Improved Treatment for the Infrared-Divergence Problem in Quantum Electrodynamics*

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Although the conceptual difficulties associated with infrared divergences in quantum electrodynamics have long since been resolved, a convenient technique for identifying the appropriate infrared-finite part of Feynman graphs has been lacking. Such a technique is presented here. The polarization sums for both real and virtual photons are rearranged into two parts. One of these (called the K-polarization sum) resembles a gauge transformation whose structure permits a simple demonstration of infrared factorization and exponentiation. It is easy to see by inspection that the residual factors (with G-polarization sums) are infrared free. This is done for each separate graph, in contrast to earlier treatments where contributions from sets of graphs had to be considered together. The technique is illustrated by a detailed treatment of the radiative corrections to lowest-order potential scattering, and generalizations to other processes are indicated.

I. INTRODUCTION

The infrared-divergence problem in quantum electrodynamics has a long history which we shall not review here.[1-3] It is known that the infrared-divergent contributions to an observable cross section cancel in any given order of perturbation theory, provided all appropriate real- and virtual-photon corrections are taken into account together. Aside from the leading divergence in each order, the complete proof of this cancellation is rather complicated. In practical calculations, one often uses a photon-mass or other infrared cutoff at intermediate stages of the work and then does the infrared cancellation by hand at the very end. Clearly it would be desirable to have a procedure whereby only infrared-free expressions would have to be evaluated.

Recently several authors have given reformulations of quantum electrodynamics using the concept of coherent photon states.[4] While these reformulations may improve the logical foundations of the subject, they so far seem to do very little for actual practical problems. In effect, they seem to justify the use of a minimum photon momentum in actual calculations. The conceptual problems treated by these authors arise because one would like to be able to define asymptotic in and out fields. In the real world, any experiment is carried out during a finite time interval so the emission of very soft photons is necessarily inhibited. But no physical result is sensitive to the way this happens (precisely because there is an infrared cancellation), and it is not interesting to make a detailed analysis of the dependence on the time duration of an experiment, just as the dependence of scattering results on the details of the wave packets has not been a very practical subject even if it is conceptually important. Therefore, in the present paper, we shall avoid these conceptual issues and simply assume that there is some convenient infrared cutoff.

Our aim is to present a more transparent demonstration of the infrared separation into divergent factors and infrared-free expressions. The technique will be to rewrite the polarization sums for both real and virtual photons as a sum of two modified polarization sums. One of these polarization sums (K type) will include the infrared-divergent terms which very conveniently factor out and exponentiate when summed over all numbers of photons. The other (G type) will lead to infrared-free expressions which no longer require a cutoff. The infrared factors depend only on the momenta of external charged particles and are already well known.[2] When real and virtual corrections are suitably combined, they lead to infrared-finite results, as is also well known.

The difficulty of giving such a demonstration is to show that it works to all orders, including the less-divergent terms which are contained within some high-order infrared divergence. In our approach, the infrared factorization becomes a simple problem in algebra and combinatorics; it is separated from the actual analysis of integrals. The demonstration that the residual factor is infrared-free is also rather simple as it involves only looking at, but not carrying out, the integrations. This is to be contrasted with the complicated proof given in YFS (Ref. 2) in which sets of integrals had to be combined in order to obtain the infrared factorization. In the present demonstration, sets of integrals are again combined, but in a simpler manner because the K-type polarization sum has a resemblance to a gauge transformation.

In Sec. II, the simplest case, virtual radiative corrections to potential scattering, is discussed. This already has most of the features of more complicated processes. Section III is concerned with real-photon emission and how it compensates the infrared divergence from virtual photons. Section IV deals with various generalizations and refinements, and a summary is given in Sec. V.

II. RADIATIVE CORRECTIONS TO THE SCATTERING VERTEX

The basic techniques for manipulating infrared divergences will first be developed for elastic scattering in some detail as this is the least complicated case to analyze. We will find that they can be readily extended to handle other processes as well, several of which will be discussed in Sec. IV. The specific problem being considered here is electron scattering from some local potential, which is to be treated in lowest order. The exact nature of the potential is immaterial; it need not correspond to an electromagnetic interaction. In the initial discussion we exclude graphs containing closed charged loops. Photons coupling to such loops will not contribute to an infrared divergence since the loop provides a strongly vanishing contribution when any external photon has vanishing four-momentum (this includes the photon self-energy after renormalization). However, radiative corrections to such graphs can have infrared divergences due to other photons. We shall return to this point in Sec. IV and describe how to extract the correct infrared factors from graphs with charged loops.

A. Preliminaries

We first state a convenient rule for obtaining the correct external-line renormalizations. In the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula a factor $Z_2^{-1/2}$ is required for each external electron line (wave-function renormalization). On the other hand, the sum of all external electron-self-energy insertions yields a factor of Z_2 for each external electron line (propagator renormalization). Thus, there is a net factor of Z_2 for each electron line passing entirely through a graph. The rule is simply to omit the wave-function renormalizations and at the same time drop self-energy insertions on the outgoing electron line. This rule has no physical significance, of course, but it will prove to be a useful computational device.

In the analysis of a higher-order vertex, it is convenient to be able to treat each photon (labeled with momentum k_i and polarization μ_i) independently of the others. Thus we want the set of graphs describing an n-photon vertex to be symmetric in the n photons; that is, it must contain graphs with all possible labelings of the photon momenta (thereby overcounting by $n!$). We remind the reader of a procedure for symmetrically inserting an additional photon in a given n-photon graph: Fix one end of the photon line (the initial point) on the external initial electron line. Then generate $(n+1)$-photon graphs by inserting the second end of the photon line (the final point) on all possible later electron lines (following the arrow on the electron line). Repeat this for all possible choices of initial point. To correspond with our rule for obtaining the correct external-line renormalization, we will refer to graphs which have no electron self-energy insertions on the final external electron line as "allowable graphs". Clearly some of the allowable graphs in a given order are obtained by symmetric insertion of an additional photon in graphs disallowed in the previous order. This circumstance will require some special discussion later on, but will not lead to any real difficulty.

B. Rearrangement of the Perturbation Series

Let us restate our objective. We wish to develop a procedure for rearranging the contribution from each order into a sum of terms, each of which is an infrared factor (some power of $\ln\lambda$) multiplying an infrared-free factor. A brute force approach to this problem was given in YFS.[2] Of course the definition of an infrared-divergent factor will not be unique since it can be changed merely by adding any finite expression. We shall use the same gauge-invariant definition as in YFS, although that may not be the best for all practical purposes. Our present approach will permit us to give an expression for the infrared-free factors by means of a simple modification of the usual Feynman rules.

We find that it is possible to accomplish the above by rewriting each polarization sum $(-g_{\mu\nu})$ as the sum of two modified polarization sums, one of which gives the appropriate infrared-finite contribution of a particular photon, while the other leads to the infrared-divergent factor. This amounts to rearranging each virtual-photon propagator according to

$$\frac{-g_{\mu\nu}}{k^2}=\frac{-(g_{\mu\nu}-bk_\mu k_\nu)}{k^2}-\frac{bk_\mu k_\nu}{k^2}\equiv\frac{-G_{\mu\nu}}{k^2}-\frac{K_{\mu\nu}}{k^2}\,, \tag{2.1}$$

where k is the photon momentum and b depends on where the photon is inserted into the graph. We define the p leg as the set of all electron lines on the incoming side of the external potential; the

p' leg is similarly defined for the outgoing side of the external potential. Then we define b for a photon of momentum k_j as

$$b^{(j)} = b^{(j)}(p_f, p_i) = \begin{cases} \dfrac{(2p_f - k_j)\cdot(2p_i - k_j)}{(k_j^2 - 2k_j\cdot p_f)(k_j^2 - 2k_j\cdot p_i)}, & \text{if } p_i \neq p_f \\ \dfrac{1}{2}\left[\dfrac{(2p_f - k_j)\cdot(2p_i - k_j)}{(k_j^2 - 2k_j\cdot p_f)(k_j^2 - 2k_j\cdot p_i)} + (k_j \to -k_j)\right], & \text{if } p_i = p_f \end{cases} \quad (2.2)$$

where p_i (p_f) is equal to p or p' according to whether the initial (final) point is in the p or p' leg.

