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Definition of Position and Spin Operators for Massive Elementary Relativistic Systems of Arbitrary Spin

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Since the important work of Newton and Wigner on the position operator and the localized states, several methods have been developed in the literature to deal with the problem of finding a position operator for relativistic systems. One of the most relevant has been the method based on the use of "canonical" transformations such as the Foldy-Wouthuysen transformation. In this paper, we strictly consider the Foldy-Wouthuysen transformation as a procedure which allows one to write the Bargmann-Wigner equations in a form which in the nonrelativistic limit leads to the Galilei-invariant Schrödinger equations for arbitrary spin. As a result of this interpretation, we derive for the position operator and the localized states of elementary systems the same expressions as Newton and Wigner; the spin operator is also obtained. Finally, the Chakrabarti transformation is considered in the same spirit.

I. INTRODUCTION

As is well known, the invariance requirements imposed on a physical theory lead to the enumeration[1] and construction[2] of the relativistic equations for elementary systems, which, in turn, are defined[3] as those whose manifold of states provide the support space of an irreducible unitary representation of the restricted Poincaré group. Within this scheme, only the expression for the dynamical quantities associated with the infinitesimal generators of the group (as the momentum, energy, and angular momentum) is provided by the theory, whereas for other observable quantities there is no straightforward procedure leading to their explicit form, which has to be found by means of other considerations. This is the case, for instance, of the position operator in relativistic quantum mechanics. In contrast, there is no ambiguity in defining the position operator in nonrelativistic quantum mechanics, where it is simply given by the multiplicative operator $\vec{x}$.[4] In this case, the physically meaningful[5-9] elementary systems are classified according to nontrivial[10] projective representations of the Galilei group, the group which is obtained (by contraction[11]) from the Poincaré group by taking the limit $c\to\infty$. Instead of the parameters (mass, spin, and sign of the energy) which characterize the elementary massive relativistic systems, the nonrelativistic ones are determined by the spin and mass (which comes through the system of factors of the projective representation), the other invariant of the Galilei group, the "internal energy," not having physical significance for a free particle. In this case, the position operator belongs to the Lie algebra of the Galilei group; more precisely, it is given by $\vec{N}/\kappa$, where $\vec{N}$ is the generator of the accelerations (Galilean boosts) and κ is to be interpreted as the mass.[7] Since Galilean boosts commute among themselves, this guarantees that the position operator is left unaltered under the action of accelerations.

In relativistic quantum mechanics, the problem of finding a position operator and its localized states has been treated in various ways. Since in this paper we shall be concerned only with the Newton-Wigner (NW) position operator and with those which can be obtained by using "canonical" transformations as the Foldy-Wouthuysen one, we shall not review them but instead refer the reader to the papers by Kálnay[12] and DeVries.[13]

The classical paper of Newton and Wigner[3] corresponds to what Kálnay[12] calls "the equivalent frames of reference approach." NW found their position operator as well as an expression for the localized states by imposing the following conditions: (NW 1, 2) the states localized at $\vec{x}=0$ at time $t=0$ form a linear manifold invariant under rotations about the origin and under reflections of both the spatial and time coordinates; (NW 3) two localized states related to each other by a spatial displacement are orthogonal; and (NW 4) (regularity condition) the infinitesimal operators of the Lorentz group have to be applicable to the localized states. NW found these axioms sufficient to determine an operator $\vec{q}_{NW}$ acting on the positive-energy solutions of the corresponding wave equation, Hermitian within the relativistic scalar product, and having a continuous spectrum. As noted by NW themselves, $\vec{q}_{NW}$ has no simple covariant meaning under the Lorentz group, a feature to be expected since (NW 1, 2) imposes the physical equivalence of only spatially rotated frames of equivalence.

The "canonical transformations approach" to the position operator, first used by Foldy and Wouthuysen,[14] makes the assumption that the representation provided by the (manifestly covariant) relativistic equation is not adequate to identify $\vec{x}$ with the position operator, and one then looks in this representation for the operator $\vec{X}$ which corresponds to $\vec{x}$ in the new (conjugate) representation obtained by means of the canonical transformation. The operator $\vec{X}$ has the important property of possessing as temporal derivative the relativistic velocity $\vec{p}/p_0$ and describes the mean position of the relativistic particle, which oscillates around it with an amplitude of the order of the Compton wavelength ($1/\kappa$ in natural units, κ being the mass).

The Foldy-Wouthuysen (FW) transformation allows the definition of a position operator in the aforementioned way as well as the definition of a mean-spin or intrinsic-spin operator which is a constant of the motion. On the other hand, this transformation provides a canonical form of the wave equation[15] which leads in a natural way to the nonrelativistic wave equation by restricting it to the subspace of positive-energy solutions and considering $|\vec{p}| \ll \kappa$. It is then not surprising to find in the usual representation a position operator with adequate properties since precisely this mean position $\vec{X}$ is the transformed one of the operator $\vec{x} = i\partial/\partial\vec{p}$, which has a precise meaning in nonrelativistic quantum mechanics. However, and in contrast to the position operator of NW, the FW mean-position operator has the inconvenience of not being observable, i.e., Hermitian within the relativistic scalar product which is defined for the solutions of the corresponding wave equation.

