Bilinear Bethe-Salpeter Equation and Conditions for a Bound on a Two-Particle Reducible Part of the Elastic Scattering Amplitude

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A high-energy bound is obtained for the full two-particle reducible amplitude in the crossed channel, provided that the Froissart bound is satisfied by the full off-mass-shell amplitude, and the Gribov finite-mass hypothesis is correct for asymptotic amplitudes. Implications of this bound are discussed.

I. INTRODUCTION

At large energies, the problems confronting particle physicists seem least formidable. Despite this simplicity, the behavior of the high-energy elastic scattering amplitude in even π-π scattering is still an open question. The most recent experimental data seem to suggest a growing total cross section consistent with saturation of the Froissart bound.[1] Some models leading to this behavior are based on eikonal iterations of "ladderlike" structures in the direct channel. Froissart-bound saturation occurs when the "ladderlike" Born term of the eikonal series grows as a power of s greater than unity.[2] The Born term in these models is usually two-particle reducible in the crossed channel. At first glance it seems that the Born term cannot grow as a power of s greater than unity without violating the Froissart bound. At second glance, however, one finds that the eikonal iteration in the direct channel eliminates the leading behavior of the Born term and allows only saturation of the Froissart bound, correcting the apparent violation. We present here a third glance at this situation.

To be specific, we show in detail that if (a) the full two-particle reducible part of the amplitude in the crossed channel has leading s behavior generated in a multiperipheral (ladderlike) way, (b) a weak version of the Froissart bound holds for off-mass-shell amplitudes, and (c) very-far-off-mass-shell effects are not important in a particular integral we consider (a rather technical assumption), then the full two-particle reducible piece of the amplitude can grow as a power of s no greater than unity in the forward direction, with a similar result for the nonforward direction if the assumptions are still true. We then show that condition (a) can be relaxed, provided that a particular series in $\ln s$ is polynomially bounded in s, where s is the energy squared. In deriving these results we use only the Bethe-Salpeter equation, which can be derived in an axiomatic framework,[3] and some results from the classic paper of Amati, Stanghellini, and Fubini.[4]

The question of whether there is a bound for off-mass-shell amplitudes is still an open one.[5] The existence of an off-mass-shell unitarity equation in some theories[6] suggests that this is at least a possibility. Our results show that if the above-mentioned eikonal models are correct and if the far-off-shell region does not dominate our expressions, then there is no bound similar to the Froissart bound for off-mass-shell amplitudes. Alternatively, if the Froissart bound can be established for off-mass-shell amplitudes in the weak sense used here, then the above-mentioned "Froissart-bound-saturating" eikonal models must be improved, or the far-off-shell region must dominate some of our integrals.

We consider here only neutral pseudoscalar mesons for simplicity. We believe that the essential ingredients of our arguments apply to a much broader class of field theories.

II. THE BETHE-SALPETER EQUATION

We use here a normalization such that

$$\operatorname{Im}T(s,t,\mu^2,\mu^2,\mu^2,\mu^2)|_{t=0}=2|\vec{k}|\sqrt{s}\,\sigma_{\text{tot}}, \qquad (2.1)$$

where $T(s,t,u_1,u_2,u_3,u_4)=T(q,p,p')$ is the scattering amplitude for two off-shell pseudoscalar mesons (see Fig. 1) and

$$\begin{aligned} &s=(p+p')^2, \quad t=q^2,\\ &u_1=(p+\tfrac{1}{2}q)^2, \quad u_2=(p-\tfrac{1}{2}q)^2,\\ &u_3=(p'-\tfrac{1}{2}q)^2, \quad u_4=(p'+\tfrac{1}{2}q)^2, \qquad (2.2)\\ &\mu=\text{pion mass},\\ &\vec{k}=\text{c.m. momentum of one pion}. \end{aligned}$$

We define $I(s,t,u_1,u_2,u_3,u_4)=I(q,p,p')$ to be the full amplitude that is two-particle irreducible in the t channel. This means that $I(q,p,p')$ gives no

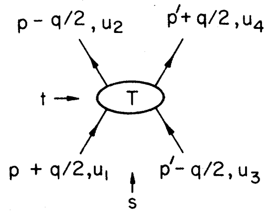

FIG. 1. Definition of variables. Here $s=(p+p')^2$, $t=q^2$, $u_1=(p+\frac{1}{2}q)^2$, $u_2=(p-\frac{1}{2}q)^2$, $u_3=(p'-\frac{1}{2}q)^2$, $u_4=(p'+\frac{1}{2}q)^2$.

contribution to the discontinuity of the cut along the real t axis which begins at the normal threshold branch point at $t=4\mu^2$. Likewise we define $T_2(s,t,u_1,u_2,u_3,u_4)=T_2(q,p,p')$ to be the full amplitude that is two-particle reducible in that channel (i.e., it is the sum of all contributions which do contribute to this cut). Both T_2 and I are crossing-symmetric in the s-u sense. T is related to I by the Bethe-Salpeter equation[3]

$$T(q,p,p')=I(q,p,p') -\tfrac{1}{2}i\int\frac{d^4p''}{(2\pi)^4}I(q,p,p'')T(q,-p'',p') \times\Delta(p''-\tfrac{1}{2}q)\Delta(p''+\tfrac{1}{2}q)\,, \tag{2.3}$$

where $\Delta(p)$ is the full renormalized pion propagator. This equation is illustrated graphically in Fig. 2. It has been derived in an axiomatic framework, and therefore any proposed solution to the field equations, for example a perturbative solution, must satisfy it. We will use (2.3), which is a statement of t-channel unitarity, in the s-channel physical region. T, T_2, and I are related by

$$T(q,p,p')=T_2(q,p,p')+I(q,p,p')\,. \tag{2.4}$$

Let us now use the Cutkosky rules to take the absorptive part in s of Eq. (2.3). We assume that the external particles and the vacuum are stable and therefore neither of the exchanged pions in the

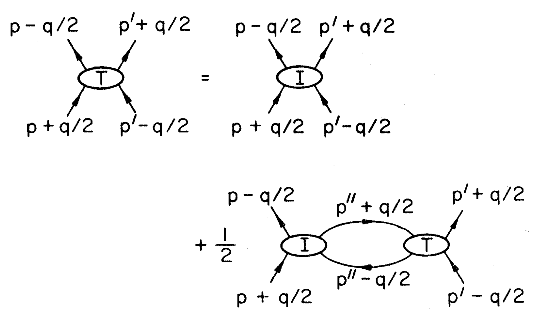

FIG. 2. The Bethe-Salpeter equation.