Equation (2.1) is to be used in the following way: All virtual-photon propagators in a given vertex diagram are first reexpressed as the sum of a G propagator [i.e., $-G_{\mu\nu}/k^2 = -(g_{\mu\nu} - bk_\mu k_\nu)/k^2$] and a K propagator ($-K_{\mu\nu}/k^2 = -bk_\mu k_\nu/k^2$). The diagram is then "expanded out" to give a sum of diagrams in which each virtual photon has *either* a G or a K propagator; for convenience we refer to the photons now as G or K photons. The K propagator resembles a gauge transformation; and, in fact, if b did not depend on how the photon is inserted, summing over all insertions would yield zero according to the usual gauge-invariance arguments. Now, however, it will turn out that when we sum over all insertions of K photons the result will be a product of infrared-divergent factors, one for each K photon, multiplying a lower-order contribution whose virtual photons have only G propagators. In order to demonstrate this, we first look at the effect of symmetrically inserting a K photon into an allowable n-photon graph.[5] Each end of a K-photon line will attach to an emission vertex $\not{k}_{n+1}$. Figure 1(a) pictures an electron line which originally had n simple vertices, for either real or virtual photons, into which we have now inserted an additional interaction vertex with an emission factor $\not{k}_{n+1}$. At each point of insertion the contribution can be written as the difference of two factors by use of the identity

$$k_{n+1}^\mu \left(\frac{1}{\not{p}_r - \not{k}_{n+1} - m}\gamma_\mu \frac{1}{\not{p}_r - m}\right) = \frac{1}{\not{p}_r - \not{k}_{n+1} - m} - \frac{1}{\not{p}_r - m}. \quad (2.3)$$

Thus when we sum over all possible insertion points, there is a pairwise cancellation of terms so that the only contributions which fail to cancel directly are left over from the first and last insertions [cf. Fig. 1(b)]. The emission factor $\not{k}_{n+1}$ disappears (denoted by the circle around the photon end) and momentum k_{n+1} flows out of the electron line either before or after the original struc-

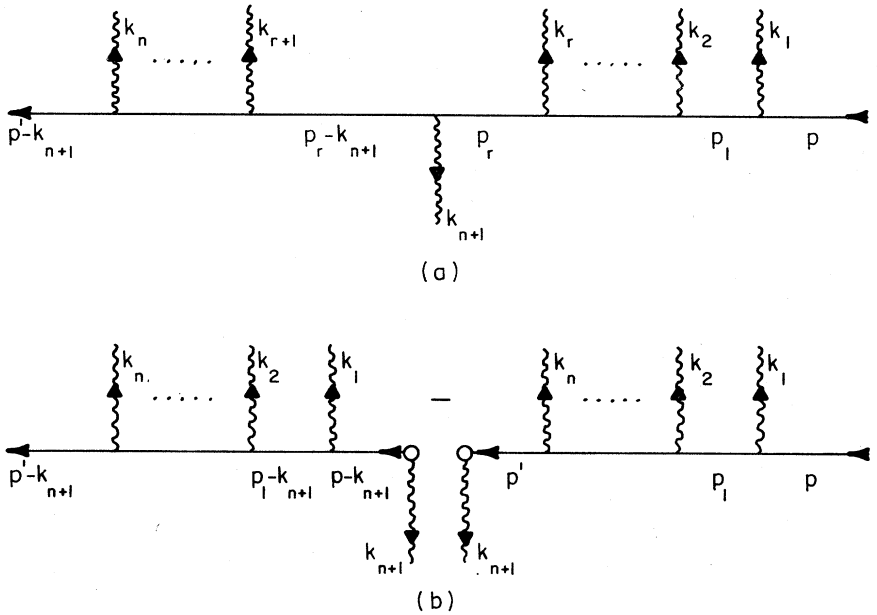

FIG. 1. Use of the Feynman identity [Eq. (2.3)] to simplify expressions when a photon has an emission vertex $\not{k}_{n+1}$. Insertion of such a vertex in all ways in an electron line as in (a) yields terms which cancel pairwise except for the remainders illustrated in (b). The small circles represent a factor of 1. Only a set of terms with the same factor of b may be grouped together in this simplification.

ture. Since the K propagator is defined according to the manner in which the photon is inserted, there will be three cases to consider when the symmetric insertion is performed.

Case 1. Both initial and final points in the p leg [factor $b(p,p)$]: Because of our choice of $b(p,p)$ (symmetric in k), the term which is left over from inserting the final point into the first electron line after the initial point will contain the factor

$$\int \frac{d^4k_{n+1}}{k_{n+1}^2} \not{k}_{n+1} b^{(n+1)}(p,p) = 0 \tag{2.4}$$

and this term may be dropped. The term which is left over from inserting the final point into the line immediately before the external potential will have the following structure: The momentum k_{n+1} flows out of the electron line at the initial point where there is still an emission factor $\not{k}_{n+1}$. It flows back in at the external potential vertex, but without the additional emission factor $\not{k}_{n+1}$ at that end. We may now sum this over all initial points, again with pairwise cancellation of terms. The term remaining from insertion on the incoming line gives no contribution because a factor $(\not{p} - m)$ acts on the initial spinor. The term left from insertion of the initial point in the line immediately before the external potential is very simple. It is just a factor

$$\frac{2i\alpha}{(2\pi)^3} \int \frac{d^4k_{n+1}}{k_{n+1}^2} b^{(n+1)}(p,p) \tag{2.5}$$

multiplying the expression from the original n-photon graph, which is independent of k_{n+1}.

Case 2. Initial point in p leg, final point in p' leg [factor $b(p',p)$]: Following a similar sequence of steps, we find that this produces a factor

$$\frac{-2i\alpha}{(2\pi)^3} \int \frac{d^4k_{n+1}}{k_{n+1}^2} b^{(n+1)}(p',p) \tag{2.6}$$

times the n-photon result.

Case 3. Both initial and final points in the p' leg [factor $b(p',p')$]: The previous arguments now give zero.

In conclusion, when we add up the contributions for all possible choices of initial and final points, we obtain the startling result that the symmetrically inserted K photon no longer appears in the graph but has been replaced by a simple multiplicative factor. Under the integral sign we can now take

$$b^j(p_f,p_i) = \frac{(2p_f - k_j)\cdot(2p_i - k_j)}{(k_j^2 - 2p_f\cdot k_j)(k_j^2 - 2p_i\cdot k_j)} \tag{2.7}$$

instead of the symmetric form of (2.2) and can further symmetrize under $p \rightleftarrows p'$ by the replacement $b(p,p) \to \frac{1}{2}[b(p,p) + b(p',p')]$.

Finally, we see that the entire effect of symmetrically inserting a K photon into an allowable n photon graph is to produce a factor (αB) multiplying the original graph, where

$$B = \frac{i}{(2\pi)^3} \int \frac{d^4k}{k^2 - \lambda^2} \left(\frac{2p' - k}{k^2 - 2p'\cdot k} - \frac{2p - k}{k^2 - 2p\cdot k} \right)^2 \tag{2.8}$$

is the well-known infrared factor of YFS and α is the fine-structure constant. We have explicitly introduced the photon mass as an infrared regulator.