In the literature the FW formalism has been extensively developed to obtain operators for relativistic particles of various spins, and other transformations, like the Chakrabarti transformation[16] have been introduced in a similar spirit (a list of references can be found in Refs. 12, 13, and 17). Repeatedly the differences arising between the NW position operator and those obtained by means of the aforementioned canonical transformations have been noticed. However, we feel that the problem of conciliating both procedures has not yet been fully treated. This is the object of the present work. In this paper we show that the strict interpretation of the FW transformation as a procedure which leads to a representation which permits one to perform trivially the nonrelativistic limit of the wave equation together with the assumption that in this new representation the position operator is given by $i\partial/\partial\vec{p}$ is sufficient to obtain for the localized states and the position operator precisely the same results as Newton and Wigner. Moreover, the localized states will also be eigenstates of the third component of the mean-spin operator derived at the same time as the NW position operator and not considered by these authors.

We shall not discuss the difficulties which arise in the search for a Lorentz-covariant position operator. These have been considered by many authors from different points of view leading to different answers according to which of the NW axioms is abandoned. We refer to the clear analysis and references of the already mentioned review by Kálnay.[12]

This paper is organized as follows: In Secs. II and III, for the sake of simplicity, we treat the simplest case of a Dirac particle giving in Sec. II the generalities of the method, and in Sec. III the position and spin operators and their eigenstates. In Sec. IV we analyze the arbitrary spin case by means of the Foldy-Wouthuysen-Pursey (FWP) transformation[18,17] for the Bargmann-Wigner (BW) equations[2] and derive the NW operator, the spin operator, and the localized states for (massive) elementary systems of any spin.[19] Finally, Sec. V is devoted to the consideration of the Chakrabarti transformation for any spin.[20,17] As the FWP and Chakrabarti transformations are given by different operators leaving invariant different scalar products, we obtain in Sec. V a slightly different form for the position and spin operators and the localized states corresponding to the "second" scalar product that can be defined for the BW equations. The connection between the operators derived in Secs. IV and V and the "mean" operators which are obtained by means of the FWP and Chakrabarti transformations[17] is also explicitly shown in both cases.

II. OBSERVABLES IN THE DIRAC AND PAULI REPRESENTATIONS (SPIN $\frac{1}{2}$)

The Dirac equation

$$(\gamma^\mu p_\mu - \kappa)\psi_D = 0 \tag{2.1a}$$

can be written in the Hamiltonian form

$$p^0\psi_D = (\vec{\alpha}\cdot\vec{p} + \beta\kappa)\psi_D \equiv H\psi_D. \tag{2.1b}$$

For its solutions $\psi_D(p, \xi)$ (ξ being the four-valued spin variable), one usually defines the positive-definite invariant scalar product

$$(\psi_D, \psi_D) = \int \frac{d^3p}{\omega^2} \psi_D^\dagger \psi_D\,, \tag{2.2a}$$

which is equivalent to[21]

$$(\psi_D, \psi_D)' = \frac{\epsilon}{\kappa} \int \frac{d^3p}{\omega} \psi_D^\dagger \gamma^0 \psi_D\,, \tag{2.2b}$$

where $\epsilon = \mathrm{sgn}\, p^0$, $\omega = +(\kappa^2 + \vec{p}^2)^{1/2}$. We shall consider here (2.2a), since it is the scalar product which is left invariant by the FW transformation.

The FW representation is obtained by applying to (2.1b) the FW transformation with the result

$$p^0 \psi_{\mathrm{FW}} = \beta \omega \psi_{\mathrm{FW}}\,, \tag{2.3}$$

where

$$\psi_{\mathrm{FW}} = F \psi_D \tag{2.4}$$

and $[\vec{\alpha} = \gamma^0 \vec{\gamma}, \beta = \gamma^0]$

$$\begin{aligned} &F = [2\omega(\omega+\kappa)]^{-1/2}(\omega + \kappa + \beta \vec{\alpha} \cdot \vec{p})\,, \\ &F^\dagger F = I\,, \end{aligned} \tag{2.5}$$

which satisfies, as mentioned,

$$(\psi_D, \psi_D) = (\psi_{\mathrm{FW}}, \psi_{\mathrm{FW}})\,. \tag{2.6}$$

Introducing the projectors to the positive- and negative-energy states in the FW representation

$$\Lambda^{(\pm)} = \tfrac{1}{2}(1 \pm \beta)\,, \tag{2.7}$$

Eq. (2.3) splits into

$$p^0 \psi_{\mathrm{FW}}^{(\pm)} = \pm\, \omega \psi_{\mathrm{FW}}^{(\pm)}\,, \quad \psi_{\mathrm{FW}}^{(\pm)} = \Lambda^{(\pm)} \psi_{\mathrm{FW}}\,. \tag{2.8}$$

The equation satisfied by the positive-energy spinor $\psi_{\mathrm{FW}}^{(+)}$ easily provides the nonrelativistic limit. In fact, the spinor defined by

$$\psi_P = \omega^{-1} \psi_{\mathrm{FW}}^{(+)}\,, \tag{2.9}$$

with norm

$$\langle \psi_P, \psi_P \rangle \equiv \int d^3p\, \psi_P^\dagger \psi_P = (\psi_{\mathrm{FW}}^{(+)}, \psi_{\mathrm{FW}}^{(+)})\,, \tag{2.10}$$

satisfies the equation

$$p^0 \psi_P = \omega \psi_P\,, \tag{2.11}$$

which in the nonrelativistic limit $|\vec{p}| \ll \kappa$ reduces to

$$E \psi_{\mathrm{NR}} = (\vec{p}^{\,2}/2\kappa) \psi_{\mathrm{NR}} \tag{2.12}$$

or Schrödinger equation for a particle of spin $\frac{1}{2}$ $(E = p^0 - \kappa)$.

