Quantum Model of Photodetection

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We consider the probability $P^{(n)}(t_0, t)$ of counting n photoelectrons during any time interval $[t_0, t]$ in a photodetection model built in the general framework of quantum field theory. Owing to an equilibrium assumption on the detector state, $P^{(n)}(t_0, t)$ is evaluated from the incident field and some electron-correlation functions. The conditions for $P^{(n)}(t_0, t)$ to be a compound Poisson distribution as expected are discussed. This leads to the definition of coherence properties for the electrons of the detector in connection with the reliability of the detection operation in counting experiments.

I. INTRODUCTION

The completely quantum-mechanical treatments of the photoelectric detection[1-5] lead to counting probabilities which are, as in classical and semiclassical treatments,[6,7] given by the compound Poisson distribution. Some postulates and approximations are necessary to obtain this result. Consider the photocathode as constituted by a set of independent atoms[1] or a set of one-dimensional harmonic oscillators[2] or split the time interval of counting into small subintervals containing not more than one event.[3] Suppose that the emission electrons do not interact with one another and that the single-electron correlation functions are different from zero only for very small space-time spacing.[5]

We present a new approach to this problem in a general quantum-field theory formalism and use some statistical mechanics techniques. The general idea is not to determine the measured quantities for realistic experimental conditions.[8] It is rather to relate the "apparent statistics" given by the detector system to the "true statistics" of an incident-ideal electromagnetic field in order to investigate the possible effects of the quantum nature of the detection mechanism on the counting distribution. To this end, we idealize this mechanism to its elementary quantum characteristics and do not consider secondary emissions, multiphoton absorptions,[9] or source-field interactions.

We introduce an equilibrium property for the detector during the counting and get, for any time interval, the expression of the apparent statistics in a recursive form. This shows which rigorous conditions are needed to obtain a compound Poisson distribution. The theoretical interest of these conditions, given on some electron correlation functions, is to show in the model quantum states for which the detector can be considered as "classical" in the counting operation. The realistic experimental situations are not very different from these ones.

II. THE MODEL OF PHOTODETECTION

We start from the following Hamiltonian

$$H(t) = H_D + H_\gamma + \int_D \mathcal{H}(x)\, d\vec{x},$$

where H_D is the Hamiltonian of the unperturbed detector, H_γ is the free Hamiltonian of the electromagnetic field, and $\mathcal{H}(x) = J^\mu(x) A_\mu(x)$ is the interaction density in its general form. Here, $J(x)$ is the current density in the detector, and the interaction Hamiltonian is obtained by integration over all active electrons in the detector. We ignore the source-field interaction term.[10]

In the "detector" we include not only the cathode (by definition any extended system capable of ejecting electrons when it is illuminated by a suitable radiation field), but also the photoelectrons emitted by this cathode and kept in a "collector." In other words, the one-particle space of one active electron in the detector is of the form

$$K = K^b \oplus K^f . \tag{2.1}$$

K^b is the one-particle space of the active electrons of the cathode. They are bound in some static external potential (for instance, mean potential due to the nuclei of the system). K^f is the one-particle space of photoelectrons. They will be considered as free particles, with respect to the previous case, in our description. K^b and K^f are orthogonal subspaces of K. We shall suppose that the transition probability from a bound state to a free state is different from zero only in the presence of the radiation field. By this assumption, we neglect in particular thermionic emission.

Now, a general quantum state of the detector is some element in the Fock space $\sigma(K)$ built on K. It is sufficient to consider the states of $\sigma(K)$ of the form

$$|\Phi\rangle = |\Phi^b \wedge \Phi^f\rangle\,,$$

where

$$\Phi^b \in \sigma(K^b) \subset \sigma(K)\,, \qquad \Phi^f \in \sigma(K^f) \subset \sigma(K)\,, \tag{2.2}$$

and $\wedge$ represents the antisymmetrization operation in $\sigma(K)$. Already notice that

$$\langle \Phi_1^b \wedge \Phi_1^f | \Phi_2^b \wedge \Phi_2^f\rangle = \langle \Phi_1^b | \Phi_2^b\rangle \langle \Phi_1^f | \Phi_2^f\rangle \tag{2.3}$$

from the orthogonality property of K^b and K^f.

The time development operator in the interaction picture is given by

$$V(t, t_0) = T \exp\left\{ -i \int_{t_0}^{t} \int_D \mathcal{H}(x)\, dx \right\}, \tag{2.4}$$

where T is Dyson's time-ordering operator. If the total system is, at time t_0, in a state $|i_D \otimes i_\gamma\rangle$, where $|i_D\rangle$ is the state of the detector and $|i_\gamma\rangle$ is the state of the photon field, the probability that the system can be found at time $t > t_0$ in the final state $|f_D \otimes f_\gamma\rangle$ is given by

$$P_{i\to f}(t_0, t) = |\langle f_D \otimes f_\gamma | V(t, t_0) | i_D \otimes i_\gamma\rangle|^2\,. \tag{2.5}$$

We are interested in the probability that n photoelectrons will be registered in the time interval $[t_0, t]$. We assume that every time one photon falls on the cathode and is absorbed, one photoelectron is created and registered. This efficiency hypothesis is not essential for our problem. If we state that, on the average, only a certain part of the total number of absorbed photons gives a photoelectric effect, the statistical laws which are proved in the following are not modified in their general form. Hence, we only deal with the elementary process shown in Fig. 1, and, in order to describe our counting experiment, we consider the relation (2.5) when the final state is experimentally obtained from the initial state $|i_D \otimes i_\gamma\rangle$ by n repetitions of the event (see Fig. 1) during the time interval $[t_0, t]$.