We still have to account for the allowable $(n+1)$-photon graphs which are obtained from insertions into disallowed n-photon graphs. Specifically, these involve insertions into n-photon graphs which have electron self-energy insertions on the outgoing electron line; they are pictorially represented in Fig. 2(a), where the self-energy part is shown as a shaded blob. It is easy to see that this set of diagrams gives a vanishing contribution. The terms remaining after the pairwise cancellation described above are represented in Fig. 2(b). The circle on the photon line again means that there is no interaction vertex $\not{k}_{n+1}$ associated with it; the momentum k_{n+1} simply flows in at that point. Thus for this type of graph there will be an electron self-energy part which is proportional to

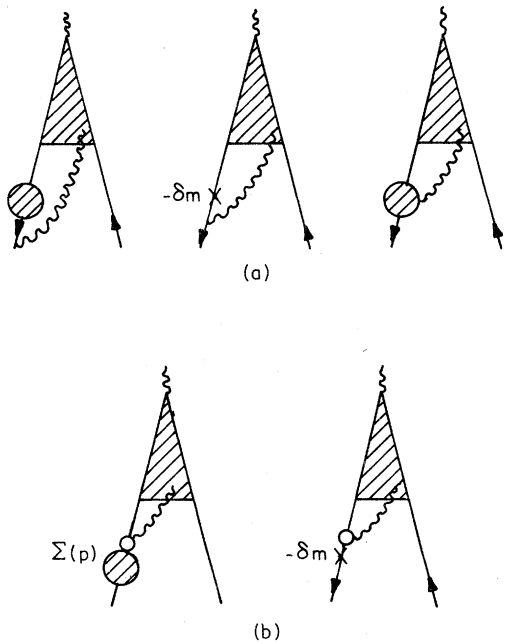

FIG. 2. (a) Allowable $(n+1)$-photon graphs which are obtained from insertions in disallowed n-photon graphs. (b) The net sum of K-photon contributions from (a), taking into account pairwise cancellations.

$(\not{p}' - m)$ directly next to the final spinor $\bar{u}_{p'}$, whence it gives zero.

It is now a simple matter to demonstrate exponentiation of the infrared-divergent factors. Consider the fully symmetrized set of all n-photon vertex graphs, M_n. M_n must of course be divided by $n!$ to correct for the overcounting due to symmetrization. According to Eq. (2.1) each virtual photon propagator can be written as the sum of a G propagator and a K propagator. Each graph in the set can thus be "expanded out" as a sum of graphs in which each virtual photon has only a K or a G propagator. The result is a decomposition into contributions m_{n_G,n_K} $(n_G + n_K = n)$ with n_G G photons and n_K K photons inserted in all possible ways. Each distinct graph clearly arises $n!/n_G!n_K!$ ways, hence we have

$$\sum_{n=0}^{\infty} \frac{1}{n!} M_n = \sum_{n=0}^{\infty} \sum_{n_K=0}^{n} \frac{1}{n_K!} \frac{1}{(n-n_K)!} m_{n_G,n_K} = \sum_{n_K=0}^{\infty} \sum_{n_G=0}^{\infty} \frac{1}{n_K!} \frac{1}{n_G!} m_{n_G,n_K} . \tag{2.9}$$

We have just shown that $m_{n_G,n_K} = (\alpha B)^{n_K} m_{n_G,0} \equiv (\alpha B)^{n_K} m_{n_G}$, which when inserted in (2.9) yields

$$\sum_{n=0}^{\infty} \frac{1}{n!} M_n = \sum_{n_K=0}^{\infty} \frac{(\alpha B)^{n_K}}{n_K!} \sum_{n_G=0}^{\infty} \frac{1}{n_G!} m_{n_G} = e^{\alpha B} \sum_{n=0}^{\infty} \frac{1}{n!} m_n . \tag{2.10}$$

The result (2.10) exhibits the infrared exponentiation found by YFS, according to whom the m_n should be free of infrared divergences. We will shown in Sec. IIC that not only are the m_n infrared finite but that each separate diagram contributing to m_n is as well. We thus have an explicit prescription for projecting out the non-infrared-divergent part of any single vertex diagram: Simply replace each virtual-photon propagator by $-G_{\mu\nu}/k^2 = -(g_{\mu\nu} - bk_\mu k_\nu)/k^2$, with b given by Eq. (2.2). The terms in these graphs which would produce an infrared-divergent part are canceled *before* any integrations are performed and thus no (cancelling) divergent parts occur in the intermediate steps of calculation.

C. Demonstration of Finite Remainder

Let us first look at several lower-order vertex graphs to illustrate how infrared divergences arise. We shall use power counting as the criterion for deciding whether an expression has an infrared divergence. This criterion will be applied to each combination of subintegrations, as well as to the over-all integral. The second-order vertex, shown in Fig. 3(a), has the form

$$\Lambda_\mu^{(1)} \sim \bar{u}_{p'} \int \frac{d^4k}{k^2} \gamma_\alpha \frac{(\not{p}' - \not{k} + m)}{a_k'} \gamma_\mu \frac{(\not{p} - \not{k} + m)}{a_k} \gamma_\beta u_p g^{\alpha\beta} , \tag{2.11}$$

where

$$a_k = k^2 - 2p \cdot k \qquad a_k' = k^2 - 2p' \cdot k \ . \tag{2.12}$$

The small-k behavior of the integrand

$$\Lambda_\mu^{(1)} \sim \int \frac{d^4k}{k^2} \frac{1}{p \cdot k \, p' \cdot k} \times [\gamma_\alpha(\not{p}' + m)\gamma_\mu(\not{p} + m)\gamma^\alpha + O(k)] \tag{2.13}$$

indicates that the integral will diverge logarithmically according to the power-counting criterion (we ignore ultraviolet divergences for the moment). Terms with one or more powers of k in the numerator will be infrared-finite.

Now let us see how replacing $g_{\alpha\beta}$ by $G_{\alpha\beta}$ in Eq. (2.11) leads to an infrared-convergent integral:

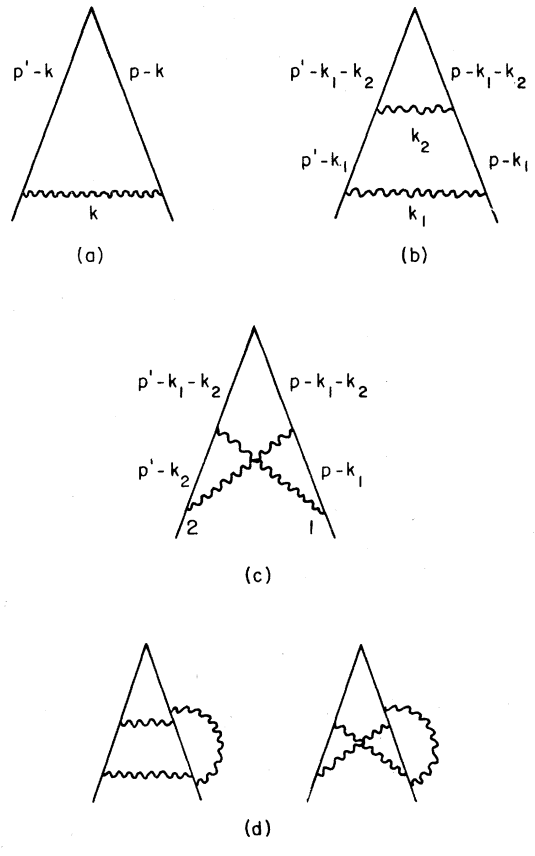

FIG. 3. Various infrared-divergent graphs. (a), (c), and (d) have only an over-all divergence, while (b) has a subdivergence.

$$\Lambda_\mu^{(1)\mathrm{NIR}} \propto \bar{u}_{p'} \int \frac{d^4k}{k^2}\left[\gamma_\alpha \frac{(\not p' - \not k + m)}{a_k'}\gamma_\mu \frac{(\not p - \not k + m)}{a_k}\gamma^\alpha - b(p',p)\not k \frac{(\not p' - \not k + m)}{a_k'}\gamma_\mu \frac{(\not p - \not k + m)}{a_k}\not k\right] u_p$$

$$= \bar{u}_{p'} \int \frac{d^4k}{k^2}\left[\left(\frac{(2p'-k)_\alpha}{a_k'} - \frac{\frac{1}{2}[\gamma_\alpha, \not k]}{a_k'}\right)\gamma_\mu \left(\frac{(2p-k)^\alpha}{a_k} - \frac{\frac{1}{2}[\not k, \gamma^\alpha]}{a_k}\right) - b(p',p)\gamma_\mu\right] u_p$$

$$= \bar{u}_{p'} \int \frac{d^4k}{k^2} \frac{[\not k, \not p']\gamma_\mu + \gamma_\mu[\not p', k] - \frac{1}{4}[\gamma_\alpha, \not k]\gamma_\mu[\gamma^\alpha, \not k]}{a_k a_k'} u_p \,, \tag{2.14}$$

which is clearly infrared-finite. The term in (2.11) which gives rise to the infrared divergence is explicitly canceled by the K subtraction. Interestingly but not obviously, (2.14) is also ultraviolet-finite.