It should be clear that step (2.9) (redefinition of the amplitudes) is an essential one in our procedure. It allows the definition of the nonrelativeisticlike scalar product $\langle , \rangle$ within which the operator $\vec{x}$ is Hermitian and thus can be adopted as the position operator in the representation defined by ψ_P (Sec. III).

The presence of the spin in (2.12) through the variable ξ $[\psi_P = \psi_P(\vec{p}, \xi)]$ is of course not at all surprising; it is a well-known fact, particularly stressed by Lévy-Leblond,[8,22] that the spin is not a relativistic effect. In fact (2.12) is what could be called the canonical form of the Schrödinger equation[23] for a particle of mass κ and spin $\frac{1}{2}$ for which the internal energy $E - \vec{p}^{\,2}/2\kappa$ has been taken equal to zero. Thus, the transition to the nonrelativistic equation and to the corresponding scalar product has been accomplished by means of the following steps: (a) the FW transformation, (b) restriction to the subspace of positive-energy states, (c) redefinition of the amplitudes (multiplication by ω^{-1}), and (d) the condition $|\vec{p}| \ll \kappa$. The representation (2.10) is obtained by (a), (b), and (c) and has qualitatively [except for condition (d)] all the properties of the nonrelativistic limit: It satisfies the Schrödinger-like equation (2.11) and its corresponding scalar product is defined by (2.10). Hereafter we shall call this representation the Pauli or nonrelativisticlike representation.

We consider now the form of the observables in the different representations introduced.[24] This is done by means of the following.

Proposition: Let O_P be a Pauli observable and ψ_P be an eigenstate corresponding to the eigenvalue σ:

$$O_P \psi_P = \sigma \psi_P\,. \tag{2.13}$$

Then, the Dirac observable O_D is given by

$$O_D = F^{-1} \omega O_P \omega^{-1} \Lambda^{(+)} F \tag{2.14}$$

and satisfies $(\psi_D = \omega F^{-1} \psi_P)$

$$O_D \psi_D = \sigma \psi_D\,, \tag{2.15}$$

$$(\psi_D, O_D \psi_D) = \langle \psi_P, O_P \psi_P \rangle\,, \tag{2.16}$$

O_D being Hermitian within (2.2a). (The demonstration is trivial.) By construction, O_D acts only on the positive-energy states.

III. POSITION AND SPIN OBSERVABLES IN THE DIRAC REPRESENTATION

The position operator for a massive nonrelativistic particle [Eq. (2.12)] is given in momentum space by

$$\vec{x} = i\partial/\partial\vec{p}\,. \tag{3.1}$$

As discussed in the Introduction (3.1) behaves adequately under Galilean transformations. *We now make the assumption that the position operator in the relativistic case is again given by* $\vec{q} = i\partial/\partial\vec{p}$ *in the Schrödinger-like (Pauli) representation* defined by (2.11) and (2.10) which, as discussed in Sec. II, has formally all the properties of the nonrelativistic limit (except $|\vec{p}| \ll \kappa$). It is clear that in the passage from the Galilei to the Poincaré group $\vec{q}$ will still transform as a vector under the rotation part of the latter group, but its transformation properties under Lorentz boosts will be modified. For instance, $[N_1, q_2] \neq 0$, meaning that a Lorentz transformation in the direction of the first axis changes the components of $\vec{q}$ perpendicular to it.[25] Accordingly, a Lorentz-covariant meaning cannot be attached to $\vec{q}$, and this feature will appear again in the Dirac representation where, as we are going to show, we shall find the NW operator $\vec{q}_{NW}$.

Let us consider now the spin operator. In the nonrelativistic case $\vec{s}$ is given simply by the Pauli matrices $\vec{s} = \frac{1}{2}\vec{\sigma}$. In the relativistic case, the spin operator in the nonrelativisticlike representation is written

$$\vec{s} = \tfrac{1}{2}\vec{\sigma}\Lambda^{(+)}, \tag{3.2}$$

where, in terms of the Dirac γ matrices, $\sigma^k = i\gamma^i\gamma^j (i, j, k \text{ cycl})$. Strictly speaking, the projector $\Lambda^{(+)}$ could be omitted since $\vec{s}$ operates on the two nonzero components of ψ_P, which is a positive-energy function [(2.9)].

The observables $\vec{q}$ and $\vec{s}$ commute, and therefore the solutions of Eq. (2.11), eigenstates of the two operators, can be written in the form $(\omega^2 - \vec{p}^{\,2} = \kappa^2)$

$$\psi_{P,m}(\vec{y}, t; \vec{p}, \xi) = (2\pi)^{-3/2} e^{-i(\vec{p}\cdot\vec{y} - \omega t)} v_m(\xi), \tag{3.3}$$

the two functions $v_m(\xi)$ $(m = +\frac{1}{2}, -\frac{1}{2})$ satisfying the conditions

$$s_3 v_m(\xi) = m\, v_m(\xi), \tag{3.4a}$$

$$v_m^\dagger v_m = 1, \tag{3.4b}$$

$$\beta v_m = v_m, \tag{3.4c}$$

the last one due to the fact that $\beta\psi_P = \psi_P$. In fact, it is evident that

$$\vec{q}\psi_{P,m}(\vec{y}, t; \vec{p}, \xi) = \vec{y}\psi_{P,m}(\vec{y}, t; \vec{p}, \xi), \tag{3.5}$$

$$s_3\psi_{P,m}(\vec{y}, t; \vec{p}, \xi) = m\psi_{P,m}(\vec{y}, t; \vec{p}, \xi). \tag{3.6}$$

