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Calculation of the Graviton Self-Energy Using Dimensional Regularization

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One-loop contributions to the graviton self-energy are calculated, in the context of covariant quantization, by employing a modified technique of dimensional regularization and by extending Goldberg's version of the Einstein Lagrangian to n dimensions. It is shown that the sum of the graviton and fictitious-particle contributions to the graviton propagator satisfies Slavnov-Ward identities and that the finite part of this sum can be extracted in a manner consistent with these identities. There are no infrared problems in off-mass-shell Green's functions, and tadpole contributions can consistently be set equal to zero.

I. INTRODUCTION

The beauty and apparent simplicity of Einstein's theory of general relativity give little indication of the difficult problems which one encounters in the quantization of the gravitational field, for instance in the derivation of Feynman rules. To derive these rules, one may pursue one of several approaches to quantization, such as the technique of canonical quantization[1] or covariant quantization. The general program of covariant quantization for the gravitational field has previously been studied by Feynman,[2] Mandelstam,[3] DeWitt,[4] Faddeev and Popov,[5] and Fradkin and Tyutin.[6] In particular, these authors emphasized the necessity of introducing *fictitious* particles in order to construct a unitary and gauge-invariant S matrix. Despite all this theoretical work, there is a marked absence of explicit calculations in quantum gravity.

It is the purpose of this article to suggest a regularization technique and to apply it to the *graviton self-energy* working to lowest order in the gravitational coupling constant K ($K^2 = 32\pi G$). Following the technique of 't Hooft, we derive appropriate Slavnov-Ward identities[7] for the two-point graviton Green's function.

In view of the somewhat technical nature of the present paper a few comments are in order on its physical background. In the classical theory of gravitation one is led fairly naturally (although not uniquely) to the Einstein Lagrangian which is in general a nonlinear function of a single field, namely the metric tensor, and which involves two derivatives. From the relativist's point of view there is little motivation, either theoretical or experimental, for trying to quantize this Lagrangian except as a possible way of escaping gravitational collapse. However, for the quantum-field theorist it is natural to think of the curved space of general relativity as being due to the propagation of "gravitons" in a flat background metric. While such a picture appears to be consistent for tree graphs, higher-order contributions lead to divergent Feynman diagrams containing graviton loops and this necessitates renormalization. The Einstein Lagrangian is also interesting to the quantum-field theorist as an example of a non-Abelian gauge theory possessing a high degree of symmetry.

The Einstein Lagrangian contains two derivatives in the interaction part making the resulting Feynman diagrams highly divergent; for this reason the problem of renormalizing the Lagrangian to all orders is still unresolved. However, by analogy with quantum electrodynamics, we might expect that the use of a *gauge-invariant* regularizing technique would remove at least some of the leading divergences. In this way a field theory—which on power-counting arguments alone might be expected to be nonremormalizable—can in fact be shown to be renormalizable through the extra restrictions imposed by gauge invariance. In this paper we demonstrate that dimensional regularization successfully solves the first problem in the renormalization program: It provides a regularizing technique which is gauge-invariant in the sense of *preserving the Slavnov-Ward identities*. Although the completion of this program, i.e., the program of providing a completely renormalizable quantized theory of gravity, is far from being realized, such a program could have interesting consequences. For instance it would be possible to test the conjecture[4] that gravity—by smearing the light-cone singularities—provides a universal regulator for other quantum-field theories as well. Quantum effects might also play a crucial role in gravitational collapse.

The outline of the paper is as follows. In Sec. II we use a particular form of the n-dimensional Einstein Lagrangian (developed in Appendix A) to derive the graviton and fictitious-particle Feynman rules. The remodeled Lagrangian, written in terms of the tensor density $\tilde{g}^{\alpha\beta}=\sqrt{-g}\,g^{\alpha\beta}$, leads to a set of Feynman rules which is considerably less complicated than the corresponding set in terms of the metric tensor $g^{\alpha\beta}$ (see Sec. V C). In Sec. III we derive the crucial Slavnov-Ward identity [Eq. (3.2)], which holds to each order in the gravitational coupling constant K.

The explicit calculation of the graviton self-energy and our suggested technique for regularizing it are outlined in Secs. IV and V. In the first part of Sec. IV we compute in considerable detail the contribution from the fictitious-particle loop. The analogous derivation for the graviton loop is given in Sec. IV B. We emphasize that, due to the particular regularizing technique employed, there are no infrared-divergence problems arising in the off-mass-shell Green's functions.

To evaluate the various momentum-space integrals, we employ a modified version of the method of dimensional regularization[8-10] which is particularly well suited for gauge theories and whose principal features may be summarized as follows. First we define each integral over a 2ω-dimensional Euclidean space (ω in general complex) and evaluate each integral for general ω. We then expand the resulting expressions in a Laurent series about the pole $\omega=2$ (four-dimensional space-time). Pole terms in the Laurent expansion may be canceled by inserting appropriate counterterms in the interaction Lagrangian. The value of each integral is given by the remaining part of the expansion, continued to Minkowski space.

We begin Sec. V by showing that the total amplitude $T_{\alpha\beta\alpha'\beta'}(p)$ for the graviton self-energy satisfies the Slavnov-Ward identities [Eqs. (5.11) and (5.12)]. In Sec. V B we construct with this self-energy $T_{\alpha\beta\alpha'\beta'}$, the connected Green's function $Q_{\nu\sigma\mu\lambda}$. The latter satisfies the Slavnov-Ward identity (5.10), which is independent of ω. Hence, expanding $Q_{\nu\sigma\mu\lambda}$ about $\omega=2$ and continuing analytically to Minkowski space, we find that $Q_{\nu\sigma\mu\lambda}$ decomposes into a pole term and the finite (physical) part of the connected Green's function.

We shall work almost entirely in Euclidean space; only in Sec. V B do we change from Euclidean to Minkowski space. We employ natural units, $\hbar=c=1$, throughout this paper, in which case the gravitational constant $K^2=32\pi G\simeq 4\times 10^{-44}(m_e)^{-2}\simeq 4\times 10^{-38}\ \text{GeV}^{-2}$, where G is the Newtonian constant and m_e the mass of the electron.

II. FEYNMAN RULES

In this section we summarize the relevant Feynman rules (see, for instance, Ref. 6). The Lagrangian density is given by

$$\mathcal{L}=+\frac{2}{K^2}\sqrt{-g}\,g^{\mu\nu}R_{\mu\nu}\,, \tag{2.1}$$

where $g^{\mu\nu}$ is the metric tensor. $R_{\mu\nu}$ is defined by

$$R_{\mu\nu}=\Gamma^{\rho}_{\mu\rho,\nu}-\Gamma^{\rho}_{\mu\nu,\rho}-\Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho}+\Gamma^{\rho}_{\sigma\nu}\Gamma^{\sigma}_{\mu\rho}\,, \tag{2.2}$$

where

$$\Gamma^{\rho}_{\mu\nu}=\tfrac{1}{2}g^{\rho\sigma}(g_{\mu\sigma,\nu}+g_{\sigma\nu,\mu}-g_{\mu\nu,\sigma})\,. \tag{2.3}$$

It is convenient to define

$$\tilde{g}^{\mu\nu}=\sqrt{-g}\,g^{\mu\nu}\,, \tag{2.4}$$

which enables us to write $\mathcal{L}$ in the form (see Appendix A)

$$\mathcal{L}=\frac{1}{2K^2}\left(\tilde{g}^{\rho\sigma}\tilde{g}_{\lambda\mu}\tilde{g}_{\kappa\nu}-\frac{1}{n-2}\tilde{g}^{\rho\sigma}\tilde{g}_{\mu\kappa}\tilde{g}_{\lambda\nu}-2\delta^{\sigma}_{\kappa}\delta^{\rho}_{\lambda}\tilde{g}_{\mu\nu}\right)\times\tilde{g}^{\mu\kappa}{}_{,\rho}\,\tilde{g}^{\lambda\nu}{}_{,\sigma}\,, \tag{2.5}$$

where n is the dimension of the space. Next, consider the generating functional (see Appendix B)

$$Z[j_{\mu\nu}]=\int d[\tilde{g}^{\mu\nu}]\,\Delta[\tilde{g}^{\mu\nu}]\times\exp\left\{i\int dx\left[\mathcal{L}+\frac{1}{K}g^{\mu\nu}j_{\mu\nu}-\frac{1}{K^2\alpha}(\partial_\mu\tilde{g}^{\mu\nu})^2\right]\right\}, \tag{2.6}$$

where $\Delta[\tilde{g}^{\mu\nu}]$ is the fictitious-particle contribution and $-(K^2\alpha)^{-1}(\partial_\mu\tilde{g}^{\mu\nu})^2$ the gauge-breaking term. $Z[j_{\mu\nu}]$ leads to an S matrix which is unitary as well as independent of the gauge, specified by the parameter α.

