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Geodesic Synchrotron Radiation*

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This paper presents the results and methods of computing the high-frequency radiation emitted by freely falling particles moving in circular geodesic orbits in a spherically symmetric gravitational field. The high-frequency radiation, to which the methods of this paper apply, is the principal part of radiated energy only in the case of a particle moving in a highly relativistic, and therefore unstable, circular geodesic. The geodesic synchrotron radiation emitted in this case shows as main features excitation of high-frequency harmonics and a narrow angular distribution. A Green's-function solution of the scalar wave equation is obtained using WKB methods. For application to relativistic circular orbits a parabolic WKB approximation is required and yields solutions in terms of parabolic cylinder functions. Although only scalar radiation is treated explicitly in this paper, the Green's functions constructed here for the solution of the scalar wave equation also suffice for the vector and tensor calculations.

I. INTRODUCTION

This paper presents the results and methods of computing the high-frequency radiation emitted by freely falling particles moving in circular geodesic orbits in a spherically symmetric gravitational field. Some of the results have been reported previously,[1] and further results have been quoted from drafts of this paper in other work which has already appeared.[2-5] Although only scalar radiation is treated explicitly in this paper, the Green's functions constructed here for the solution of the scalar wave equation also suffice for the vector and tensor calculations just cited.

The high-frequency radiation, to which the methods of this paper apply, is the principal part of the radiated energy only in the case of a particle moving in a highly relativistic, and therefore unstable, circular geodesic. For particles moving in stable circular orbits, the main radiation is the low-frequency (fundamental) mode, for which numerical computations are required,[6] and the computations in Sec. IV of this paper give the asymptotic form of this spectrum at high frequencies in which only a negligible fraction of the radiated energy is to be found.

There are very few examples known in the theory of general relativity where the radiation emitted by a well-defined source is computed. The only widely known such calculation is the computation[7] of the gravitational radiation emitted by a nonrelativistically oscillating quadrupole in the linearized theory of gravity. Other computations include the radiation from a vibrating neutron star,[8] gravitational brehmsstrahlung,[9] the radiation emitted by a small mass falling radially into a Schwarzschild black hole,[10] and an unpublished computation by Zee[11] of electromagnetic radiation emitted by a charge in a Schwarzschild circular orbit. As an introduction to the concept of radiation emitted from a freely falling particle,[12] and thus as a counterexample to uncritical attempts to apply the principle of equivalence outside its domain of validity, the present example is much preferable to those cited previously. The previous computations of radiation from relativistic sources require a much more complicated formalism (vector and tensor fields) and, because they deal with low or fundamental frequencies, also lead to a requirement for numerical solution of the differential equations. In the present example many complications are avoided by focusing on a scalar field, and the high-frequency approximation allows the differential equations to be solved analytically in a WKB approximation. The result is a "textbook" example of radiation emitted by a freely falling particle in a strong gravitational field.

The specific problem we consider is that of a particle moving in a circular geodesic orbit about a black hole described by the Schwarzschild solution of Einstein's equations of general relativity. The standard coordinates, and the metric signature shown, will be used:

$$ds^2 = -(1-2Mr^{-1})dt^2 + (1-2Mr^{-1})^{-1}dr^2 + r^2 d\Omega^2\,,$$
$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\varphi^2\,. \quad (1.1)$$

We furthermore set $G=1=c$. Let M be the mass of the black hole, μ the mass of the moving particle, and μf the coupling (scalar charge) of the small moving particle to the scalar field. The equations are solved in the approximation

$$\mu/M \ll 1, \quad \mu f/M \ll 1, \tag{1.2}$$

which makes the geodesic motion appropriate for the small particle, and which allows the neglect, in Einstein's equations, of the stress-energy tensor of the scalar field. The geodesic motion would also give rise to gravitational radiation as well as scalar radiation in this problem, but the computations of these two forms of radiation are not coupled to each other, and only the scalar radiation is calculated in this paper. The high-frequency parts of the gravitational radiation have been calculated by Breuer, Ruffini, Tiomno, and Vishveshwara[3] using methods based in part on those of this paper.