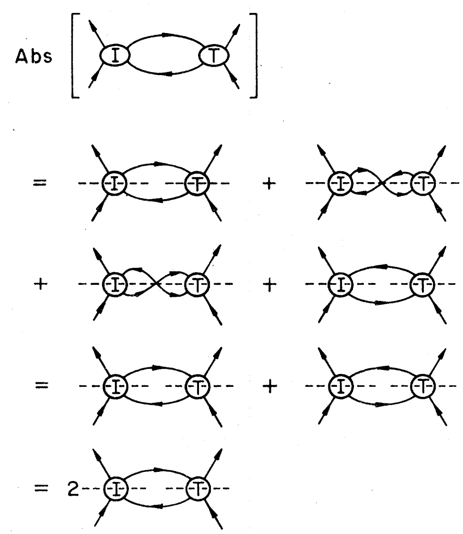

FIG. 3. Cutkosky rules for taking the s-channel absorptive part of a t-channel iteration of crossing-invariant objects.

integral (2.3) can ever be put on shell. All particle lines which are put on shell in this process must be inside either I or T. This clearly causes the integrand to become a product of the absorptive parts of I and T in the subenergies. Both I and T are invariant under crossing; therefore each has a left-hand cut in s which contributes to the absorptive part of the integral. If we let dashed lines through a graph represent only the right-hand discontinuity in the subenergy which is cut by the dashed line, then we can include the contribution from the left-hand cut by simply reversing the momenta of the two virtual pions. This is illustrated in Fig. 3. The two surviving terms are identical since the exchanged pions are integrated over all momenta and the two amplitudes are invariant under crossing. This scheme simply yields a factor of 2 times an integral over only the right-hand absorptive parts. This factor of 2 nicely cancels the factor of $\frac{1}{2}$ outside the integral in (2.3), yielding (see Fig. 4)

$$\mathrm{Abs}T(q,p,p') =\mathrm{Abs}I(q,p,p') +\int\frac{d^4p''}{(2\pi)^4}\mathrm{Abs}I(q,p,p'')\,\mathrm{Abs}T(q,-p'',p') \times\Delta(p''-\tfrac{1}{2}q)\Delta(p''+\tfrac{1}{2}q)\,. \tag{2.5}$$

Here $\mathrm{Abs}T=2\mathrm{Im}T$ on shell in the s-channel physical region, and the integration is only over positive subenergies $(p+p'')^2$ and $(p'-p'')^2$. Equation (2.5) is the famous ABFST equation studied by Amati, Bertocchi, Fubini, Stanghellini, and Tonin

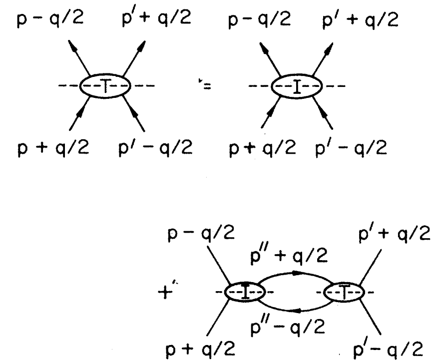

FIG. 4. The s-channel absorptive part of Fig. 2, the ABFST equation.

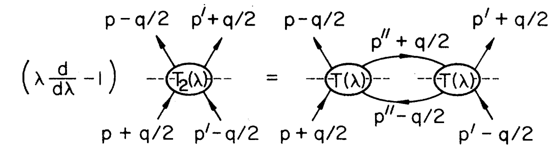

FIG. 5. The bilinear equation associated with Fig. 4.

in the ladder approximation.

For convenience, we define the following operation $\times_t$ for the various 4-point functions that we will be working with:

$$a\times_t b = \int \frac{d^4p''}{(2\pi)^4}\, a(q,p,p'')b(q,-p'',p') \times\Delta(p''+\tfrac{1}{2}q)\Delta(p''-\tfrac{1}{2}q)\,, \qquad (2.6)$$

where a and b are understood to have only right-hand support in the variables $(p+p'')^2$ and $(p'-p'')^2$ respectively when they represent absorptive parts, and also

$$a\times_t a = a^2, \qquad a\times_t a^{n-1} = a^{n-1}\times_t a = a^n\,. \qquad (2.7)$$

Then (2.3) and (2.5) become

$$T = I - \tfrac{1}{2}iI\times_t T\,, \qquad (2.8)$$

$$\text{Abs}T = \text{Abs}I + \text{Abs}I\times_t \text{Abs}T\,. \qquad (2.9)$$

These equations are solved by iteration yielding

$$T = 2i\sum_{n=1}^{\infty}(-\tfrac{1}{2}i)^n I^n\,, \qquad (2.10)$$

$$\text{Abs}T = \sum_{n=1}^{\infty}(\text{Abs}I)^n\,, \qquad (2.11)$$

$$T_2 = 2i\sum_{n=2}^{\infty}(-\tfrac{1}{2}i)^n I^n\,, \qquad (2.12)$$

$$\text{Abs}T_2 = \sum_{n=2}^{\infty}(\text{Abs}I)^n\,. \qquad (2.13)$$

We emphasize that the above relations for absorptive parts are valid only for values of the external masses below the two-particle threshold because we have used a stability condition in deriving them.

