Wilson's Theory of Critical Phenomena and Callan-Symanzik Equations in $4-\epsilon$ Dimensions

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(Received 15 January 1973)

The relevance of the Gell-Mann–Low eigenvalue condition to the description of the critical behavior in statistical mechanics is discussed in detail, using the natural framework of the Callan-Symanzik equations, in $4-\epsilon$ dimensions. Wilson's method relying on the existence of a bare coupling constant adjusted to yield scaling laws in perturbation theory is justified and Wilson's results for critical exponents are rederived using renormalized perturbation theory.

I. INTRODUCTION

Wilson's theory of critical phenomena allows one to calculate the critical exponents as power series in $\epsilon = 4 - d$, where d is the dimension of the system.[1-4] These exponents are easily related to the field-theoretic anomalous dimensions introduced by Wilson,[5] either for the field itself or for composite operators made of powers of the field. The calculations which have been made using Wilson's Feynman-diagram method[2] consist in constructing an asymptotically scale-invariant[4] theory in $4-\epsilon$ dimensions. The existence of such a theory is *a priori* somewhat surprising, since when d is not equal to four, the coupling constant has a dimension, and therefore one expects below four dimensions the Green's functions to behave, for large momenta, like their Born terms. We have studied here why such a theory may exist by examining the Gell-Mann–Low eigenvalue condition[6] using the more rigorous framework of Callan-Symanzik equations.[7]

It is found that there is a solution for the coupling constant to this eigenvalue condition, but this solution is not "attractive," i.e., it is only for that particular value of the coupling constant that one obtains the asymptotic scale invariance. One has then to understand why critical phenomena are described by this particular value of the coupling constant. It is shown that this value corresponds to a limit in which the bare coupling constant goes to infinity (although the renormalizations are finite in $4-\epsilon$ dimensions). In fact, in statistical mechanics there is a natural momentum cutoff Λ much larger than the physical "mass" and momenta,[8] which is provided by something like the inverse of the lattice spacing, and the bare coupling constant is measured in units of Λ^{ϵ}.[9]

Therefore the universality of critical phenomena, that is to say, the independence of the critical exponents with respect to the interaction, comes from the fact that one has a large bare coupling constant for which the renormalized coupling constant is automatically fixed at the solution of the eigenvalue condition. Finally, since the solution for the coupling constant to the eigenvalue problem is small when ϵ is small, the calculation of the corresponding values of the anomalous dimensions may be done using the renormalized perturbation theory in $4-\epsilon$ dimensions.[10]

The explanation proposed here differs somewhat from the one given previously by Mack and Schroer.[11] These authors considered the massless theory (the critical point) in the long-distance limit for which the solution to the eigenvalue condition becomes attractive. In our work, scaling laws are shown to hold in the vicinity of the critical point, for momenta much larger than the mass. In physical terms, this means that the power behavior of the correlation functions at the critical temperature T_c holds in the vicinity of T_c for distances much larger than the lattice spacing, but small compared to the correlation length. In a subsequent work the same ideas will be applied to obtain the equation of state and corrections to scaling laws.[12]

The setup of this article is the following: in Sec. II we briefly summarize the results of the Callan-Symanzik theory in four dimensions and give the modifications which appear in $4-\epsilon$ dimensions. In Sec. III we show how to calculate explicitly by this method the expansion in powers of ϵ of the critical exponents. Section IV is a discussion of the possibility of having an asymptotic scaling behavior in $4-\epsilon$ dimensions; the connection with statistical mechanics is examined.

In Sec. V we illustrate our previous considerations by the model in which the number of components of the field $\varphi_{\alpha}(x)$ becomes infinite, and for which the exact solution is known.

The last section contains some speculations in an attempt to show that it is unlikely that the eigenvalue condition may be satisfied in four dimensions for an attractive value of the coupling constant if the function $\beta(g)$ is analytic. A short appendix gives the values of the relevant Feynman diagram in $4-\epsilon$ dimensions.