Our next example is the fourth-order ladder vertex of Fig. 3(b), which has the structure

$$\Lambda_\mu^{(2b)} \sim \int \frac{d^4k_1}{k_1^2}\frac{d^4k_2}{k_2^2}\left[\gamma_\alpha \frac{(\not p' - \not k_1 + m)}{a_1'}\gamma_{\alpha'} \frac{(\not p' - \not k_1 - \not k_2 + m)}{a_1' + a_2' + c_{12}}\gamma_\mu \frac{(\not p - \not k_1 - \not k_2 + m)}{a_1 + a_2 + c_{12}}\gamma_{\beta'} \frac{(\not p - \not k_1 + m)}{a_1}\gamma_\beta\right] g^{\alpha\beta} g^{\alpha'\beta'} \tag{2.15}$$

with

$$c_{12} = 2k_1 \cdot k_2 \,. \tag{2.16}$$

We see that the two electron denominators next to the external lines, a_1 and a_1', can simultaneously vanish as $k_1 \to 0$ and thus produce an infrared divergence, but that the internal electron propagators cannot vanish unless both k_1 and k_2 approach zero. For an nth-order ladder vertex with photon momenta, $k_1, \ldots, k_n$, labeled from the "outside in", no divergence can occur as $k_m \to 0$ unless all $k_i \to 0$, for $i < m$, also. Thus the outermost photon in the ladder controls the divergence, and a factor of k_1 in the numerator will render the integral finite in all photon momenta.

Before showing how the replacement of $g_{\alpha\beta}$ by $G_{\alpha\beta}$ in all photon lines renders the ladder graphs free of infrared divergence, we consider the fourth-order crossed vertex shown in Fig. 3(c), which has the structure

$$\Lambda_\mu^{(2c)} \sim \int \frac{d^4k_1}{k_1^2}\frac{d^4k_2}{k_2^2}\left[\gamma_{\alpha'} \frac{(\not p' - \not k_2 + m)}{a_2'}\gamma_\alpha \frac{(\not p' - \not k_1 - \not k_2 + m)}{a_1' + a_2' + c_{12}}\gamma_\mu \frac{(\not p - \not k_1 - \not k_2 + m)}{a_1 + a_2 + c_{12}}\gamma_{\beta'} \frac{(\not p - \not k_1 + m)}{a_1}\gamma_\beta\right] g^{\alpha\beta} g^{\alpha'\beta'} . \tag{2.17}$$

Now both k_1 and k_2 must approach zero simultaneously to produce an infrared divergence. Consequently, either a factor of k_1 or k_2 in the numerator makes the integral finite. Thus if a diagram contains an outermost set of m mutually crossed photons, any numerator factor of k_i for $i \in [1, m]$ makes the entire integral finite. We see then that the set controls the divergence in the same way that the single outermost photon did in the ladder vertices.

We can extend this notion of a set controlling the infrared divergence to include virtual photons in the p or p' legs, provided that they span two or more photons which cross between the two legs. The simplest examples are shown in Fig. 3(d). Here all three photon momenta must go to zero simultaneously in order to produce a divergence.

The general specification of the controlling set is quite simple, namely, it is the set of photon lines which make up the skeleton of any graph (the skeleton is the graph which is obtained when all divergent subgraphs are removed). This may be seen by noting that when all these photon momenta are simultaneously small the electron denominators will be linear in these momenta and the over-all power count for small $\{k_i\}$ will be zero. If any single photon is held fixed, at least three electron denominators will be prevented from vanishing as the other momenta go to zero. Hence there cannot be an infrared divergence associated with any subset. Any factor of $\not k_i$ in the numerator will make the entire integral infrared-free. If we now introduce vertices or electron self-energies into a skeleton (not into the incoming line), it is immediately clear that the momenta inside these insertions need not be small in order for the controlling momenta to have an infrared divergence. On the other hand, these insertions can participate in the infrared divergence and hence produce higher-order divergences.

Now it is a simple matter to show that the skeleton graphs cannot have an infrared divergence when all photons are G photons. To show this, we have merely to ignore all powers of the momenta k_i in the numerator. For a photon which connects the p and p' legs this will lead to a factor such as

$$\bar{u}_{p'} \cdots (\not p' + m)\gamma_\alpha (\not p' + m) \cdots (\not p + m)\gamma_\beta (\not p + m) \cdots u_p$$

$$= \bar{u}_{p'} \cdots (\not p' + m) 2p_\alpha' \cdots 2p_\beta (\not p + m) \cdots u_p \,, \tag{2.18}$$

where we would have arrived at the same result

had γ_α or γ_β stood directly next to a spinor. A similar result obtains for p or p' leg photons which belong to the controlling set. Consequently, each factor $G_{\alpha\beta}$ is multiplied by two powers of external momentum according to

$$p_f^\alpha p_i^\beta G_{\alpha\beta} = p_f \cdot p_i - k \cdot p_f k \cdot p_i b(p_f, p_i) . \quad (2.19)$$

In all cases, these expressions vanish linearly in k, thereby cancelling out the infrared divergence.

We now need to establish that the insertion of vertex parts or electron self-energies into a skeleton graph does not invalidate the above argument. The momenta inside such insertions need not be small in order to have an infrared divergence in other momenta. However, these cases are also easily dispensed with. Suppose, for example, that we have an electron self-energy insertion in the p leg (but not in the external line). The momentum flowing through it will be $p+K$, where K is a linear combination of momenta from the controlling set. Inserting the self-energy increases the number of electron denominators, *apparently* tending to worsen the infrared divergence. It is necessary to study the self-energy contribution for $K \to 0$. Since it is composed of G photons, if we set $K=0$, it will itself be infrared-finite and reduce to a multiple of $(\not{p} - m)$ after mass renormalization. (We note in passing that it differs from the usual electron self-energy in that it has two distinct arguments, $p+K$ and p, the latter coming from the G photons.) Terms linear in K would be of the form $\not{K}$ or $K \cdot p$. In any case, there will be a factor of $(\not{p} + \not{K} + m)$ on either side of the self-energy giving

$$\begin{aligned}(\not{p} + \not{K} + m)[\, a(\not{p} - m) + b\not{K} + cK \cdot p\,](\not{p} + \not{K} + m) \\ = (\not{p} + m)(b\not{K} + cK \cdot p)(\not{p} + m) + O(K^2) \\ = 2K \cdot p\,(b + cm)(\not{p} + m) + O(K^2) . \quad (2.20)\end{aligned}$$

The factor $K \cdot p$ cancels the extra denominator, and the electron self-energy insertion accordingly does not upset the analysis. This is true also for multiple self-energy insertions.