Two localized states in different points $\vec{y}$ and $\vec{y}'$ are orthogonal:

$$\langle \psi_{P,m}(\vec{y}, t; \vec{p}, \xi), \psi_{P,m'}(\vec{y}', t; \vec{p}, \xi)\rangle = \delta(\vec{y} - \vec{y}')\delta_{mm'}, \tag{3.7}$$

and, in coordinate representation, the states localized at $\vec{y}$ at $t = 0$ are written

$$\psi_{P,m}(\vec{y}; \vec{x}, \xi) = \frac{1}{(2\pi)^{3/2}} \int d^3p\, e^{-i\vec{p}\cdot\vec{x}} \psi_{P,m}(\vec{y}; \vec{p}, \xi)$$
$$= \delta(\vec{x} - \vec{y}) v_m(\xi). \tag{3.8}$$

We now proceed to find the form of these observables in the Dirac representation. In accordance with (2.14), this is given by

$$\vec{q}_D = F^{-1}\omega i \partial_{\vec{p}}\, \omega^{-1}\Lambda^{(+)}F, \tag{3.9}$$

$$\vec{s}_D = F^{-1}\omega \vec{s}\, \omega^{-1}\Lambda^{(+)}F, \tag{3.10}$$

which, after some algebra, can be written in the form

$$\vec{q}_D = E(1+\beta)\frac{\omega^{3/2}}{(\omega+\kappa)^{1/2}}\, i\frac{\partial}{\partial\vec{p}}\, \frac{\omega^{-1/2}}{(\omega+\kappa)^{1/2}}E, \tag{3.11}$$

$$\vec{s}_D = \left[\frac{\kappa}{\omega}\frac{\vec{\sigma}}{2} + \frac{(\vec{\sigma}\cdot\vec{p})\vec{p}}{2\omega(\omega+\kappa)} - \frac{i\beta(\vec{\alpha}\times\vec{p})}{2\omega}\right]E, \tag{3.12}$$

where E is the projector for positive-energy states

$$E = \tfrac{1}{2}\left(1 + \frac{\vec{\alpha}\cdot\vec{p} + \beta\kappa}{\omega}\right) = \frac{H+\omega}{2\omega}. \tag{3.13}$$

The operator $\vec{q}_D$ is nothing other than the NW operator $\vec{q}_{NW}$ for spin $\frac{1}{2}$, and $\vec{s}_D$ is, except for the projector E, the mean-spin operator $\vec{S}$ obtained from the FW representation. The process followed then leads to the position and spin operators (3.11) and (3.12) which are the physical observables in the Dirac representation. On the contrary, the direct application of the FW transformation to the operator $i\partial/\partial\vec{p}$ leads to the FW mean-position operator $\vec{X}$ which is not Hermitian within (2.2a) and which is not restricted to positive-energy states as is $\vec{q}_D \equiv \vec{q}_{NW}$. In fact, the FW operators $\vec{X}$ and $\vec{S}$ are related to $\vec{s}_D$ and $\vec{q}_D$ by the expressions

$$\vec{q}_D = (\vec{X} - i\vec{p}/\omega^2)E, \tag{3.14}$$

$$\vec{s}_D = \vec{S}E; \tag{3.15}$$

the term $-i\vec{p}/\omega^2$ in (3.14) is necessary to obtain a Hermitian operator (see Sec. IV and Ref. 26).

The eigenstates of the operators $\vec{q}_D$ and $\vec{s}_D$ are given by (2.15)

$$\psi_{D,m}(\vec{y}, t; \vec{p}, \xi) = (2\pi)^{-3/2}\omega\, e^{-(\vec{p}\cdot\vec{y} - \omega t)}\; F^{-1}v_m(\xi); \tag{3.16}$$

obviously, by Eq. (2.10), two states localized at $\vec{y}$ and $\vec{y}'$ are orthogonal:

$$(\psi_{D,m}(\vec{y}, t; \vec{p}, \xi), \psi_{D,m'}(\vec{y}, t; \vec{p}, \xi)) = \delta(\vec{y} - \vec{y}')\delta_{mm'}. \tag{3.17}$$

We show now the identity of the localized states (3.16) with those obtained by Newton and Wigner. Taking into account the relation (3.4c) and that

$$F^{-1}\beta v_m = \frac{(\omega+\kappa)\beta + \vec{\alpha}\cdot\vec{p}}{[2\omega(\omega+\kappa)]^{1/2}} v_m$$
$$= \frac{2\omega}{[2\omega(\omega+\kappa)]^{1/2}} E v_m, \tag{3.18}$$

one finally obtains

$$\psi_{D,m}(\vec{y}, t; \vec{p}, \xi) = (2\pi)^{-3/2} e^{-i(\vec{p}\cdot\vec{y} - \omega t)}$$
$$\times 2^{1/2}\omega^{3/2}(\omega+\kappa)^{-1/2} E v_m(\xi), \tag{3.19}$$

which, with

$$V_m(\vec{p}, \xi) \equiv E v_m(\xi), \tag{3.20}$$

can be written in the form

$$\psi_{D,m}(\vec{y}, t; \vec{p}, \xi) = (2\pi)^{-3/2} e^{-i(\vec{p}\cdot\vec{y} - \omega t)} \times 2^{1/2}\omega^{3/2}(\omega+\kappa)^{-1/2} V_m(\vec{p}, \xi), \tag{3.21}$$

which coincides at $\vec{y}=0, t=0$ with the expression of the NW localized states for $s=\frac{1}{2}$.