II. CALLAN-SYMANZIK EQUATIONS IN $4-\epsilon$ DIMENSIONS

We shall study the asymptotic behavior of the Green's functions in $d=4-\epsilon$ dimensions for the field theory, whose Lagrangian density is

$$\mathcal{L}(x) = -\frac{Z_3}{2}\sum_{\alpha=1}^{n}\left[\sum_{i=1}^{d}(\partial_i\varphi^\alpha)^2 + m^2(\varphi^\alpha)^2\right] - g\frac{Z_1}{4!}[\vec{\varphi}^2(x)]^2 - \frac{Z_3}{2}\delta m^2\vec{\varphi}^2(x), \tag{1}$$

where $\varphi_\alpha(x)$ is a field with n components and

$$\vec{\varphi}^2(x) = \sum_{\alpha=1}^{n}\varphi_\alpha{}^2(x).$$

Since we shall limit ourselves to Euclidean values of the external momenta, we have chosen the conventions of statistical mechanics, where the time is purely imaginary and the metric is Euclidean. Let us note that in $4-\epsilon$ dimensions Z_1 and Z_3 are finite quantities. Infinities only appear in the mass counterterm which is of no interest here.

The natural framework to study the asymptotic behavior of the theory is given by the Callan-Symanzik equations.[7] Let us first briefly summarize the results in four dimensions. The one-particle irreducible Green's functions (or vertex functions) satisfy the equation

$$\Gamma^{(s)}(\lambda p_1\cdots\lambda p_s; m, g) = \lambda^{4-s}\,\Gamma^{(s)}(p_1\cdots p_s; m/\lambda, g), \tag{2}$$

as can be seen by a simple dimensional analysis.

Therefore the asymptotic behavior in λ may be obtained by studying the zero-mass limit of the theory. The functions $\Gamma^{(s)}$ satisfy the Callan-Symanzik equations

$$\left[m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} - \tfrac{1}{2}s\gamma_3(g)\right]\Gamma^{(s)} = \Delta\Gamma^{(s)}, \tag{3}$$

where the $\Delta\Gamma^{(s)}$ can be asymptotically neglected (at least order by order in perturbation theory). For the asymptotic part $\Gamma^{(s)}_{\rm as}$ of $\Gamma^{(s)}$, this equation yields

$$\Gamma^{(s)}_{\rm as}(p_i; m/\lambda, g) = \left[\exp\left(-\tfrac{1}{2}s\int_g^{g(\lambda)}dg'\,\frac{\gamma_3(g')}{\beta(g')}\right)\right] \times \Gamma^{(s)}_{\rm as}(p_i; m, g(\lambda)), \tag{4}$$

where $g(\lambda)$ is defined by

$$\int_g^{g(\lambda)}\frac{dg'}{\beta(g')} = \ln\lambda. \tag{5}$$

If, in addition one assumes that $\beta(g)$ has a single zero g_∞ with $\beta'(g_\infty)<0$, then $g(\lambda)$ has a finite limit g_∞ and, for a continuous range of values of g, one has

$$\Gamma^{(s)}_{\rm as}(\lambda p_i; m, g) \underset{\lambda\to\infty}{\sim} \lambda^{4-s-s\,\gamma_3(g_\infty)/2}. \tag{6}$$

If $\beta(g)$ vanishes at a point g_∞ where $\beta'(g_\infty)$ is positive, then one obtains an asymptotically scale-invariant theory only when the coupling constant has exactly the particular value g_∞; for that special theory, one has

$$\Gamma^{(s)}_{\rm as}(\lambda p_i; m, g_\infty) \underset{\lambda\to\infty}{\sim} \lambda^{4-s-s\,\gamma_3(g_\infty)/2}. \tag{7}$$

Let us now discuss how things are modified in $d=4-\epsilon$ dimensions. Since the coupling constant g is no longer dimensionless—g has the dimension m^ϵ—Eq. (2) does not hold any more. It is thus convenient to introduce the dimensionless parameter u as

$$g = m^\epsilon u, \tag{8}$$

with which one recovers the analog of Eq. (2):

$$\Gamma^{(s)}(\lambda p_i; m, u) = \lambda^{d-s(d-2)/2}\,\Gamma^{(s)}(p_i; m/\lambda, u).$$

Then the Callan-Symanzik equations have their usual form [Eq. (3)] if g is replaced by u, where now β and γ are functions of u (and of d).