The transition probability for some particular final state of the previous type can be written as

$$P^{(n)}_{i\to f}(t_0, t) = |\langle f_D \otimes f_\gamma | \hat{V}^{(n)}(t, t_0) | i_D \otimes i_\gamma\rangle|^2\,, \tag{2.6}$$

with

$$\hat{V}^{(n)}(t_0, t) = \frac{(-i)^n}{n!} \int_D \cdots \int_D \int_{t_0}^{t} \cdots \int_{t_0}^{t} N[\hat{\mathcal{H}}(x_1) \cdots \hat{\mathcal{H}}(x_n)] \times dx_1 \cdots dx_n\,,$$

where N is the normal-product operator and $\hat{\mathcal{H}}(x)$ is the "effective" Hamiltonian of this process.

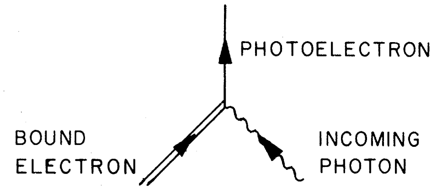

FIG. 1. The elementary process.

The decomposition (2.1) permits us to express the field operators on $\sigma(K)$ in the form

$$\psi^{\pm}(x) = \psi_b^{\pm}(x) + \psi_f^{\pm}(x)\,,$$

$$\overline{\psi}^{\pm}(x) = \overline{\psi}_b^{\pm}(x) + \overline{\psi}_f^{\pm}(x)\,,$$

with the usual anticommutation relations in $\sigma(\mathcal{H}^b)$ $[\sigma(\mathcal{H}^f)]$ for $\psi_b^{\pm}$ and $\overline{\psi}_b^{\pm}$ $[\psi_f^{\pm}$ and $\overline{\psi}_f^{\pm}]$, and every anticommutator between an operator indexed by b and an operator indexed by f equal to zero.

Under these conditions, the operator $\hat{\mathcal{H}}(x)$ is given by

$$\hat{\mathcal{H}}(x) = -e\,\overline{\psi}_f^{-}(x)\gamma^\mu \psi_b^{+}(x) A_\mu^{+}(x)\,. \tag{2.7}$$

Now, in a counting experiment we are not interested in the final state of our system. We only measure the counting rate. Thus we have to sum the relation (2.6) over a complete set of final states. We obtain

$$P_i^{(n)}(t_0, t) = \sum_{f_i} P^{(n)}_{i\to f_i}(t_0, t) = \langle i_D \otimes i_\gamma | \hat{V}^{(n)}(t, t_0)^* \hat{V}^{(n)}(t, t_0) | i_D \otimes i_\gamma\rangle\,.$$

Further, the initial state of the total system generally is not a pure state, but a mixture described by a density matrix $\rho(t_0) = \rho_D(t_0)\rho_\gamma(t_0)$, and the expression for the counting probability becomes

$$P^{(n)}(t_0, t) = \mathrm{Tr}\{\rho_D(t_0)\rho_\gamma(t_0) \times \hat{V}^{(n)}(t, t_0)^* \hat{V}^{(n)}(t, t_0)\}\,. \tag{2.8}$$

III. THE INITIAL STATE OF THE SYSTEM

We shall use for $\rho_\gamma(t_0)$ the diagonal representation[11] in the form

$$\rho_\gamma(t_0) = \int P_{t_0}\{\varphi\}\, |\varphi_{\mathrm{coh}}\rangle\langle\varphi_{\mathrm{coh}}|\, dm\{\varphi\}\,, \tag{3.1}$$

where $\varphi = \{\varphi_\mu(x)\}$ is a quadridimensional function, element of the one-photon space and

$$dm\{\varphi\} = \prod_k d(\mathrm{Re}\,\alpha_k)\, d(\mathrm{Im}\,\alpha_k)\,,$$

where the α_k's are the components of the function φ in some basis of the one-photon space. $|\varphi_{\mathrm{coh}}\rangle$ is the coherent state generated by φ, such that[11]

$$A_\mu^{+}(x)|\varphi_{\mathrm{coh}}\rangle = \varphi_\mu(x)|\varphi_{\mathrm{coh}}\rangle\,. \tag{3.2}$$

Finally, $P_{t_0}\{\varphi\}$, when it is considered as a function of the α_k's, is an element of the $\mathcal{S}$ Schwartz space[12].