The form of the localized states in coordinate space is now not that of a δ function.[3]

IV. THE ARBITRARY SPIN CASE

The reasoning in Sec. III for $s=\frac{1}{2}$ can be extended without difficulties to the general case. For arbitrary spin, we consider the Bargmann-Wigner (BW) equations[2]

$$(\gamma^{\mu(r)}p_\mu - \kappa)\psi_{\rm BW} = 0, \quad r = 1, \ldots, 2s \tag{4.1a}$$

in the form

$$p^0\psi_{\rm BW} = (\vec{\alpha}^{(r)}\cdot\vec{p} + \beta^{(r)}\kappa)\psi_{\rm BW}. \tag{4.1b}$$

The solutions $\psi_{\rm BW}(\vec{p}, \xi_1 \cdots \xi_{2s})$ are symmetric in the $2s$ four-valued variables. Considering $\psi_{\rm BW}$ as an element of the 4^{2s}-dimensional space obtained by taking the Kronecker product of the $2s$ 4-dimensional spaces, the $\gamma^{\mu(r)}$, $\vec{\alpha}^{(r)}$, etc., can be written in the form

$$\gamma^{\mu(r)} = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \gamma^\mu \otimes 1 \otimes \cdots \otimes 1 \otimes 1 \text{ (}2s\text{ factors)}, \tag{4.2}$$

the term different from unity being placed in the rth position. For the solutions of the BW equations we shall consider the equivalent scalar products

$$(\psi_{\rm BW}, \psi_{\rm BW}) = \int \frac{d^3p}{\omega^{2s+1}} \psi_{\rm BW}^\dagger \psi_{\rm BW}, \tag{4.3a}$$

$$(\psi_{\rm BW}, \psi_{\rm BW})' = \left(\frac{\epsilon}{\kappa}\right)^{2s} \int \frac{d^3p}{\omega} \psi_{\rm BW}^\dagger \prod_{r=1}^{2s} \gamma^{0(r)} \psi_{\rm BW}, \tag{4.3b}$$

expressions which, for $s=\frac{1}{2}$, reproduce (2.2a) and (2.2b) [in (4.3b), $\prod_{r=1}^{2s}\gamma^{0(r)} = \gamma^0 \otimes \gamma^0 \otimes \cdots \otimes \gamma^0$, $2s$ factors.]

Applying the FWP transformation defined by [18,17] (the new index s refers to the spin)

$$F^s = \prod_{r=1}^{2s} F^{(r)} \tag{4.4}$$

to Eqs. (4.1b), one obtains

$$p^0\psi_{\rm FWP}^s = \beta^{(r)}\omega\psi_{\rm FWP}^s, \quad r = 1, \ldots, 2s. \tag{4.5}$$

Introducing the projectors[17]

$$\Lambda^{s(\pm)} = \prod_{r=1}^{2s} [\tfrac{1}{2}(1 \pm \beta^{(r)})], \tag{4.6}$$

we see that (4.5) split into

$$p^0\psi_{\rm FWP}^{s(\pm)} = \pm\omega\psi_{\rm FWP}^{s(\pm)}, \tag{4.7a}$$

with the nondynamical constraints[18]

$$\beta^{(r)}(\psi_{\rm FWP}^{(+)} + \psi_{\rm FWP}^{(-)}) = \beta^{(s)}(\psi_{\rm FWP}^{(+)} + \psi_{\rm FWP}^{(-)}) \text{ for all } r, s, \tag{4.7b}$$

which are identities due to the relations $\beta^{(r)}\Lambda^{s(+)} = \Lambda^{s(+)}$, $\beta^{(r)}\Lambda^{s(-)} = -\Lambda^{s(-)}$.

Proceeding as in Sec. II we define the nonrelativisticlike or Pauli spinor by

$$\psi_P^s = \omega^{-(s+1/2)}\psi_{\rm FWP}^{s(+)}. \tag{4.8}$$

This spinor satisfies the Schrödinger-like equation

$$p^0\psi_P^s = \omega\psi_P^s, \tag{4.9}$$

which, for $|\vec{p}| \ll \kappa$, reduces to the Schrödinger equation for a particle of mass κ and spin s.[27] The FWP transformation (4.4) is unitary within (4.3a), and therefore we find again

$$(\psi_{\rm FWP}^{s(+)}, \psi_{\rm FWP}^{s(+)}) = \int d^3p\, \psi_P^{s\dagger}\psi_P \equiv \langle\psi_P^s, \psi_P^s\rangle. \tag{4.10}$$

The observable $O_{\rm BW}^s$ associated with O_P^s is given now by the expression

$$O_{\rm BW}^s = (F^s)^{-1}\omega^{s+1/2} O_P^s \omega^{-(s+1/2)}\Lambda^{s(+)} F^s, \tag{4.11}$$

and again the relation

$$(\psi_{\rm BW}, O_{\rm BW}^s\psi_{\rm BW}) = \langle\psi_P^s, O_P^s\psi_P^s\rangle \tag{4.12}$$

and the expressions corresponding to (2.15) are seen to hold.