A most intriguing radiation spectrum arises in the case where the particle motion corresponds to a highly relativistic (but unstable) circular orbit approximating the $r=3M$ null circular orbit in the Schwarzschild solution. In this case one obtains an analog of familiar electromagnetic synchrotron radiation that arises from relativistic motion of charges in circular orbits in flat space. As in that familiar example, the radiation from a highly relativistic circular geodesic is emitted at very high multiples of the fundamental orbit frequency, and the radiation is beamed very strongly in the plane of the orbit. (A subsequent calculation by Hughes[13] shows that this radiation is also strongly beamed in the azimuthal direction so that it comes out in the form of a rotating lighthouse beacon, as is also well known to be the case for a single charged particle in a relativistic circular orbit in flat space.) The investigations reported here of geodesic synchrotron radiation (GSR) were initiated in the hope that extensions of this phenomenon to cases involving rotating Kerr black holes and noncircular "dive-in" orbits could, in the case of gravitational radiation, shed some light on Weber's observations.[14] This hope was not borne out and there do not appear to be any reasonable prospects for interpreting Weber's observations along these lines, as was described in reviews by Misner.[15] The analysis of related phenomena for Kerr black holes is given in papers by Bardeen, Press, and Teukolsky,[16] Chrzanowski and Misner,[17] and Chrzanowski.[18]

II. CIRCULAR ORBITS

We are interested in calculating the power radiated by a test particle executing circular geodesic motion in the Schwarzschild geometry. Therefore, we begin with a review of the key formulas that describe circular geodesics.

The geodesic equations for a particle moving in the $\theta=\pi/2$ plane in a Schwarzschild background are

$$\left(\frac{dr}{d\tau}\right)^2 + V(r) = 0, \tag{2.1a}$$

$$V(r) = \left(1-\frac{2M}{r}\right)\left(\frac{h^2}{r^2}+1\right) - \gamma^2, \tag{2.1b}$$

$$\frac{d\varphi}{d\tau} = \frac{h}{r^2}, \tag{2.2}$$

$$\frac{dt}{d\tau} = \gamma\left(1-\frac{2M}{r}\right)^{-1}. \tag{2.3}$$

Here τ is the proper time and both γ and h are conserved quantities;

$$\gamma = -p_t = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} \tag{2.4}$$

is the energy per unit rest mass of the test particle as measured at infinity. When $\gamma<1$ ($\gamma>1$), the particle is in a bound (unbound) orbit. The constant

$$h = p_\varphi = r^2\,\frac{d\varphi}{dt} \tag{2.5}$$

is the conserved angular momentum per unit rest mass of the particle. The angular frequency of the particle motion is given by

$$\omega_0 = \frac{d\varphi}{dt} = \frac{h(1-2M/r)}{r^2\gamma}. \tag{2.6}$$

Equation (2.1) can be simplified for circular particle motion, for then

$$-V(r_0) = 0 = \gamma^2 - \left(1-\frac{2M}{r_0}\right)\left(\frac{h^2}{r_0^{\,2}}+1\right) \tag{2.7}$$

and

$$-\frac{dV}{dr}(r_0) = 0 = \frac{1}{r_0^{\,4}}\,(2h^2r_0 - 2Mr_0^{\,2} - 6Mh^2), \tag{2.8}$$

where r_0 is the radius of the orbit. These last two equations may be solved simultaneously for γ and h in terms of r_0. We find

$$\gamma = \left(1-\frac{2M}{r_0}\right)\left(1-\frac{3M}{r_0}\right)^{-1/2}, \tag{2.9}$$

$$\bar{b} \equiv \frac{h}{\gamma} = \pm(Mr_0)^{1/2}\left(1-\frac{2M}{r_0}\right)^{-1}. \tag{2.10}$$

The quantity $\bar{b}$ is the angular momentum per unit energy at infinity; it is defined for all orbits. A closely related quantity,

$$b = \frac{h}{(\gamma^2-1)^{1/2}} = \pm r_0\left(\frac{4M}{r_0}-1\right)^{-1/2}, \tag{2.11}$$

is more convenient for unbound ($\gamma>1$) orbits, where b is the impact parameter for the scattering orbits into which the unstable circular orbits with $3M< r_0< 4M$ are converted by small perturbations. No circular orbits exist with $r_0< 3M$, but $r_0=3M$, where γ becomes infinite, is the radius of an unstable circular photon orbit.

By using (2.9) and (2.10), we may simplify many formulas of interest, eliminating γ and h and re-expressing the relations as functions of r_0 only. In particular, the angular frequency becomes

$$\omega_0=\frac{d\varphi}{dt}=\left(\frac{M}{r_0^3}\right)^{1/2}, \tag{2.12}$$

while $u^0=dt/d\tau$ reduces to

$$\frac{dt}{d\tau}=\left(1-\frac{3M}{r_0}\right)^{-1/2}. \tag{2.13}$$

Although circular particle orbits exist for all radii $r_0>3M$, not all of these orbits are stable. Stability occurs only when $V''(r_0)\geq 0$. From Eq. (2.1b) we see that the last stable orbit is at $r_0=6M$. All orbits with $3M<r_0<6M$ are unstable.