If we define the graviton field $\phi^{\mu\nu}$ by

$$\tilde{g}^{\mu\nu}=\delta^{\mu\nu}+K\phi^{\mu\nu}, \tag{2.7}$$

where $\delta^{\mu\nu}$ is the n-dimensional Kronecker δ, then

$$\tilde{g}_{\mu\nu}=\delta_{\mu\nu}-K\phi_{\mu\nu}+K^2\phi_{\mu\alpha}\phi_{\alpha\nu}-K^3\phi_{\mu\alpha}\phi_{\alpha\beta}\phi_{\beta\nu}+O(K^4)\,. \tag{2.8}$$

Due to definition (2.7), there is now no need to distinguish between upper and lower indices on $\phi_{\mu\nu}$. Writing the Lagrangian $\mathcal{L}$ as

$$\mathcal{L}=\sum_{j=2}^{\infty}K^{j-2}\mathcal{L}_{(j)}\,, \tag{2.9}$$

we find that

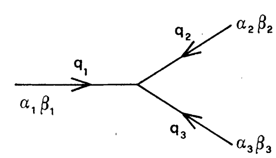

FIG. 1. The three-graviton vertex.

$$\mathcal{L}_{(2)}(x)=\tfrac{1}{2}\partial_\mu\phi_{\nu\lambda}\partial_\mu\phi_{\nu\lambda}(x)-\frac{1}{2(n-2)}\partial_\mu\phi_{\nu\nu}\partial_\mu\phi_{\rho\rho}(x)-\partial_\mu\phi_{\mu\nu}\partial_\rho\phi_{\rho\nu}(x)\,. \tag{2.10}$$

We fix the gauge by choosing $\alpha=-1$, which results in the considerably simplified free propagator

$$D_{\alpha\beta,\lambda\mu}(x)=\tfrac{1}{2}(\delta_{\alpha\lambda}\delta_{\beta\mu}+\delta_{\alpha\mu}\delta_{\beta\lambda}-\delta_{\alpha\beta}\delta_{\lambda\mu})D(x)\,, \tag{2.11a}$$

where $D(x)=(4\pi x^2)^{-1}$ is the massless scalar propagator, or, in momentum space,

$$D_{\alpha\beta,\lambda\mu}(p)=\frac{1}{2p^2}(\delta_{\alpha\lambda}\delta_{\beta\mu}+\delta_{\alpha\mu}\delta_{\beta\lambda}-\delta_{\alpha\beta}\delta_{\lambda\mu})\,. \tag{2.11b}$$

The inverse of the fictitious-particle factor $\Delta[\tilde{g}^{\mu\nu}]$ is shown in Appendix B to be

$$(\Delta[\tilde{g}^{\mu\nu}])^{-1}=\int d[\xi_\lambda]d[\eta_\nu]\exp\left\{i\int dx\,\eta_\nu[\delta_{\nu\lambda}\Box-K(\phi_{\mu\nu,\lambda\mu}-\phi_{\mu\rho}\delta_{\nu\lambda}\partial_\mu\partial_\rho-\phi_{\mu\rho,\mu}\delta_{\nu\lambda}\partial_\rho+\phi_{\mu\nu,\mu}\partial_\lambda)]\xi_\lambda\right\}, \tag{2.12}$$

where η_ν and ξ_λ are the fictitious particles whose Feynman propagator is

$$\langle T(\xi_\mu\eta_\nu)\rangle=\frac{\delta_{\mu\nu}}{p^2}\,. \tag{2.13}$$

[We also observe that $\langle T(\xi_\mu\xi_\nu)\rangle=0=\langle T(\eta_\mu\eta_\nu)\rangle$.] The graviton-$\eta$-$\xi$ vertex, in momentum space, is given by [see Fig. 2(c)]

$$V_{\alpha\beta,\lambda,\mu}(k_1,k_2,k_3)=K[-\delta_{\lambda(\alpha}k_{1\beta)}k_{2\mu}+\delta_{\lambda\mu}k_{2(\alpha}k_{3\beta)}]\,, \tag{2.14}$$

with the notation

$$A_{(\alpha}B_{\beta)}\equiv\tfrac{1}{2}(A_\alpha B_\beta+A_\beta B_\alpha)\,. \tag{2.15}$$

From Eqs. (2.5), (2.7), and (2.8) we find for $\mathcal{L}_{(3)}$

$$\mathcal{L}_{(3)}=\tfrac{1}{2}\left[\phi_{\sigma\rho}\left(\phi_{\mu K,\rho}\phi_{\mu K,\sigma}-\frac{1}{n-2}\phi_{\mu\mu,\rho}\phi_{\nu\nu,\sigma}\right)+2\phi_{\mu\tau}\left(\phi_{\mu\sigma,\tau}\phi_{\lambda\tau,\sigma}-\phi_{\mu K,\rho}\phi_{\tau K,\rho}+\frac{1}{n-2}\phi_{\mu\tau,\rho}\phi_{\nu\nu,\rho}\right)\right]. \tag{2.16}$$

With the labeling shown in Fig. 1, Eq. (2.16) implies the momentum vertex

$$\begin{aligned}U(q_1,q_2,q_3)_{\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3}=-\frac{K}{2}\Big[&q^2_{(\alpha_1}q^3_{\beta_1)}\left(2\delta_{\alpha_2(\alpha_3}\delta_{\beta_3)\beta_2}-\frac{2}{n-2}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}\right)\\
&+q^1_{(\alpha_2}q^3_{\beta_2)}\left(2\delta_{\alpha_1(\alpha_3}\delta_{\beta_3)\beta_1}-\frac{2}{n-2}\delta_{\alpha_1\beta_1}\delta_{\alpha_3\beta_3}\right)\\
&+q^1_{(\alpha_3}q^2_{\beta_3)}\left(2\delta_{\alpha_1(\alpha_2}\delta_{\beta_2)\beta_1}-\frac{2}{n-2}\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\right)+2q^3_{(\alpha_2}\delta_{\beta_2)(\alpha_1}\delta_{\beta_1)(\alpha_3}q^2_{\beta_3)}\\
&+2q^1_{(\alpha_3}\delta_{\beta_3)(\alpha_2}\delta_{\beta_2)(\alpha_1}q^3_{\beta_1)}+2q^2_{(\alpha_1}\delta_{\beta_1)(\alpha_3}\delta_{\beta_3)(\alpha_2}q^1_{\beta_2)}\\
&+q^2\cdot q^3\left(\frac{2}{n-2}\delta_{\alpha_1(\alpha_2}\delta_{\beta_2)\beta_1}\delta_{\alpha_3\beta_3}+\frac{2}{n-2}\delta_{\alpha_1(\alpha_3}\delta_{\beta_3)\beta_1}\delta_{\alpha_2\beta_2}-2\delta_{\alpha_1(\alpha_2}\delta_{\beta_2)(\alpha_3}\delta_{\beta_3)\beta_1}\right)\\
&+q^1\cdot q^3\left(\frac{2}{n-2}\delta_{\alpha_2(\alpha_1}\delta_{\beta_1)\beta_2}\delta_{\alpha_3\beta_3}+\frac{2}{n-2}\delta_{\alpha_2(\alpha_3}\delta_{\beta_3)\beta_2}\delta_{\alpha_1\beta_1}-2\delta_{\alpha_2(\alpha_1}\delta_{\beta_1)(\alpha_3}\delta_{\beta_3)\beta_2}\right)\\
&+q^1\cdot q^2\left(\frac{2}{n-2}\delta_{\alpha_3(\alpha_1}\delta_{\beta_1)\beta_3}\delta_{\alpha_2\beta_2}+\frac{2}{n-2}\delta_{\alpha_3(\alpha_2}\delta_{\beta_2)\beta_3}\delta_{\alpha_1\beta_1}-2\delta_{\alpha_3(\alpha_1}\delta_{\beta_1)(\alpha_2}\delta_{\beta_2)\beta_3}\right)\Big].\end{aligned} \tag{2.17}$$

III. SLAVNOV-WARD IDENTITY

In this section we derive the Slavnov-Ward identity[7] following the technique of 't Hooft.[11] We first note that the functional $Z[j_{\mu\nu}]$ of Eq. (B5) with $j^{\mu\nu}=0$ is, in fact, independent of $B^\nu(x)$. Hence, instead of using $\rho(B)$ [see Eq. (B10) of Appendix B] as a weight function, we can use $\rho(B^\nu-J^\nu)$, where $J^\nu(x)$ is some arbitrary function of x. Hence,

$$Z[j_{\mu\nu}=0]=\int d[\tilde{g}^{\mu\nu}]\Delta[\tilde{g}^{\alpha\beta}]\exp\left\{i\int\left[\mathcal{L}-\frac{1}{\alpha K^2}(\partial_\alpha\tilde{g}^{\alpha\nu}-J^\nu)^2\right]dx\right\} \tag{3.1}$$

is independent of $J^\nu(x)$.