With the form (3.1) of $\rho_\gamma(t_0)$ and using (3.2), we find the counting probability (2.8) to be

$$P^{(n)}(t_0,t)=\int P_{t_0}\{\varphi\}\,\mathrm{Tr}\{\rho_D(t_0)\,\hat{V}_c^{(n)}(t,t_0)^*\times\hat{V}_c^{(n)}(t,t_0)\}\,dm\{\varphi\}, \tag{3.3}$$

where $\hat{V}_c^{(n)}$ is obtained from $\hat{V}^{(n)}$ by the substitution $A_\mu^+\to\varphi_\mu$ in (2.7).

We are now interested in the quantity

$$P_\varphi^{(n)}(t_0,t)=\mathrm{Tr}\{\rho_D(t_0)\,\hat{V}_c^{(n)}(t,t_0)^*\times\hat{V}_c^{(n)}(t,t_0)\}, \tag{3.4}$$

which corresponds to the counting probability in the case where the radiation field is, at time t_0, in the coherent state $|\varphi_{\mathrm{coh}}\rangle$.

The "quasiclassical" properties of coherent states allow us to work out an analogy of our problem with an external field problem. Indeed, we can consider that (3.4) is obtained from a situation where the detector interacts in an external classical field, with the effective interaction Hamiltonian:

$$\hat{\mathcal{H}}_\varphi(x)=-e\bar{\psi}_f^-(x)\gamma^\mu\psi_b^+(x)\,\varphi_\mu(x)\,.$$

It is natural and usual to assume that at time 0, when the interaction with the radiation field is turned on, the cathode is in thermodynamical equilibrium (at a sufficiently low temperature that thermionic emission is negligible), and that there is no photoelectron in the collector.

The density matrix on $\sigma(K^b)$ corresponding to the thermodynamical equilibrium of the cathode is of the Gibbs form

$$\rho_D^b(0)=e^{-\beta_0(H_D^b-\mu N_D^b)}\times\mathrm{Tr}\{e^{-\beta_0(H_D^b-\mu N_D^b)}\}^{-1},$$

where H_D^b is the Hamiltonian of the unperturbed cathode and N_D^b is the number operator of bound electrons.

Introducing the polar decomposition of $\rho_D^b(0)$:

$$\rho_D^b(0)=\sum_i p_i\,|\,\Phi_i^b\rangle\langle\Phi_i^b|\,,$$

where $\{\Phi_i^b\}_i$ is some basis in $\sigma(K^b)$, the density matrix at time 0 of the detector can be written as

$$\rho_D(0)=\sum_i p_i\,|\Phi_i^b\wedge\Omega^f\rangle\,\langle\Phi_i^b\wedge\Omega^f|\,, \tag{3.5}$$

where Ω^f is the vacuum state of $\sigma(K^f)$.

We introduce the following notation: If $\rho^b=\sum_i b_i\,|\,\Phi_i^b\rangle\langle\Phi_i^b|$ is a density matrix on $\sigma(K^b)$ and $\rho^f=\sum f_i\,|\Phi_i^f\rangle\langle\Phi_i^f|$ is a density matrix on $\sigma(K^f)$, then

$$\rho^b\wedge\rho^f=\sum_{ij}b_i f_j\,|\,\Phi_i^b\wedge\Phi_j^f\rangle\langle\Phi_i^b\wedge\Phi_j^f\,| \tag{3.6}$$

is a density matrix on $\sigma(K)$. Hence

$$\rho_D(0)=\rho_D^b(0)\wedge P_{\Omega^f}\,, \tag{3.7}$$

P_{Ω^f} being the projector on Ω^f.

If in relation (3.4) the time t_0 is $t_0=0$, the counting probability $P_\varphi^{(n)}(0,t)$ must be calculated with the form (3.7) of the density matrix $\rho_D(0)$.

However, in the present work we are only interested in the possible effects of the quantum nature of the detector upon the counting probabilities. Our aim is to relate the "true statistics" of the incident radiation field to the "apparent statistics" given by the detector, when this detector is in its stable working state. We do not want to consider the transient effects which can occur when the interaction is established or which may result from some instabilities of the system[13]. For these reasons, it is convenient to calculate $P_\varphi^{(n)}(t_0,t)$ not at $t_0=0$ but in an interval $[t_0,t]$ during the stable working of the detector.

At this point, we clearly need some quantum characterization of this "working state" of the detector. We shall give it for the case where the radiation field is coherent and stationary. In this case, we insist the counting probabilities (3.4) be invariant under translation of the time origin, in order to guarantee the reproducibility in time of the experiments. This is equivalent to requiring time invariance for the density matrix ρ_D, which consequently can be considered as describing some equilibrium state of the total system (detector in interaction with the external electromagnetic field, as previously stated, and, furthermore with the complete apparatus). We shall be more precise on this point later; for our present discussion, only the general idea is relevant: ρ_D is the density matrix of some equilibrium state of a fermion system in a certain external classical field.