We are particularly interested in unbound orbits with $\gamma>>1$. These orbits correspond to high-energy particles with impact parameters $b\simeq 3\sqrt{3}\,M$ scattered by the black hole through angles $\Delta\varphi>>2\pi$. Letting $r_0=(3+\delta)M$ with $\delta<<1$, we find

$$\gamma^2\simeq\frac{1}{3\delta} \tag{2.14}$$

and

$$(u^0)^2=\left(\frac{dt}{d\tau}\right)^2=\frac{3}{\delta} \tag{2.15}$$

III. THE SCALAR EQUATION

In order to develop the simplest possible model of GSR, we consider the emission of scalar waves by a test particle in a relativistic circular orbit. The interaction between a scalar field ϕ and a test particle of mass μ is described by the action

$$I=-\frac{1}{8\pi}\int d^4x\sqrt{-g}\;\phi_{,\mu}\phi^{,\mu} -\mu\int d\tau(1+f\phi)(-\dot{z}^{\mu}\dot{z}_{\mu})^{1/2}. \tag{3.1}$$

The test particle follows a world line $z^\mu(\tau)$ with parameter τ, which we may choose to be the proper time. The constant f measures the strength of the "scalar charge" of the particle. The action I reproduces Newtonian gravitational interactions in the static limit when $f=\sqrt{G}$. By standard methods we obtain the energy-momentum tensor

$$T^{\mu\nu}=\frac{1}{4\pi}(\phi^{,\mu}\phi^{,\nu}-\tfrac{1}{2}g^{\mu\nu}\phi_{,\eta}\phi^{,\eta}) +\mu(-g)^{-1/2}\int d\tau(1+f\phi)\delta^4(x-z)\dot{z}^\mu\dot{z}^\nu \tag{3.2}$$

and the field equation for ϕ

$$\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\,g^{\mu\nu}\frac{\partial\phi}{\partial x^\nu}\right)=4\pi f\mu\left(\frac{dz^0}{d\tau}\right)^{-1}\delta^3(x-z(t)). \tag{3.3}$$

For a particle moving in a circular orbit with radius r_0 and angular velocity ω_0, we may write

$$\delta^3(x-z(t))=\delta(r-r_0)\delta(\theta-\pi/2)\delta(\varphi-\omega_0 t). \tag{3.4}$$

Exploiting the spherical symmetry of the Schwarzschild metric, we write the scalar field in separated form:

$$\phi=\sum_{m=-\infty}^{\infty}\sum_{l=|m|}^{\infty}r^{-1}u_{lm}(r)\,Y_l^m(\theta,\varphi)e^{-im\omega_0 t}, \tag{3.5}$$

with the reality conditions

$$u_{l,-m}=(-1)^m\bar{u}_{l,m}, \qquad Y_l^{-m}=(-1)^m\,\bar{Y}_l^m. \tag{3.6}$$

Insertion of ϕ into the scalar wave equation (3.3) leads to a radial equation

$$-\frac{d^2u_{lm}}{dr^{*2}}+\left[\left(1-\frac{2M}{r^3}\right)\left(\frac{2M}{r^3}+\frac{l(l+1)}{r^2}\right)-m^2\omega_0^2\right]u_{lm} =C_{lm}\delta(r^*-r_0^*), \tag{3.7}$$

in which the Regge-Wheeler coordinate $r^*=r-3M+2M\ln(r/M-2)$ is used. The homogeneous part of the wave equation was separated by Matzner[19] and Price,[20] who investigated the scattering of scalar waves. The strength of the inhomogeneous term is

$$C_{lm}=-4\pi(u^0r_0)^{-1}f\mu Y_l^m(\pi/2,0). \tag{3.8}$$

Wave equations for source particles coupling to electromagnetic[21] and gravitational[22] fields have also been studied. The radial equations can always be brought into the form of a Schrödinger equation

$$-u''+(V-E)u=0 \tag{3.9}$$

with $E=m^2\omega_0^2$ in each case, but with different values of V. In the limit $l>>1$, all of the potentials turn out to be identical:

$$V=(1-2m/r)r^{-2}l(l+1)+O(l^0). \tag{3.10}$$

The odd and even tensor potentials differ by terms of $O(l^{-2})$. In the vector case, $O(l^0)\equiv 0$.

The solution of Eq. (3.7) is, apart from the factor C_{lm}, just a Green's function. It is obtained by matching two solutions of the homogeneous equation at r_0^*, with a discontinuity in the derivative as specified by the δ function. The boundary conditions imposed are as follows:

(1) pure outgoing radiation at $r^*\sim r\to\infty$,

(2) pure ingoing radiation (towards the horizon of the black hole) at $r\to 2M$ or $r^*\to-\infty$.

The details of the derivation are given in Appendix A.