Let us now introduce the position and spin operators in the Pauli representation. Since in the nonrelativistic case [Eq. (4.9) for $|\vec{p}| \ll \kappa$] $\vec{x}$ is given by $i\partial/\partial\vec{p}$ for any spin, we take for the position operator in the Pauli representation

$$\vec{q}^s = i\partial/\partial\vec{p} \tag{4.13}$$

and, in an analogous way, we take for the spin operator

$$\vec{S}^s = \tfrac{1}{2}\vec{\sigma}^s\Lambda^{s(+)}, \tag{4.14}$$

where

$$\sigma^{s,k} = i\sum_{r=1}^{2s}\gamma^{i(r)}\gamma^{j(r)}, \quad i, j, k \text{ cycl}. \tag{4.15}$$

The solutions of Eq. (4.9), eigenstates of the two operators $\vec{q}$ and s_3^s, now adopt the form

$$\psi_{P,m}^s(\vec{y}, t; \vec{p}, \xi_1 \cdots \xi_{2s}) = (2\pi)^{-3/2} e^{-i(\vec{p}\cdot\vec{y} - \omega t)} \times v_m(\xi_1 \cdots \xi_{2s}), \tag{4.16}$$

where the $v_m(\xi_1 \cdots \xi_{2s})$, symmetric in the ξ's, are the $2s+1$ independent functions satisfying the equations[3]

$$\beta^{(r)}v_m = v_m\,,\ r=1,\ldots,2s\,, \tag{4.17a}$$

$$s_3^s v_m = m v_m\,,\ m=-s,\ldots,0,\ldots,s\,, \tag{4.17b}$$

$$v_m^\dagger v_m = 1\,. \tag{4.17c}$$

Obviously,

$$\vec{q}^s\,\psi^s_{P,m} = \vec{y}\psi^s_{P,m}\,, \tag{4.18}$$

$$s_3\psi^s_{P,m} = m\psi^s_{P,m}\,, \tag{4.19}$$

and

$$\langle \psi^s_{P,m}(\vec{y},t;\vec{p},\xi_1\cdots\xi_{2s}),\psi^s_{P,m'},(\vec{y}',t;\vec{p},\xi_1\cdots\xi_{2s})\rangle = \delta(\vec{y}-\vec{y}')\delta_{mm'}\,. \tag{4.20}$$

The expressions for the observables in the representation (4.1) are now obtained as in Sec. III, with the result

$$\vec{q}_{BW} = E^s \prod_{r=1}^{2s}(1+\gamma^{0(r)})\frac{\omega^{2s+1/2}}{(\omega+\kappa)^s}i\frac{\partial}{\partial\vec{p}}\frac{\omega^{-1/2}}{(\omega+\kappa)^s}E^s\,, \tag{4.21}$$

$$\vec{s}_{BW} = \left[\frac{\kappa}{\omega}\frac{\vec{\sigma}^s}{2} + \frac{(\vec{\sigma}^s\cdot\vec{p})\vec{p}}{2\omega(\omega+\kappa)} - \frac{i(\beta\vec{\alpha}^s\times\vec{p})}{2\omega}\right]E^s\,, \tag{4.22}$$

where in Eqs. (4.21) and (4.22) E^s is given by

$$E^s \equiv \prod_{r=1}^{2s} E^{(r)} \tag{4.23}$$

[with $E^{(r)}$ given by (3.13) and (4.2)] and, in (4.22),

$$\vec{\sigma}^s = \sum_{r=1}^{2s}\vec{\sigma}^{(r)}\,,\quad \vec{\alpha}^s = \sum_{r=1}^{2s}\vec{\alpha}^{(r)}\,. \tag{4.24}$$

Again, $\vec{q}_{BW}$ coincides with the NW position operator $\vec{q}_{NW}$,[3] and $\vec{s}_{BW}$ is, except for the energy projector E^s, the FWP mean-spin operator[17] $\vec{S}^s$. $\vec{q}_{BW}$ is Hermitian within (4.3a) and is related to the FWP mean-position operator (which is not, Ref. 26)

$$\vec{X}^s = \vec{x} + \frac{i\beta\vec{\alpha}^s}{2\omega} - \frac{(\vec{\sigma}^s\times\vec{p})\omega + i\beta(\vec{\alpha}^s\cdot\vec{p})\vec{p}}{2\omega^2(\omega+\kappa)} \tag{4.25}$$

through the expression

$$\vec{q}_{NW} = \vec{X}^s - i(s+\tfrac{1}{2})\frac{\vec{p}}{\omega^2}\,E^s\,; \tag{4.26}$$

for the spin operator the analogous equation to (3.15) is

$$\vec{s}_{BW} = \vec{S}^s E^s\,. \tag{4.27}$$

Let us now turn to the localized states. These are given by

$$\psi_{BW,m} = (2\pi)^{-3/2}\omega^{(s+1/2)}e^{-i(\vec{p}\cdot\vec{y}-\omega t)} \times (F^s)^{-1}v_m(\xi_1\cdots\xi_{2s}) \tag{4.28}$$

and again, comparing $(F^s)^{-1}(\prod_{r=1}^{2s}\beta^{(r)})$ with E^s and taking into account Eq. (4.17a), one obtains for the $\psi_{BW,m}$ localized states the $(2s+1)$ functions:

$$\psi_{BW,m}(\vec{y},t;\vec{p},\xi_1\cdots\xi_{2s}) = (2\pi)^{-3/2}2^s\,\omega^{2s+1/2}(\omega+\kappa)^{-s}e^{-i(\vec{p}\cdot\vec{y}-\omega t)}E^s\,v_m(\xi_1\cdots\xi_{2s})\,, \tag{4.29}$$