Expanding (3.1) as a power series in $J^\nu(x)$, all coefficients of J^ν (except the zeroth order) must therefore vanish. Setting the coefficient of $J^{\nu}(z)J^{\beta}(y)$ to zero, we obtain the Slavnov-Ward identity

$$\frac{2}{\alpha}\langle T\phi_{\mu\nu,\mu}(z)\phi_{\lambda\beta,\lambda}(y)\rangle = -\delta_{\nu\beta}\delta(z-y), \qquad \alpha \neq 0\,. \tag{3.2}$$

Equation (3.2) must be true for each order in K and for any gauge specified by the parameter α. We note that Eq. (3.2) does not depend on the dimensionality. From Eqs. (2.6) and (2.10) we obtain for the graviton propagator

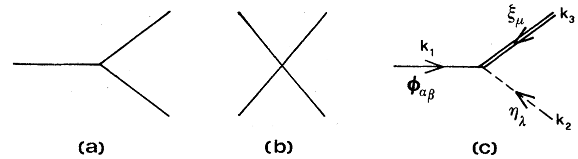

FIG. 2. (a) Three-graviton vertex. (b) Four-graviton vertex. (c) Graviton–fictitious-particle vertex. The two different fictitious particles ξ and η have momentum labels k_3 and k_2 and polarization labels μ and λ, respectively.

$$D_{\mu\nu,\lambda\beta}(x) = -\tfrac{1}{2}\Big[(2+\alpha)\delta_{\mu\nu}\delta_{\lambda\beta} - \delta_{\mu\lambda}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\lambda} - 2(\alpha+1)\Big(\delta_{\mu\nu}\frac{\partial_\lambda\partial_\beta}{\partial^2} + \delta_{\lambda\beta}\frac{\partial_\mu\partial_\nu}{\partial^2}\Big)$$
$$+(\alpha+1)\Big(\delta_{\mu\lambda}\frac{\partial_\nu\partial_\beta}{\partial^2} + \delta_{\mu\beta}\frac{\partial_\nu\partial_\lambda}{\partial^2} + \delta_{\nu\lambda}\frac{\partial_\mu\partial_\beta}{\partial^2} + \delta_{\nu\beta}\frac{\partial_\mu\partial_\lambda}{\partial^2}\Big)\Big]D(x)\,. \tag{3.3}$$

If we substitute Eq. (3.3) into the left-hand side of Eq. (3.2), we find that the latter identity is indeed valid to lowest order in K. Furthermore, Eq. (3.2) implies, for the lowest-order self-energy contribution $T_{\alpha\beta\alpha'\beta'}$,

$$p_\mu p_\nu D_{\mu\lambda,\gamma\beta}(p)T_{\gamma\beta\gamma'\beta'}(p)D_{\gamma'\beta',\nu\sigma}(p) = 0\,. \tag{3.4}$$

IV. LOWEST-ORDER GRAVITON SELF-ENERGY INSERTIONS

The vertices relevant to our discussion are shown in Fig. 2, while all possible self-energy corrections to order K^2 are depicted in Fig. 3. In this paper we calculate explicitly only the contributions from diagrams 3(a) and 3(b). The massless tadpole diagram 3(c) leads to an integral of the form $\int d^{2\omega}q\,(q^2)^{-1}$, which can be treated by a suitable redefinition of the 2ω-dimensional Gaussian integral over momentum space (2ω denotes the total number of dimensions). For further details we refer the reader to some previous work.[10,12] It turns out that such tadpole diagrams may consistently (within the framework of dimensional regularization) be equated to zero. A similar conclusion applies to diagram 3(f), which corresponds to a $\delta^4(0)$ term. These $\delta^4(0)$ terms are due to the presence of more than one derivative in the nonlinear interaction Lagrangian, and can likewise be shown to vanish in the context of dimensional regularization.[10,12] Diagrams 3(d) and 3(e), containing zero-momentum propagators of mass zero, are more difficult to evaluate than 3(c) and 3(f). However, both 3(d) and 3(e) satisfy the Slavnov-Ward identities to be discussed in Sec. V, so we shall ignore these diagrams here.

A. The Fictitious-Particle Self-Energy Loop

The evaluation of the fictitious-particle contribution to the graviton self-energy diagram (Fig. 4) is less complicated algebraically than the corresponding calculation of the graviton loop (Fig. 5). Hence, we explain the derivation of the fictitious contribution in some detail to demonstrate the general technique.

From the Feynman rules developed in Sec. II and Appendix B, the contribution from the fictitious loop is given by

$$F_{\alpha\beta\alpha'\beta'}(p) = -K^2\int d^{2\omega}q\; V_{\alpha\beta,\lambda,\sigma}(p,-q,q-p)D_{\lambda\lambda'}(-q)V_{\alpha'\beta',\sigma',\lambda'}(-p,p-q,q)D_{\sigma\sigma'}(q-p) \tag{4.1a}$$

$$= -K^2\int d^{2\omega}q\,[q^2(q-p)^2]^{-1}[\delta_{\lambda\sigma}(q-p)_{(\alpha}q_{\beta)} - \delta_{\lambda(\beta}p_{\alpha)}q_\sigma][\delta_{\lambda'\sigma'}q_{(\alpha'}(q-p)_{\beta')} - \delta_{\sigma'(\beta'}p_{\alpha')}(p-q)_{\lambda'}]\delta_{\lambda\lambda'}\delta_{\sigma\sigma'}, \tag{4.1b}$$

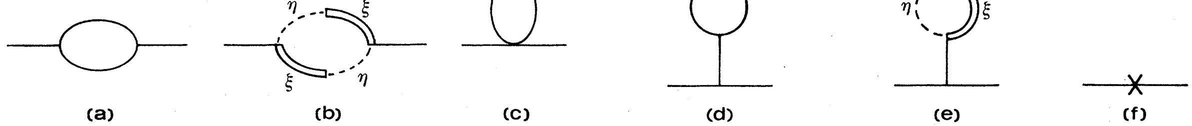

FIG. 3. Lowest-order contribution to the graviton self-energy.

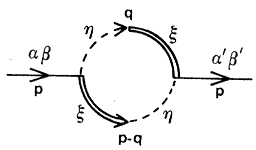

FIG. 4. Fictitious-particle loop.

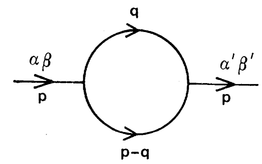

FIG. 5. Graviton loop.

where, following the dimensional-regularization technique of 't Hooft and Veltman, and Ashmore,[8] the momentum-space integration is defined over a 2ω-dimensional Euclidean space. The *regulating* parameter 2ω (in general complex) now replaces the integer n of Sec. II. Before integrating the right-hand side of Eq. (4.1b), we shall simplify the integrand by noting that $\delta_{\mu\mu}=2\omega$, and obtain

$$F_{\alpha\beta\alpha'\beta'}(p)=-K^2\int d^{2\omega}q[q^2(q-p)^2]^{-1}[p_\alpha p_\beta p_{\alpha'}q_{\beta'}+q_\alpha p_\beta p_{\alpha'}p_{\beta'}-q_\alpha q_\beta p_{\alpha'}p_{\beta'}$$
$$+(1-2\omega)q_\alpha q_\beta q_{\alpha'}p_{\beta'}-(1+2\omega)p_\alpha q_\beta q_{\alpha'}q_{\beta'}$$
$$+(2\omega-1)p_\alpha q_\beta p_{\alpha'}q_{\beta'}+2\omega q_\alpha q_\beta q_{\alpha'}q_{\beta'}]\ . \tag{4.2}$$