Now suppose that one of the controlling photons enters the electron line through a vertex. Again, since that (lower-order) vertex has G photons, it will not be infrared-divergent and we may expand it in powers of the momenta of the controlling photon. Only the zeroth power need be considered in discussing the infrared divergence. This is a vertex whose electron momenta are both p (or p'). Its structure is then necessarily of the form

$$a' \gamma_\alpha + b' p_\alpha + c' p_\alpha (\not{p} - m) . \quad (2.21)$$

The term with $\not{p} - m$ multiplies either u_p or $(\not{p} + \not{K} + m)$; in either case the leading infrared-divergent term cancels. The first two terms will always yield a factor p_α multiplying $G^{\alpha\beta}$, and hence the infrared divergence cancels as before.

The remaining situation to discuss is a vertex insertion at the external potential (such as the ladder graphs). Now, because the controlling photons are G photons and (2.18) and (2.19) are valid, there can be no infrared divergence.

Finally, it is necessary to discuss briefly the relationship between the infrared and ultraviolet divergences, in order to insure that the latter have not been made more severe by our procedure for treating the former. From the fact that B is convergent in the ultraviolet, it should be obvious that this has not happened. However, it is equally clear that since b depends on the external momenta of the graph as a whole, explicit calculation of higher-order vertices would be considerably more difficult. For example, if we imitate the usual procedure of defining self-energies and subvertices, they will generally have an explicit dependence on p and/or p' as well as on the internal momenta. However, since the factors of b do not affect the degree in k, the usual power-counting rules are not altered.

Now it is easy to see the role of b in the ultraviolet region. Since it then becomes independent of p and p', it corresponds to a gauge transformation; in fact, for large k, a G photon assumes the Landau gauge. This means that if we were to carry out the usual subtractions for renormalization, the ultraviolet-divergent part of the subtraction constants would not have a spurious p, p' dependence. They would be just the cutoff-dependent part of Z_1 and Z_2 in the Landau gauge. The residual finite parts, however, would have this dependence.

It is easily shown that for electron self-energies and vertices defined in terms of G photons the Ward identity is still satisfied at the mass shell, and that Z_1 and Z_2 will be equal and, of course, infrared-finite.

III. REAL-PHOTON EMISSION

A. Rearrangement of the Cross-Section Series

Let us generalize the preceding discussion to include the emission of unobserved real photons. We suppose that in addition to the unobserved photons, there may be a certain number of photons, say n_0, which are detected. We assign to these the momenta $q_1, \ldots, q_{n_0}$. However, in writing the amplitude, we will explicitly show only the dependence on the n undetected photons and suppress the virtual- and observed real-photon parameters (with suitable normalization):

$$\tilde{M}_n = \epsilon_1^{\mu_1}\epsilon_2^{\mu_2}\cdots\epsilon_n^{\mu_n}\rho^n_{\mu_1\cdots\mu_n}(k_1\cdots k_n)e^{\alpha B}. \quad (3.1)$$

Here $\epsilon_i^{\mu_i}$ is the polarization vector of the photon with momentum k_i. This expression is of course symmetric in its dependence on the real-photon variables (including, in fact, the detected ones). The infrared factor from the virtual photons has been separated out of the amplitude, according to the analysis of Sec. II B. We shall argue later that the presence of the real photons does not upset the fact that ρ^n has no infrared divergence associated with the virtual photons. Since we are considering an external potential problem, ρ^n contains the Fourier transform of the potential evaluated at

$$\vec{q} = \vec{p} - \vec{p}' - \sum_{i=1}^{n}\vec{k}_i - \sum_{j=1}^{n_0}\vec{q}_j \ .$$

Energy conservation is to be applied in the cross section.

Let us recall how the infrared divergence arises in the emission of real photons. If a photon is emitted from an incoming line, the matrix element will contain a factor

$$\cdots\frac{(\not{p}-\not{k}+m)}{-2k\cdot p}\not{\epsilon}\,u_p \cong \cdots(-)\frac{p\cdot\epsilon}{k\cdot p}u_p$$

so that ρ is of order $1/k$. Since the cross section contains the phase space $d^3k/2k$, the integral over the photon energy is logarithmically divergent. We shall now present a brief indication of how this divergence could be studied directly in the amplitude, but we shall actually do it a different way later. We rearrange the emission factor according to

$$\not{\epsilon} \to \left(\not{\epsilon} - \not{k}\frac{\epsilon\cdot p_e}{k\cdot p_e}\right) + \not{k}\frac{\epsilon\cdot p_e}{k\cdot p_e}\ , \quad (3.2)$$

where p_e is p or p' according to whether the emission is in the p or the p' leg. Insertion of the second term in all ways leads to a factor

$$\left(\frac{\epsilon\cdot p'}{k\cdot p'} - \frac{\epsilon\cdot p}{k\cdot p}\right) \quad (3.3)$$

multiplying an expression whose k dependence occurs only in the value of $\vec{q}$. By arguments similar to those given in Sec. II, the first term in (3.2) cannot lead to an infrared divergence. Thus in the cross section, the infrared divergence is associated solely with the square of (3.3). The interference terms in the cross section from the two parts of (3.2) are not infrared-divergent.

Rather than pursue this approach, we separate the infrared divergence in the cross section. We first replace the transverse polarization sum by a covariant one using

$$\sum_{\text{pol}}\epsilon^{\mu_i}\epsilon^{\nu_i} = -g^{\mu_i\nu_i}\ , \quad (3.4)$$

which of course uses the gauge-invariance property of ρ^n. The cross section may then be written [$d\vec{p}' \equiv d^3p/(E/M)$, $d\vec{q}_i \equiv d^3q_i/2\omega_i$]

$$\frac{d\sigma_n}{d\vec{p}'d\vec{q}_1\cdots d\vec{q}_{n_0}} = \frac{1}{n!}e^{2\alpha B}\int\delta\left(E-E'-\sum_{j=1}^{n_0}q_j-\sum_{i=1}^{n}\omega_i\right)\rho^{n\dagger}_{\nu_1\cdots\nu_n}\rho^n_{\mu_1\cdots\mu_n}\prod_{i=1}^{n}(-g^{\mu_i\nu_i})\frac{d^3k_i}{2\omega_i}\ . \quad (3.5)$$

Our prescription can now be given in complete analogy with the virtual-photon case, Eqs. (2.1) and (2.2). As before, we rearrange each of the polarization sums in Eq. (3.5) as the sum of two modified polarization sums:

$$-g^{\mu_i\nu_i} = -(g^{\mu_i\nu_i} - \tilde{b}^{(i)}k_i^{\mu_i}k_i^{\nu_i}) - \tilde{b}^{(i)}k_i^{\mu_i}k_i^{\nu_i}$$
$$\equiv -\tilde{G}^{\mu_i\nu_i} - \tilde{K}^{\mu_i\nu_i}\ . \quad (3.6)$$

The factor $\tilde{b}$ is defined as

$$\tilde{b}^{(i)} = \tilde{b}^{(i)}(p_a,p_b) \equiv \frac{p_a\cdot p_b}{k_i\cdot p_a\, k_i\cdot p_b}\ , \quad (3.7)$$

where p_a (p_b) is p or p' according to whether the emission was from the p leg or the p' leg in ρ^n ($\rho^{n\dagger}$). No modification of the polarization sums is made for photons emitted from a closed electron loop.

As in the virtual-photon case, we will "expand out" Eq. (3.5) in terms of $\tilde{G}$ and $\tilde{K}$ photons. Again the Feynman identity can be used to show that the insertion of an additional $\tilde{K}$ photon into $\rho^{n\dagger}\rho^n$ merely produces a factor

$$-\frac{\alpha}{2\pi^2}[\tilde{b}^{(n+1)}(p,p) + \tilde{b}^{(n+1)}(p',p') - 2\tilde{b}^{(n+1)}(p',p)]$$
$$= -\frac{\alpha}{2\pi^2}\left(\frac{p'}{k_{n+1}\cdot p'} - \frac{p}{k_{n+1}\cdot p}\right)^2 \quad (3.8)$$

multiplying $\rho^{n\dagger}\rho^n$ and alters the momentum transfer to the potential by $-\vec{k}_{n+1}$. The allowable $(n+1)$-photon graphs obtained by insertions into disallowed n-photon graphs again cause no difficulty.