which, in terms of the spin functions

$$V_m(\vec{p},\xi_1\cdots\xi_{2s}) \equiv E^s\,v_m(\xi_1\cdots\xi_{2s})\,, \tag{4.30}$$

can be written as

$$\psi_{BW,m}(\vec{y},t;\vec{p},\xi_1\cdots\xi_{2s}) = (2\pi)^{-3/2}2^s\omega^{2s+1/2}(\omega+\kappa)^{-s} \times e^{-i(\vec{p}\cdot\vec{y}-\omega t)}V_m(\vec{p};\xi_1\cdots\xi_{2s})\,, \tag{4.31}$$

which is the expression of the NW localized states. Obviously the $2s+1$ functions given by (4.30) are solutions of Eqs. (4.1) as they should be, the $V_m(p,\xi_1\cdots\xi_{2s})$ being normalized through

$$\sum |V_m|^2 = \left(\frac{\omega+\kappa}{2\omega}\right)^{2s}. \tag{4.32}$$

Thus, we have shown that the transition to the nonrelativisticlike equations provided by the FW (or FWP) transformation allows us to obtain in a naive and easy way the relativistic NW position operator and the localized states. Moreover, the localized states are eigenstates of the third component of the spin operator, which is also obtained at the same time.

V. A SECOND FORM FOR THE NW POSITION OPERATOR

We consider now the definition of a position operator when the scalar product (4.3b) is used instead of (4.3a). This can be accomplished following a similar procedure to that used in Sec. IV by means of a generalization of the Chakrabarti transformation. This transformation, which is given by the Lorentz boost that carries the vector p^μ to $(\kappa,\vec{0})$, diagonalizes the "mass operator" [$\gamma^\mu p_\mu$ for the spin-$\frac{1}{2}$ case] and gives rise to a realization of the $[m,s]$ representation which corresponds to the canonical one introduced by Wigner[1]; it leaves (4.3b) invariant and allows the definition of new operators in a way similar to the FW case. In this section we intend to show that theory already developed is also applicable to this case, the essential difference being the use of (4.3b) instead of (4.3a). We consider directly the problem for any spin.

The Chakrabarti transformation for arbitrary spin is defined by[20,17]

$$Q^s = \prod_{r=1}^{2s} Q^{(r)}\,, \tag{5.1}$$

where

$$Q^{(r)}=[2m(m+p^0)]^{-1/2}[p^0+m-\vec{\alpha}^{(r)}\cdot\vec{p}], \qquad (5.2)$$

$\epsilon=\operatorname{sgn}p^0$, and $m=\epsilon(p^\mu p_\mu)^{1/2}$. Under the action of (5.1), the Eqs. (4.1a) take the form

$$(m\gamma^{0(r)}-\kappa)\psi_Q^s=0, \qquad (5.3)$$

where

$$\psi_Q^s=Q^s\psi_{BW}. \qquad (5.4)$$

Here, as well, the introduction of the projectors $\Lambda^{s(\pm)}$ [Eq. (4.6)] allows one to obtain from (5.3) the equations

$$m\psi_Q^{s(\pm)}=\pm\kappa\psi_Q^{s(\pm)}, \qquad (5.5)$$

where the ± sign refers again to the energy. Defining

$$\psi_P'^s=\omega^{-1/2}\kappa^{-s}\psi_Q^{s(+)} \qquad (5.6)$$

and taking into account that

$$\int\frac{d^3p}{\omega}\psi_Q^{s(+)\dagger}\left(\prod_{r=1}^{2s}\gamma^{0(r)}\right)\psi_Q^{s(+)}=\int\frac{d^3p}{\omega}\psi_Q^{s(+)\dagger}\psi_Q^{s(+)} \qquad (5.7)$$

because

$$\left(\prod_{r=1}^{2s}\gamma^{0(r)}\right)\Lambda^{s(+)}=\Lambda^{s(+)}, \qquad (5.8)$$

we find

$$(\psi_Q^{s(+)},\psi_Q^{s(+)})'=\int d^3p\,\psi_P'^{s\dagger}\psi_P'^s$$

$$=\langle\psi_P'^s,\psi_P'^s\rangle, \qquad (5.9)$$

where the prime refers to (4.3b).

As $\psi_{FWP}^{s(+)}=(\omega/\kappa)^s\psi_Q^{s(+)}$,[17] one verifies immediately that also ψ_P' satisfies the Schrödinger-like equation (4.9). Accordingly, we define the observables for this case by means of the expression

$$O_{BW}'=(Q^s)^{-1}\omega^{1/2}O_P^s\omega^{-1/2}\Lambda^{s(+)}Q^s, \qquad (5.10)$$

by which definition it is verified that

$$(\psi_{BW},O_{BW}'\psi_{BW})'=\langle\psi_P'^s,O_P^s\psi_P'^s\rangle. \qquad (5.11)$$