Let us now consider the various integrals in Eq. (4.2). The *basic* integral I_1,

$$I_1=\int d^{2\omega}q[q^2(q-p)^2]^{-1}, \tag{4.3}$$

can be evaluated by using the following parametrization of the momentum-space propagators

$$(q^2)^{-1}=\int_0^\infty \exp(-\alpha q^2)d\alpha\ ,\quad q^2>0, \tag{4.4}$$

together with the formula

$$\int d^{2\omega}q\,\exp(-aq^2+2b\cdot q)=\left(\frac{\pi}{a}\right)^\omega \exp\left(\frac{b^2}{a}\right),\quad a>0\ . \tag{4.5}$$

The result reads

$$I_1=\pi^\omega\Gamma(2-\omega)\int_0^1 d\xi[p^2\xi(1-\xi)]^{\omega-2} \tag{4.6}$$

$$=\pi^\omega[\Gamma(2\omega-2)]^{-1}\Gamma(2-\omega)\Gamma(\omega-1)\Gamma(\omega-1)(p^2)^{\omega-2}, \tag{4.7}$$

where we have used the following definition of the Γ function:

$$\Gamma(z)=\int_0^\infty dt\,t^{z-1}e^{-t},\quad \mathrm{Re}(z)>0\ . \tag{4.8}$$

Before the various Γ functions in (4.7) can be expanded about $\omega=2$, it is imperative to continue them analytically to other values of ω by means of the partial-fraction expansion,[13]

$$\Gamma(1-\omega)=\sum_{n=0}^\infty\frac{(-1)^n}{n!(n+1-\omega)}+\int_1^\infty dt\,t^{-\omega}e^{-t}\ . \tag{4.9}$$

The concept of "analytic continuation in the number of dimensions" is the most important single feature in the technique of dimensional regularization.

The remaining integrals in Eq. (4.2) can readily be obtained by differentiating Eq. (4.5) partially with respect to b_μ (this vector is also defined over a space of 2ω dimensions). Thus

$$\int d^{2\omega}q[q^2(q-p)^2]^{-1}q_\alpha=p_\alpha I_2, \tag{4.10}$$

$$\int d^{2\omega}q[q^2(q-p)^2]^{-1}q_\alpha q_\beta=\delta_{\alpha\beta}I_3+p_\alpha p_\beta I_4, \tag{4.11}$$

$$\int d^{2\omega}q[q^2(q-p)^2]^{-1}q_\alpha q_\beta q_\gamma=p_\alpha p_\beta p_\gamma I_5+E_{\alpha\beta\gamma}I_6, \tag{4.12}$$

$$\int d^{2\omega}q\,[q^2(q-p)^2]^{-1}q_\alpha q_\beta q_\gamma q_\sigma = p_\alpha p_\beta p_\gamma p_\sigma I_7 + G_{\alpha\beta\gamma\sigma}I_8 + H_{\alpha\beta\gamma\sigma}I_9\,, \tag{4.13}$$

where

$$E_{\alpha\beta\gamma} \equiv \delta_{\alpha\beta}p_\gamma + \delta_{\beta\gamma}p_\alpha + \delta_{\gamma\alpha}p_\beta\,, \tag{4.14a}$$

$$G_{\alpha\beta\gamma\sigma} \equiv \delta_{\alpha\beta}p_\gamma p_\sigma + \delta_{\beta\gamma}p_\alpha p_\sigma + \delta_{\beta\sigma}p_\alpha p_\gamma + \delta_{\alpha\gamma}p_\beta p_\sigma + \delta_{\alpha\sigma}p_\beta p_\gamma + \delta_{\gamma\sigma}p_\alpha p_\beta\,, \tag{4.14b}$$

$$H_{\alpha\beta\gamma\sigma} \equiv \delta_{\alpha\beta}\delta_{\gamma\sigma} + \delta_{\alpha\sigma}\delta_{\beta\gamma} + \delta_{\beta\sigma}\delta_{\alpha\gamma}\,. \tag{4.14c}$$

Substitution of Eqs. (4.10)–(4.14c) into Eq. (4.2) yields

$$\begin{aligned} F_{\alpha\beta\alpha'\beta'}(p) = -K^2[&p_\alpha p_\beta p_{\alpha'}p_{\beta'}I_2 + (2\omega-1)(p_\alpha p_{\alpha'}\delta_{\beta\beta'}I_3 + p_\alpha p_\beta p_{\alpha'}p_{\beta'}I_4) - (2\omega+1)(p_\alpha p_\beta p_{\alpha'}p_{\beta'}I_5 + p_{(\alpha}F_{\beta)\alpha'\beta'}I_6) \\ &- (2\omega-1)(p_\alpha p_\beta p_{\alpha'}p_{\beta'}I_5 + p_{(\alpha'}F_{\beta')\alpha\beta}I_6) - p_{\alpha'}p_{\beta'}(\delta_{\alpha\beta}I_3 + p_\alpha p_\beta I_4) \\ &+ 2\omega(p_\alpha p_\beta p_{\alpha'}p_{\beta'}I_7 + G_{\alpha\beta\alpha'\beta'}I_8 + H_{\alpha\beta\alpha'\beta'}I_9) + p_\alpha p_\beta p_{\alpha'}p_{\beta'}I_2]\,. \end{aligned} \tag{4.15}$$

Each of the integrals $I_2, \ldots, I_9$ has the following simple structure in terms of I_1 [Eq. (4.7)]:

$$I_2 = (2)^{-1}I_1\,, \tag{4.16}$$

$$I_3 = -[4(2\omega-1)]^{-1}p^2 I_1\,, \tag{4.17}$$

$$I_4 = \omega[2(2\omega-1)]^{-1}I_1\,, \tag{4.18}$$

$$I_5 = (\omega+1)[4(2\omega-1)]^{-1}I_1\,, \tag{4.19}$$

$$I_6 = -[8(2\omega-1)]^{-1}p^2 I_1\,, \tag{4.20}$$

$$I_7 = (\omega+1)(\omega+2)[4(4\omega^2-1)]^{-1}I_1\,, \tag{4.21}$$

$$I_8 = -(\omega+1)[8(4\omega^2-1)]^{-1}p^2 I_1\,, \tag{4.22}$$

$$I_9 = [16(4\omega^2-1)]^{-1}(p^2)^2 I_1\,. \tag{4.23}$$

The following comment is now in order. If the integrals $I_2, \ldots, I_9$ are evaluated "naively," then they are only valid in *nonoverlapping* regions of the ω plane. For instance, the original integrals for I_7, I_8, and I_9 [cf. Eqs. (4.21)–(4.23)] hold in the nonoverlapping ranges $1<\mathrm{Re}\,\omega<2$, $0<\mathrm{Re}\,\omega<1$, respectively. If regularization is therefore performed in this "naive" way, there is no *unique* analytic continuation to the neighborhood $\omega=2$ and it is then impossible to evaluate the physical amplitude unambiguously. It was shown in Ref. 10 that an analytic continuation is indeed possible, provided we replace Eq. (4.5) by the definition

$$\int d^{2\omega}q\,\exp(-aq^2+2b\cdot q) \equiv \left(\frac{\pi}{a}\right)^{\omega}\exp\left[\left(\frac{b^2}{a}\right)-af(\omega)\right]\,, \quad a>0\,, \tag{4.5$'$}$$

where $f(\omega)$ is a nonzero analytic function. It was furthermore shown that the final integrals expanded about $2\omega = 3, 4, 5, \ldots$ can be made independent of the exact form of $f(\omega)$ by placing reasonable conditions on $f(\omega)$ [e.g., $f(\omega)$ and its derivative must vanish for 2ω equal to an integer]. Since the final results turn out to be *independent* of $f(\omega)$ anyway, we do not consider this question any further here, but instead refer the reader to Ref. 10.