From Eq. (A8) it follows that the asymptotic solution of the wave equation (3.3) is

$$\phi\sim\begin{cases}\sum_{lm}\tfrac{1}{2}i\,r^{-1}|m\omega_0|^{-1/2}C_{lm}L(r_0^*)Y_l^m(\theta,\varphi)e^{im\omega_0(r^*-t)}, & r^*\to+\infty \quad (3.11a)\\ \sum_{lm}\tfrac{1}{2}i\,r^{-1}|m\omega_0|^{-1/2}C_{lm}R(r_0^*)Y_l^m(\theta,\varphi)e^{-im\omega_0(r^*+t)}, & r^*\to-\infty\,. \quad (3.11b)\end{cases}$$

Here $L(r^*)$ and $R(r^*)$ are solutions of the homogeneous equation (3.7) in which the incident wave travels to the left (toward $r^*\to-\infty$) and right (toward $r^*\to+\infty$), respectively, with "unit flux" normalization as prescribed in Eq. (A4).

The total power radiated outward to infinity (or down the black hole) is obtained by integrating over solid angles the outward (or inward) flux of energy:

$$P^{\text{out}}=-\int d\Omega\, r^2\, T^r{}_t=-\frac{1}{4\pi}\int d\Omega\, r^2\,\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial r}\,,\qquad r>r_0 \qquad (3.12a)$$

$$P^{\text{down}}=+\int d\Omega\, r^2\, T^r{}_t=+\frac{1}{4\pi}\int d\Omega\, r^2\,\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial r}\,,\qquad r<r_0\,. \qquad (3.12b)$$

Applying this formula to the asymptotic solution (3.11), and using the orthogonality of spherical harmonics, we find for the radiated power

$$P^{\text{out}}=\frac{1}{8\pi}\sum_{m=0}^{\infty}\sum_{l=|m|}^{\infty}m\omega_0|C_{lm}|^2|L(r_0^*)|^2, \qquad (3.13a)$$

$$P^{\text{down}}=\frac{1}{8\pi}\sum_{m=0}^{\infty}\sum_{l=|m|}^{\infty}m\omega_0|C_{lm}|^2|R(r_0^*)|^2. \qquad (3.13b)$$

From Eq. (3.8), we may compute the value of the coefficient C_{lm}. We find that C_{lm} vanishes unless $l-|m|\equiv 2q$ is even, and that

$$\begin{aligned}C_{|m|+2q,m}&=4\pi f\mu(u^0r_0)^{-1}(4\pi^3)^{-1/4}(-1)^{m+q+1}\\&\quad\times(|m|+2q+\tfrac{1}{2})^{1/2}(2q\,!)^{1/2}\\&\quad\times(2^q q!)^{-1}(q+|m|)^{-1/4}\,,\end{aligned} \qquad (3.14)$$

with $q=0,1,2,\ldots$ and $|m|>>q$. We shall see later that only the $q=0$ term contributes significantly to the power spectrum. In the approximation of neglecting all terms where $l\neq|m|$, an important relationship is

$$Y_m^m(\theta,0)=(-1)^m|m|^{1/4}(4\pi^3)^{-1/4}(1+1/4m)\sin^{|m|}\theta, \qquad (3.15)$$

which describes the distribution of the $l=|m|$ mode in polar angles for large m.

The power formulas (3.12) are based on the "energy current"

$$J^\mu\equiv-T^{\mu\nu}\xi_\nu=-T^\mu{}_t \qquad (3.16)$$

constructed from the stress-energy tensor equation (3.2) and from the Killing vector $\vec{\xi}=(\partial/\partial t)$ or $\xi^\mu=\delta^\mu_t$ which displays the time-translation symmetry characterizing the static nature of the Schwarzschild geometry. Because J^μ is a *vector* satisfying the conservation law $J^\mu{}_{;\mu}=0$, it gives rise to a global (integral) conservation law, although in more general situations one has only a local conservation law from $T^{\mu\nu}{}_{;\nu}=0$. It should be particularly noted that this conservation law for the quantity

$$E\equiv-\int T^t{}_t(-g)^{1/2}\,dr\,d\theta\, d\varphi \qquad (3.17)$$

shows that the energy radiated away as computed from Eqs. (3.12) must result in a corresponding decrease in the value of $\mu\gamma$ for the particle which is the source of the radiation.

Let us now turn to the explicit calculation of the power spectra.

IV. WKB SOLUTIONS: OFF-PEAK INTENSITIES

Because of Eq. (3.13), the problem of calculating the scalar power radiated by a point particle in a circular geodesic orbit reduces to finding solutions of the Schrödinger-type equation

$$-\frac{d^2u(r^*)}{dr^{*2}}+[V(r)-E]u(r^*)=0\,, \tag{4.1}$$

where

$$E=\omega^2=m^2\omega_0^{\,2}=\frac{Mm^2}{r_0^{\,3}}\,, \tag{4.2a}$$

$$V(r)=\left(1-\frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right]. \tag{4.2b}$$

The solutions of interest, denoted by $L(r^*)$ and $R(r^*)$, satisfy the boundary conditions (A4). When l is small, numerical integration techniques must be used to solve Eq. (4.1). Such solutions will not be investigated here.