Applying (5.10) to the operators $\vec{q}^s$ and $\vec{s}^s$ defined by (4.13) and (4.14), one obtains

$$\vec{q}_{BW}'=E'^s\left(\prod_{r=1}^{2s}(1+\gamma^{0(r)})\right)\frac{\kappa^{2s}\omega^{1/2}}{(\omega+\kappa)^s}i\frac{\partial}{\partial\vec{p}}\frac{\omega^{-1/2}}{(\omega+\kappa)^s}E'^s, \qquad (5.12)$$

$$\vec{s}_{BW}'=\frac{1}{2m}\left[p^0\vec{\sigma}+i(\vec{\alpha}^s\times\vec{p})-\frac{(\vec{\sigma}^s\cdot\vec{p})\vec{p}}{(p^0+m)}\right]E'^s, \qquad (5.13)$$

where E'^s is now the covariant projector

$$E'^s=\prod_{r=1}^{2s}\left(\frac{\gamma^\mu p_\mu+\kappa}{2\kappa}\right)^{(r)}, \qquad (5.14)$$

which is Hermitian within the metric defined by $\prod_{r=1}^{2s}\gamma^{0(r)}$, i.e.,

$$E'^{s\dagger}\left(\prod_{r=1}^{2s}\gamma^{0(r)}\right)=\left(\prod_{r=1}^{2s}\gamma^{0(r)}\right)E'^s. \qquad (5.15)$$

The connection between the operators (5.12) and (5.13) and the mean operators obtained by use of the transformation (5.1) is given by

$$\vec{q}_{BW}'=(\vec{X}'^s-i\vec{p}/2\omega^2)E'^s, \qquad (5.16)$$

$$\vec{s}_{BW}'=\vec{S}'^sE'^s, \qquad (5.17)$$

where

$$\vec{X}'^s=\vec{x}-\frac{i\vec{\alpha}^s}{2m}+\frac{1}{2m(m+p^0)}\left[(\vec{\sigma}^s\times\vec{p})+\frac{i(\vec{\alpha}^s\cdot\vec{p})\vec{p}}{p^0}\right]. \qquad (5.18)$$

Again, $\vec{X}'^s$ is not Hermitian within (4.3), in contrast with $\vec{q}_{BW}'$ which is Hermitian.

Finally, the eigenstates for the operators (5.16) and (5.17) are given by

$$\psi_{BW,m}'=(2\pi)^{-3/2}\omega^{1/2}e^{-i(\vec{p}\cdot\vec{y}-\omega t)}(Q^s)^{-1}\kappa^s v_m(\xi_1\cdots\xi_{2s}) \qquad (5.19)$$

$$(\psi_{BW,m}'(\vec{y},t;\vec{p},\xi_1\cdots\xi_{2s}),\psi_{BW,m'}'(\vec{y}',t;\vec{p},\xi_1\cdots\xi_{2s}))$$
$$=\delta(\vec{y}'-\vec{y})\delta_{mm'}. \qquad (5.20)$$

Comparing the effect of $(Q^s)^{-1}$ and E'^s on the v_m functions, (5.19) can be written in the form

$$\psi_{BW,m}'=(2\pi)^{-3/2}2^s\omega^{1/2}\kappa^{2s}(\omega+\kappa)^{-s}$$
$$\times e^{-i(\vec{p}\cdot\vec{y}-\omega t)}V_m'(p,\xi_1\cdots\xi_{2s}), \qquad (5.21)$$

where

$$V_m'\equiv E'^sv_m. \qquad (5.22)$$

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$$[U(b, \vec{a}, \vec{v}, R)\psi](\vec{p}) = \exp[i(Eb - \vec{p}\cdot\vec{a})]\, D^{1/2}(R)\psi(R^{-1}(\vec{p} - \kappa\vec{v})), \tag{1}$$

where $(b, \vec{a}, \vec{v}, R)$ is the element of the group which carries $(\vec{x}, t)$ to $(\vec{x}\,', t')$, $\vec{x}\,' = R\vec{x} + \vec{v}t + \vec{a}$, $t' = t + b$. The noncanonical form of the equation is simply given (Ref. 22) by the nonrelativistic limit of (2.1a):

$$\begin{pmatrix} E & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 2\kappa \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = 0, \quad \Phi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{2}$$

which possesses nontrivial solutions only if $E = \vec{p}^{\,2}/2\kappa$; φ and χ are related through $\chi = -(\vec{\sigma}\cdot\vec{p}/2\kappa)\varphi$. The canonical form is obtained in the usual way (see, for instance, Ref. 19). Making use of the Galilean boost $(0, 0, \vec{v} = \vec{p}/\kappa, I)$ given by

$$V = \begin{pmatrix} 1 & 0 \\ -\vec{\sigma}\cdot\vec{p}/2\kappa & 1 \end{pmatrix}, \tag{3}$$

one gets $\Phi_{can} = V^{-1}\Phi = \binom{\varphi}{0}$; the covariant scalar product for (2), which was $\Phi^\dagger V^{-1\dagger} V^{-1}\Phi (= \varphi^\dagger\varphi)$ is now evaluated directly and the new (canonical) equation satisfied by φ is (2.12) by simply identifying ψ_{NR} with φ.

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[26]The non-Hermiticity of $\vec{X}$ is most easily seen by noting that

$$F\vec{q}_{NW}F^{-1} = [\vec{x} - i(s + \tfrac{1}{2})\, \vec{p}/\omega^2]\, \Lambda^{s\,(+)}$$

[compare with (4.26)]; the term $i(s + \frac{1}{2})\vec{p}/\omega^2$ makes the operator Hermitian.

[27]More precisely, to the "canonical" form of the Schrödinger equations for a particle of mass κ and spin s, which can be obtained from that corresponding to $s = \frac{1}{2}$ following the BW procedure (Ref. 22). The form (4.9) can be derived from these by a similar method to that outlined in Ref. 23.