Remembering that

$$\delta_{\alpha\alpha} = 2\omega, \tag{4.24}$$

we obtain the following useful identities from Eqs. (4.14a)–(4.14c):

$$p_{(\alpha}E_{\beta)\alpha'\beta'} + p_{(\alpha'}E_{\beta')\alpha\beta} = G_{\alpha\beta\alpha'\beta'}\,, \tag{4.25}$$

$$E_{\lambda\lambda\alpha} = 2(\omega+1)p_\alpha\,, \tag{4.26}$$

$$p_\lambda E_{\lambda\alpha\beta} = 2p_\alpha p_\beta + p^2\delta_{\alpha\beta}\,, \tag{4.27}$$

$$G_{\lambda\lambda\alpha\beta} = \delta_{\alpha\beta}p^2 + 2(\omega+2)p_\alpha p_\beta\,, \tag{4.28}$$

$$H_{\lambda\lambda\alpha\beta} = 2(\omega+1)\delta_{\alpha\beta}\,. \tag{4.29}$$

Let us continue with Eq. (4.15). Substituting the various integrals (4.16)–(4.23) into the right-hand side of (4.15) and simplifying the resulting expression by means of Eqs. (4.14a)–(4.14c) and Eqs. (4.25)–(4.29), we obtain the following results for the fictitious-particle contribution:

$$\begin{aligned} F_{\alpha\beta\alpha'\beta'}(p) = +K^2[&p_\alpha p_\beta p_{\alpha'}p_{\beta'}F_1(p^2) + \delta_{\alpha\beta}\delta_{\alpha'\beta'}F_2(p^2) + (\delta_{\alpha\alpha'}\delta_{\beta\beta'} + \delta_{\beta\alpha'}\delta_{\alpha\beta'})F_3(p^2) + (\delta_{\alpha\beta}p_{\alpha'}p_{\beta'} + \delta_{\alpha'\beta'}p_\alpha p_\beta)F_4(p^2) \\ &+ (\delta_{\alpha\alpha'}p_\beta p_{\beta'} + \delta_{\beta\alpha'}p_\alpha p_{\beta'} + \delta_{\alpha\beta'}p_\beta p_{\alpha'} + \delta_{\beta\beta'}p_\alpha p_{\alpha'})F_5(p^2)]\,, \end{aligned} \tag{4.30}$$

where

$$F_1 = -[2(4\omega^2-1)]^{-1}(\omega^3+3\omega^2-2\omega-2)I_1\,, \tag{4.31}$$

$$F_2 = F_3 = -[8(4\omega^2-1)]^{-1}\omega(p^2)^2 I_1\,, \tag{4.32}$$

$$F_4 = -[8(4\omega^2-1)]^{-1}(2\omega^2+2\omega+1)p^2 I_1 , \tag{4.33}$$

$$F_5 = -[16(4\omega^2-1)]^{-1}p^2 I_1 . \tag{4.34}$$

The explicit form of these F's will be required in Sec. V. to verify the Slavnov-Ward identities.

B. The Graviton Self-Energy Loop

The procedure outlined in Sec. IV A for the fictitious loop may now be applied to the graviton particle loop shown in Fig. 5. Using the Feynman rules of Sec. II [Eqs. (2.17) and (2.11b)], we have for the graviton self-energy amplitude

$$R_{\alpha\beta\alpha'\beta'}(p) = \tfrac{1}{2}K^2 \int d^{2\omega}q\, U(p,-q,q-p)_{\alpha\beta\alpha_1\beta_1\alpha_2\beta_2} D(q)_{\alpha_1\beta_1\alpha_1'\beta_1'} D(p-q)_{\alpha_2\beta_2\alpha_2'\beta_2'} U(q,p-q,-p)_{\alpha_1'\beta_1'\alpha_2'\beta_2'\alpha'\beta'} . \tag{4.35}$$

The factor $\frac{1}{2}$ appears due to the way in which we have defined the vertex for three identical particles. The reduction of the above integrand is straightforward though rather tedious and will not be repeated here. We merely state the final result of this vast algebraic manipulation:

$$\begin{aligned}
R_{\alpha\beta\alpha'\beta'}(p) = \frac{K^2}{32}\int d^{2\omega}q \Big(& 16\{4p_\alpha p_\beta p_{\alpha'} p_{\beta'} - 2[p_\alpha p_\beta p_{(\alpha'} q_{\beta')} + q_{(\alpha} p_{\beta)} p_{\alpha'} p_{\beta'}] \\
& + (2\omega+3)(p_\alpha p_\beta q_{\alpha'} q_{\beta'} + q_\alpha q_\beta p_{\alpha'} p_{\beta'}) - \omega(2\omega+1)[p_{(\alpha} q_{\beta)} q_{\alpha'} q_{\beta'} + q_\alpha q_\beta q_{(\alpha'} p_{\beta')}] \\
& + (2\omega^2 - 3\omega - 2) p_{(\alpha} q_{\beta)} p_{(\alpha'} q_{\beta')} + (2\omega+1)\omega q_\alpha q_\beta q_{\alpha'} q_{\beta'}\} \\
& + 16\{p^2[-(2\omega+4) p_{(\alpha}\delta_{\beta)(\alpha'} p_{\beta')} - 2\omega p_{(\alpha}\delta_{\beta)(\alpha'} q_{\beta')} - 2\omega q_{(\alpha}\delta_{\beta)(\alpha'} p_{\beta')} + 4\omega q_{(\alpha}\delta_{\beta)(\alpha'} q_{\beta')}] \\
& + 4(\omega+1) p\cdot q\, p_{(\alpha}\delta_{\beta)(\alpha'} p_{\beta')} - 4(\omega+1) q^2 p_{(\alpha}\delta_{\beta)(\alpha'} p_{\beta')}\} \\
& + \frac{8}{\omega-1}\delta_{\alpha(\alpha'}\delta_{\beta')\beta}[(q^2)^2(4\omega^2-4\omega-2) + q^2 p\cdot q(-8\omega^2+8\omega+4) + q^2p^2(8\omega^2+4\omega-22) \\
& + p^2 p\cdot q(-8\omega^2+12\omega-8) + (p\cdot q)^2(4\omega^2-20\omega+28) + (p^2)^2(4\omega^2-5)] \\
& + \frac{4}{\omega-1}\delta_{\alpha\beta}\delta_{\alpha'\beta'}[(q^2)^2(4\omega-2) - q^2 p\cdot q(8\omega-4) - p^2q^2(6\omega-6) \\
& + p^2 p\cdot q(10\omega-12) + 4(p\cdot q)^2 + (p^2)^2(-2\omega^2-4\omega+9)] \\
& + \frac{8}{(\omega-1)^2}\{p^2\delta_{\alpha\beta}[4(\omega-1)p_{\alpha'}p_{\beta'} + (2\omega^3-3\omega^2-2\omega+2)p_{(\alpha'}q_{\beta')} + (-2\omega^3+\omega^2+5\omega-3)q_{\alpha'}q_{\beta'}] \\
& + p\cdot q\,\delta_{\alpha\beta}[-(2\omega-2)p_{\alpha'}p_{\beta'} + (-4\omega^2+6\omega)(\omega-1)p_{(\alpha'}q_{\beta')} + (4\omega+2)(\omega-1)^2 q_{\alpha'}q_{\beta'}] \\
& + q^2\delta_{\alpha\beta}[-(2\omega-3)(\omega-1)p_{\alpha'}p_{\beta'} + (4\omega+2)(\omega-1)^2 p_{(\alpha'}q_{\beta')} \\
& + (-4\omega-2)(\omega-1)q_{\alpha'}q_{\beta'}] + (\alpha\leftrightarrow\alpha';\beta\leftrightarrow\beta')\}\Big) .
\end{aligned} \tag{4.36}$$

Our next task is to integrate the right-hand side of Eq. (4.36) with the aid of Eqs. (4.7) and (4.10)–(4.13) and then to simplify the various terms, as before, using Eqs. (4.14a)–(4.29). The final form of the graviton loop contribution to the self-energy (4.35) reads [cf. Eq. (4.30)]:

$$\begin{aligned}
R_{\alpha\beta\alpha'\beta'}(p) = K^2[& p_\alpha p_\beta p_{\alpha'} p_{\beta'} R_1(p^2) + \delta_{\alpha\beta}\delta_{\alpha'\beta'} R_2(p^2) + (\delta_{\alpha\alpha'}\delta_{\beta\beta'} + \delta_{\beta\alpha'}\delta_{\alpha\beta'})R_3(p^2) + (\delta_{\alpha\beta}p_{\alpha'}p_{\beta'} + \delta_{\alpha'\beta'}p_\alpha p_\beta)R_4(p^2) \\
& + (\delta_{\alpha\alpha'}p_\beta p_{\beta'} + \delta_{\beta\alpha'}p_\alpha p_{\beta'} + \delta_{\alpha\beta'}p_\beta p_{\alpha'} + \delta_{\beta\beta'}p_\alpha p_{\alpha'})R_5(p^2)] ,
\end{aligned} \tag{4.37}$$

where

$$R_1 = [8(2\omega-1)]^{-1}(\omega^3-\omega^2+24\omega-8)I_1 , \tag{4.38}$$

$$R_2 = [32(\omega-1)^2(2\omega-1)]^{-1}(-7\omega^3+2\omega^2+13\omega)(p^2)^2 I_1 , \tag{4.39}$$

$$R_3 = [32(2\omega-1)]^{-1}(8\omega^2+5\omega-8)(p^2)^2 I_1 , \tag{4.40}$$

$$R_4=[32(2\omega-1)(\omega-1)]^{-1}(2\omega^3-2\omega^2+20\omega+4)p^2I_1, \tag{4.41}$$

$$R_5=[32(2\omega-1)]^{-1}(-8\omega^2-5\omega+10)p^2I_1. \tag{4.42}$$

Equations (4.38)–(4.42) for the R's are essential for verifying the Slavnov-Ward identities (Sec. V).