We choose to renormalize the theory at zero momentum[13] by imposing

$$\Gamma^{(2)}(p, -p; m, u)|_{p^2=0} = m^2, \tag{9a}$$

$$\frac{\partial}{\partial p^2}\Gamma^{(2)}(p, -p; m, u)\Big|_{p^2=0} = 1, \tag{9b}$$

$$\Gamma^{(4)}(0,0,0,0; m, u) = g. \tag{9c}$$

Standard arguments give for β and γ the following expressions:

$$\beta(u) = m\frac{\partial u}{\partial m}\Big|_{g_0}, \tag{10}$$

$$\gamma_3(u) = m\,\frac{\partial \ln Z_3(u)}{\partial m}\Big|_{g_0}, \tag{11}$$

where the bare coupling constant g_0 is defined by

$$g_0 = g\,\frac{Z_1(u)}{Z_3{}^2(u)}. \tag{12}$$

In terms of u these formulas become

$$\beta(u) = -\epsilon\Big/\frac{d}{du}\ln\frac{uZ_1(u)}{Z_3{}^2(u)}, \tag{13}$$

$$\gamma_3(u) = \beta(u)\,\frac{d\ln Z_3(u)}{du}. \tag{14}$$

III. EXPANSION IN POWERS OF ϵ; CRITICAL EXPONENTS

The functions β and γ are then calculated as power series in u. An important difference with four dimensions, where $\beta(g)$ is proportional to g^2 for small g, is that $\beta(u)$ has a term linear in u (with a coefficient of order ϵ). This fact will explain that the function $\beta(u)$ has a zero u_∞ of order ϵ and justifies the use of perturbation theory for u close to u_∞ and for small values of ϵ. The calculations will be performed in order to obtain Wilson's anomalous dimensions[4,5] of the fields φ and φ^2 (which determine the two critical exponents η and γ) up to order ϵ^3 and ϵ^2, respectively.

The conditions (9b) and (9c) determine the coefficients of the expansion of Z_1 and Z_3 in powers of u. Some details about the calculations of the relevant diagrams may be found in the Appendix. The results are expressed in terms of the values of the four diagrams a, b, c, and d, given in the Appendix and of the geometrical coefficient

$$S=\frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{1}{(2\pi)^d}\,. \tag{15}$$

One finds

$$Z_1=1+Su\,\frac{n+8}{6}a+(Su)^2\left[\frac{n^2+26n+108}{36}a^2-\frac{(5n+22)c}{9}\right]+O(u^3), \tag{16}$$

$$Z_3=1+\frac{n+2}{18}(Su)^2b+\frac{(n+2)(n+8)}{54}(Su)^3(ab-\tfrac{1}{2}d)+O(u^4). \tag{17}$$

In these expressions the coefficients of u^p are of order $(1/\epsilon)^p$, and these divergences when ϵ goes to zero are not surprising since Z_1 and Z_3 are infinite in four dimensions. However those singularities should cancel when one calculates $\beta(u)$ [Eq. (13)] since β is finite in four dimensions. Indeed the result is

$$\beta(u)=-\epsilon u\left\{1-\frac{n+8}{6}a(uS)+\tfrac{2}{9}(uS)^2\left[(n+2)b+(5n+22)(c-\tfrac{1}{2}a^2)\right]\right\}+O(u^4),$$

and when we replace the diagrams by their explicit values,

$$\beta(u)=-u\left[\epsilon-\frac{n+8}{6}(1-\epsilon/2)(uS)+\frac{3n+14}{12}(uS)^2\right]+O(u^4). \tag{18}$$

As stated previously, this function has a zero of order ϵ:

$$Su_\infty=\frac{6}{n+8}\epsilon\left[1+\epsilon\left(\tfrac{1}{2}+\frac{3(3n+14)}{(n+8)^2}\right)\right]+O(\epsilon^3). \tag{19}$$

