_1 = (\ln 2 - \tfrac{1}{2})(t - q^2 - q'^2)\tilde{B}(1-q^2)\tilde{B}(1-q'^2), \tag{6.8}$$

where we introduce the symbol

$$\tilde{B}(\nu) = \int_0^{\pi} d\theta(\sin\theta)^{\nu} = 2^{\nu}B\left(\frac{\nu+1}{2}, \frac{\nu+1}{2}\right) = B\left(\frac{1+\nu}{2}, \frac{1}{2}\right). \tag{6.9}$$

In R_1, the factor $(\ln 2 - \frac{1}{2})$ signals it is only the $(\lambda = 1)$ b set of oscillators that contribute. This is because there is a built-in signature factor in the amplitude when Fig. 3 and Fig. 4 are added together that eliminates $\alpha_s = $ odd poles from the amplitude when λ is even, e.g., $\lambda = 0$ and $\lambda = 2$. The origin of this factor is easy to understand from the expression

$$\int_0^{\pi} d\theta\, e^{iq\cdot\overline{V}(\theta,\tau)} = \int_0^{\pi/2} d\theta\, e^{iq\cdot\overline{V}(\theta,\tau)} + \int_0^{\pi/2} d\psi\, e^{iq\cdot\overline{V}(\pi-\psi,\tau)},$$

where $\psi = \pi - \theta$. Now,

$$\cos n\theta = (-1)^n\cos n\psi,$$

and, more generally,

$$C_m^{\lambda}(-x) = (-1)^m C_m^{\lambda}(x)$$

implies that

$$\phi_n^{(\lambda)}(\theta) = (-1)^{n-\lambda}\phi_n^{(\lambda)}(\psi).$$

These terms appear in the expansion of $\phi_\mu^{(\lambda)}(\theta,\tau)$

as coefficients of $e^{\pm in\tau}$. Thus the factor $(-1)^n$ may be absorbed by shifting $\tau \to \tau + \pi$ in each term. However, the factors $(-1)^\lambda$ cannot be absorbed in the same way. Instead, one has

$$\bar{V}_\mu(\theta, \tau) \equiv W_\mu(\psi, \tau) = \sum_n \left(\frac{2}{n}\right)^{1/2} (a_{n\mu} e^{in(\tau+\pi)} + a_{n\mu}^\dagger e^{-in(\tau+\pi)}) \cos n\psi + \sum_\lambda (-1)^\lambda \beta_\lambda \phi_\mu^{(\lambda)}(\tau+\pi, \psi).$$

Now, if in the contributions to the 2-current amplitude, the currents lie on opposite sides of $\pi/2$, the contraction of the fields to produce a Green's function will produce a modification of the energy denominator relative to the case when they are on the same side,

$$e^{iL_0(\tau'-\tau)} \to e^{iL_0(\tau'-\tau)}(-1)^{L_0},$$

which has the required signature effect $(-1)^N$ at a pole. However, the $(-1)^\lambda$ factors do not have a similar effect, and we get poles for odd N unless λ is even. To get properly signatured poles, we should always choose λ even. For our present purpose of illustration, this is disregarded.

To examine the positivity of R_1, pass to the kinematics of an elastic scattering, $q^2 = q'^2 = M^2$, $p^2 = p'^2 = \mu^2$, with c.m. scattering angle

$$\cos\phi = 1 + 2st[s^2 - 2s(\mu^2 + M^2) + (\mu^2 - M^2)^2]^{-1}.$$

The residue at the first pole should have the form

$$R = g_0^2(M^2)P_0(t) + g_1^2(M^2)P_1(t),$$

where $g_i^2(M^2)$ are form factors, and $P_i(t)$ are Legendre polynomials. Thus the constant term and coefficient of t should be positive. Under these kinematic conditions, we have

$$R_1 = (\ln 2 - \tfrac{1}{2})(t - 2M^2)[\tilde{B}(1 - M^2)]^2. \qquad (6.10)$$

Since the $\tilde{B}$ functions are real, $\tilde{B}^2$ are positive. Thus we see that the vector excitation is a "good" state, but the scalar daughter is a ghost for $M^2 > 0$. Why is there any "good" particle at all in this case? The reason is that the $\lambda = 1$ oscillators enter V_μ with coefficient $\beta_1^2 = 2\ln 2 - 1 > 0$ [see Eq. (6.3)]. It is not necessary for *all* the new oscillators to have wrong metric in order to satisfy the constraints following Eq. (4.5).