V. THE PHYSICAL AMPLITUDE

A. Total Loop Contribution

Adding Eqs. (4.30) and (4.37) we obtain the total contribution from the graviton and fictitious particle loops (Figs. 5 and 4, respectively)

$$T_{\alpha\beta\alpha'\beta'}(p)=K^2[p_\alpha p_\beta p_{\alpha'}p_{\beta'}T_1(p^2)+\delta_{\alpha\beta}\delta_{\alpha'\beta'}T_2(p^2)+(\delta_{\alpha\alpha'}\delta_{\beta\beta'}+\delta_{\beta\alpha'}\delta_{\alpha\beta'})T_3(p^2)$$
$$+(\delta_{\alpha\beta}p_{\alpha'}p_{\beta'}+\delta_{\alpha'\beta'}p_\alpha p_\beta)T_4(p^2)+(\delta_{\alpha\alpha'}p_\beta p_{\beta'}+\delta_{\beta\alpha'}p_\alpha p_{\beta'}+\delta_{\alpha\beta'}p_\beta p_{\alpha'}+\delta_{\beta\beta'}p_\alpha p_{\alpha'})T_5(p^2)], \tag{5.1}$$

where

$$T_1=[8(4\omega^2-1)]^{-1}(+2\omega^4-5\omega^3+35\omega^2+16\omega)I_1, \tag{5.2}$$

$$T_2=[32(\omega-1)^2(4\omega^2-1)]^{-1}(-14\omega^4-7\omega^3+36\omega^2+9\omega)(p^2)^2I_1, \tag{5.3}$$

$$T_3=[32(4\omega^2-1)]^{-1}(+16\omega^3+18\omega^2-15\omega-8)(p^2)^2I_1, \tag{5.4}$$

$$T_4=[32(\omega-1)(4\omega^2-1)]^{-1}(+4\omega^4-10\omega^3+38\omega^2+32\omega+8)p^2I_1, \tag{5.5}$$

$$T_5=[32(4\omega^2-1)]^{-1}(-16\omega^3-18\omega^2+15\omega+8)p^2I_1. \tag{5.6}$$

In order to verify the Slavnov-Ward identities from Eq. (3.4), we must first construct the connected Green's function

$$Q_{\nu\sigma,\mu\lambda}(p)=D_{\nu\sigma,\alpha\beta}(p)T_{\alpha\beta\alpha'\beta'}(p)D_{\alpha'\beta',\mu\lambda}(p), \tag{5.7}$$

where $D_{\nu\sigma,\alpha\beta}$ and $T_{\alpha\beta\alpha'\beta'}$ are given by Eqs. (2.11b) and (5.1), respectively. The result is

$$Q_{\nu\sigma\mu\lambda}(p)=[4(p^2)^2]^{-1}\{a_{1\nu\sigma\mu\lambda}T_1+(\omega-1)^2a_{2\nu\sigma\mu\lambda}T_2+[a_{3\nu\sigma\mu\lambda}+(\omega-2)a_{2\nu\sigma\mu\lambda}]T_3+(\omega-1)a_{4\nu\sigma\mu\lambda}T_4+a_{5\nu\sigma\mu\lambda}T_5\}, \tag{5.8}$$

with

$$a_{1\nu\sigma\mu\lambda}=4p_\nu p_\sigma p_\mu p_\lambda-2p^2\delta_{\mu\lambda}p_\nu p_\sigma-2p^2\delta_{\nu\sigma}p_\mu p_\lambda+(p^2)^2\delta_{\nu\sigma}\delta_{\mu\lambda}, \tag{5.9a}$$

$$a_{2\nu\sigma\mu\lambda}=4\delta_{\nu\sigma}\delta_{\mu\lambda}p^2, \tag{5.9b}$$

$$a_{3\nu\sigma\mu\lambda}=4(\delta_{\mu\nu}\delta_{\sigma\lambda}+\delta_{\nu\lambda}\delta_{\sigma\mu})(p^2)^2, \tag{5.9c}$$

$$a_{4\nu\sigma\mu\lambda}=4(p^2\delta_{\nu\sigma}\delta_{\mu\lambda}-\delta_{\mu\lambda}p_\nu p_\sigma-\delta_{\nu\sigma}p_\mu p_\lambda)p^2, \tag{5.9d}$$

$$a_{5\nu\sigma\mu\lambda}=4(\delta_{\mu\nu}p_\sigma p_\lambda+\delta_{\sigma\mu}p_\nu p_\lambda+\delta_{\nu\lambda}p_\sigma p_\mu+\delta_{\sigma\lambda}p_\nu p_\mu-2\delta_{\nu\sigma}p_\mu p_\lambda-2\delta_{\mu\lambda}p_\nu p_\sigma+p^2\delta_{\mu\lambda}\delta_{\nu\sigma})p^2. \tag{5.9e}$$

The Slavnov-Ward identity (3.4) can be written as

$$p_\mu p_\nu Q_{\nu\sigma\mu\lambda}(p)=0, \tag{5.10}$$

or equivalently,

$$T_3+p^2T_5=0, \tag{5.11}$$

$$(p^2)^2T_1+4(\omega-1)^2T_2+4(\omega-1)(T_3-p^2T_4)=0. \tag{5.12}$$

Equations (5.11) and (5.12) are valid for *all* values of the regulating parameter ω (in particular for $\omega=2$) as may be verified directly from Eqs. (5.2)–(5.6).

B. The Connected Green's Function

Our principal aim in this section is to obtain the *finite* part of the connected Green's function $Q_{\nu\sigma\mu\lambda}$ by expanding the *entire* right-hand side of Eq. (5.8) about $\omega=2$ and then separating the various pole terms from the real and imaginary parts. To accomplish this we express the amplitudes T_j first in the form

$$T_j=\Gamma(2-\omega)(p^2)^{\omega-2}f_j(\omega),\quad j=1,\ldots,5, \tag{5.13}$$

expand each T_j about $\omega = 2$ giving

$$T_j = \frac{f_j(2)}{2-\omega} + [\psi(1) f_j(2) - f_j(2)\ln p^2 - f_j{}'(2)] + O((\omega-2)^2), \tag{5.14}$$

and then continue $\ln p^2$ analytically from Euclidean to Minkowski space ($p^2 = p_0{}^2 - \vec{p}\,^2$). Consequently,

$$T_j = \frac{f_j(2)}{2-\omega} + [\psi(1) f_j(2) - f_j(2)\ln|p^2| - f_j{}'(2)] + i\pi f_j(2) + O((\omega-2)^2), \quad j = 1, \ldots, 5 \tag{5.15}$$

where the prime denotes differentiation with respect to ω and $\psi(\omega) = (d/d\omega)[\ln\Gamma(\omega)]$. Before proceeding, we note that Eq. (5.10), in terms of the connected Green's function Q, does not explicitly depend on ω. On the other hand, if the Slavnov-Ward identities are written in terms of the self-energy amplitudes T_j, then there *is* an explicit dependence on ω. Therefore, in order to extract an invariant result, it is essential to expand $Q_{\nu\sigma\mu\lambda}$ and *not* the self-energy $T_{\alpha\beta\alpha'\beta'}$. With this in mind we write the total amplitude for the graviton self-energy conveniently as

$$Q_{\nu\sigma\mu\lambda}(p) = [4(p^2)^2]^{-1}[Q^{\text{Pole}}_{\nu\sigma\mu\lambda} + Q^{\text{Real}}_{\nu\sigma\mu\lambda} + iQ^{\text{Im}}_{\nu\sigma\mu\lambda}], \tag{5.16}$$