For large l (≥ 10), WKB methods may be used to find analytic formulas for $L(r_0^*)$ and $R(r_0^*)$, and hence the high-frequency power spectrum. The calculation is analogous to that for barrier penetration of plane wave of energy $E=\omega^2$ incident upon a wide potential barrier $V(r)$. Using the boundary conditions (A4) and the WKB approximation, we find the value of $L(r^*)$ and $R(r^*)$ in the region under the potential barrier, where $V(r)\gg E$:

$$L(r^*)=\frac{e^{-i\pi/4}}{1-\frac{1}{4}e^{-\omega\tau}}\left\{\frac{e^{-\omega\theta(r^*)}}{[\omega\kappa(r^*)]^{1/2}}-\frac{ie^{-\omega[2\tau-\theta(r^*)]}}{2[\omega\kappa(r^*)]^{1/2}}\right\}$$

$$\simeq\frac{e^{-i\pi/4}e^{-\omega\theta(r^*)}}{[\omega\kappa(r^*)]^{1/2}}\,, \tag{4.3}$$

$$R(r^*)=\frac{e^{i\pi/4}}{1-\frac{1}{4}e^{-\omega\tau}}\left\{\frac{e^{-\omega[\tau-\theta(r^*)]}}{[\omega\kappa(r^*)]^{1/2}}+\frac{ie^{-\omega[\tau+\theta(r^*)]}}{2[\omega\kappa(r^*)]^{1/2}}\right\}. \tag{4.4}$$

These formulas are written in terms of the parameters

$$\omega\kappa(r^*)=[V(r)-E]^{1/2}\,, \tag{4.5a}$$

$$\theta(r^*)=\int_{r^*}^{r_+^*}\kappa(r^{*\prime})dr^{*\prime}, \tag{4.5b}$$

$$\tau=\theta(r_-^*)=\int_{r_-^*}^{r_+^*}\kappa(r^*)dr^*, \tag{4.5c}$$

where r_+^* and r_-^* are the classical turning points at the outer and inner edges of the potential barrier.

In the high-l approximation, Eq. (4.2) takes the form

$$V(r)-E\simeq\omega^2\left\{\left(1-\frac{2M}{r}\right)\times\left[1+\frac{4q+1}{m}+O\left(\frac{q^2}{m^2}\right)\right]\frac{r_0^3}{Mr^2}-1\right\}, \tag{4.6}$$

where r_0 is the radius of the circular orbit and $2q=l-m$. Since $r_0\geq 3M$, $V(r_0)>E$, so that Eqs. (3.13) and (4.3) give the high-frequency tail of the power spectrum,

$$P_{lm}^{\text{out}}(\omega)=\frac{|C_{lm}|^2}{8\pi}\,\frac{\exp[-2\omega\theta(r_0)]}{(r_0/M-3)^{1/2}}\,, \tag{4.7}$$

$$P^{\text{out}}(\omega)=\sum_{q=0}^{\infty}\frac{|C_{m+2q,m}|^2}{8\pi}\,\frac{\exp[-2\,\omega\theta(r_0)]}{(r_0/M-3)^{1/2}}\,, \tag{4.8}$$

in which $\theta(r_0)$ and $|C_{m+2q,m}|^2$ are given by Eqs. (4.5) and (3.4), respectively.

In order to determine how fast the sum over q converges in (4.8), define

$$\bar\theta(r_0)=\int_{r_0^*}^{r_+^*}\left[\left(1-\frac{2M}{r'}\right)\frac{r_0^3}{Mr'^2}-1\right]^{1/2}\frac{dr^*}{dr'}\,dr'\,. \tag{4.9}$$

Then

$$2\omega\theta(r_0)\simeq 2\omega\bar\theta(r_0)+[1+4q+O(q^2/m)]I\,, \tag{4.10}$$

where

$$I=\int_1^{r_+/r_0}\frac{dx}{[-(M/r_0)x^4+x^2-2(M/r_0)x]^{1/2}}\,. \tag{4.11}$$

$I\geq O(1)$ except possibly when $r_0\simeq 3M$. As $r_0\to\infty$, I approaches a complete elliptic integral of the first kind and takes the value

$$I\simeq\left(\frac{r_0}{M}\right)^{1/4}\frac{K(\frac{1}{2})}{\sqrt{2}}\simeq 1.31\left(\frac{r_0}{M}\right)^{1/4}. \tag{4.12}$$