Next, consider the $\alpha_s = 2$ level. The signature effect now decouples the $\lambda = 1$ oscillator, but we still have the effect of the even-λ extra modes. Explicitly, in the kinematics of an elastic scattering, we find

$$\tilde{R}_2 \propto + t^2[1 + (1 - M^2)^2] + tM^2(9M^2 - 4M^4 - 8) + 2M^4(2M^4 - 5M^2 + 4).$$

It is easy to verify that for $M^2 > 0$, the vector particle is a ghost. However, tensor and scalar parts have a positive sign. This illustrates two points: (1) The "a" modes continue to produce good particles on the leading trajectory, even if the theory is only Möbius-invariant, as in the ordinary Veneziano model; (2) a more detailed analysis shows the scalar daughter receives positive contributions from *both* the "a" mode and the $\lambda = 2$ "b" mode. Thus, just as the "a" modes produce good leading particles, but may produce bad daughters, the "b" mode that couples with $\beta^2 < 0$ produces a bad leading particle, but may produce good daughters.

We also note that the "a" mode itself contributes to the negative residue of the vector particle. It can be made positive for $M^2 < 0$, but this is not satisfactory for our "current" theory. The b-mode contributes to make this part of the residue even more negative. Thus there does not seem to be much hope of contriving cancellations between "a" and the ghost "b" particles so all residues are positive.

D. Asymptotic Behavior

It is possible to show that our T_2 amplitude has Regge behavior in all the proper limits. In particular, it does not grow as $u \to \infty$, which was a problem in the phenomenological dual model of Nambu.[2] However, we will only exhibit the s-channel asymptotic behavior, since it displays a fixed pole.[32]

Consider the limit $s \to \infty$ with t held fixed. As can be seen from (6.4), any exact evaluation of the integral representation for T_2 involves pole terms. For this reason it is necessary to keep s off the real axis when calculating the asymptotic behavior, and we shall take $\alpha_s \to -\infty$, a standard procedure in dual resonance models.[30] Following the discussion of Nambu,[2] write

$$T_2 = \int_0^1 dx\, x^{-\alpha_s - 1} F(x). \qquad (6.11)$$

The usual analysis indicates the integrand will be maximal near $x = 1$, $\theta, \theta' = 0$. However, notice that according to (6.1) $F(x = 1) \neq 0$. Thus

$$T_2 \underset{\alpha_s \to -\infty}{\sim} (-\alpha_s)^{-1}[f_1(t, q^2, q'^2) + \alpha_s^{-1} f_2(t, q^2, q'^2) + \cdots]. \qquad (6.12)$$

This may indicate the presence of a fixed pole, provided the series does not add up to give Regge asymptotic behavior. We shall see that from the second term onward it must add up this way, but the fixed pole remains.

To study these effects in detail, write $x = e^{-\tau}$, and for $\tau\approx 0$, $\theta\approx 0$, and $\theta'\approx 0$, introduce spherical coordinates

$$\begin{aligned}\tau&=\rho\cos\alpha\,, \quad 1\gg\rho_0>0\,,\\ \theta&=\rho\sin\alpha\,\cos\beta\,, \quad \tfrac{1}{2}\pi>\alpha>0\,,\\ \theta'&=\rho\sin\alpha\,\sin\beta\,, \quad \tfrac{1}{2}\pi>\beta>0\,.\end{aligned} \tag{6.13}$$

The variable w, (3.11), becomes

$$w=\csc^2\alpha\,\csc^2\beta\begin{cases}\approx 1, & \alpha=\tfrac{1}{2}\pi\,,\quad \beta=\tfrac{1}{4}\pi\,,\\ \to\infty & \text{otherwise}\,.\end{cases} \tag{6.14}$$

In the same approximation,

$$F(x,\theta,\theta')\to\rho^{-t}(\sin\alpha\,\cos\beta)^{-q^2}(\sin\alpha\,\sin\beta)^{-q'^2}\times(1+\sin^2\alpha\,\sin^2\beta)^{-aq\cdot q'}(1-\sin^2\alpha\,\sin^2\beta)^{-bq\cdot q'}\exp[-2bq\cdot q'Q_1(w)]\,. \tag{6.15}$$

The asymptotic behavior can be extracted straightforwardly from this expression. However, it is necessary to keep a close watch on the angular integrations in order to see that the fixed pole cannot coexist with simultaneous poles in both current lines. Except near $\alpha=\frac{1}{2}\pi$, $\beta=\frac{1}{4}\pi$, F can be expanded further.

$$F(\rho,\alpha,\beta)\approx\rho^{-t}(\sin\alpha\,\cos\beta)^{-q^2}(\sin\alpha\,\sin\beta)^{-q'^2}\times\sum_{nmk}(\sin^2\alpha\,\sin^2\beta)^{n+m}\binom{-aq\cdot q'}{n}\binom{-bq\cdot q'}{m}\frac{(-2bq\cdot q')^k}{k!}\sum_p\Lambda_{kp}(\sin^2\alpha\,\sin^2\beta)^{k+p}\,. \tag{6.16}$$