where

$$Q^{\text{Pole}}_{\nu\sigma\mu\lambda} = \frac{\pi^2}{120(p^2)^2(2-\omega)}(+328a_{1\nu\sigma\mu\lambda} - 59a_{2\nu\sigma\mu\lambda} + 81a_{3\nu\sigma\mu\lambda} + 104a_{4\nu\sigma\mu\lambda} - 81a_{5\nu\sigma\mu\lambda}), \tag{5.17}$$

$$\begin{aligned} Q^{\text{Real}}_{\nu\sigma\mu\lambda} = \pi^2[120(p^2)^2]^{-1}\{&[\psi(1) - \ln|p^2|](+328a_{1\nu\sigma\mu\lambda} - 59a_{2\nu\sigma\mu\lambda} + 81a_{3\nu\sigma\mu\lambda} + 104a_{4\nu\sigma\mu\lambda} - 81a_{5\nu\sigma\mu\lambda}) \\ &+ (30)^{-1}(896a_{1\nu\sigma\mu\lambda} + 1517a_{2\nu\sigma\mu\lambda} - 1143a_{3\nu\sigma\mu\lambda} + 598a_{4\nu\sigma\mu\lambda} + 1143a_{5\nu\sigma\mu\lambda})\}, \end{aligned} \tag{5.18}$$

$$Q^{\text{Im}}_{\nu\sigma\mu\lambda} = +\pi^3[120(p^2)^2]^{-1}(+328a_{1\nu\sigma\mu\lambda} - 59a_{2\nu\sigma\mu\lambda} + 81a_{3\nu\sigma\mu\lambda} + 104a_{4\nu\sigma\mu\lambda} - 81a_{5\nu\sigma\mu\lambda}). \tag{5.19}$$

We have therefore reached the major goal of this calculation, namely the extraction from $Q_{\nu\sigma\mu\lambda}$ of the finite part $Q^{\text{Real}} + iQ^{\text{Im}}$ in a manner consistent with the Slavnov-Ward identities (5.11) and (5.12). Furthermore, each of the terms Q^{Pole}, Q^{Real}, and Q^{Im} satisfies the identities (5.11) and (5.12) *separately*. The proof is straightforward and will not be given here.

C. Lagrangian Density in Two Dimensions

A few remarks are in order concerning the Einstein Lagrangian (A14) in *two* dimensions (i.e., $n=2$ or $\omega=1$). Equation (A14) exhibits a simple pole at $n=2$ which arises from the fact that the original Einstein Lagrangian (2.1) is meaningless in two dimensions, reducing to a surface integral. We note that the Lagrangian (A14) leads to a simple pole $(n-2)^{-1}$ in the *vertex* (2.17) but *not* in the propagator (2.11a). On the other hand, if instead of $\tilde{g}^{\alpha\beta}$ we use the metric tensor $g^{\alpha\beta}$ as our basic field, we find that the simple pole $(n-2)^{-1}$ now occurs in the graviton *propagator*[14] [cf. Eq. (2.11a)]

$$D_{\alpha\beta,\lambda\mu} = \tfrac{1}{2}\left(\delta_{\alpha\lambda}\delta_{\beta\mu} + \delta_{\alpha\mu}\delta_{\beta\lambda} - \frac{2}{n-2}\delta_{\alpha\beta}\delta_{\lambda\mu}\right)D(x), \tag{5.20}$$

rather than in the graviton vertex. Pole terms appear, therefore, regardless of the choice of field. Another way of seeing this is to note that the field transformation defined by Eqs. (A5) and (A8) is singular in two dimensions. Our reason for working with $\tilde{g}^{\alpha\beta}$, rather than $g^{\alpha\beta}$, is simply the fact that the corresponding expressions for the graviton vertices are much simpler. Moreover, if we then consider connected Green's functions (i.e., diagrams with propagators attached to the graviton vertices), all terms proportional to $(n-2)^{-1}$ disappear. This does *not* mean, of course, that the use of $\tilde{g}^{\alpha\beta}$ would enable us to renormalize such a two-dimensional theory.

VI. CONCLUSION

We have succeeded in *regularizing* the lowest-order contributions to the graviton Green's function—consisting of the single graviton and fictitious-particle loops—by employing the method of dimensional regularization, suitably modified to cope with tadpole terms. In order to obtain simple vertex functions, the n-dimensional Einstein Lagrangian (2.1) is rewritten in terms of the tensor density $\tilde{g}^{\alpha\beta}$ (see Appendix A). Since the regularized Green's function (5.8) satisfies the Slavnov-Ward identity (5.10) the Laurent expansion about $\omega=2$ (physical space) gives pole terms and a finite part [see Eqs. (5.18) and (5.19)] consistent with this identity. The pole terms in the *connected* Green's function may be canceled by appropriate counterterms in the Lagrangian.

It is not clear at this stage whether or not higher order contributions can be treated in a similar fashion. Also the question of the renormalizability of the gravitational Lagrangian remains open.

We finally note that the invariant amplitudes $T_1, \ldots, T_5$ given in Eqs. (5.2) to (5.6) also satisfy Ward identities. The latter relate the two-point graviton-fictitious particle Green's function to the graviton fictitious particle vertex. The results of this calculation are contained in a separate paper.[15]

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APPENDIX A

In this appendix we derive a useful form of the Einstein Lagrangian (2.1) in an n-dimensional space (n a positive integer). In the classical theory of general relativity one introduces a metric tensor $g^{\mu\nu}$ and a Lagrangian density $\mathfrak{L}$ given by[16]

$$\mathfrak{L} = \frac{2}{K^2}(-g)^{1/2} R_{\mu\nu}\, g^{\mu\nu}, \tag{A1}$$

where

$$g = \det g_{\mu\nu}\,, \tag{A2}$$

$g_{\mu\nu}$ being defined via $g_{\mu\nu} g^{\nu\beta} = \delta_\mu^\beta$, and where

$$R_{\mu\nu} = \Gamma^\rho_{\mu\rho,\nu} - \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} + \Gamma^\rho_{\sigma\nu}\Gamma^\sigma_{\mu\rho}\,, \tag{A3}$$

with

$$\Gamma^\alpha_{\beta\gamma} = \frac{g^{\alpha\omega}}{2}\,(g_{\beta\omega,\gamma} + g_{\omega\gamma,\beta} - g_{\beta\gamma,\omega})\,. \tag{A4}$$

The above equations are all equally valid in n dimensions.

We can derive a more useful form of the Lagrangian (2.1) by defining the tensor density $\tilde{g}^{\alpha\beta}$ (weight +1)

$$\tilde{g}^{\alpha\beta} = \sqrt{-g}\, g^{\alpha\beta} \tag{A5}$$

so that

$$\det \tilde{g}^{\mu\nu} = \det(\sqrt{-g}\, g^{\mu\nu}) = (-g)^{n/2} \det g^{\mu\nu} \tag{A6}$$

and

$$\det g^{\mu\nu} \det g_{\alpha\beta} = 1\,. \tag{A7}$$

Hence,

$$g = -(-\det \tilde{g}^{\mu\nu})^{2/(n-2)}. \tag{A8}$$

From the properties of determinants,

$$\partial_\rho(\det \tilde{g}^{\mu\nu}) = \tilde{g}_{\alpha\beta}\tilde{g}^{\alpha\beta}{}_{,\rho} \det \tilde{g}^{\mu\nu} \tag{A9}$$

and therefore

$$\partial_\rho g = \frac{2}{n-2}\, g\, \tilde{g}_{\alpha\beta}\tilde{g}^{\alpha\beta}{}_{,\rho}\,. \tag{A10}$$