Anomalous Dimensions of φ

We are now in position to obtain $\gamma_3(u_\infty)$ up to order ϵ^3. First we have from Eq. (14),

$$\gamma_3(u)=-\frac{\epsilon(n+2)}{9}(uS)^2\left[b+(uS)\frac{n+8}{6}(2ab-\tfrac{3}{2}d)\right]+O(u^4),$$

and if we insert in that formula the value (19) of u_∞ then

$$\gamma_3(u_\infty)=\frac{n+2}{2(n+8)^2}\epsilon^2\left[1+\epsilon\left(\frac{6(3n+14)}{(n+8)^2}-\tfrac{1}{4}\right)\right]+O(\epsilon^4). \tag{20}$$

Wilson's anomalous dimension d_φ of the field φ, is simply related to $\gamma_3(u_\infty)$ as shown by Eq. (7) since in terms of d_φ, the vertex function $\Gamma^{(s)}$ should behave as

$$\Gamma^{(s)}(\lambda p_i)\underset{\lambda\to\infty}{\sim}\frac{\lambda^{sd_\varphi-(s-1)d}}{\lambda^{s(2d_\varphi-d)}}=\lambda^{d-sd_\varphi}\,.$$

This means that the relation is

$$2d_\varphi=d-2+\gamma_3(u_\infty)\,. \tag{21}$$

In statistical mechanics the two-point correlation function at the critical temperature is characterized by a critical exponent η which is given in terms of d_φ (Ref. 4) by $2d_\varphi=d-2+\eta$. Therefore one has simply

$$\eta=\gamma_3(u_\infty)\,. \tag{22}$$

Anomalous Dimension of φ^2

One needs to know only two critical exponents to determine the others by scaling laws. The divergence of the magnetic susceptibility at the critical temperature is characterized by an exponent γ which is related to the anomalous dimension of φ^2 according to

$$\gamma=\frac{2-\eta}{d-d_{\varphi^2}}\,. \tag{23}$$

To calculate d_{φ^2} we shall consider the one-particle irreducible Green's function, where one operator is the composite operator $\varphi^2(x)$. An additional renormalization is needed for the φ^2 vertex, and we shall call $Z_4(u)$ the corresponding renormalization constant determined by

$$\int e^{i(px+qy)}\langle \varphi^2(x)\varphi(y)\varphi(0)\rangle\Big|_{p=q=0}=1\,, \qquad (24)$$

where the last renormalization associated to the vertex φ^2 is given by

$$\langle \varphi^2(x)\varphi(y)\varphi(0)\rangle \equiv Z_4(u)\langle \varphi^2(x)\varphi(y)\varphi(0)\rangle_{\text{bare}}\,. \qquad (25)$$

This implies immediately that the function

$$\Gamma^{(1,s)}(q,p_1\cdots p_s;m,u) = \int e^{i(qx+p_1x_1+\cdots+p_sx_s)} \times \langle \varphi^2(x)\varphi(x_1)\cdots\varphi(x_s)\rangle_{\text{one particle irreducible}} \qquad (26)$$

satisfies the Callan-Symanzik equation

$$\left[m\frac{\partial}{\partial m}+\beta(u)\frac{\partial}{\partial u}-(\tfrac{1}{2}s-1)\gamma_3(u)-\gamma_4(u)\right]\Gamma^{(1,s)}=\Delta\Gamma^{(1,s)}\,, \qquad (27)$$

where

$$\gamma_4=m\frac{\partial}{\partial m}\ln Z_4\Big|_{g_0}\equiv\beta(u)\frac{d\ln Z_4}{du}\,. \qquad (28)$$

For $u=u_\infty$, the asymptotic behavior of $\Gamma^{(1,s)}$ is given by

$$\Gamma^{(1,s)}(\lambda q;\lambda p_i)\sim\lambda^{-(s-2)(d-2)/2-(s/2-1)\gamma_3(u_\infty)-\gamma_4(u_\infty)}\,,$$

and since in terms of d_φ and d_{φ^2} it reads

$$\Gamma^{(1,s)}(\lambda q;\lambda p_i)\sim\lambda^{-sd_\varphi+d_{\varphi^2}}\,,$$

we obtain

$$d_{\varphi^2}=2d_\varphi-\gamma_4(u_\infty)\,. \qquad (29)$$

Therefore, we just have to calculate the expansion of $Z_4(u)$ up to order u^2 to obtain γ up to order ϵ^2.