We now would like to introduce the following generating functions, which prove to be very useful:

$$\tfrac{1}{2}T(\lambda) = i\sum_{n=1}^{\infty}\lambda^n(-\tfrac{1}{2}i)^n I^n\,, \qquad (2.14)$$

$$\tfrac{1}{2}T_2(\lambda) = i\sum_{n=2}^{\infty}\lambda^n(-\tfrac{1}{2}i)^n I^n\,, \qquad (2.15)$$

$$\text{Abs}T(\lambda) = \sum_{n=1}^{\infty}\lambda^n(\text{Abs}I)^n\,, \qquad (2.16)$$

$$\text{Abs}T_2(\lambda) = \sum_{n=2}^{\infty}\lambda^n(\text{Abs}I)^n\,. \qquad (2.17)$$

Varying λ, as we see it, corresponds to varying the strength of the basic t-channel force. It plays the same role here as the coupling constant in the ladder approximation to the Bethe-Salpeter equation. It might seem pointless at present to introduce these unphysical functions; however, they enable us to do some rather complex manipulations quite easily. The actual physical amplitudes are gotten by taking $\lambda = 1$ at the end of the calculation. The functions (2.14)–(2.17) enable us to derive and work with the following relations, which are the cornerstone of this paper (see Fig. 5):

$$\tfrac{1}{2}\left(\lambda\frac{\partial}{\partial\lambda}-1\right)T_2(\lambda) = \tfrac{1}{2}\left(\lambda\frac{\partial}{\partial\lambda}-1\right)T(\lambda) = -\tfrac{1}{4}iT(\lambda)\times_t T(\lambda)\,, \qquad (2.18)$$

$$\left(\lambda\frac{\partial}{\partial\lambda}-1\right)\text{Abs}T_2(\lambda) = \left(\lambda\frac{\partial}{\partial\lambda}-1\right)\text{Abs}T(\lambda) = \text{Abs}T(\lambda)\times_t\text{Abs}T(\lambda)\,. \qquad (2.19)$$

From (2.18) and (2.19) one can derive the following useful Taylor expansions for $T(\lambda)$ and $\text{Abs}T(\lambda)$:

$$\tfrac{1}{2}T(\lambda) = \tfrac{1}{2}T(\lambda=1) + i\sum_{n=1}^{\infty}(\lambda-1)^n\{[-\tfrac{1}{2}iT(\lambda=1)]^n + [-\tfrac{1}{2}iT(\lambda=1)]^{n+1}\}\,, \qquad (2.20)$$

$$\text{Abs}T(\lambda) = \text{Abs}T(\lambda=1) + \sum_{n=1}^{\infty}(\lambda-1)^n\{[\text{Abs}T(\lambda=1)]^n + [\text{Abs}T(\lambda=1)]^{n+1}\}\,. \qquad (2.21)$$

See Appendix A for the derivation of (2.21); the same derivation also applies to (2.20). For completeness, we write the integrals (2.18) and (2.19) out in full:

$$\left(\lambda\frac{\partial}{\partial\lambda}-1\right)T_2(q,p,p';\lambda)=-\tfrac{1}{2}i\int\frac{d^4p''}{(2\pi)^4}T(q,p,p'';\lambda)T(q,-p'',p';\lambda)\Delta(p''+\tfrac{1}{2}q)\Delta(p''-\tfrac{1}{2}q)\,, \tag{2.22}$$

$$\left(\lambda\frac{\partial}{\partial\lambda}-1\right)\mathrm{Abs}T_2(q,p,p';\lambda)=\int\frac{d^4p''}{(2\pi)^4}\mathrm{Abs}T(q,p,p'';\lambda)\,\mathrm{Abs}T(q,-p'',p';\lambda)\Delta(p''-\tfrac{1}{2}q)\Delta(p''+\tfrac{1}{2}q)\,. \tag{2.23}$$

We emphasize once more that (2.22) can be justified for all values of the external momenta, whereas (2.23) is limited to the region where all the external particle masses are below $(2\mu)^2$.

Integrals such as those occurring in (2.22) and (2.23) have been studied by ABFST to leading order in s and in the forward direction $t=q^2=0$. In studying these integrals they assume that the leading behavior comes from the integration over the subenergies $(p+p'')^2$ and $(p''-p')^2$, while the integrations over the masses $(p''+\frac{1}{2}q)^2$ and $(p''-\frac{1}{2}q)^2$ are held relatively close to the mass shell by damping coming from the various terms in the integrand. With this assumption the leading behavior of (2.22) or (2.23) can be gotten quite nicely. It seems reasonable that high-energy off-shell amplitudes should be damped in the external masses since we like to think of hadrons as being composite particles having form factors which die off quickly. Moreover, this assumption (sometimes called the Gribov finite-mass hypothesis) is the basis of a number of theoretical techniques, and if it leads to an unusual state of affairs here then this is perhaps not an uninteresting result. At any rate we shall make this finite-mass hypothesis in what follows. From here on we work with the equations for the absorptive parts, although the equations for the complex amplitudes could equally well be used.

III. THE ABFST INTEGRAL

We wish to study the integral on the right-hand side of (2.23). To do this it is convenient to make the following change of variables:

$$\begin{aligned}W&=\int\frac{d^4p''}{(2\pi)^4}\mathrm{Abs}T(q,p,p'';\lambda)\,\mathrm{Abs}T(q,-p'',p';\lambda)\Delta((p''-\tfrac{1}{2}q)^2)\Delta((p''+\tfrac{1}{2}q)^2)\\&=\int ds_0ds'du_1'du_2'\,\mathrm{Abs}T(s_0,t,u_1,u_2,-u_1',-u_2';\lambda)\mathrm{Abs}T(s',t,-u_1',-u_2',\mu^2,\mu^2;\lambda)\\&\quad\times\Delta(-u_1')\Delta(-u_2')K(s,s_0,s',t,u_1',u_2',u_1,u_2)\,,\end{aligned} \tag{3.1}$$

with $(p'\pm\frac{1}{2}q)^2=\mu^2$. K is given by

$$\begin{aligned}K(s,s_0,s',t,u_1',u_2',u_1,u_2)&=\frac{1}{(2\pi)^4}\int d^4p''\delta(s_0-(p+p'')^2)\delta(s'-(p'-p'')^2)\,\delta(u_1'+(p''-\tfrac{1}{2}q)^2)\delta(u_2'+(p''+\tfrac{1}{2}q)^2)\\&=\frac{1}{8(2\pi)^4}\frac{1}{s}\frac{\theta(J)}{\sqrt{J}}\end{aligned} \tag{3.2}$$

and J is given by

$$J=\begin{vmatrix}-\frac{1}{2}t & \frac{1}{2}(u_1'-u_2') & \frac{1}{2}(u_2-u_1) & 0\\ \frac{1}{2}(u_1'-u_2') & u_1'+u_2'+\frac{1}{2}t & \frac{1}{2}(-u_1-u_2+u_1'+u_2'+t)+s_0 & s'-\mu^2+\frac{1}{2}(u_1'+u_2')\\ \frac{1}{2}(u_2-u_1) & \frac{1}{2}(-u_1-u_2+u_1'+u_2'+t)+s_0 & -u_1-u_2+\frac{1}{2}t & s'-\mu^2-\frac{1}{2}(u_1+u_2)\\ 0 & s'-\mu^2+\frac{1}{2}(u_1'+u_2'+t) & s-\mu^2+\frac{1}{2}(t-u_1-u_2) & 2\mu^2-\frac{1}{2}t\end{vmatrix}. \tag{3.3}$$