Using Eqs. (4.10) and (4.7), we find

$$\frac{P_{l=m+2q,m}^{\text{out}}(\omega)}{P_{l=m,m}^{\text{out}}(\omega)}=e^{-4qI}\,, \tag{4.13}$$

so that nearly all of the power is radiated in the $l=m$ mode. Thus, to very high accuracy, the total power radiated at frequency ω can be shown to be

$$P^{\text{out}}(\omega)=P_{l=m,m}^{\text{out}}(\omega)$$

$$\simeq\frac{f^2\mu^2}{M^2}\left(\frac{M}{r_0}\right)^{1/4}\left[\frac{r_0-3M}{M\pi}\right]^{1/2}\times(M\omega)^{1/2}\exp[-2\omega\bar\theta(r_0)]\,. \tag{4.14}$$

This expression follows from combining Eqs. (2.15) and (3.14) with Eqs. (4.7). Equation (4.9) for $\bar\theta(r_0)$ must be evaluated numerically. We find that the value

$$\bar{\theta}(r_0) \simeq 3(r_0 - 3M) \tag{4.15}$$

is an excellent fit to the results of the numerical computations. Also, from Eqs. (4.3) and (4.4), it follows that

$$\frac{P^{\text{in}}(\omega)}{P^{\text{out}}(\omega)} \simeq \exp\{-2\omega[\tau - 2\bar{\theta}(r_0)]\} . \tag{4.16}$$

Numerical calculations give

$$\tau - 2\bar{\theta}(r_0) \simeq 4(r_0 - 3M), \tag{4.17}$$

so that this ratio (4.16) approaches unity as $r_0 \to 3M$ and dies off exponentially as $r_0 \to \infty$.

From Eq. (4.15), $\bar{\theta}(r_0) \to 0$ as $r_0 \to 3M$, indicating that considerable power may be radiated at high frequencies when $r_0 \simeq 3M$. To investigate this case more carefully, define

$$r_0 = (3+\delta)M, \quad \delta \ll 1 . \tag{4.18}$$

It is straightforward to verify that

$$r_+ \simeq 3M + \sqrt{3}M\left(\frac{1+4q+\delta m}{m}\right)^{1/2}. \tag{4.19}$$

Since $V(r)$ has a maximum at $r = 3M$ and we are interested in the potential in only a small region around $r = 3M$, we may approximate $V(r)$ by a parabola. Integrating (4.5) with this approximation, we have

$$\omega\theta(r_0) \simeq 3\int_{r_0}^{r_+} \sqrt{\kappa}\, dr \simeq \frac{\pi}{4}\left(1 + 4q + \frac{4}{\pi}\frac{m}{m_{\text{crit}}}\right), \tag{4.20}$$

where $m_{\text{crit}} \equiv 4/\pi\delta$. It follows that

$$P^{\text{out}}_{l=m+2q,m}(\omega) = \frac{|C_{lm}|^2}{8\pi}\frac{e^{-\pi\epsilon/2}}{\sqrt{\epsilon}}\, m^{1/2}$$
$$= \frac{4}{27}\frac{f^2\mu^2}{m^2\sqrt{\pi}}\left|\frac{m}{m_{\text{crit}}}\right|\frac{e^{-\pi/2\epsilon}}{\sqrt{\epsilon}}\frac{(2q)!}{(q!)^2 2^{2q}}, \tag{4.21}$$

with

$$\epsilon = 1 + 4q + \frac{4}{\pi}\left|\frac{m}{m_{\text{crit}}}\right| .$$

When $q=0$ and $m < m_{\text{crit}}$, $P^{\text{out}}_{l,m}(\omega)$ becomes very large since the barrier-penetration factor (4.20) is small. Under these circumstances the WKB approximation is no longer valid anywhere under the potential barrier, so that parabolic WKB methods, developed by Ford, Hill, Wakano, and Wheeler,[23] must be used to compute $L(r_0^*)$ in order to analyze the peak of the power spectrum.

V. PEAK INTENSITIES

In order to compute the most important contributions to the power spectrum—the modes with $l \simeq m$ and $m \leq m_{\text{crit}}$—we must use the parabolic WKB methods described in Appendix B. In that appendix it is shown that

$$|L(r_0^*)|^2 = |R(r_0^*)|^2$$
$$= \frac{3\sqrt{3}\,M}{4\pi m^{1/2}}\, e^{-\pi\epsilon/4}|\Gamma(\tfrac{1}{4} + \tfrac{1}{4}i\epsilon)|^2 \tag{5.1}$$

when $r_0 = (3+\delta)M$ and $\delta \ll 1$. As defined in Sec. IV,

$$\epsilon = 1 + 4q + \frac{4}{\pi}\left(\frac{m}{m_{\text{crit}}}\right), \tag{5.2}$$

with $2q = l - m$ and $m_{\text{crit}} = 4/\pi\delta = 12\gamma^2/\pi$. Using Eqs. (5.1), (3.13), and (3.14), we find that the peak of the power spectrum is given by

$$P_{lm}(\omega) = f^2\left(\frac{\mu}{M}\right)^2 \frac{(2q)!}{27\pi^{5/2}2^{2q}(q!)^2}\frac{m}{m_{\text{crit}}}$$
$$\times e^{-\pi\epsilon/4}|\Gamma(\tfrac{1}{4} + \tfrac{1}{4}i\epsilon)|^2 . \tag{5.3}$$

The power radiated down the black hole, which in this case is equal to the power radiated to infinity, is also given by Eq. (5.3). When $\epsilon \gg 1$, the power reduces to the form given in Eq. (4.19), which we derived using ordinary WKB methods.