In terms of the same integrals (given in Appendix), the result is

$$Z_4^{-1}=1-\frac{n+2}{6}a(Su)+\frac{n+2}{6}(c-a^2)(Su)^2+O(u^3)\,, \qquad (30)$$

for which we obtain

$$\gamma_4(u)=\frac{n+2}{6}\epsilon\,(Su)[-a+(Su)(2c-a^2)]+O(u^3)\,. \qquad (31)$$

Again, here the coefficients of Z_4^{-1} diverge when ϵ goes to zero, but the same cancelation occurs so that $\gamma_4(u)$ has a finite limit. This gives

$$\gamma_4(u_\infty)=-\frac{n+2}{n+8}\epsilon\left[1+\frac{6\epsilon(n+3)}{(n+8)^2}\right]+O(\epsilon^3)\,, \qquad (32)$$

and using (23) and (29), we end up with

$$\gamma=1+\tfrac{1}{2}\epsilon\frac{(n+2)}{n+8}+\tfrac{1}{4}\epsilon^2\frac{(n+2)}{(n+8)^3}(n^2+22n+52)+O(\epsilon^3)\,, \qquad (33)$$

in agreement, as expected, with Wilson's result.[2]

IV. DISCUSSION OF THE EXISTENCE OF ASYMPTOTIC SCALE INVARIANCE IN $4-\epsilon$ DIMENSIONS—CONNECTION WITH STATISTICAL MECHANICS

Though the function β has a nontrivial zero in $4-\epsilon$ dimensions, it is important to note that $\beta(u)$ is negative for small u's, and therefore vanishes at $u=u_\infty$ near the origin, but with a positive derivative. Therefore this point u_∞ is not attractive, that is to say, the ultraviolet asymptotic behavior of the theory with u small but not equal to u_∞ is not governed by the dimension $\gamma_3(u_\infty)$.

In fact, since the coupling constant has a dimension, one does not expect this theory to exhibit scale-invariance; its asymptotic behavior is in general simply given by the Born term. It is only for that particular value u_∞ of u that the singular behavior of the renormalization constants generates the asymptotic scale invariance as will be discussed more precisely below.

It is therefore necessary to understand why statistical mechanics for which the scaling behavior in the vicinity of the critical point is independent of the interactions, leads to a renormalized theory with this prescribed value u_∞ of the renormalized coupling constant.

In order to understand this point, let us study the behavior of the various relevant functions in the vicinity of u_∞; $\beta(u)$ has a single zero with a positive slope: if

$$\beta(u)=\omega(u-u_\infty)\,,\quad \omega\equiv\beta'(u_\infty)>0$$

then

$$Z_3\sim(u-u_\infty)^{[\gamma_3(u_\infty)]/\omega} \qquad (34)$$

and

$$\frac{uZ_1}{Z_3^{\,2}}\sim(u-u_\infty)^{-\epsilon/\omega}\,. \qquad (35)$$

Therefore $g_0=m^\epsilon uZ_1/Z_3^{\,2}$ becomes infinite when $u=u_\infty$. But it is to be recalled that in statistical mechanics there is a natural cutoff given by the inverse of the lattice spacing which, in the vicinity of the critical point, is much bigger than all the relevant mass and momenta.

Therefore we shall reintroduce a cutoff Λ in the

Feynman diagrams although they are convergent, in order to study the divergence of g_0 when $u \to u_\infty$. The first correction to the finite part of a diagram is at least of order $(m/\Lambda)^2$. Therefore, with the cutoff Λ, one has

$$g_0 = m^\epsilon \frac{u Z_1(u, m/\Lambda)}{Z_3^{\,2}(u, m/\Lambda)} \tag{36}$$

which, for u in the vicinity of u_∞, behaves like

$$g_0 \approx m^\epsilon \frac{1}{(u-u_\infty)^{\epsilon/\omega} + B(m/\Lambda)^\epsilon}, \tag{37}$$

where B is some constant.