Then T_2 is approximately

$$T_2\approx\int_0^{\rho_0}d\rho\,\rho^2\int_0^{\pi/2}d\alpha\,\sin\alpha\,e^{\alpha_s\rho\cos\alpha}\times\int_0^{\pi/2}d\beta\,F(\rho,\alpha,\beta)\,. \tag{6.17}$$

With a steepest-descents estimate, we have

$$\int d\rho\,e^{\alpha_s\rho\cos\alpha}\rho^{2-t}\underset{\substack{\alpha_s\to-\infty\\ \cos\alpha\neq 0}}{\sim}\frac{(-\alpha_s)^{t-3}}{3-t}(\cos\alpha)^{t-3}\,, \tag{6.18}$$

which is Regge behavior with $\alpha_t=t-3$, independent of the values of q^2 and q'^2. Now examine the remaining angle integrals.

$$\begin{aligned}\int d\alpha\,\sin\alpha\int d\beta\,F(\alpha,\beta)(\cos\alpha)^{t-3}&\equiv\int d\alpha\int d\beta(\sin\alpha)^{1-q^2-q'^2}(\cos\alpha)^{t-3}\\&\quad\times(\sin\beta)^{-q'^2}(\cos\beta)^{-q^2}\\&\quad\times\sum_{l=0}^{\infty}C_l(q^2,q'^2,t)(\sin^2\alpha\,\sin\beta\cos\rho)^l\,.\end{aligned}$$

$C_l=C_l(q^2,q'^2,t)$ is a polynomial in its arguments and will not be singular as a function of any of them. The β integration can be carried out at once. Strictly, the same formula should not be used for the α integration, since we have noted that $\alpha=\frac{1}{2}\pi$ causes the expansion (6.16) to fail. However, the upper end of the integration region in α affects only the $(\cos\alpha)^{t-3}$ term, which is singular for $t<0$, an unphysical result. If we set that term equal to unity, consistent with (6.18), we do not affect the q^2 and q'^2 singularity structure, which is of interest. Thus

$$T_2\approx\sqrt{\pi}\,\frac{(-\alpha_s)^{t-3}}{3-t}\sum_l C_l(q^2,q'^2,t)\times\frac{\Gamma(\frac{1}{2}(l+1-q^2))\,\Gamma(\frac{1}{2}(l+1-q'^2))}{\Gamma(\frac{1}{2}(2l+3-q^2-q'^2))}\,. \tag{6.19}$$

This combination of Γ functions allows simultaneous poles in q^2 and q'^2, for any positive integer values of these quantities. The denominator produces dips for $q^2+q'^2=2n+3$, $n=0,1,\ldots$; this will eliminate simultaneous poles if q^2 and q'^2 are not both even or both odd for sufficiently large q^2, q'^2, an interesting effect.

In the region where $\alpha\approx\frac{1}{2}\pi$, a different set of approximation is appropriate, and it is here that we will pick up the fixed pole. Looking back at the coordinates (6.13), we see that for $\psi\equiv\frac{1}{2}\pi-\alpha\approx 0$,

$$\begin{aligned}\tau&\approx\rho\psi\,,\\ \theta&\approx\rho\cos\beta\,,\\ \theta'&\approx\rho\sin\beta\,.\end{aligned} \tag{6.20}$$

The characteristic feature of this domain is that τ and *one* of the angles θ, θ', can be second order small, but the other angle is simultaneously only first order small.

We now have

$$T_2\approx\int d\rho\,\rho^{2-t}\int d\psi\,e^{\alpha_s\rho\sin\psi}\int d\beta\,F(\beta,\psi^2)\,. \tag{6.21}$$

The integrand may be expanded,

$$F(\beta,\psi^2)=F(\beta,0)+\psi^2\left.\frac{\partial F}{\partial\psi^2}\right|_{\psi^2=0}+\cdots\,. \tag{6.22}$$

Since only powers of ψ^2 appear in the expansion, the second term is already contained in the alternate expression (4.31), and we need only keep $F(\beta, 0)$ to define $T_{\text{f.p.}}$.

Now introduce

$$y \equiv \rho\psi \tag{6.23}$$

and examine

$$\int_0^{\rho_0} d\rho \int_0^{\rho\psi_0} dy\, e^{\alpha_s y} \rho^{1-t} = (\alpha_s)^{-1} \int_0^1 d\rho\, \rho^{1-t}(e^{\alpha_s \rho\psi_0} - 1). \tag{6.24}$$

The ρ integration has been extended to $\rho_0 \sim O(1)$. For $(\psi_0)_{\max} \sim |\alpha_s|^{-1}$, we obtain

$$A_{\text{f.p.}} \approx (-\alpha_s)^{-1}(t-3)^{-1} \int_0^{\pi/2} d\beta\, F(\beta, 0). \tag{6.25}$$

The β integral is a pole term, since

$$F(\beta, 0) = (\cos\beta)^{-q^2}(\sin\beta)^{-q'^2}(1+\sin 2\beta)^{-a q\cdot q'} \times (1-\sin 2\beta)^{-b q\cdot q'} \exp[-2b\, q\cdot q' Q_1(w(\beta))], \tag{6.26}$$

with $w(\beta) = \csc 2\beta$.