Now, it is a known result in statistical mechanics that the equilibrium states of a fermion system in an external field are "quasifree" states[14]: Their $2n$-correlation function can be expressed in the form of a sum of products of 2-correlation functions, such that

$$\mathrm{Tr}\{\rho_D A_1\cdots A_{2n}\}=\sum\chi\,\mathrm{Tr}\{\rho_D A_{i_1}A_{j_1}\}\cdots\times\mathrm{Tr}\{\rho_D A_{i_n}A_{j_n}\}\,, \tag{3.8}$$

the sum being extended to all two-by-two arrangement of 1, 2, ..., $2n$ such that $i_k<j_k$ and $i_1<i_2$

$<\ldots< i_n$. The A_i's are field operators and χ is the parity of the permutation $(1, 2, \ldots, 2n) \to (i_1, j_1, \cdots, i_n, j_n)$. Furthermore

$$\operatorname{Tr}\{\rho_D A_1 \cdots A_{2n+1}\} = 0 \quad \text{for every } n, \tag{3.9}$$

$$\operatorname{Tr}\{\rho_D A_k A_l\} = 0 \quad \text{if } A_k \text{ and } A_l \tag{3.10}$$

are two creators or two annihilators.

We shall use another equivalent form of (3.5) in the following:

$$\operatorname{Tr}\left\{\rho_D \prod_{j=1}^{2n} A_j\right\} = \sum_{i=1}^{2n-1} (-1)^{i+1} \operatorname{Tr}\{\rho_D A_i A_{2n}\} \times \operatorname{Tr}\left\{\rho_D \prod_{\substack{j=1\\ j\neq i}}^{2n-1} A_j\right\}. \tag{3.11}$$

Furthermore we assume that ρ_D, the density matrix on $\sigma(K)$, can be written, using the notation (3.6)

$$\rho_D = \rho_D^b \wedge \rho_D^f, \tag{3.12}$$

where ρ_D^b (ρ_D^f) is a density matrix on $\sigma(K^b)$ $[\sigma(K^f)]$.

The practical significance of this assumption is the following: The relation (3.12) is [using (2.3)] equivalent to

$$\operatorname{Tr}_{\sigma(K)}\{\rho_D O_b O_f\} = \operatorname{Tr}_{\sigma(K^b)}\{\rho_D^b O_b\} \times \operatorname{Tr}_{\sigma(K^f)}\{\rho_D^f O_f\}, \tag{3.13}$$

where O_b $[O_f]$ is some element of the algebra of field operators on $\sigma(K^b)$ $[\sigma(K^f)]$.[15]

If ρ_D is quasifree, it is evident from (3.13) that ρ_D^b and ρ_D^f are also quasifree and satisfy the relation (3.11) for suitable field operators. Consequently ρ_D^b and ρ_d^f can be considered as equilibrium states of the subsystems cathode (bound electrons) and collector (free electrons).

In conclusion, in our model the initial state of the detection system is an equilibrium state of the detector characterized by the fact that both the subsystems cathode and collector are themselves in an equilibrium state. That is equivalent in physical terms to saying that, when the detector is in its working state, every photoemission process is so closely connected with the return to the electrical neutrality for the cathode and the collector by interaction with the apparatus, that the states are not notably modified. Hence, the complete apparatus has a fundamental role in the previous equilibrium hypothesis. However, the corresponding interactions are of classical type, and do not prevent us from using the theorem leading to the quasifree properties of these different equilibrium states.

Let us remark that in relation (3.7) $\rho_D^b(0)$ and P_{Ω^f} also satisfy the factorization relation (3.8). Indeed, $\rho_D^b(0)$ is quasifree because it describes the thermodynamical equilibrium of a system with only external interactions[14] and P_{Ω^f} satisfies (3.8) by direct application of the Wick theorem. So the developments which follow are also valuable in their general form in the time interval $[0, t]$, without a supplementary hypothesis.

IV. THE COUNTING PROBABILITY

The preceding assumptions allow us to reexpress the counting probability (3.4). Expressing $P_\varphi^{(n)}(t_0, t) = P_\varphi^{(n)}(t - t_0) = P_\varphi^{(n)}(\tau)$,

$$P_\varphi^{(n)}(\tau) = \left(\frac{e^n}{n!}\right)^2 \int_D \cdots \int_D \int_\tau \cdots \int_\tau \prod_{i=1}^{2n} dx_i \prod_{j=1}^{n} \gamma_{\alpha_j \beta_j}^{\mu_j *} \varphi_{\mu_j}^*(x_j) \prod_{k=n+1}^{2n} \gamma_{\alpha_k \beta_k}^{\mu_k} \varphi_{\mu_k}(x_k) \times \operatorname{Tr}\left\{\rho_D \prod_{i=1}^{n} \overline{\psi}_b^{-,\alpha_i}(x_i) \prod_{i=1}^{n} \psi_f^{+,\beta_i}(x_i) \prod_{j=n+1}^{2n} \overline{\psi}_f^{-,\alpha_j}(x_j) \prod_{j=n+1}^{2n} \psi_b^{+,\beta_j}(x_j)\right\}, \tag{4.1}$$

we have, using (3.13), (3.10), and (3.11)