The cross section for emission of a given set of unobserved photons of momentum $\{k_i\}$ is thus given by a sum of terms, in each of which a given photon is assigned to be a $\tilde{G}$ or a $\tilde{K}$ photon. Consider any such assignment and let S_K (S_G) be the set of k_i's which are to be K photons (G photons). Let the number of photons in each set be n_K and n_G, respectively. Then we have

$$\prod_{i=1}^{n}(-g^{\mu_i\nu_i})\rho^{n\dagger}_{\mu_1}\cdots_{\mu_n}\rho^{n}_{\mu_1}\cdots_{\mu_n}=\sum_{S_K}\prod_{i\in S_K}\left[-\frac{\alpha}{2\pi^2}\left(\frac{p'}{k_i\cdot p'}-\frac{p}{k_i\cdot p}\right)^2\right]\prod_{j\in S_G}(-\tilde{G}^{\mu_j\nu_j})\rho^{n_G\dagger}_{\cdots\mu_j\cdots}\rho^{n_G}_{\cdots\nu_j\cdots}\,. \tag{3.9}$$

A given decomposition $n_K+n_G=n$ occurs in $n!/n_G!n_K!$ ways so that (3.5) may be rewritten as

$$\frac{d\sigma_n}{d\vec{p}'d\vec{q}_1\cdots d\vec{q}_{n_0}}=e^{2\alpha B}\sum_{n_K=0}^{n}\int\delta\left(E-E'-\sum_{l=1}^{n_0}q_l-\sum_{j=n_K+1}^{n}k_j-\sum_{i=1}^{n_K}\omega_i\right)\frac{1}{n_K!}\prod_{i=1}^{n_K}\frac{d^3k_i}{2\omega_i}\left[-\frac{\alpha}{2\pi^2}\left(\frac{p'}{k_i\cdot p'}-\frac{p}{k_i\cdot p}\right)^2\right]$$

$$\times\frac{1}{n_G!}\prod_{j=n_K+1}^{n}\frac{d^3k_j}{2\omega_j}(-\tilde{G}^{\mu_j\nu_j})\,\rho^{n_G\dagger}_{\cdots\mu_j}\cdots\rho^{n_G}_{\cdots\nu_j\cdots}\,, \tag{3.10}$$

where

$$\vec{q}=\vec{p}-\vec{p}'-\sum_{l=1}^{n_0}\vec{q}_l-\sum_{j=n_K+1}^{n}\vec{k}_j-\sum_{i=1}^{n_K}\vec{k}_i\,. \tag{3.11}$$

In Eq. (3.10), ρ^{n_G} depends on k_i, $i\in[1,n_K]$ only through the value of q. To make this dependence more explicit, we introduce

$$1=\int d^3q\,\delta\left(\vec{q}-\vec{p}+\vec{p}'+\sum_{l=1}^{n_0}\vec{q}_l+\sum_{j=n_K+1}^{n}\vec{k}_j+\sum_{i=1}^{n_K}\vec{k}_i\right) \tag{3.12}$$

and use $\vec{q}$ as an argument of $\rho^{n_G\dagger}\rho^{n_G}$ rather than the right-hand side of (3.11). If the photon momenta which comprise the infrared-divergent factors did not also appear in the δ function, we could sum (3.10) over n and exponentiate the infrared dependence. We can still achieve the exponentiation by use of a standard technique,

$$\delta^4\left(f-q-\sum_{j=n_K+1}^{n}k_j-\sum_{i=1}^{n_K}k_i\right)=\int\frac{d^4x}{(2\pi)^4}\exp\left[i\left(f-q-\sum_{j=n_K+1}^{n}k_j\right)\cdot x\right]\prod_{i=1}^{n_K}e^{-ik_i\cdot x}\,, \tag{3.13}$$

where

$$f=p-p'-\sum_{l=1}^{n_0}q_l\,. \tag{3.14}$$

The $n=0$ term in (3.10) vanishes as B is negative and infinite. We will extract the $n_G=0$ term and treat it separately as was done in Ref. 6, using the identity $1=\sum_{i=1}^{n_K}k_i/f_0$. Then Eq. (3.10) becomes

$$\frac{d\sigma}{d\vec{p}'d\vec{q}_1\cdots d\vec{q}_{n_0}}=\int\frac{d^4x}{(2\pi)^4}\,d^3q\,e^{2\alpha B+2\alpha\hat{B}(x)}e^{i(f-q)\cdot x}$$

$$\times\left[\int d^3k_1\tilde{S}(k_1)e^{-ik_1\cdot x}\frac{\rho^{0\dagger}\rho^{0}}{f_0}+\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{j=1}^{n}\int\frac{d^3k_j}{2k_j}e^{-ik_j\cdot x}(-\tilde{G}^{\mu_j\nu_j})\rho^{n\dagger}_{\nu_1\cdots\nu_n}\rho^{n}_{\mu_1\cdots\mu_n}\right], \tag{3.15}$$

with

$$\tilde{S}(k)=-\frac{\alpha}{(2\pi)^2}\left(\frac{p'}{k\cdot p'}-\frac{p}{k\cdot p}\right)^2 \tag{3.16}$$

and

$$2\alpha\hat{B}(x)=\int\frac{d^3k}{\omega}\tilde{S}(k)e^{-ik\cdot x}\,. \tag{3.17}$$

The infrared-divergent part of $\alpha\tilde{B}(x)$ can be decoupled from the x integration in the following way:

$$2\alpha\hat{B}(x)=2\alpha\tilde{B}(\epsilon)+\alpha\tilde{D}(x,\epsilon)\,, \tag{3.18}$$

where

$$2\alpha\tilde{B}(\epsilon)=\int\frac{d^3k}{\omega}\tilde{S}(k)\theta(\epsilon-\omega) \tag{3.19}$$

and

$$\alpha\tilde{D}(x,\epsilon)=\int\frac{d^3k}{\omega}\,\tilde{S}(k)[e^{-ik\cdot x}-\theta(\epsilon-\omega)]\,. \tag{3.20}$$

The constant ϵ is to be chosen at some convenient value as will be discussed later. Finally, we substitute (3.18) in Eq. (3.15) to get

$$\frac{d\sigma}{d\vec{p}'d\vec{q}_1\cdots d\vec{q}_{n_0}}=e^{2\alpha B+2\alpha\tilde{B}(\epsilon)}\int\frac{d^4x}{(2\pi)^4}\,d\vec{q}\,e^{\alpha\tilde{D}(x,\epsilon)}e^{i(f-q)\cdot x}$$

$$\times\left[\int d^3k_1\tilde{S}(k_1)e^{-ik_1\cdot x}\frac{\rho^{0\dagger}\rho^{0}}{f_0}+\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{j=1}^{n}\int\frac{d^3k_j}{2k_j}e^{-ik_j\cdot x}(-\tilde{G}^{\mu_j\nu_j})\rho^{n\dagger}_{\nu_1\cdots\nu_n}\rho^{n}_{\mu_1\cdots\mu_n}\right]. \tag{3.21}$$

The well-known combination $2\alpha B+2\alpha\tilde{B}(\epsilon)$ is infrared-finite.[2] We shall not study this combination further except to note that it contains a term

$$\alpha A \ln\frac{\epsilon}{E} \tag{3.22}$$

so that the whole exponential vanishes as

$$\left(\frac{\epsilon}{E}\right)^{\alpha A} \tag{3.23}$$

as $\epsilon \to 0$. Here

$$\alpha A = \vec{k}^2 \int d\Omega\, \tilde{S}(\vec{k}) \cong \frac{2\alpha}{\pi}\left(\ln\frac{2p\cdot p'}{m^2}-1\right) \quad (p\cdot p' \gg m^2). \tag{3.24}$$

This apparent ϵ dependence is of course precisely compensated by the integral.