Thus, we see that $g_0(u_\infty)$ is proportional to Λ^ϵ. Conversely, in statistical mechanics g_0, which is measured in terms of a parameter which is much larger than all the physical quantities which describe the critical point, is of the form:

$$g_0 = u_0 \Lambda^\epsilon .$$

Equation (36) shows that this behavior is only possible for u near u_∞. More exactly, solving Eq. (37), we get

$$u - u_\infty \approx \left(\frac{m}{\Lambda}\right)^\omega \left(\frac{1}{u_0} - C\right)^{\omega/\epsilon} . \tag{38}$$

So when (m/Λ) goes to zero, which corresponds to the critical point, u goes to u_∞ independently of the value of u_0.

Now, if one wants to perform a calculation in perturbation theory, one has to take u strictly equal to u_∞. By Eq. (37) we see that this corresponds to choosing a particular value for g_0: $g_0 = \Lambda^\epsilon / B$. Therefore we understand that, in order to calculate in perturbation theory with the bare coupling constant g_0 as is done in Wilson's method, a cutoff has to be introduced.

In order to achieve the comparison with Wilson's method, let us calculate the value $g_0(\epsilon)$ which generates scaling in perturbation theory, by expanding all quantities in powers of ϵ. Using $Z_1/Z_3^{\,2}$ up to first order in ϵ:

$$\frac{Z_1}{Z_3^{\,2}} = 1 - \epsilon\left(\tfrac{1}{2} + \ln\frac{m}{\Lambda}\right), \tag{39}$$

we obtain

$$\begin{aligned}\frac{g_0(\epsilon)}{\Lambda^\epsilon} &= u_\infty \left(\frac{m}{\Lambda}\right)^\epsilon \frac{Z_1}{Z_3^{\,2}} \\ &= u_\infty\left(1 + \epsilon \ln\frac{m}{\Lambda}\right)\left[1 - \epsilon\left(\tfrac{1}{2} + \ln\frac{m}{\Lambda}\right)\right] \\ &\quad + O(\epsilon^3) \\ &= u_\infty(1 - \tfrac{1}{2}\epsilon) + O(\epsilon^3).\end{aligned}$$

This yields for $g_0(\epsilon)$ the result

$$g_0(\epsilon) = \frac{1}{S}\frac{6}{n+8}\,\epsilon\left[1 + \epsilon\ln\Lambda + \frac{3(3n+14)}{(n+8)^2}\right] + O(\epsilon^3) \tag{40}$$

in complete agreement with the value which has been used in the previous calculations.[3,4]

V. LIMIT OF INFINITE n

We consider now the situation in which the number n of components of the field $\varphi_\alpha(x)$ becomes large, and the coupling constant goes to zero as $1/n$. It has been shown[14] that, in this limit, the geometrical series of the "bubble" graphs gives the dominant contribution to the 4-point function $\Gamma^{(4)}$ and therefore Z_1 and Z_4, and that Z_3 equals unity.

Summing up the series, one obtains:

$$\begin{aligned}\Gamma^{(4)}_{ijkl}(p_i) &= \delta_{ij}\delta_{kl}A[(p_1+p_2)^2] + \delta_{ik}\delta_{jl}A[(p_1+p_3)^2] \\ &\quad + \delta_{il}\delta_{jk}A[(p_1+p_4)^2], \\ A(k^2) &= \tfrac{1}{3}\frac{1}{1/g_0 + \frac{1}{6}nm^{-\epsilon}I(k/m)} \\ &= \tfrac{1}{3}\frac{m^\epsilon}{1/u + \frac{1}{6}n[I(k/m) - I(0)]}, \\ g_0 &= m^\epsilon Z_1 u, \\ Z_1 &= Z_4 \\ &= \frac{1}{1 - \frac{1}{6}nuI(0)},\end{aligned} \tag{41}$$

and

$$I(k) = \int \frac{d^dp}{(2\pi)^d}\frac{1}{p^2+1}\frac{1}{(p+k)^2+1} .$$

For large k, $I(k)$ behaves like $k^{-\epsilon}$ and in terms of our previous notations, $I(0) = Sa$.