Near $\beta = \frac{1}{4}\pi$, $Q_1(w)$ cancels the apparent singularity in $(1-\sin 2)^{b q\cdot q'}$. Thus the only singularities are near $\beta = 0$, $\frac{1}{2}\pi$, and

$$\int d\beta\, F(\beta, 0) = \sum_l C_l(q^2, q'^2, t) \times B(\tfrac{1}{2}(1+l-q^2), \tfrac{1}{2}(1+l-q'^2)). \tag{6.27}$$

As in the ordinary Veneziano case, the B function has singularities in q^2 or q'^2, but not simultaneously in both.

We find, then, that the fixed pole arises if the triple limit τ, θ, $\theta' \to 0$ is performed in such a way that an "equal time" configuration must exist prior to θ and θ' going to zero. If any small "time" difference is allowed to persist as the vertices approach the edge, a Regge pole occurs instead of a fixed pole, and both currents may convert to hadronic states. *Thus, there are two distinct physical ways of approaching the limit.* An extrapolation of this result would be that the equal-time commutator of currents determined as (see Fig. 7). An extrapolation of this result would be that the equal-time commutator of currents determined as the limit of an "almost-equal-time" commutator is not unique; however, this extrapolation can only be made meaningful be a detailed analysis which is beyond the scope of this paper.
the "c-number" part of $\rho(0, \tau)$. The density excitations produce contributions to the spectrum, as mentioned in Sec. III, but these are always positive norm scalars. They also contribute to the real

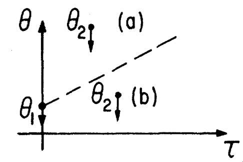

FIG. 7. As θ, θ', τ all go to zero, two situations are possible. In (a), (θ_2, τ) approaches $(\theta_1, 0)$, and then θ_1, θ_2 go to zero; in (b) it is possible for $\theta_1 = \theta_2 = 0$, $\tau \neq 0$.

part of the amplitude in a way discussed in Appendix C.

To study how these excitations alter the Regge behavior, recall the relevant Green's function has the form

$$\frac{1}{2}\frac{\partial^2}{\partial\theta\partial\theta'} \ln\frac{w+1}{w-1} \equiv \frac{f(\alpha,\beta)}{\rho^2},$$

$$f(\alpha,\beta) = \frac{\cos 2\alpha - \sin^2\alpha \sin 2\beta}{(1+\sin^2\alpha\sin 2\beta)^2} + \frac{\cos 2\alpha + \sin^2\alpha\sin 2\beta}{(1-\sin^2\alpha\sin 2\beta)^2}. \tag{6.28}$$

This result has been written in the polar coordinates (6.13). It modifies (6.15) to read

$$F(x,\theta,\theta') \to \frac{F(x,\theta,\theta')f(\alpha,\beta)}{\rho^2}.$$

The new factor ρ^{-2} effectively shifts $t \to t+2$, which changes the Regge behavior to $\alpha(t) = t-1$. Thus we have the asymptotic form

$$T_2 \approx \frac{1}{-\alpha_s}\frac{F(q^2, q'^2)}{1-t} + (-\alpha_s)^{t-1}\beta(t, q^2, q'^2). \tag{6.29}$$

It is straightforward to verify that the angular function $f(\alpha,\beta)$ in Eq. (6.28) does not alter the arguments we gave regarding the "decoupling" of the fixed pole from the purely hadronic amplitude.

VII. SUMMARY AND DISCUSSION

The premise on which this paper was based is that a useful approximation to the electrodynamics of hadrons can be obtained in a semiclassical framework. It has been shown in a number of works that the "free" fields of DRM's result from a special choice of coordinate gauge in which to express the action of a classical free particle. Nambu's hypothesis for the vector current provides a natural electromagnetic interaction term in this view, since it formally leads to the Lorentz equation with V_μ playing the role of space-time coordinate.

The first retreat from this aesthetically appealing framework occurs in making allowance for a nontrivial q^2 dependence of the matrix elements of the current. To achieve this, a regulator field, which allows cancellation of infinities in the operator e^{iqV}, and leads to the Green's function structure of the Rebbi-Drummond model, was introduced. It is not the only possible way to proceed. A "dual loop" approach possibly has an advantage of greater internal consistency[10]; but the machinery needed for calculations is so cumbersome as to detract from its attractiveness as a phenomenological theory, which is all we are claiming for the DRM in this paper.

The extra-field method allows for straightforward calculation at the phenomenological level, but one must concede at the outset that ghosts will be present in the spectrum. This is very unpleasant, for the good features which the "free-particle" formulation could lead one to hope for are already sacrificed.