The important underlying features of Eq. (3.21) can be more easily seen by specializing to the case of very small f_0. For such a kinematic situation the cross section is dominated by the first term in square brackets. Also, since the total momentum carried off by unobserved photons is not greater than f_0, it is a reasonable approximation to replace (3.11) by $\vec{q}=\vec{f}$. The function of $\alpha\tilde{D}$ thus will not depend on $\vec{k}$ so that (3.21) reduces to

$$\frac{d\sigma}{d\vec{p}'d\vec{q}_1\cdots d\vec{q}_{n_0}} = e^{2\alpha B+2\alpha\tilde{B}(\epsilon)}\int\frac{dx_0}{(2\pi)}e^{\alpha\tilde{D}(x_0,\epsilon)} \times\int d^3k_1\tilde{S}(k_1)e^{i(f^0-k_1^0)x_0}\frac{|\rho_0(\vec{q}=\vec{f})|^2}{f_0}. \tag{3.25}$$

In the above approximation, which is equivalent to that of YFS, the x_0 integration may be performed exactly. Here, however, we expand $e^{\alpha\tilde{D}}$ and integrate term by term. The second term (i.e., $\alpha\tilde{D}$) can be made to vanish by the choice $\epsilon=f_0$, giving

$$\frac{d\sigma}{d\vec{p}'d\vec{q}_1\cdots d\vec{q}_{n_0}} = e^{2\alpha[B+\tilde{B}(f_0)]} \times\frac{\alpha A}{f_0}\left[1-\tfrac{1}{12}\pi^2(\alpha A)^2+\cdots\right]|\rho^0|^2 \propto \frac{\alpha A}{f_0^{1-\alpha A}}|\rho^0|^2. \tag{3.26}$$

Thus inclusion of the radiative correction softens the singularity as $f_0\to 0$, thereby making the cross section integrable.

B. Demonstration of Finite Cross Section

To simplify the discussion of the x and $\vec{q}$ integrations, we define the quantity

$$F(K)=\theta(f_0-K_0)\int d^3q\left\{\int d^3k_1\tilde{S}(k_1)\frac{|\rho^0(\vec{q})|^2}{f_0}\delta^4(f-q-k_1-K) + \sum_{n=1}^{\infty}\frac{1}{n!}\prod_{j=1}^{n}\int\frac{d^3k_j}{2\omega_j}(-\tilde{G}^{\mu_j\nu_j})\rho^{n\dagger}_{\nu_1\cdots\nu_n}\rho^{n}_{\mu_1\cdots\mu_n}\delta^4\left(f-q-\sum_{i=1}^{n}k_i-K\right)\right\}. \tag{3.27}$$

Then in Eq. (3.21) we can expand $e^{\alpha\tilde{D}(x,\epsilon)}$ and perform the x and $\vec{q}$ integrations term by term to obtain

$$\frac{d\sigma}{d\vec{p}'d\vec{q}_1\cdots d\vec{q}_{n_0}} = e^{2\alpha B+2\alpha\tilde{B}(\epsilon)}\left\{F(0)+\int\frac{d^3k_1'}{\omega_1'}\tilde{S}(k_1')[F(k_1')-F(0)\theta(\epsilon-\omega_1')] + \int\frac{d^3k_1'}{\omega_1'}\frac{d^3k_2'}{\omega_2'}\tilde{S}(k_1')\tilde{S}(k_2')[F(k_1'+k_2')-F(k_1')\theta(\epsilon-\omega_2')-F(k_2')\theta(\epsilon-\omega_1') + F(0)\theta(\epsilon-\omega_1')\theta(\epsilon-\omega_2')]+\cdots\right\}. \tag{3.28}$$

Clearly, the differences in the square brackets which appear in each order (of $\tilde{D}$) eliminate the infrared divergence, provided the function F itself is finite.

We will now argue that the k_i integrations of Eq. (3.21) are finite, so that F is well defined. Equations (3.6) and (3.7) imply that

$$\tilde{G}_{\alpha\beta}p_a^{\alpha}p_b^{\beta}=0, \tag{3.29}$$

a result similar to that of (2.19).

We first consider real-photon emission only; the addition of virtual corrections will be discussed later. It is not necessary to look at the detailed structure of the graphs contributing to ρ and $\rho^{\dagger}$; rather, we will utilize the general feature that each rationalized electron propagator will have a numerator structure of the form $(\not{p}-\not{K}+m)$ for those in the p leg and $(\not{p}'+\not{K}+m)$ for those in the p' leg, where K is some linear combination of the k_i's. Consider, for example, the contribution from some pair of graphs in ρ and $\rho^{\dagger}$. If at first all numerator factors of k_i are neglected, each

$\tilde{G}$ photon will appear in the form

$$\rho^\dagger\rho \propto [\cdots(\not{p}_b+m)\gamma_\beta(\not{p}_b+m)\times\cdots(\not{p}_a+m)\gamma_\alpha(\not{p}_a+m)\cdots]\tilde{G}^{\alpha\beta}$$
$$=[\cdots 2p_{b\beta}(\not{p}_b+m)\cdots 2p_{a\alpha}(\not{p}_a+m)\cdots]\tilde{G}^{\alpha\beta}$$
$$=0. \qquad (3.30)$$

A similar result obtains if either γ_α or γ_β stands next to a spinor.

Thus there must be some factors of k_i in the numerator (in ρ and/or $\rho^\dagger$) in order to have a nonvanishing contribution to the cross section. We will designate as S_0 the set of photons whose momenta appear somewhere in the numerator of a term in either ρ or $\rho^\dagger$. These numerator factors suffice to assure that the photons in S_0 cannot contribute to an infrared divergence. In addition, the momentum of a photon in S_0, call it k_s, flows through all electron lines lying between the emission point of that photon and the external vertex, and thus none of these electron propagators can vanish independently of k_s. Accordingly, the integral is also finite in the momenta of photons emitted internal of any photon in S_0. (This can occur in ρ and/or $\rho^\dagger$.) We expand the original set to include these additional photons and call the result S_1. By the same arguments, any additional photons emitted internal to members of S_1 will be infrared-free. We continue the above process for identification of infrared-finite photons until it closes on the largest set, S_m. If S_m contains all of the n real photons, the integral in question will be infrared-finite in all photon momenta. On the other hand, any photons not in S_m must be entirely external to S_m (otherwise, it would belong to S_m), and would thus have the structure (3.30), which gives zero. It follows that all of the k_i, $i\in[1,n]$ integrations in Eq. (3.21) are infrared-finite.

Finally, we must consider the role of virtual photons. If there are no internal vertices or self energies in ρ or $\rho^\dagger$, the virtual-photon momenta can be treated on the same footing as real-photon momenta, and the above arguments can be again carried through. Vertices and self-energy insertions can be treated as in Sec. II C. We conclude that F is completely free of infrared divergences.

IV. GENERALIZATIONS AND REFINEMENTS

In the previous sections we have discussed the infrared structure of radiative corrections to potential scattering treated in lowest order. Infrared factorization and exponentiation were achieved through manipulation of G and K photons, a concept we would now like to extend to more complicated processes. One particularly simplifying feature of the case already treated was that the external four-momentum entered the electron line at a single vertex. We thus had a natural way to separate the electron line into p and p' legs, thereby defining unambiguously the G- and K-photon propagators. Such a feature does not exist, for instance, when there are multiple interactions with the external potential. In this case, momentum is transferred to the electron line at a number of different points and the distinction between p and p' legs thus becomes somewhat ambiguous. A similar problem arises in Compton scattering and in vertices with internal fermion loops, such as the one shown in Fig. 4. As we shall show, this difficulty is easily dealt with for these cases. Only in a process such as electron-electron scattering do we have to introduce additional techniques.