The total phase space in the integral (3.1), as a consequence of the finite-mass hypothesis, is growing logarithmically with s. The region where the inequalities $s_0/s<\epsilon$, $s'/s<\epsilon$, all masses $<\epsilon s_0, \epsilon s'$, are true also has phase space which grows logarithmically with s for any ϵ. This logarithmic growth comes from the rapidity available to the virtual pions in the integral. The regions of the integral which do not satisfy the above inequalities have phase space which is constant as s increases. Therefore, for very large s, if we integrate only over that region where the above inequalities are satisfied the mistake we make will be down by a factor $\ln s$ from this leading integration region, provided that the integrand in the other region is not on the average $\ln s$ bigger than in this leading region. This does not happen if the leading term grows as a power of s (modulo $\ln s$). ABFST have considered this case carefully, and their results can be gotten by taking the above inequalities to be true in the entire integration region. So long as $\mathrm{Abs}T(s,t;\lambda)$ is polynomially bounded in s and grows faster than $1/s$, this procedure will lead to the correct leading behavior, provided the integration over the masses does not give zero. Even if this happens, this procedure will give a bound on the behavior of the integral for large s, and this is all we need in this paper.

We are interested in amplitudes which satisfy the finite-mass hypothesis and are polynomially bounded in s, and so we are free to take the inequalities above to be true inside the integral. Therefore we need only keep the leading term of $\mathrm{Abs}T$ in the integrand. We choose ϵ to be a small number and therefore neglect terms of order ϵ in the integrand compared to unity. We emphasize that as far as the leading behavior of the integral (3.1) is concerned this is not an approximation. Taking the inequalities

$$s \gg s', \quad s \gg s_0; \quad s_0, s' \gg \text{all masses} \tag{3.4}$$

to be true inside the integral of (3.1) we find after some work

$$W=\frac{1}{8(2\pi)^4}\frac{1}{s}\int_{4\mu^2}^{s} ds_0 ds' \int_0^\infty du_1' du_2' \mathrm{Abs}T(s_0,t,u_1,u_2,-u_1',-u_2';\lambda)$$
$$\times \mathrm{Abs}T(s',t,-u_1',-u_2',\mu^2,\mu^2;\lambda)\Delta(-u_1')\Delta(-u_2')\frac{\theta(H)}{\sqrt{H}}\,, \tag{3.5}$$

with

$$H=-\tfrac{1}{4}(u_1'-u_2')^2+|t|\tfrac{1}{2}(u_1'+u_2')-\tfrac{1}{4}|t|^2-|t|(s_0 s'/s)\,. \tag{3.6}$$

We notice that, as far as the leading behavior of W is concerned, the kernel no longer depends on the external masses u_1 and u_2.

IV. IMPLICATIONS FOR THE OFF-MASS-SHELL FROISSART BOUND

Let us suppose that a weak form of the Froissart bound holds for $\mathrm{Abs}T(q,p,p')$ for unphysical values of the external masses, as long as these are small compared to the total energy $\sqrt{s}$, i.e.,

$$|\mathrm{Abs}T(q,p,p')|=|\mathrm{Abs}T(s,t,u_1,u_2,u_3,u_4)|$$
$$<\rho(t,u_1,u_2,u_3,u_4)s^{1+\epsilon}\,, \tag{4.1}$$

where ϵ is an arbitrarily small positive number and ρ is an arbitrary positive function of t, u_1, u_2, u_3, u_4, strongly dampled in these masses. First, we generate an inequality from Eq. (2.23):

$$\left|\left(\lambda\frac{\partial}{\partial\lambda}-1\right)\mathrm{Abs}T_2(q,p,p';\lambda)\right| \le \int\frac{d^4p''}{(2\pi)^4}|\mathrm{Abs}T(q,p,p'';\lambda)|\,|\mathrm{Abs}T(q,-p'',p';\lambda)|\,|\Delta((p''-\tfrac{1}{2}q)^2)|\,|\Delta((p''+\tfrac{1}{2}q)^2)|. \tag{4.2}$$

Next we put the Froissart bound (4.1) into the integrand of (4.2) at $\lambda=1$ and use the expression (3.5) of the previous section for the leading behavior for large s. We get

$$\left|\left(\lambda\frac{\partial}{\partial\lambda}-1\right)\mathrm{Abs}T_2(q,p,p';\lambda)\right|_{\lambda=1}\left|<\frac{1}{8(2\pi)^4}\frac{1}{s}\int_{4\mu^2}^{s} ds' ds_0\int_0^\infty du_1' du_2'|s_0^{1+\epsilon}\rho(t,\mu^2,\mu^2,-u_1',-u_2')|\right.$$
$$\times|s'^{1+\epsilon}\rho(t,-u_1',-u_2',\mu^2,\mu^2)|\,|\Delta(-u_1')|\,|\Delta(-u_2')|\frac{\theta(H)}{\sqrt{H}}\,. \tag{4.3}$$

Equation (4.3) can be rewritten, after one integration, as

$$\left|\left(\lambda\frac{\partial}{\partial\lambda}-1\right)\mathrm{Abs}T_2(q,p,p';\lambda)\Big|_{\lambda=1}\right|<(\ln s)s^{1+\epsilon}\int_0^\infty dp\,p^{1+\epsilon}\int_0^\infty du_1'du_2'|\rho(t,\mu^2,\mu^2,-u_1',-u_2')|\,|\rho(t,-u_1',-u_2',\mu^2,\mu^2)|$$
$$\times|\Delta(-u_1')|\,|\Delta(-u_2')|\frac{\theta(H(p))}{[H(p)]^{1/2}}+O(s^{1+\epsilon}),\qquad(4.4)$$

where $H(p)$ is given by

$$H(p)=-\tfrac{1}{4}(u_1'-u_2')^2+|t|\tfrac{1}{2}(u_1'+u_2')-\tfrac{1}{4}|t|^2-|t|p\,.\qquad(4.5)$$

The integral in (4.4) is essentially an arbitrary positive function of t since ρ is an arbitrary function. We thus get the important relation

$$\left|\left(\lambda\frac{\partial}{\partial\lambda}-1\right)\mathrm{Abs}T_2(s,t;\lambda)\Big|_{\lambda=1}\right|<f(t)(\ln s)s^{1+\epsilon}\qquad(4.6)$$

for arbitrary ϵ and s large.