Hence one obtains $\beta(u)$:

$$\beta(u) = -\epsilon u[1 - \tfrac{1}{6}nuI(0)] . \tag{42}$$

Therefore

$$u_\infty = \frac{6}{nI(0)} . \tag{43}$$

On this example we see that $\Gamma^{(4)}$ behaves asymptotically as a constant, except for the special value u_∞ of u, for which

$$A(k^2) \sim k^\epsilon , \quad k \to \infty$$

and g_0 is infinite. We shall introduce a cutoff in order to compute g_0. Using

$$I(0,\Lambda) = I(0) - \frac{1}{\epsilon}\left(\frac{m}{\Lambda}\right)^\epsilon , \tag{44}$$

we obtain

$$g_0 = \Lambda^\epsilon u_\infty (1 - \epsilon/2) \tag{45}$$

for

$$u = u_\infty .$$

Conversely, if one takes $g_0 = \Lambda^\epsilon u_0$, one sees that $A(k^2, \Lambda)$ is independent of u_0 for $\Lambda \gg k \gg m$ and then has a scaling behavior. But if one expands $A(k^2, \Lambda)$ in powers of g_0, one shall only be able to exhibit the scaling behavior for the special value $g_0(u_\infty)$ of g_0.

VI. SPECULATIONS ABOUT THE NONEXISTENCE OF SOLUTIONS TO THE EIGENVALUE CONDITION IN FOUR DIMENSIONS

We shall try now to argue that if the function $\beta(g)$ is nonsingular for some domain of g around $g=0$ in four dimensions, then $\beta(g)$ cannot vanish for g positive with a negative derivative. The argument goes as follows: In $4-\epsilon$ dimensions it seems impossible for $\beta(u, \epsilon)$ to vanish with a negative derivative for a value u^* of u in a domain where β would be nonsingular; indeed if such a u^* would exist then[7] one would have asymptotic scale invariance for some continuous range of values of u around u^*. But, as argued before, in $4-\epsilon$ dimensions scale invariance may only occur for discrete values of u, which are values for which $Z_3(u)$ is singular. Thus it is hard to imagine how such a u^* could exist. Now we let ϵ go to zero and assume that $\beta(u, \epsilon)$ is continuous, as is indeed the case order by order in powers of u. Then if there is no attractive zero in $4-\epsilon$ dimensions, it will be also true in four dimensions.

Of course, this argument relies on the hypothetical assumption that first $\beta(g)$ has some nonvanishing domain of analyticity, and second that one can still use perturbative arguments to describe the theory beyond the first nonvanishing zero of $\beta(u, \epsilon)$ given in Eq. (19). If these speculations have some relevance, these "arguments" which are only based on the nonexistence of a universal asymptotic behavior in less than four dimensions, should apply to other renormalizable field theories as well.

APPENDIX

One calculates the following diagrams, using the Feynman parametrization, and expanding in powers of ϵ.

The relevant Feynman diagrams a, b, c, and d, extracting the geometrical factor S defined previously in Sec. III Eq. (15), have the following expressions:

$$Sa = \frac{1}{(2\pi)^d} \int \frac{d^d q}{(q^2+1)^2} ,$$

$$S^2 b = \frac{1}{(2\pi)^{2d}} \frac{d}{dp^2}\bigg|_{p^2=0} \times \int \frac{d^d q_1 d^d q_2}{(q_1^2+1)(q_2^2+1)[(p+q_1+q_2)^2+1]} ,$$

$$S^2 c = \frac{1}{(2\pi)^{2d}} \int \frac{d^d q_1 d^d q_2}{(q_1^2+1)^2(q_2^2+1)[(q_1+q_2)^2+1]} ,$$

$$S^3 d = \frac{1}{(2\pi)^{3d}} \frac{d}{dp^2}\bigg|_{p^2=0} \times \int \frac{d^d q_2}{(p+q_2)^2+1} \left\{ \int \frac{d^d q_1}{(q_1^2+1)[(q_1+q_2)^2+1]} \right\}^2 .$$