$$\begin{aligned}
&\operatorname{Tr}\left\{\rho_D \prod_{i=1}^{n} \overline{\psi}_b^{-}(x_i) \prod_{i=1}^{n} \psi_f^{+}(x_i) \prod_{j=n+1}^{2n} \overline{\psi}_f^{-}(x_j) \prod_{j=n+1}^{2n} \psi_b^{+}(x_j)\right\} \\
&= \operatorname{Tr}\left\{\rho_D^b \prod_{i=1}^{n} \overline{\psi}_b^{-}(x_i) \prod_{j=n+1}^{2n} \psi_b^{+}(x_j)\right\} \operatorname{Tr}\left\{\rho_D^f \prod_{i=1}^{n} \psi_f^{+}(x_i) \prod_{j=n+1}^{2n} \overline{\psi}_f^{-}(x_j)\right\} \\
&= \sum_{k=1}^{n} (-1)^{k+1} \operatorname{Tr}\left\{\rho_D^b \overline{\psi}_b^{-}(x_k) \psi_b^{+}(x_{2n})\right\} \operatorname{Tr}\left\{\rho_D^b \prod_{\substack{i=1\\ i\neq k}}^{n} \overline{\psi}_b^{-}(x_i) \prod_{i=n+1}^{2n-1} \psi_b^{+}(x_i)\right\} \\
&\times \sum_{l=1}^{n} (-1)^{l+1} \operatorname{Tr}\left\{\rho_D^f \psi_f^{+}(x_l) \overline{\psi}_f^{-}(x_{2n})\right\} \operatorname{Tr}\left\{\rho_D^f \prod_{\substack{j=1\\ j\neq l}}^{n} \psi_f^{+}(x_j) \prod_{j=n+1}^{2n-1} \overline{\psi}_f^{-}(x_j)\right\}.
\end{aligned} \tag{4.2}$$

We separate this sum into two parts, isolating the terms where $x_k = x_l$:

$$\mathrm{Tr}\left\{\cdots\right\}=\sum_{k=1}^{n}\mathrm{Tr}\left\{\rho_D^b\bar{\psi}_b^-(x_k)\psi_b^+(x_{2n})\right\}\mathrm{Tr}\left\{\rho_D^f\psi_f^+(x_k)\bar{\psi}_f^-(x_{2n})\right\}$$

$$\times\mathrm{Tr}\left\{\rho_D^b\prod_{\substack{i=1\\ j\neq k}}^{n}\bar{\psi}_b^-(x_i)\prod_{i=n+1}^{2n-1}\psi_b^+(x_i)\right\}\mathrm{Tr}\left\{\rho_D^f\prod_{\substack{j=1\\ j\neq k}}^{n}\psi_f^+(x_j)\prod_{j=n+1}^{2n-1}\bar{\psi}_f^-(x_j)\right\}+C(x_1,\ldots,x_n;x_{n+1},\ldots,x_{2n})$$

$$=\sum_{k=1}^{n}\mathrm{Tr}\left\{\rho_D\bar{\psi}_b^-(x_k)\psi_f^+(x_k)\bar{\psi}_f^-(x_{2n})\psi_b^+(x_{2n})\right\}\mathrm{Tr}\left\{\rho_D\prod_{\substack{i=1\\ i\neq k}}^{n}\bar{\psi}_b^-(x_i)\prod_{\substack{i=1\\ i\neq k}}^{n}\psi_f^+(x_i)\prod_{j=n+1}^{2n-1}\bar{\psi}_f^-(x_j)\prod_{j=n+1}^{2n-1}\psi_b^+(x_j)\right\}+C(\{x_i\})$$

$$=C_0(\{x_i\})+C(\{x_i\}). \tag{4.3}$$

Each term of the sum C_0 gives under the integrations the same contribution in (4.1) and $P_\varphi^{(n)}(t_0,t)$ becomes

$$P_\varphi^{(n)}(\tau)=n\left(\frac{e^n}{n!}\right)^2\int_D\int_D\int_\tau\int_\tau dy_1\,dy_2\,\gamma_{\eta_1\zeta_1}^{\lambda_1 *}\varphi_{\lambda_1}^*(y_1)\gamma_{\eta_2\zeta_2}^{\lambda_2}\varphi_{\lambda_2}(y_2)\,\mathrm{Tr}\left\{\rho_D\bar{\psi}_b^{-,\eta_1}(y_1)\psi_f^{+,\zeta_1}(y_1)\bar{\psi}_f^{-,\eta_2}(y_2)\psi_b^{+,\zeta_2}(y_2)\right\}$$

$$\times\int_D\cdots\int_\tau\prod_{i=1}^{2n-2}dx_i\prod_{j=1}^{n-1}\gamma_{\alpha_j\beta_j}^{\mu_j *}\varphi_{\mu_j}^*(x_j)\prod_{k=n}^{2n-2}\gamma_{\alpha_k\beta_k}^{\mu_k}\varphi_{\mu_k}(x_k)$$