Equation (4.6) holds only at the point $\lambda=1$ and for any positive ϵ. It would be useful if, for a given ϵ, we could establish this bound in some neighborhood of $\lambda=1$. The following argument shows that this is in fact possible. Let us consider again Eq. (2.21):

$$\mathrm{Abs}T(\lambda)=\mathrm{Abs}T(\lambda=1)+\sum_{n=1}^{\infty}(\lambda-1)^n\{[\mathrm{Abs}T(\lambda=1)]^n+[\mathrm{Abs}T(\lambda=1)]^{n+1}\}\,.\qquad(4.7)$$

Taking the absolute value of each side of this equation yields

$$|\mathrm{Abs}T(\lambda)|\leq|\mathrm{Abs}T(\lambda=1)|+\sum_{n=1}^{\infty}|\lambda-1|^n\{|[\mathrm{Abs}T(\lambda=1)]^n|+|[\mathrm{Abs}T(\lambda=1)]^{n+1}|\}\,.\qquad(4.8)$$

We now put the asymptotic bound (4.1) into the right-hand side of (4.8). Arguments similar to those above (see Appendix B) show that

$$|[\mathrm{Abs}T(\lambda=1)]^n|<|(s^{1+\epsilon}\rho)^n|\leq c_{n-1}s^{1+\epsilon}(\ln s)^{n-1}\,.\qquad(4.9)$$

Let us choose c_n so that this inequality is satisfied for s greater than some lower limit s_L, for all n. Then (4.8) becomes

$$|\mathrm{Abs}T(\lambda)|<c_0s^{1+\epsilon}+\sum_{n=1}^{\infty}|\lambda-1|^n[c_{n-1}s^{1+\epsilon}(\ln s)^{n-1}+c_ns^{1+\epsilon}(\ln s)^n]\,,\quad s>s_L\,.\qquad(4.10)$$

Now let us define c_n' by the inequality

$$c_n'(\ln s)^n\geq c_{n-1}(\ln s)^{n-1}+c_n(\ln s)^n,\quad s>s_L\,.\qquad(4.11)$$

Equation (4.10) then becomes

$$|\mathrm{Abs}T(\lambda)|<\sum_{n=0}^{\infty}|\lambda-1|^nc_n'(\ln s)^ns^{1+\epsilon}\,.\qquad(4.12)$$

We emphasize that this result follows from our two assumptions, the finite-mass hypothesis and the Froissart bound off shell, and is not a general asymptotic result for the Bethe-Salpeter equation. We now make a rather technical assumption which we hope will not disturb the reader too much. We assume that the c_n' can be chosen so that the right-hand side is polynomially bounded in s for at least one value of $\lambda\neq1$, and for $s>s_L$. This seems reasonable since ρ is an arbitrary, strongly damped function of the external masses, and it would seem that we could choose it so that this would be true. Moreover if the kernel I of the Bethe-Salpeter equation is bounded by a polynomial in s, as we strongly expect it to be, then we would expect the solution of this equation to be polynomially bounded even for $\lambda\neq1$. Therefore we feel that this is a minimal assumption which takes the form

$$\sum_{n=0}^{\infty}|\lambda_0-1|^nc_n'(\ln s)^n<s^M,\quad s>s_L,\ \lambda_0\neq1\qquad(4.13)$$

for some M no matter how large, and for some λ_0. For arbitrary λ we can write

$$\sum_{n=0}^{\infty}|\lambda-1|^nc_n'(\ln s)^n=\sum_{n=0}^{\infty}|\lambda_0-1|^nc_n'[\ln(s^{|\lambda-1|/|\lambda_0-1|})]^n\qquad(4.14)$$

and therefore

$$\sum_{n=0}^{\infty}|\lambda-1|^nc_n'(\ln s)^n<s^{M|\lambda-1|/|\lambda_0-1|},\quad s^{|\lambda-1|/|\lambda_0-1|}>s_L\,.\qquad(4.15)$$

Putting this back into (4.12) yields (choosing units in which $s_L=1$)

$$|\mathrm{Abs}T(\lambda)|<s^{1+\epsilon+M|\lambda-1|/|\lambda_0-1|},\quad s>1\,.\qquad(4.16)$$

Equation (4.18) shows that, for a given δ,

$$|\mathrm{Abs}T(\lambda)|<s^{1+\delta}\,,\qquad(4.17)$$

in some neighborhood of $\lambda=1$, asymptotically. The size of this neighborhood will of course depend on δ. This is precisely the condition we need to continue the bound (4.6) to a neighborhood of $\lambda = 1$. This follows from (2.23), (4.2), (4.3), (4.4), (4.6), and (4.17) by relaxing the condition $\lambda = 1$ in the above steps and replacing it by $\lambda \in N$, some neighborhood of $\lambda = 1$. Thus we obtain

$$\left|\left(\lambda \frac{\partial}{\partial \lambda} - 1\right) \mathrm{Abs}T_2(s, t; \lambda)\right| < f(t)(\ln s) s^{1+\epsilon} \quad (4.18)$$

in some neighborhood of $\lambda = 1$ for any positive ϵ.

Relations (4.6) and (4.18) are the basis for our results. They are bounds on an operator acting on $\mathrm{Abs}T_2(s, t; \lambda)$, given the stated conditions. The remainder of this paper will be devoted to establishing a bound on $\mathrm{Abs}T_2(s, t; \lambda)|_{\lambda=1}$, all by itself. In the next section we consider a very special and simple case.