Employing Eqs. (A4) and (A10), together with

$$g_{\alpha'\beta',\rho} = -g_{\beta\beta'} g_{\alpha\alpha'} g^{\alpha\beta}{}_{,\rho}\,, \tag{A11}$$

we obtain

$$\Gamma^\alpha_{\beta\gamma} = -\tfrac{1}{2}\Big(\tilde{g}_{\beta\mu}\tilde{g}^{\mu\alpha}{}_{,\gamma} + \tilde{g}_{\gamma\mu}\tilde{g}^{\alpha\mu}{}_{,\beta} - \tilde{g}^{\alpha\lambda}\tilde{g}_{\beta\mu}\tilde{g}_{\gamma\nu}\tilde{g}^{\mu\nu}{}_{,\lambda} - \frac{1}{n-2}\delta^\alpha_\beta \tilde{g}_{\lambda\delta}\tilde{g}^{\lambda\delta}{}_{,\gamma} - \frac{1}{n-2}\delta^\alpha_\gamma \tilde{g}_{\lambda\delta}\tilde{g}^{\lambda\delta}{}_{,\beta} + \frac{1}{n-2}\tilde{g}^{\alpha\mu}\tilde{g}_{\beta\gamma}\tilde{g}_{\lambda\delta}\tilde{g}^{\lambda\delta}{}_{,\mu}\Big). \tag{A12}$$

Using Eqs. (A1), (A3), and (A12), together with integration by parts, we finally obtain the Lagrangian

$$L \equiv \int \mathfrak{L}\, d^n x \tag{A13}$$

$$= \frac{1}{2K^2}\int d^n x\,[\tilde{g}^{\rho\sigma}\tilde{g}_{\lambda\alpha}\tilde{g}_{\kappa\tau}\tilde{g}^{\alpha\kappa}{}_{,\rho}\tilde{g}^{\lambda\tau}{}_{,\sigma} - (n-2)^{-1}\tilde{g}^{\rho\sigma}\tilde{g}_{\alpha\kappa}\tilde{g}_{\lambda\tau}\tilde{g}^{\alpha\kappa}{}_{,\rho}\tilde{g}^{\lambda\tau}{}_{,\sigma} - 2\tilde{g}_{\alpha\tau}\tilde{g}^{\alpha\kappa}{}_{,\rho}\tilde{g}^{\rho\tau}{}_{,\kappa}], \tag{A14}$$

which clearly exhibits the dependence on n. In four dimensions the Lagrangian (A14) reduces to the form previously obtained by Goldberg.[17]

APPENDIX B

The gravitational fictitious-particle contribution in a general gauge has previously been derived by Fradkin and Tyutin.[6] Here we give a simpler derivation, valid in n dimensions, which mainly follows the approach outlined by Faddeev and Popov.[5] Consider the generating functional

$$Z[j_{\mu\nu}] = \int \chi[\tilde{g}^{\mu\nu}]\, d[\tilde{g}^{\mu\nu}] \exp\Big[i\int(\mathfrak{L} + \tilde{g}^{\mu\nu} j_{\mu\nu})\, dx\Big], \tag{B1}$$

where $\chi[\tilde{g}^{\mu\nu}]d[\tilde{g}^{\mu\nu}]$ is the local gauge-invariant measure.[18] χ comes from having a nonlinear interaction Lagrangian (containing more than one derivative) and gives rise to $\delta^4(0)$ terms. Since such terms can consistently be set equal to zero within the context of the regularizing technique used here,[10] we do not consider the $\chi[\tilde{g}^{\mu\nu}]$ factor further.

We are concerned here with a non-Abelian gauge theory which gives rise to two problems in constructing a well-defined perturbation theory. Firstly, the functional in (B1) contains an infinite volume which arises from integrating over points in the function space of $\tilde{g}^{\mu\nu}$, some of which are related by the gauge group $G(\Omega)$. For quantum gravity—a non-Abelian gauge theory—it is essential to explicitly remove this volume integral over the gauge group. Secondly, there is the usual problem of inverting the free part of a gauge-invariant Lagrangian.

To solve the first problem we reduce the integration over the entire $\tilde{g}^{\mu\nu}$ space to an integral over a hypersurface defined by $\partial_\mu \tilde{g}^{\mu\nu}(x) = B^\nu(x)$ (B^ν arbitrary) and the integral over the group $G(\Omega)$. To this end we define a functional $\Delta[\tilde{g}^{\mu\nu}]$ via

$$\Delta[\tilde{g}^{\mu\nu}] \int d[\Omega]\, \delta[\partial_\mu \tilde{g}^{\mu\nu}_\Omega - B^\nu] = \text{constant}, \tag{B2}$$

where $\tilde{g}^{\mu\nu}_\Omega$ is the result of operating on $\tilde{g}^{\mu\nu}$ with Ω, and then insert this constant into (B1), viz.,

$$Z[j_{\mu\nu}] = \int d[\tilde{g}^{\mu\nu}]d[\Omega]\Delta[\tilde{g}^{\mu\nu}]\,\delta[\partial_\mu \tilde{g}^{\mu\nu}_\Omega - B^\nu] \times \exp\left[i \int (\mathcal{L} + \tilde{g}^{\mu\nu} j_{\mu\nu})dx\right]. \tag{B3}$$

Making the inverse transformation,

$$\tilde{g}^{\mu\nu} \to \tilde{g}^{\mu\nu}_{\Omega^{-1}}, \tag{B4}$$

we obtain

$$\begin{aligned} Z[j_{\mu\nu}] &= \int d[\Omega] \int d[\tilde{g}^{\mu\nu}]\Delta[\tilde{g}^{\mu\nu}]\delta[\partial_\mu \tilde{g}^{\mu\nu} - B^\nu]\exp\left[i\int (\mathcal{L} + \tilde{g}^{\mu\nu}j_{\mu\nu})dx\right] \\ &= \int d[\tilde{g}^{\mu\nu}]\Delta[\tilde{g}^{\mu\nu}]\delta[\partial_\mu \tilde{g}^{\mu\nu} - B^\nu]\exp\left[i\int (\mathcal{L} + \tilde{g}^{\mu\nu}j_{\mu\nu})dx\right], \end{aligned} \tag{B5}$$

where we have used the fact that the integration over Ω now leads to an irrelevant constant. Because of the δ functional in Eq. (B5), we only need to evaluate the integral in (B2) over the hypersurface defined by $\delta[\partial_\mu \tilde{g}^{\mu\nu} - B^\nu]$. We thus obtain

$$\begin{aligned} (\Delta[\tilde{g}^{\mu\nu}])^{-1} &= \int d[\xi_\lambda]\delta[\delta(\partial_\mu \tilde{g}^{\mu\lambda})] \\ &= \int d[\xi_\lambda]d[\eta_\nu]\exp\left\{i\int \eta_\nu[\delta_{\nu\lambda}\Box - K(\phi_{\mu\nu,\lambda\mu} - \phi_{\mu\rho}\delta_{\nu\lambda}\partial_\rho\partial_\mu - \phi_{\mu\rho,\mu}\delta_{\nu\lambda}\partial_\rho + \phi_{\mu\nu,\mu}\partial_\lambda)]\xi_\lambda dx\right\}, \end{aligned} \tag{B6}$$

where we have used the exponential representation of the δ functional,

$$\delta[a] = \int d[c]\exp\left[i\int a(x)c(x)dx\right]. \tag{B7}$$

The functional (B6) can be interpreted as representing fictitious particles ξ_λ and η_μ with only an ξ-η propagator. The fact that we have evaluated Δ^{-1} rather than the required Δ, simply necessitates the introduction of a factor -1 for each closed loop of fictitious particles. If we make a change of variable in Eq. (B2) and use the functional determinant representation

$$\det A = \exp(\mathrm{Tr}\ln A) \tag{B8}$$

instead of Eq. (B6), we obtain

$$(\Delta[\tilde{g}^{\mu\nu}])^{-1} = \exp\{-i\,\mathrm{Tr}\ln[\delta_{\nu\lambda} - K(\phi_{\mu\nu,\lambda\mu} - \phi_{\mu\beta}\delta_{\nu\lambda}\partial_\rho\partial_\mu - \phi_{\mu\rho,\mu}\delta_{\nu\lambda}\partial_\rho + \phi_{\mu\nu,\mu}\partial_\lambda)\Box^{-1}]\}. \tag{B9}$$

Equation (B9) agrees with (4.63) of Fradkin and Tyutin.[6]

We now consider the second problem, that of inverting the free part of the Lagrangian. The δ functional in (B5) fixes the gauge but, except for the case $B^\nu = 0$, it is not in a very convenient form. However, we observe that (B5) is in fact independent of B^ν, so that we may introduce a suitable weight function $\rho(B)$ (cf. Ref. 11) and integrate over B^ν. The effect of such a $\rho(B)$ is to add a gauge-breaking term to the free Lagrangian. In the main text of the paper [see Eq. (2.6)] we choose

$$\rho[B] = \exp\left(-\frac{i}{\alpha K^2}\int [B(x)]^2 dx\right). \tag{B10}$$

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