$$\times\mathrm{Tr}\left\{\rho_D\prod_{i=1}^{n-1}\bar{\psi}_b^{-,\alpha_i}(x_i)\prod_{i=1}^{n-1}\psi_f^{+,\beta_i}(x_i)\prod_{j=n}^{2n-2}\bar{\psi}_f^{-,\alpha_j}(x_j)\prod_{j=n}^{2n-2}\psi_b^{+,\beta_j}(x_j)\right\}$$

$$+\left(\frac{e^n}{n!}\right)^2\int_D\cdots\int_\tau\prod_{i=1}^{2n}dx_i\prod_{j=1}^{n}\gamma^{\mu_j *}\varphi_{\mu_j}^*(x_j)\prod_{k=n+1}^{2n}\gamma^{\mu_k}\varphi_{\mu_k}(x_k)\,C(x_1,\ldots,x_n;x_{n+1},\ldots,x_{2n})$$

$$=\frac{1}{n}\int_D\int_D\int_\tau\int_\tau dy_1dy_2\,\mathrm{Tr}\left\{\rho_D\hat{\mathcal{H}}_\varphi^*(y_1)\hat{\mathcal{H}}_\varphi(y_2)\right\}P_\varphi^{(n-1)}(\tau)+C_\varphi^{(n)}(\tau).$$

To calculate $C_\varphi^{(n)}$, we decompose the n^2-n remaining terms of (4.2) following (3.11). We keep again the terms leading to separate integrations in (4.1) and continue the decomposition for the other terms and so on.

We obtain for $P_\varphi^{(n)}$ a recursive expression of the form

$$P_\varphi^{(n)}=\frac{1}{n}D_1P_\varphi^{(n-1)}-\frac{1}{n(n-1)}D_2P_\varphi^{(n-2)}+\sum_{p=3}^{n}(-1)^{p+1}\Theta(\{D_p\})P_\varphi^{(n-p)}, \tag{4.4}$$

where

$$D_p=e^{2p}\int_D\cdots\int_\tau\prod_{i=1}^{2p}dx_i\prod_{j=1}^{p}\gamma^{\mu_j *}\varphi_{\mu_j}^*(x_j)\prod_{j=p+1}^{2p}\gamma^{\mu_j}\varphi_{\mu_j}(x_j)\prod_{i=1}^{p}\left[\mathrm{Tr}\left\{\rho_D^b\bar{\psi}_b^-(x_i)\psi_b^+(x_{i+p})\right\}\mathrm{Tr}\left\{\rho_D^f\psi_f^+(x_{\sigma i})\bar{\psi}_f^-(x_{i+p})\right\}\right], \tag{4.5}$$

with

$$\sigma_i=\begin{cases}i+1 & \text{for } 1\leq i\leq p-1,\\ 1 & \text{for } i=p\end{cases}$$

and $\Theta(\{D_p\})$ is a polynomial of the general form

$$\Theta(\{D_p\})=\sum\lambda_{i_1\cdots i_k}^{j_1\cdots j_k}D_{i_1}^{j_1}\cdots D_{i_k}^{j_k},$$

the sum being taken on all possibilities such that $\sum_{l=1}^{k}i_l\,j_l=p$ and $\lambda_1^p=0$ (the term D_1^p never exists).

The counting probability $P_\varphi^{(n)}(t_0,t)$ was defined in (3.4) in the case where the radiation field is in the coherent state $|\varphi_{\mathrm{coh}}\rangle$.

Hence, the true statistics, i.e., the real statistics of the incident light is a Poisson statistic, as is well known for a pure coherent state. It is clear in (4.4) that in our model the apparent statistics given by the detection operation is not in general a Poisson statistic. The incident statistics are modified by the multiple-correlation effects in the cathode and in the collector, connected by the electromagnetic field as shown in the expression (4.5).

We obtain a Poisson distribution if we neglect in the sum (4.4) all the terms with $p>1$:

$$P_\varphi^{(n)}=\frac{1}{n}D_1P_\varphi^{(n-1)}. \tag{4.6}$$

After normalization, we have in this case

$$P_\varphi^{(n)}(\tau) = \frac{D_1^2}{n!} e^{-D_1}, \quad (4.7)$$

where

$$D_1 = \int_D \int_D \int_\tau \int_\tau dx_1 dx_2 \, \mathrm{Tr}\left\{\rho_D \hat{\mathcal{K}}_\varphi^*(x_1) \hat{\mathcal{K}}_\varphi(x_2)\right\}$$

$$= \mathrm{Tr}\left\{\rho_D D_\varphi^*(\tau) D_\varphi(\tau)\right\}, \quad (4.8)$$

with

$$D_\varphi(\tau) = \int_D \int_\tau \hat{\mathcal{K}}_\varphi(x)\, dx$$

$$= -e \int_D \int_\tau \overline{\psi}_f^-(x) \gamma^\mu \psi_b^+(x) \varphi_\mu(x)\, dx,$$

and in the general situation of a radiation field described by the density matrix (3.1) we obtain from (3.3)

$$P^{(n)}(t_0, t_0+\tau) = \int P_{t_0}\{\varphi\} \frac{[\mathrm{Tr}\{\rho_D D_\varphi^*(\tau) D_\varphi(\tau)\}]^n}{n!} \times e^{-\mathrm{Tr}\{\rho_D D_\varphi^*(\tau) D_\varphi(\tau)\}} dm\{\varphi\}, \quad (4.9)$$

that is, as expected, a compound Poisson distribution.