In Compton scattering external momentum enters (leaves) the electron line at two different points, the emission or absorption vertices of the two hard real photons. As we have mentioned, neither hard photon can be considered more fundamental in distinguishing between p and p' legs of the electron line. We could, for instance, select the incoming photon as the distinguishing one and define b of Eq. (2.2) accordingly. On the other hand, we could use the emission vertex of the final photon to separate the electron line, giving another definition of b. In either case, the same infrared factorization is achieved. The residual infrared-finite series will differ for the two choices only in the definition of G propagators for those photons which terminate on electron lines which lie between the emission vertices of the two hard photons. As the (nonvanishing) momentum of the hard photons runs through these electron lines, the corresponding electron propagators cannot vanish. Thus photons which terminate on lines between the two hard photons cannot participate in an infrared divergence. This means that the contributions of particular graphs to the remainder series in the two cases differ only by infrared-finite terms. The sum of contributions in each order will necessarily be identical [e.g., the m_n in Eq. (2.10)]. Thus the results of Sec. III can be immediately applied to Compton scattering. In fact, we have essentially done so in Sec. III by allowing hard photons in the final state. We need only reinterpret the external vertex as the absorption vertex of the incoming photon and add a δ function for momentum conservation in that development.

We proceed in a similar manner for vertices with internal fermion loops such as that shown in Fig. 4. For this case momenta q_i, $i\in[1,m]$ enter the electron line at different points with the con-

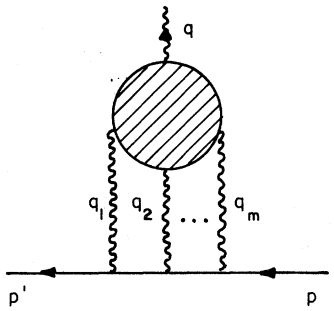

FIG. 4. An example where the p and p' legs cannot be unambiguously defined. As explained in the text, any of the exchanged photons may be selected to separate the electron line into two legs.

straint $\sum_{i=1}^{m} q_i = q$. Since the four-momentum entering each point is now a variable, there is no apparent way to distinguish one photon as carrying a large momentum transfer which can be used to distinguish p and p' legs. However, it is a property of closed electron loops in QED (quantum electrodynamics) (due to gauge invariance) that the corresponding function vanishes as the momentum of any photon entering the loop vanishes. A consequence of this is that any photon connecting points on the electron line that runs through the graph cannot participate in an infrared divergence if one of those points lies between the vertices of photons coming from the closed loop. Thus we may arbitrarily select one of the photons of momentum q_i as the distinguishing one for the same reasons as in Compton scattering. Again different choices of the distinguishing photon produce the same infrared factorization and identical contributions to the remainder series order by order. The contributions of individual graphs to the remainder series will, of course, differ by infrared-finite terms in each case.

Electron-electron scattering is slightly more problematic. Suppose there are m virtual photons exchanged between the two electron lines, labeled with momenta q_i such that $\sum_{i=1}^{m} q_i = q$, where q is the net four-momentum transfer between the two lines. Again, the amount of momentum each exchanged photon brings into one of the electron lines varies across the domain of the q_i integration. In addition, the exchanged photons do participate in the infrared divergences and we must therefore expect that it will be necessary to modify their propagators also. Thus none of the exchanged photons can be singled out as the distinguishing one over the entire domain of integration without first decoupling that photon from the infrared divergences. We can do this in the following simple way: The denominator structure due to the exchanged photon propagators will be of the form

$$\int \prod_{k=1}^{m} \frac{d^4 q_k}{q_k^2}\, \delta^4\left(q - \sum_{l=1}^{m} q_l\right), \tag{4.1}$$

which can be rewritten as

$$\int \prod_{k=1}^{m} \frac{d^4 q_k}{q_k^2} \frac{\left(\sum_{i=1}^{m} q_i\right)^2}{q^2}\, \delta^4\left(q - \sum_{l=1}^{m} q_l\right) = \frac{1}{q^2}\left\{\sum_{i=1}^{m} \int \prod_{k\neq i}^{m} \frac{d^4 q_k}{q_k^2} + \sum_{\substack{i,j \\ i>j}}^{m} \int \prod_{k\neq i,j}^{m} \frac{d^4 q_k}{q_k^2} \int \frac{d^4 q_i}{q_i^2} \frac{d^4 q_j}{q_j^2}\, 2 q_i \cdot q_j\, \delta^4\left(q - \sum_{l=1}^{m} q_l\right)\right\}. \tag{4.2}$$

In the first sum, the ith photon can no longer participate in the infrared divergence. It may therefore be selected to separate the p and p' legs in both electron lines. In the second sum, either the ith or jth photon may be selected for that purpose. Only the photons labeled here with a k need to have their polarization sums rearranged into G and K types. The combinatorics works out straightforwardly to give infrared exponentiation from the K photons, times an infrared-free expression. The factors p_a and p_b in the definition of b are now to be assigned the momenta of the electron leg to which the photon attaches.

The techniques of the preceding paragraph can also be used to treat multiple scattering from a Coulomb potential. The appropriate definition of b is obtained by taking the infinite-mass limit for the target.

V. SUMMARY AND CONCLUSIONS

In this paper we have presented a more transparent demonstration of the exponentiation and subsequent cancellation of infrared-divergent contributions to observable cross sections. Our technique of separating polarization sums into G and K sums leads very directly to the factorization of infrared-divergent terms on the one hand, and to a simple prescription for obtaining the infrared-finite remainder on the other. Most of our attention has been devoted to the treatment of potential scattering in lowest order, so as to present the basic techniques as simply as possible. The few illustrative generalizations we have given should indicate the kind of modifications needed in order to apply these techniques to other processes.

It is appropriate at this point to comment on the

utility of these techniques, or rather the lack of it, in performing higher-order calculations. Since the remainder series is completely infrared-finite, graph-by-graph computation is certainly simplified in that artificial cutoffs on photon momenta can be avoided. A severe problem is that the factors b, in terms of which the G propagators are defined, introduce extra denominators which require additional integrations when Feynman parameters are introduced. It is not surprising that such a difficulty arises. Suppose, for example, one were to attempt to make an infrared-divergent integral finite by subtracting from the integrand its value for small momenta. The over-all integral could thus be made finite, although each term separately would still be divergent. Such a subtracted form is usually acceptable for numerical integrations. The two terms, which in general would have an entirely different denominator structure, could be combined with a common denominator to give an explicitly infrared-finite expression. The new expression would necessarily have more denominators, however. In this spirit, the expressions for the contributions from graphs with G photons are in a combined form and accordingly have extra denominators. One could, of course, expand out graphs in terms of $g_{\mu\nu}$ and $bk_\mu k_\nu$ propagators, apply the identity (2.3) wherever possible, and thus remove the extra denominators. The result would be an infrared-finite sum of separately divergent terms, a form which might conceivably be useful in numerical work.

Let us review those features of our method which are or are not desirable in calculations. The use of G-photons gives a simple prescription for identifying what to "subtract" from a given integral in order to make it infrared-finite. In addition, we know in advance the contribution of the divergent terms we have subtracted off so that we never need actually calculate them. Use of a more arbitrary subtraction method would require the analytic calculation of the terms subtracted off to assure that a proper infrared cancellation had been effected and to find the remaining finite contribution due to these terms. Thus one main feature we would like to retain in any reformulation of our method is that the subtractions be defined so as to add up to give a known infrared structure. The number of subtraction terms associated with each graph must be kept small if the method is to be useful, however. In addition, they must have a denominator structure similar to that of the original graph for the reasons discussed above. Finally, a simple prescription for obtaining the subtracted (infrared-finite) result, such as a modified set of Feynman rules, is certainly desirable. It seems doubtful our method can be made to satisfy all the above criteria by a trivial redefinition of b. It is true, however, that by combining related sets of graphs a simplified infrared structure is obtained. Thus it is conceivable that a judicious choice of b for different sets of graphs can produce an infrared subtraction scheme amenable to actual computations.

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