V. THE MULTIPERIPHERAL CASE

The arguments here will apply to models which have $\mathrm{Abs}T_2(s, t; \lambda)$ generated multiperipherally. To state this precisely we consider the following proposition: Suppose

$$\mathrm{Abs}T_2(s, t; \lambda) = \beta(t, \lambda)(\ln s)^m s^{\alpha(t, \lambda)} + (\text{nonleading terms}), \quad (5.1)$$

where m is arbitrary, but independent of s. Moreover, we assume that

$$\left.\frac{\partial \alpha(t, \lambda)}{\partial \lambda}\right|_{\lambda=1} \neq 0 \quad (5.2)$$

at the particular value of t we are considering, and also that

$$\frac{\partial}{\partial \lambda} \mathrm{Abs}T_2(s, t; \lambda) = \frac{\partial}{\partial \lambda}[\beta(t, \lambda)(\ln s)^m s^{\alpha(t, \lambda)}] + (\text{nonleading terms}). \quad (5.3)$$

Plugging this into Eq. (4.6) and keeping only leading terms in s we find

$$|\beta(t, \lambda)| \left|\frac{\partial \alpha(t, \lambda)}{\partial \lambda}\right| (\ln s)^{m+1} |s^{\alpha(t, \lambda)}| \Big|_{\lambda=1} < f(t)(\ln s) s^{1+\epsilon} . \quad (5.4)$$

This inequality must be satisfied for all $\epsilon > 0$. Thus we get the following condition on α:

$$\mathrm{Re}\, \alpha(t, \lambda = 1) \leq 1 , \quad (5.5)$$

and therefore, in this case,

$$|\mathrm{Abs}T_2(s, t)| < f(t) s^{1+\epsilon} \quad (5.6)$$

asymptotically for all ϵ.

This case is not completely academic, since many popular models (for example ladder models) have $\mathrm{Abs}T_2(s, t)$ being produced multiperipherally and satisfying (5.1)–(5.3). Equation (5.6) states that if the Froissart bound is satisfied off the mass shell by the full amplitude, and the finite-mass hypothesis is correct, then $\mathrm{Abs}T_2(s, t)$ cannot be produced in a simple multiperipheral way and still grow as a power of s greater than unity. Making use of dispersion relations for $T_2(s, t)$ and crossing symmetry, we can extend the bound to the full amplitude

$$|T_2(s, t)| < f(t) s^{1+\epsilon} . \quad (5.7)$$

In the next section we consider the general case, and present arguments why (5.7) should be true in general under our stated conditions.

VI. THE GENERAL CASE

We will use Eqs. (2.13) and (4.18) in this section, and hence we restate them here:

$$\mathrm{Abs}T_2(s, t; \lambda) = \sum_{n=2}^{\infty} \lambda^n (\mathrm{Abs}I)^n , \quad (6.1)$$

$$\left|\left(\lambda \frac{\partial}{\partial \lambda} - 1\right) \mathrm{Abs}T_2(s, t; \lambda)\right| < f(t) s^{1+\epsilon}, \quad \lambda \in N \quad (6.2)$$

where we have dropped the factor $\ln s$ in (6.2) since it is irrelevant. Recall that in order to derive (6.2) we had to use an unproved assumption of polynomial-boundedness. Equation (6.2) is satisfied for all positive ϵ and in some neighborhood of $\lambda = 1$. We note two things about (6.1). First, it has no term linear or constant in λ, and second, it is a polynomial in λ for fixed s. The second result follows from unitarity and the absence of zero-mass particles in our picture.[7]

Let us write $\mathrm{Abs}T_2(s, t; \lambda)$, with λ in the above neighborhood of unity, as a Mellin transform:

$$\mathrm{Abs}T_2(s, t; \lambda) = \frac{1}{2\pi i} \int_c d\beta\, F(\beta, \lambda, t) s^{\beta} , \quad (6.3)$$

$$F(\beta, \lambda, t) = \int_0^{\infty} ds\, \mathrm{Abs}T_2(s, t, \lambda) s^{-\beta-1} , \quad (6.4)$$

where c is parallel to the imaginary β axis and is to the right of all singularities in β of $F(\beta, \lambda, t)$, and $F(\beta, \lambda, t)$ is analytic along c. Now move the contour to the left, picking up possible pole and cut contributions in so doing. We consider for simplicity the case where there are singularities in the β plane at real β only, a leading branch point and poles. The general case of complex branch point and poles will be essentially no different.

$$\mathrm{Abs}T_2(s,t;\lambda)=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}d\beta\,F(\beta,\lambda,t)s^{\beta}+\frac{1}{2\pi i}\int_{b}^{\alpha_c}d\beta s^{\beta}\,\mathrm{Disc}F(\beta,\lambda,t)+\sum_i s^{\beta_i}\mathrm{Res}F(\beta_i,\lambda,t)\,; \quad (6.5)$$

see Fig. 6. We choose $b=1$; if any piece of $\mathrm{Abs}T_2(s,t;\lambda)$ grows as a power of s greater than unity, then this piece must be either a cut or pole contribution above, with no contribution from the background term.

We now define the following function, which is the contribution to $\mathrm{Abs}T_2(s,t;\lambda)$ from all singularities to the right of $1+\epsilon$ in the β plane:

$$K_\epsilon(s,t,\lambda)=\frac{1}{2\pi i}\int_{1+\epsilon}^{\alpha_c}d\beta s^{\beta}\,\mathrm{Disc}F(\beta,\lambda,t)+\sum_{i,\beta_i>1+\epsilon}s^{\beta_i(\lambda,t)}\,\mathrm{Res}F(\beta_i,\lambda,t)\,. \quad (6.6)$$

The inequality (6.2) says simply that this function must be annihilated by the operator $\lambda(\partial/\partial\lambda)-1$,

$$\left(\lambda\frac{\partial}{\partial\lambda}-1\right)K_\epsilon(s,t,\lambda)=0\,. \quad (6.7)$$