With new questions to consider, the problem of harmonic-oscillator wave functions, Gaussian form factors, can also be dealt with. The method by which this has been achieved deserves further study on its own. Physically, Eq. (3.8) may be thought of as the equation of motion for a string with variable density. Since the end points have divergent "masses," except if $\lambda=0$ or 1, the central region is de-emphasized. Because of the wrong metric, we *subtract away* the rapidly falling contribution to the form factor from that region. [It may also be possible to consider Eq. (3.8) as representing a string spinning end over end.]

Further discussion of the lack of scaling in νW_2 is hardly required here. Yet, in spite of the shortcomings of the model in providing a definite answer to the problem posed in the opening paragraph of the paper, limited progress can be claimed. Some is concrete, as in the discussion of the vector-dominance aspects of the Compton amplitude; the elaboration of the origin of the fixed pole in the strip domain; and the evaluation of the effects of a quantized internal current.

Perhaps more important, however, is that detailed evaluation of this model points, in different ways, to a need for a broader internal structure in the basic formalism of the DRM. For example, the form-factor calculation is non-unique because many possible λ values for the auxiliary fields are allowed: A group which includes a spectrum of SL$(2,R)$ values has a better chance of solving the problem uniquely.

Use of a group which requires a larger number of internal dimensions is an even more interesting possibility. If one concedes that the two-dimensional domain of the DRM is a surface in physical space-time, and not an abstract space, it is more natural to attempt a new start directly with more internal dimensions, instead of covering regions of four-space by distorting the two-surface ("nonplanar-loops"). To the extent that harmonic oscillators are needed for quantization of the field, and not in the solution of the field equations themselves, it is not necessary to think of such a model as an "elastic solid," anymore than one conceives of the photon field to be such. One could, for example, consider compressional modes along the length of the string.

In this regard, it is interesting to note that in the "parton" interpretation of the DRM, the string model corresponds to the collective motion of infinitely many partons due to "soft" interactions between nearest neighbors in rapidity.[33] Frye, Kogut, and Susskind have recently drawn attention to the possibility that in addition to this soft interaction, there may also be a phenomenologically important "hard" interaction, very singular at short distances, giving forces of long-range in rapidity.[34] A vector-gluon theory would have such an effect. It is very challenging to construct a model which consistently incorporates both these "soft" and "hard" features. A generalization to a hypothetical $q\bar{q}$ sea of the degenerate electron gas problem, with its collective density fluctuations and screened Coulomb force, immediately comes to mind; but, as is true for the DRM itself, incorporation of the requirements of relativity is far from trivial.

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APPENDIX A: ALTERNATE REALIZATION OF SPACE OF SL(2,R) REPRESENTATIONS

In the upper half z plane, define linear differential operators

$$\begin{aligned}2iU_1 &= -(1+z^2)\frac{\partial}{\partial z}+\text{c.c.}\,,\\ 2iU_2 &= (1-z^2)\frac{\partial}{\partial z}+\text{c.c.}\,,\\ iU_3 &= z\frac{\partial}{\partial z}+\text{c.c.}\end{aligned}\tag{A1}$$

These obey the Lie algebra of the SL$(2,R)$ group. The Casimir invariant is

$$\Delta = -(U_1^2 - U_2^2 - U_3^2)$$
$$= (z-\bar{z})^2 \frac{\partial^2}{\partial z \partial \bar{z}} .$$

Since the Casimir operates in a representation as a multiple of the identity, the eigenfunctions of Δ transform according to representations of SL(2, R). This eigenvalue equation is just (2.4).

Define the scalar product

$$(f,g) = \int \frac{d^2z}{(z-\bar{z})^2} f^*(z)g(z), \tag{A2}$$

and the adjoint of U_i with respect to this scalar product,

$$(f, U_i g) = (U_i^\dagger f, g) . \tag{A3}$$

The operators iU_i are self-adjoint: $(iU_i)^\dagger = iU_i$. That is,

$$\int d^2z\,(z-\bar{z})^{-2} f^*(iU_j g) = \int d^2z\,(z-\bar{z})^{-2}(iU_j f)^* g .$$

It is shown in Ref. 20 that the eigenfunctions of Δ are vectors in a Hilbert space which is a valid realization of the (abstract) space in which the action of the group SL(2, R) is unitarily implemented, provided λ is in the interval $1 > \lambda > 0$ (supplementary series); or if $\lambda = \frac{1}{2}i\sigma - 1$, σ real (principal series). An element of the group acts on the functions $\phi^{(\lambda)}(z)$ according to

$$T_g\,\phi^{(\lambda)}(z) = e^{i\vec{\alpha}\cdot\vec{U}}\phi^{(\lambda)}(z)$$
$$= \phi^{(\lambda)}\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right), \tag{A4}$$

where

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,R), \quad \det g = 1$$

and U_i are the differential operators in (A1).