Expanded in powers of ϵ, their values are:

$$a = \frac{1}{\epsilon}(1 - \tfrac{1}{2}\epsilon) + O(\epsilon),$$

$$b = -\frac{1}{8\epsilon}(1 - \tfrac{1}{4}\epsilon) - \tfrac{1}{8} I + O(\epsilon),$$

$$c = \frac{1}{2\epsilon^2}[1 - \tfrac{1}{2}\epsilon + O(\epsilon^2)],$$

$$d = -\frac{1}{6\epsilon^2}[1 - \tfrac{1}{4}\epsilon + O(\epsilon^2)] - \frac{1}{4\epsilon} I ,$$

where

$$I = \int_0^1 dx \left\{ \frac{1}{1-x(1-x)} + \frac{\ln[x(1-x)]}{[1-x(1-x)]^2} \right\} .$$

The integral I, which is renormalization-dependent, disappears from the results for the critical exponents. For example, if one decides to calculate in the zero-mass theory, with the s-point vertex functions renormalized at the point

$$p_i \cdot p_j = \frac{u^2}{s-1}(s\delta_{ij} - 1),$$

the results for d_φ and d_{φ^2} are the same, but a, b, c, and d are modified and I does not appear.

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[8]We use here the language of field theory, and the connection with statistical mechanics is discussed in Ref. 4. In particular, the mass is the inverse of the correlation length and therefore vanishes at the critical point.
[9]The bare coupling constant is regular at the critical point ($m=0$) and the only dimensional parameter left there is Λ. Therefore g_0 has to be proportional to Λ^{ϵ}.
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PHYSICAL REVIEW D VOLUME 8, NUMBER 2 15 JULY 1973

Quantization of a String of Spinning Material–Hamiltonian and Lagrangian Formulations*

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(Received 2 April 1973)

The dynamics of a relativistic string made of spinning material is discussed in two different formulations. The first is a manifestly covariant formulation under a gauge transformation. The second is a Hamiltonian formalism which enables us to make the transition from the classical to the quantum description in a coherent way. The Lagrangian is also constructed. The results of our investigations are as follows: (1) The mass spectra here coincide with those of the Neveu-Schwarz model. (2) The model is ghost-free. (3) The Poincaré generators $\mathfrak{M}^{\mu\nu}$ and $\mathcal{P}^{\mu}$ are constructed. The quantization is shown to be consistent with Lorentz covariance if the dimension of space-time is 10 and the Regge intercept is $\frac{1}{2}$.

I. INTRODUCTION

The string picture[1,2] of dual models has two important aspects. The first is the substantiality of the picture, which helps us to figure out intuitive images and gain an insight into a dynamical mechanism. The second is the mathematical refinement of the formulation. In particular, in the treatment of gauge invariance, which is the fundamental clue in understanding dual models, one can take advantage of techniques developed in gravitation and Yang-Mills theories.[3,4] In a recent work Goddard, Goldstone, Rebbi, and Thorn[5] have greatly improved our understanding of the Veneziano model. One improvement is the simplification of the ghost-eliminating mechanism by Brower, and Goddard and Thorn,[6] which is now understood as given by the existence of a certain gauge where no ghost appears.[7] Another is the relation between Lorentz covariance and the dimensionality of space-time, d. In their string model they have shown that the quantization is consistent with Lorentz covariance if $d=26$.

In view of these aspects, it is a challenging problem to extend the string picture to the Neveu-Schwarz model[8] (NSM). In a previous paper[9] we showed a manifestly gauge-invariant formalism of the string model, which reduces to the NSM in a special gauge. In the present article, we further develop the argument and give the Hamiltonian and the Lagrangian formulations, which are useful for various purposes, i.e., the quantization of the string motion, the incorporation of the interaction with external sources, etc.

The model we consider is the one based on a string on which Lorentz vector quantities (spins) are continuously distributed.[10] The system is invariant under a gauge group. The generalized Hamiltonian formalism developed by Dirac,[3] then, enables us to provide a quantization procedure.