To find the total power radiated at a given frequency, the expression (5.3) must be summed from $q=0$ to infinity keeping the value of m fixed. The series very rapidly converges because of the factor $e^{-\pi\epsilon/4}|\Gamma(\frac{1}{4}+\frac{1}{4}i\epsilon)|^2$ in the power formula; in fact, over 99.9% of the power is radiated in the $q=0$ mode, so that

$$P_l(\omega) = \frac{f^2\mu^2}{27\pi^{5/2}M^2}\frac{m}{m_{\text{crit}}}\, e^{-\pi\epsilon/4}|\Gamma(\tfrac{1}{4} + \tfrac{1}{4}i\epsilon)|^2. \tag{5.4}$$

The power is plotted as a function of frequency in Fig. 1.

Clearly, the power is proportional to the frequency ω up to a cutoff frequency given by

$$\omega_{\text{crit}} = m_{\text{crit}}\,\omega_0 = \frac{4\omega_0}{\pi\delta} = \frac{12\omega_0\gamma^2}{\pi}, \tag{5.5}$$

and above the cutoff the power spectrum is exponentially damped. To calculate the total power radiated, Eq. (5.4) must be summed over m. As noted in the caption to Fig. 1, this summation can be converted to an integral over frequency. Numerically we find that

$$P_{\text{total}} = f^2(\mu\gamma/M)^2(3.9\times 10^{-3}). \tag{5.6}$$

Here P_{total} is dimensionless and should be multiplied by $1=c^5/G=3.63\times10^{59}$ erg/sec $=2.03\times10^5$ $M_\odot c^2$/sec to give the value in other units.

At a fixed frequency $\omega=|m|\omega_0$ the angular distribution of the radiation is easily computed since only a single conjugate pair of terms in the series expansion (3.5) is significant: that with $\pm m=l=|\omega/\omega_0|$. The amplitude ϕ is therefore proportional to $|Y_m^m(\theta,\varphi)|$, and for the radiated power one consequently has

$$\frac{dP(\omega)}{d\Omega}=P(\omega)|Y_m^m(\theta,\varphi)|^2\,. \qquad (5.7)$$

The angular distribution would be observed only by a detector of such a very narrow bandwidth ($Q \gg |m|$) that, when excited, it rings for times $\gg\omega_0^{-1}$. For more wide-banded receivers the angular distribution must reflect interference between terms in Eq. (3.5) corresponding to different frequencies (different m). As has been shown by a different method,[13] the angular distribution then has a strong φ dependence which Eq. (5.7) lacks. [Hughes[13] shows that the particle actually emits a narrow cone of radiation in the forward direction, which is observed at infinity as a searchlight rotating at frequency ω_0.] Equation (5.7) becomes correct even for a broader-banded detector, however, if interpreted as an average of the power over azimuthal angles φ (or over time).

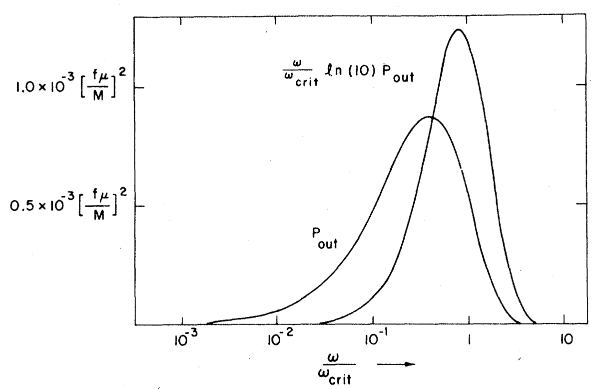

FIG. 1. Power radiated out to infinity in GSR in the limit $\delta=(3\gamma^2)^{-1}\ll1$. Here $P_{\rm out}(m)=\sum_{l=m}^{\infty}P_{\rm out}(l,m)$ for scalar radiation is given as a function of $\omega/\omega_{\rm crit}=\frac{1}{4}\pi m\delta$ for each frequency harmonic $m=\omega/\omega_0\gg1$. The total power emitted in all harmonics is $P_{\rm tot}=\sum_{m=0}^{\infty}P_{\rm out}(m)\simeq\int_0^\infty P_{\rm out}\,dm$. Since this can be written $P_{\rm tot}=\int(2\pi/\omega_0)P_{\rm out}d\nu$, the power emitted in a unit frequency interval is $dP_{\rm out}/d\nu=(2\pi/\omega_0)P_{\rm out}$. To obtain $P_{\rm tot}$ in terms of the area under a curve in a semilog plot where the abscissa is really $\log_{10}(\omega/\omega_{\rm crit})$, one writes