The remarkable feature of (A4) is that the functions $\phi^{(\lambda)}(z)$ do not acquire any multiplier function under the action of the group, regardless of the value of λ (provided it is in the supplementary or principal series).

The explanation for this phenomenon is that one is dealing with a *different* realization of the space of functions that transform as representations from the one commonly used in the discussion of the DRM.

A very interesting feature of the new realization is that the Casimir equation is the field equation for the field transforming according to a representation. In the usual realization, the Casimir operator is trivially a constant, and a Lagrangian for fields of arbitrary SL(2, R) "spin" cannot be written.

A difficulty, on the other hand, is that the feasibility of this realization has been demonstrated in Ref. 20 *only* for the principal and supplementary series. The mapping from the familiar homogeneous-function space to the new function space we are discussing has not been carried out for integer point representations by these authors, in spite of the fact these representations are also unitary.

Nevertheless, it is possible to continue the solutions of Eq. (3.8) in λ from the supplementary series to the positive integer values of λ without encountering any pathological behavior. In particular, it has been checked explicitly to second order in α, β, γ, that, for $\lambda = 2$,

$$e^{i\alpha U_3}\phi(z) = \phi(e^{\alpha}z),$$
$$e^{i\beta(U_1+U_2)}\phi(z) = \phi(z/(1+\beta z)),$$
$$e^{i\gamma(U_1-U_2)}\phi(z) = \phi(z-\gamma).$$

If repeated application of the operators U_i onto the functions $\phi^{(\lambda)}(z)$ were to lead one into a new class of functions, it would show up in second order for $\lambda = 2$.

All one can honestly say is that the status of the integer point representations in the new realization is an open, and very interesting question which merits further investigation.

For completeness, we note that a differential form of the generators in the strip domain, obtained directly from (A1) by change of variables, has been given in Eq. (3.20). The operator

$$J^2 = \tfrac{1}{2}\{L_+, L_-\} - L_0^2,$$

which is the Casimir-invariant, reduces to

$$\sin^2\theta(\partial_\tau{}^2 - \partial_\theta{}^2).$$

To study the action of these generators on the eigenfunctions, the following identity is useful, when combined with the normalization integral (3.9c):

$$(m\cos\theta \pm \sin\theta\,\partial_\theta)\phi_m(\theta)$$
$$= \left[\frac{m(m\pm\lambda)(m\mp\lambda\pm 1)}{m\pm 1}\right]^{1/2}\phi_{m\pm 1}(\theta).$$

APPENDIX B: CONSTRUCTION OF GREEN'S FUNCTION

Treating τ as a time variable, we seek the Green's function satisfying the (hyperbolic) equation

$$\left(\frac{\partial^2}{\partial\theta^2} - \frac{\partial^2}{\partial\tau^2} - \frac{\lambda(\lambda-1)}{\sin^2\theta}\right)G^{(\lambda)}(\tau\theta;\tau'\theta') = \kappa\delta(\theta-\theta')\delta(\tau-\tau'). \tag{B1}$$

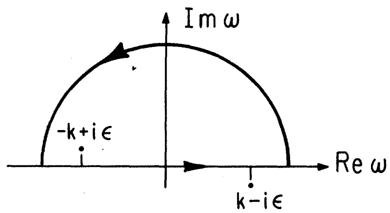

FIG. 8. Contour for evaluation of Green's function.

(κ is a normalization specifying the strength of the source.)

Expanding $G^{(\lambda)}$ in terms of its Fourier components,

$$G^{(\lambda)}(\tau\theta;\tau'\theta')=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, g^{(\lambda)}(\theta,\theta';\omega)e^{-i\omega(\tau-\tau')}, \tag{B2}$$

and then expanding $g^{(\lambda)}(\theta,\theta';\omega)$ in the complete, orthonormal eigenfunctions $\phi^{(\lambda)}(\theta)$, one obtains in a well-known fashion

$$G^{(\lambda)}=\frac{\kappa}{2\pi}\sum_k \phi_k^{(\lambda)}(\theta)\phi_k^{(\lambda)}(\theta')\int_{-\infty}^{\infty}d\omega\,\frac{e^{-i\omega(\tau-\tau')}}{\omega^2-k^2}. \tag{B3}$$

The boundary condition needed to complete the specification of G is imposed by prescribing how the integral appearing in the expression above is to be evaluated. The well-known $\lambda=0$ sum

$$G^{(\lambda=0)}=2\sum_{n=1}^{\infty}e^{in(\tau-\tau')}\frac{\cos n\theta\cos n\theta'}{n}\qquad(\tau<\tau')$$

follows from choosing the contour as in Fig. 8, with $\pm i\epsilon$ displacements as indicated, and from choosing $\kappa=+2\pi i$.