This equation must be satisfied in some neighborhood of $\lambda=1$. Therefore (6.7) is simply a differential equation for the function $K_\epsilon(s,t,\lambda)$. Its solution is

$$K_\epsilon(s,t,\lambda)=\lambda K_\epsilon(s,t)\,. \quad (6.8)$$

We can also derive this directly from (4.17) since it gives

$$\left|\mathrm{Abs}T(\lambda_1)-\frac{\lambda_1}{\lambda_2}\mathrm{Abs}T(\lambda_2)\right|<s^{1+\epsilon} \quad (6.9)$$

for arbitrary positive ϵ and λ_1,λ_2 in some neighborhood of unity. We also have

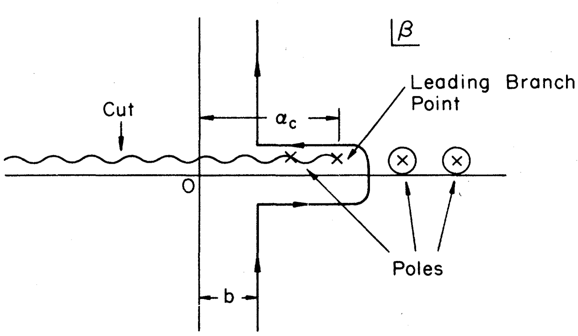

FIG. 6. The contour of integration in the Mellin transform plane.

$$\mathrm{Abs}T(\lambda)=\lambda\,\mathrm{Abs}I+\mathrm{Abs}T_2(\lambda)\,. \quad (6.10)$$

Putting this into (6.9) we find

$$\left|\mathrm{Abs}T_2(\lambda_1)-\frac{\lambda_1}{\lambda_2}\mathrm{Abs}T_2(\lambda_2)\right|<s^{1+\epsilon}\,. \quad (6.11)$$

But this clearly demands

$$K_\epsilon(s,t,\lambda_1)-\frac{\lambda_1}{\lambda_2}K_\epsilon(s,t,\lambda_2)=0\,. \quad (6.12)$$

Setting $\lambda_2=1$, and $\lambda_1=\lambda$, we arrive again at (6.8).

Comparing (6.8) to (6.1), we see that $\mathrm{Abs}T_2(s,t;\lambda)$ has no term linear in λ, and therefore its leading term $K_\epsilon(s,t,\lambda)$ cannot be linear in λ, for (6.8) demands

$$K_\epsilon(s,t,\lambda)=\frac{1}{2\pi i}\int_{c'_\epsilon}d\beta\,s^{\beta}\int_0^\infty ds' s'^{-\beta-1}\left[\sum_{n=2}^{\infty}\lambda^n(\mathrm{Abs}I)^n\right]=\lambda K_\epsilon(s,t) \quad (6.13)$$

to be true, for each positive ϵ, in some finite neighborhood of $\lambda=1$ [with c'_ϵ a contour surrounding all singularities to the right of $1+\epsilon$ yielding (6.6)]. Clearly (6.13) cannot be satisfied unless[8]

$$\lambda K_\epsilon(s,t)=K_\epsilon(s,t,\lambda)=0 \quad (6.14)$$

in this neighborhood of $\lambda=1$, and in particular at $\lambda=1$, and therefore

$$|\mathrm{Abs}T_2(s,t)|<f(t)s^{1+\epsilon} \quad (6.15)$$

for any positive ϵ. Again, it follows from crossing symmetry and dispersion relations that (6.15) is satisfied by the full amplitude which is two-particle reducible in the t channel,

$$|T_2(s,t)|<f(t)s^{1+\epsilon}\,. \quad (6.16)$$

This then is our result. It is a consequence of the t-channel unitarity of T_2 applied to the s-channel asymptotic behavior of T_2 together with our stated assumptions.

VII. CONCLUSION

We have seen that if the Froissart bound is satisfied by the off-mass-shell amplitudes, the finite-mass hypothesis is correct, and our assumption of polynomial boundedness is correct, then the full amplitude which is two-particle reducible in the t channel cannot grow as fast as $s^{1+\epsilon}$, with ϵ an arbitrary positive number. Even if the assumption of polynomial-boundedness is not correct, this result is still true if the two-particle-reducible piece is produced multiperipherally. We feel strongly that it is correct and we therefore believe that the following propositions are inconsistent:

(1) Elastic amplitudes at high energies are strongly damped off the mass shell.

(2) The Froissart bound is satisfied by the full amplitude off the mass shell.

(3) The full-two-particle-reducible piece of the amplitude in the t channel grows faster, by a power, than s^1, on and/or off the mass shell.

There are examples in perturbation theory, in some cases with certain restrictions on the coupling constants, of sets of Feynman graphs which satisfy proposition (3). The simplest case is the sum of all ladder graphs in ϕ^3 field theory. If this proposition is to be false then new graphs, which are two-particle reducible in the t channel, not heretofore studied, must be found to cancel off the leading behavior of these graphs.

If proposition (1) is false it might be interesting to study the consequences of the hypothesis that the Bjorken scaling region, rather than the Regge region considered here, dominates our integrals.

If proposition (2) is false, and this is a distinct possibility, then there may be detectable consequences in photoproduction of hadrons for off-mass-shell photons. For finite-mass photons, we might see these cross sections violate the Froissart bound asymptotically.

We cannot decide which of these propositions is least desirable and we therefore ask the reader to take his pick.

Note added in proof

1. In (4.12) c_n' can be chosen to be the maximum value of $|(\text{Abs}T)^n + (\text{Abs}T)^{n+1}|/s^{1+\epsilon}(\ln s)^n$ on the interval $s_L < s < \infty$.

2. Although we have used the results of ABFST in this paper, we have not assumed that the high-energy behavior of the irreducible kernel is small. We have also been careful not to sum only leading log terms in (4.7) and (4.8), but rather have overestimated each term separately and thus obtained (4.12).

3. We wish to make a comment regarding massive quantum electrodynamics. We have not worked this case out in detail, but we expect the following to be true. If it is assumed that the Lorentz-invariant amplitudes for $e\gamma$, $\bar{e}\gamma$, and $\gamma\gamma$ scattering satisfy the Froissart bound with photons on or off the mass shell for all spin states of the initial and final particles, then the arguments we have used in the present case are applicable. The only difference is that we must work with coupled integral equations. If it is further assumed that the above amplitudes are strongly damped off the mass shell, and that certain series analogous to (4.12) are polynomially bounded in s, then we think that it is impossible for the two-photon reducible amplitudes to grow as a power of S greater than 1 (modulo $\ln s$).