It is interesting to note that the counting probability directly involves the "effective" electron current $J_\mu^{\mathrm{eff}}(x) = -e\overline{\psi}_f^-(x)\gamma_\mu \psi_b^+(x)$ and not the photoelectric current $J_\mu^f(x) = -e\overline{\psi}_f^-(x)\gamma_\mu\psi_f^+(x)$ as in Ref. 5. However, if desired, J_μ^f can be separated from the cathode operators approximating (4.8).

One can test the accuracy of the previous approximation by introducing a less drastic one which leads again to a Poisson distribution. We also keep the term $p=2$ in the sum (4.4) and express $P_\varphi^{(n-2)}$ from (4.5) obtaining

$$P_\varphi^{(n)} = \frac{1}{n} D_1 \left(1 - \frac{D_2}{D_1^2}\right) P_\varphi^{(n-1)}. \quad (4.10)$$

Using standard inequalities, one shows directly that $|D_2| \leq D_1^2$. Higher-order approximations than (4.10) involve quantities of the form D_P/D_1^P for which we also have algebraically $|D_P| \leq D_1^P$.

Note that in the expression (4.5) of D_P the density matrix $\rho_D = \rho_D^b \wedge \rho_D^P$ depends in the general case on φ according to the previous equilibrium assumption for the total system. We can more easily than using the precise results of Ref. 14, evaluate the quantities D_p in the case $t_0 = 0$, where simply

$$\rho_D^b = e^{-\beta(H_D^b - \mu N_D^b)} \text{ and } \rho_D^f = |\Omega_f\rangle\langle\Omega_f|.$$

Then, realistic experimental conditions, which involve a very large number of active electrons as developed and used in Ref. 5 actually lead to $|D_2| \ll D_1^2$, and more generally to $|D_P|/D_1^P \ll |D_{p-1}|/|D_1^{P-1}$. This shows that the approximation (4.6) is justified in these conditions.

V. COHERENCE PROPERTIES OF THE ELECTRON SYSTEM

In the preceding part we obtained a Poisson distribution, taking only the first term of the sum (4.4), with practical experimental justifications. Let us come back now to the general exact expressions (4.1) and (4.4).

The detection function, true statistics → apparent statistics, depends essentially on the density matrix ρ_D describing the detector. The previous considerations lead to the characterization of this function by the sequence of ratios $\eta = \{\eta_p = |D_P|/D_1^p\}$. We have $\eta_1 = 1$ and $0 \leq \eta_p \leq 1$, when ρ_D is varied in the set of detector density matrices. The two following limiting cases appear:

(1) $\eta = \{1, 0, 0, \ldots, 0, \ldots\}$ The detection function is unity as directly seen on (4.4);

(2) $\eta = \{1, 1, 1, \ldots, 1, \ldots\}$ It is impossible to neglect any term in the sum (4.4). The detection function is the worst one possible.

As the correlations between the electrons of the system determine the value of the detection function, it seems interesting to introduce a coherence notion for those electrons connected with the detection function.

By pure analogy with the boson case, where the coherent states are associated with quasiclassical properties of the field, we shall say that ρ_D is "coherent" in the case (1) because the detector is then like a classical system in our counting experiments. We shall refer to as "incoherent" the density matrices ρ_D leading to the case (2). We briefly consider the two cases:

ρ_D coherent: $\eta = \{1, 0, 0, \ldots, 0, \ldots\}$.

If we take (4.3) and continue the decomposition with all the functions $C(\{x_i\})$ equal to zero, we finally obtain as a characteristic property of coherent ρ_D

$$\mathrm{Tr}\left\{\rho_D J_{\mu_1}^{\mathrm{eff}\,*}(x_1)\cdots J_{\mu_n}^{\mathrm{eff}\,*}(x_n) J_{\mu_{n+1}}^{\mathrm{eff}}(x_{n+1})\cdots J_{\mu_{2n}}^{\mathrm{eff}}(x_{2n})\right\} = \sum_{\sigma\in G_n} \prod_{i=1}^n \mathrm{Tr}\left\{\rho_D J_{\mu_i}^{\mathrm{eff}\,*}(x_i) J_{\mu_{\sigma i+n}}^{\mathrm{eff}}(x_{\sigma i+n})\right\},$$

i.e., a Gaussian-type factorization for the correlation functions of the effective current. It is interesting to note that the coherence, in the previous sense, of detector fermions corresponds to something like incoherence of currents in the usual boson terminology.

ρ_D incoherent: $\eta=\{1, 1, \ldots, 1, \ldots\}$.