Thus, in general,

$$G^{(\lambda)}=\pi\sum_k e^{ik(\tau-\tau')}\frac{\phi_k^{(\lambda)}(\theta)\phi_k^{(\lambda)}(\theta')}{k}\qquad(\tau<\tau'). \tag{B4}$$

Inserting the expressions for $\phi_k^{(\lambda)}(\theta)$ exhibited in Eq. (3.9b), we have

$$G^{(\lambda)}=2^{2\lambda-1}\Gamma(\lambda)(\sin\theta\sin\theta')e^{i\lambda(\tau-\tau')}\times\sum_{n=0}^{\infty}e^{in(\tau-\tau')}\frac{\Gamma(n+1)}{\Gamma(n+2\lambda)}C_n^\lambda(\cos\theta)C_n^\lambda(\cos\theta'), \tag{B5}$$

in which $n\equiv k-\lambda$.

To perform this sum, use the addition theorem

$$C_n^\lambda(\cos\theta)C_n^\lambda(\cos\theta')=\frac{\Gamma(n+2\lambda)}{2^{2\lambda-1}n!\,\Gamma^2(\lambda)}I_n^\lambda, \tag{B6}$$

$$I_n^\lambda\equiv\int_0^\pi d\phi(\sin\phi)^{2\lambda-1}C_n^\lambda(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\phi)\qquad(\lambda>0).$$

Further, we will need the generating function for Gegenbauer polynomials,

$$\sum_{n=0}^{\infty}z^nC_n^\lambda(w)=(1-2wz+z^2)^{-\lambda},\quad |z|<1. \tag{B7}$$

Temporarily give $(\tau-\tau')$ a small imaginary part, so that

$$z\equiv e^{i(\tau-\tau')}$$

satisfies $|z|<1$. Then using (B6) and (B7) in (B5),

$$G^{(\lambda)}=(\sin\theta\sin\theta')^\lambda z^\lambda\times\int_0^\pi d\phi\,(\sin\phi)^{2\lambda-1}(1-2z\cos\gamma+z^2)^{-\lambda}, \tag{B8}$$

$$\cos\gamma\equiv\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\phi.$$

A change of integration variable enables us to recognize

$$\int_0^\pi d\phi\,(\sin\phi)^{2\lambda-1}(1-2z\cos\gamma+z^2)^{-\lambda}=(z\sin\theta\sin\theta')^{-\lambda}Q_{\lambda-1}\left(\frac{1+z^2-2z\cos\theta\cos\theta'}{2z\sin\theta\sin\theta'}\right). \tag{B9}$$

Insertion into (B8) yields Eq. (3.11), normalized so as to agree with the known $\lambda=0$ function.

APPENDIX C: THE LOW-ENERGY THEOREM

The matrix element for Figs. 3 and 4 may be expressed in terms of

$$M_{\mu\nu}=\frac{e^2}{4}\delta^4(p+q+p'+q')\delta^4(p^2-p'^2)\times\int_0^\infty d\tau\int\int d\theta d\theta'[e^{-i\tau\alpha_s}\langle\tilde F^{p+q,q}(\theta,\tau)\tilde F_\nu^{-p',q'}(\theta',0)\rangle_0+e^{-i\tau\alpha_u}\langle\tilde F_\nu^{p+q',q'}(\theta,\tau)\tilde F_\mu^{-p',q}(\theta',0)\rangle_0], \tag{C1}$$

where

$$\tilde F_\mu^{k,q}(\theta,\tau)\equiv\{J_\alpha\partial_\alpha\overline V_\mu, e^{iq\cdot\overline V}\}+(2p_\mu^{(0)}+2k_\mu-q_\mu)J_\tau e^{iq\cdot\overline V},\qquad \overline V_\mu=V_\mu-x_\mu^{(0)}-p_\mu^{(0)}\tau.$$

The term relevant as $q,q'\to0$, in a gauge such that $\epsilon\cdot p=\epsilon'\cdot p=0$, is

$$\tilde M_{\mu\nu}=\int_0^\infty d\tau\int\int d\theta d\theta'[\langle(J_\alpha\partial_\alpha\overline V_\mu)_{\theta,\tau}(J_\beta\partial_\beta\overline V_\nu)_{\theta',0}\rangle_0+\langle(J_\alpha\partial_\alpha\overline V_\nu)_{\theta,\tau}(J_\beta\partial_\beta\overline V_\mu)_{\theta',0}\rangle_0]. \tag{C2}$$