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APPENDIX A

We wish to derive here the relation

$$\text{Abs}T(\lambda) = \text{Abs}T(\lambda=1) + \sum_{n=1}^{\infty} (\lambda-1)^n \{[\text{Abs}T(\lambda=1)^n] + [\text{Abs}T(\lambda=1)]^{n+1}\} \quad \text{(A1)}$$

from the relation

$$\left(\lambda\frac{\partial}{\partial\lambda} - 1\right)\text{Abs}T(\lambda) = \text{Abs}T(\lambda)\times_t \text{Abs}T(\lambda)\,. \quad \text{(A2)}$$

Let us consider a function of one variable which satisfies an identical equation, with $\times_t$ replaced by usual multiplication:

$$\left(x\frac{\partial}{\partial x} - 1\right)f(x) = f^2(x) \quad \text{(A3)}$$

or, letting $y = \ln x$,

$$\frac{\partial}{\partial y} f(y) = f(y) + f^2(y)\,. \quad \text{(A4)}$$

Dividing by $f(y)+f^2(y)$ and integrating we find

$$\int_0^Y dy \frac{df}{dy} \Big/ f(1+f) = Y\,, \quad \text{(A5)}$$

which can be integrated yielding

$$\ln\frac{f(y)[1+f(0)]}{f(0)[1+f(y)]} = y\,, \quad \text{(A6)}$$

and therefore

$$f(x) = \frac{xf(x=1)}{1+f(x=1)(1-x)} = x\, f(x=1)\sum_{n=0}^{\infty}(x-1)^n f^n(x=1) \quad \text{(A7)}$$

for x near unity. We can rewrite (A7) as

$$f(x) = f(x=1) + \sum_{n=1}(x-1)^n[f^n(x=1) + f^{n+1}(x=1)]\,. \quad \text{(A8)}$$

Although the manipulations we have gone through in arriving at (A8) are not correct for (A2), the final result is correct, since the nth derivative of $f(x)$ or $\text{Abs}T(\lambda)$ can be calculated from the

$(n-1)$th derivative using Eq. (A2) or (A3). Using this inductive method, one can avoid ever dividing by the functions, and therefore we can use this method to calculate a Taylor series for either $f(x)$ or $\text{Abs}T(\lambda)$. Since this method is guaranteed to yield the result (A9) for $f(x)$, it is also guaranteed to yield (A1) for $\text{Abs}T(\lambda)$. The solution can easily be checked by substituting Eq. (A1) into Eq. (A2).

APPENDIX B

We wish to show that

$$|[\text{Abs}T(s,t)]^n| < f(t)s^{1+\epsilon}(\ln s)^{n-1} \tag{B1}$$

using the postulated finite-mass condition and the off-shell Froissart bound. We proceed inductively. Suppose that (B1) is satisfied for some n; then using the results of Sec. IV we have

$$|\text{Abs}T\times_t(\text{Abs}T)^n| < \frac{1}{8(2\pi)^4}\int_{4\mu^2}^{s}\frac{ds'ds_0}{s}\int_0^\infty du_1'du_2'|s_0^{1+\epsilon}\rho(t,u_1,u_2,-u_1',-u_2')|\,|s'^{1+\epsilon}(\ln s')^{n-1}f(t,-u_1',-u_2',\mu^2,\mu^2)|$$
$$\times|\Delta(-u_1')|\,|\Delta(-u_2')|\,\frac{\theta(H(s_0s'/s))}{[H(s_0s'/s)]^{1/2}}\,. \tag{B2}$$

With the change of variables $s_0\to p=s_0s'/s$, (B2) becomes

$$|\text{Abs}T\times_t(\text{Abs}T)^n| < \frac{1}{8(2\pi)^4}\int_{4\mu^2}^{s}\frac{ds'}{s'}\int_{4\mu^2(s'/s)}^{s'}dp\,s^{1+\epsilon}(\ln s')^{n-1}J(p)\,, \tag{B3}$$

where $J(p)$ is given by

$$J(p)=\frac{p^{1+\epsilon}}{8(2\pi)^4}\int_0^\infty du_1'du_2'|\rho(t,u_1,u_2,-u_1',-u_2')|$$
$$\times|f(t,-u_1',-u_2',\mu^2,\mu^2)|$$
$$\times|\Delta(-u_1')|\,|\Delta(-u_2')|\,\frac{\theta(H(p))}{[H(p)]^{1/2}}\,, \tag{B4}$$

$$H(p)=-\tfrac{1}{4}(u_1'-u_2')^2+|t|\tfrac{1}{2}(u_1'+u_2')-\tfrac{1}{4}|t|^2-|t|p\,. \tag{B5}$$

Because of the finite-mass condition, we can set the lower limit of the p integration to zero and the upper limit to ∞ without affecting the leading behavior of (B3):

$$|\text{Abs}T\times_t(\text{Abs}T)^n| < \frac{1}{8(2\pi)^4}s^{1+\epsilon}\times\int_{4\mu^2}^{s}\frac{ds'}{s'}\int_0^\infty dp(\ln s')^{n-1}J(p)\,, \tag{B6}$$

$$|\text{Abs}T\times_t(\text{Abs}T)^n| < s^{1+\epsilon}\left\{\left[\ln\left(\frac{s}{4\mu^2}\right)\right]^n\right\}\frac{1}{n}\int_0^\infty dp J(p)\,. \tag{B7}$$

Keeping only the leading terms in s we find

$$|\text{Abs}T\times_t(\text{Abs}T)^n| < s^{1+\epsilon}(\ln s)^n\frac{1}{8(2\pi)^4}\frac{1}{n}\int_0^\infty dp J(p)\,. \tag{B8}$$

We are taking as given that Eq. (B1) is true for $n=1$. Therefore, from Eq. (B8), it follows by induction that it is true for all n.

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[7]Since $\text{Abs}I(s,t)$ must vanish for $s<4\mu^2$, we must have $(\text{Abs}I)^n=0$ if $s<(2\mu n)^2$, and therefore the series (6.1) must terminate for any s.

[8]Simply divide the equation by λ and analytically continue to $\lambda=0$. The equation itself guarantees that no singularity will be encountered.