This case is very singular and evidently without connection to practical experimental conditions. For example, an evident set of solutions is constituted by the density matrices $\rho_D=\rho_D^b \wedge \rho_D^f$ such that

$$\mathrm{Tr}\{\rho_D^b \bar{\psi}_b^{-\alpha}(x)\, \psi_b^{+\beta}(y)\} = \lambda_b^{*\alpha}(x)\, \lambda_b^{\beta}(y).$$

The quasifree states of fermions with a similar factorization property have $2n$-correlation functions equal to zero for every $n \geqslant 2$ and are essentially one-electron states. So $P_\varphi^{(n)}=0$ except for $n=0$ and 1. Let us remark only that the incoherence of detector fermions is here related to factorization properties which similarly occur for coherent bosons.

VI. CONCLUSION

The interest of this photodetection model is to furnish under idealized but precisely defined hypotheses, a general and complete formulation for the apparent statistics given by the detector and a formalism to test the deviation from the incident statistics. It would be attractive to numerically follow the modification of the Poisson law when the coherence properties of the detector are decreasing from $\eta=\{1, 0, \ldots, 0, \ldots\}$.

The density matrix ρ_D describing an equilibrium quasifree state of the analogous system detector-external electromagnetic field is a well-defined function of temperature and of field intensity.[14] Thus, it is theoretically possible to study the change of coherence properties of the detector as a function of these quantities.

Conversely, the experimental measure of a deviation of the Poisson law for coherent radiation field and the evolution of this deviation by modification of physical parameters, would be theoretically related in this model to coherence properties of the detector, i.e., to correlation effects in the electron system.

Actually these previous considerations are almost purely theoretical; it can be calculated on the relation (4.4) that, for example, if $\eta_p=10^{-p}$ for every $p>1$ the Poisson law is not significantly modified. In realistic counting experiments, where η_p is certainly smaller than 10^{-p}, the perturbations on counting distribution due to the electron correlations in the detector are really unmeasurable. Nevertheless, the theory is sufficiently general to be applied to other systems and more generally to every system where a boson is counted via some quantum transition of a fermion. The detection function of such a system would be notably different from one if, for instance, the number of active fermions is not large enough.

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Asymptotic Scale Invariance in a Massive Thirring Model with U(n) Symmetry

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Renormalized Ward identities and Callan-Symanzik equations are developed for vertex functions involving composite fields in a massive Thirring model with U(n) symmetry. The existence of a critical curve of fixed points of a renormalization group acting on a space of coupling constants is proved. On this curve all nonsoft axial-vector and scaling anomalies vanish. Attraction properties of this curve are investigated. The possibility of a second critical curve for strong coupling is discussed.

I. INTRODUCTION

The massive Thirring model with U(1) symmetry has been studied[1,2] in renormalized perturbation theory with the aid of Callan-Symanzik equations[3,4,5] (CSE) and Ward identities. If one introduces a (one-dimensional) coupling-constant space on which a renormalization group[6] acts (its differential equations are the characteristic equations of the CSE), then the main result of these studies is that every point of the coupling-constant space is a fixed point of the group. Consequently one has asymptotic scale invariance with asymptotic scale dimensions depending on the coupling constant. Aside from the intrinsic theoretical interest of the model the possible relevance of this result to statistical problems[7] has been noted.

In this paper we will study a generalization of the U(1) renormalizable massive Thirring model to the case of U(n) symmetry. The model is far more complicated, its structure considerably richer, and the results quite different from the U(1) case. It is studied with the aid of Callan-Symanzik equations[3,4,5,8] and Ward identities for Green's functions involving elementary and composite fields developed via normal-product techniques.[9,10] Powerful identities are derived connecting scaling and axial-vector anomalies. Our main result is the existence of a critical curve of fixed points of the renormalization group acting on a three-dimensional space (E^3) of coupling constants. The curve passes through the origin. The existence is established to all orders of renormalized perturbation theory. The axial-vector anomalies in this model are stronger than in the U(1) case[2]; but on the curve these anomalies, as well as scaling anomalies, disappear. As a consequence on the curve one obtains asymptotic scale invariance and axial-charge conservation. The curve is asymptotically unstable (in the Liapounoff sense); nevertheless there appear to exist ultraviolet and infrared regions of attraction. Attraction is proved up to third order in perturbation theory. These questions have been investigated with the aid of qualitative techniques which are generalizations[11] of the Liapounoff stability theory for attractive invariant sets. These results may be relevant to statistical problems.[12]

Our work was motivated by the recent interesting paper of Dashen and Frishman[13] and by the importance of the study of the renormalization group in model field theories as stressed by Wilson.[14] The former authors studied a generalization of the massless U(1) model to the case of U(n) symmetry with two coupling constants. Their method was essentially algebraic and they concluded that there existed two critical lines (in a *two*-dimensional space of coupling constants) on which scale invariance was obtained. The connection between the treatment of Dashen and Frishman and ours is at present not completely clear. In perturbation theory for a consistent renormalization one has to work in a space of *three* independent coupling constants (there exists a maximal set of three independent renormalizable interactions). These coupling constants are defined by specifying the four-point vertex functions at zero momenta, and they are related by computable finite renormalizations to certain perturbation parameters (appearing in the interaction Lagrangian). In the perturbation parameter space the critical curve of the previous paragraph is a straight line through the origin. It