Examine first

$$\tilde{M}^{(1)}_{\mu\nu}=\int_0^\infty d\tau\int\int d\theta d\theta'\left\langle\frac{\partial(\rho,\overline{V}_\mu)}{\partial(\theta,\tau)}(J_\alpha\partial_\alpha\overline{V}_\nu)_{\theta',0}\right\rangle_0$$

$$=\int\int d\theta d\theta'\left[\left\langle\left(\frac{\partial\rho}{\partial\theta}\overline{V}_\mu\right)_{\theta,\tau}(J_\alpha\partial_\alpha\overline{V}_\nu)_{\theta',0}\right\rangle_0\Big|_{\tau=0}^{\infty}\right]-\int d\tau\int d\theta'\left[\left\langle\left(\frac{\partial\rho}{\partial\tau}\overline{V}_\mu\right)_{\theta,\tau}(J_\alpha\partial_\alpha\overline{V}_\nu)_{\theta',0}\Big|_{\theta=0}^{\pi}\right]. \tag{C3}$$

From the explicit forms of the Green's functions in these expressions, such as

$$\left\langle\frac{\partial\rho}{\partial\theta}\frac{\partial\rho}{\partial\theta'}\right\rangle_0\left\langle\overline{V}_\mu(\theta,\tau)\frac{\partial\overline{V}_\nu}{\partial\tau}(\theta',0)\right\rangle_0,$$

etc., we find the only nonvanishing term is from the $\tau=0$ end point. By use of

$$\langle\overline{V}_\mu(\theta,0)\dot{\overline{V}}_\nu(\theta',0)\rangle_0=i\pi g_{\mu\nu}\delta(\theta-\theta'), \tag{C4a}$$

$$\left\langle\frac{\partial\rho(\theta,0)}{\partial\theta}\dot{\rho}(\theta',0)\right\rangle_0=-i\pi\frac{\partial}{\partial\theta}\delta(\theta-\theta'), \tag{C4b}$$

we obtain

$$\int\int d\theta d\theta'\left\langle\left(\frac{\partial\rho}{\partial\theta}\overline{V}_\mu\right)_{\theta,0}(J_\alpha\partial_\alpha\overline{V}_\nu)_{\theta',0}\right\rangle_0=-ig_{\mu\nu}\left\{1+\pi\int d\theta\frac{\partial^2Q_0(\theta,0;0,0)}{\partial\theta^2}-\pi\int\int d\theta d\theta'\frac{\partial G(\theta,0;\theta',0)}{\partial\theta'}\frac{\partial}{\partial\theta}\delta(\theta-\theta')\right\}, \tag{C5}$$

where G contains the Green's functions of all the fields in $\overline{V}_\mu$.

The first term on the right-hand side of (C5) gives rise to the correct low-energy limit of the amplitude, proportional to e^2. However, the terms involving Q_0 and G are divergent. Seagulls are required to cancel the contributions of these pieces.

Contact terms of the required form are obtained by a careful construction of H_{int}. In abbreviated form,

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_{\text{int}},$$

$$\mathcal{L}_0=\frac{1}{4\pi}[\phi'^2-\dot{\phi}^2+\mu\phi^2+\dot{\rho}^2-\rho'^2], \tag{C6}$$

$$\mathcal{L}_{\text{int}}=\frac{e}{2}\{\rho'\dot{\phi}_\mu-\dot{\rho}\phi'_\mu, A^\mu\}.$$

Introduce

$$\Pi_\mu=\frac{\delta\mathcal{L}}{\delta\dot{\phi}_\mu}=-\frac{1}{2\pi}\dot{\phi}_\mu+\frac{e}{2}\{\rho',A_\mu\};$$

$$P=\frac{\delta\mathcal{L}}{\delta\dot{\rho}}=\frac{1}{2\pi}\dot{\rho}-\frac{e}{2}\{\phi'_\mu,A^\mu\}. \tag{C7}$$

Then we have

$$H=\Pi_\mu\dot{\phi}^\mu+P\dot{\rho}-\mathcal{L}, \tag{C8a}$$

$$H_{\text{int}}=\pi e[P\{\phi'_\mu,A^\mu\}-\rho'\{\Pi_\mu,A^\mu\}]+\pi e^2[(\rho')^2A^2+\tfrac{1}{4}\{\phi'_\mu,A^\mu\}^2]. \tag{C8b}$$

The terms of order e^2 in (C8b) are the desired seagulls. Note that the second part of $\tilde{M}_{\mu\nu}$ not discussed has the same structure as $\tilde{M}^{(1)}_{\mu\nu}$; indeed $\tilde{M}_{\mu\nu}=2\tilde{M}^{(1)}_{\mu\nu}$, since only the $g_{\mu\nu}$ term is important. It is then found that the seagulls indeed cancel the required terms in (C5).

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the parameter α controls the pole structure (which may be related to Regge behavior), while β controls the asymptotic behavior. Recently, J. D. Bjorken and J. Kogut [Phys. Rev. D 8, 1341 (1973)] have attempted to relate the Regge behavior to the asymptotic behavior of the form factor. In that view, α and β are not independent. It is likely that some such relation would occur in a better dual model on grounds of symmetry. However, it must be stressed that there are no dual models at present that actually *derive* $(\cos\theta)^{\beta}$ terms.

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