$$P_{\rm tot}=\frac{\omega_{\rm crit}}{\omega_0}\int_0^\infty\frac{\omega}{\omega_{\rm crit}}(\ln10)P_{\rm out}\,d\left(\log_{10}\frac{\omega}{\omega_{\rm crit}}\right).$$

The integrand is plotted to show that the bulk (82%) of the energy is emitted in the decade $10^{-0.7}<\omega/\omega_{\rm crit}<10^{+0.3}$. Numerical integration gives $P_{\rm tot}=f^2(\mu\gamma/M)^2(3.9\times10^{-3})$. Here $P_{\rm out}$ and $P_{\rm tot}$ are dimensionless and should be multiplied by $1=c^5/G=3.63\times10^{59}$ erg/sec $=2.03\times10^5\,M_\odot c^2$/sec to obtain their values in other units.

With the aid of Eqs. (5.4) and (3.15), the angular distribution (5.7) is rewritten as

$$\frac{dP(\omega)}{d\Omega}=\frac{f^2\mu^2}{54\pi^4M^2}\;\frac{m}{m_{\rm crit}}\times e^{-\pi\epsilon/4}|\Gamma(\tfrac{1}{4}+\tfrac{1}{4}i\epsilon)|^2m^{1/2}\cos^{2m}\vartheta, \qquad (5.8)$$

where $\vartheta\equiv\pi/2-\theta$. A plot of $dP/d\Omega$ as a function of both ϑ and ω is shown in Fig. 2. The function $\cos^{2m}\vartheta$ is sharply peaked about $\vartheta=0$, so that the radiation is very strongly beamed into the plane of the orbit. The beaming is characterized by a half-width $\Delta\vartheta$ given by

$$\Delta\vartheta=m^{-1/2}\,. \qquad (5.9)$$

APPENDIX A: GREEN'S FUNCTIONS

In this appendix we solve the radial equation (3.7) by the method of Green's functions. For the sake of greater generality, we consider an inhomogeneous term which also contains the derivative of the source, as is indeed the case for the inhomogeneous vector and tensor wave equations. The equation to be solved is therefore

$$-\frac{d^2u}{dr^{*2}}+(V-\omega^2)u=C_1j(r^*)+C_2\frac{dj(r^*)}{dr^*}, \qquad (A1)$$

where C_1 and C_2 are constants and V is the potential given in Eq. (3.10). If we denote the Green's function by $G(r^*,r^{*\prime})$ then the solution of Eq. (A1) is

$$\begin{aligned}u(r^*)&=\int dr^{*\prime}\left[C_1j(r^{*\prime})+C_2\frac{dj(r^{*\prime})}{dr^{*\prime}}\right]\times G(r^*,r^{*\prime})\\&=\int dr^{*\prime}j(r^{*\prime})\left[C_1G(r^*,r^{*\prime})-C_2\frac{\partial G(r^*,r^{*\prime})}{\partial r^{*\prime}}\right]\;.\end{aligned} \qquad (A2)$$

The equation defining G is

$$-\frac{d^2G(r^*,r_0^*)}{dr^{*2}}+(V-\omega^2)G(r^*,r_0^*)=\delta(r^*-r_0^*)\,. \qquad (A3)$$

For the case considered in this paper, a pointlike test particle located at $r^*=r_0^*$, we have

$j(r^*) \propto \delta(r^*-r_0^*)$.

To solve Eq. (A3), we consider the special solutions $L(r^*)$ and $R(r^*)$ of the homogeneous equation, and match them at the singularity at $r^*=r_0^*$. The solutions L and R represent respectively scattering wave functions with unit incident flux moving to the left and right. They are characterized by transmission coefficients $\mathcal{T}_L$ and $\mathcal{T}_R$, and reflection coefficients $\mathcal{S}_L$ and $\mathcal{S}_R$. We may therefore write the asymptotic values of the solutions as boundary conditions:

$$L(r^*)\sim\begin{cases}(k_\infty)^{-1/2}(e^{-ikr^*}+\mathcal{S}_L e^{ikr^*}), & r^*\to+\infty\\ (k_{-\infty})^{-1/2}\mathcal{T}_L e^{-ikr^*}, & r^*\to-\infty\end{cases}$$
$$R(r^*)\sim\begin{cases}(k_\infty)^{-1/2}\mathcal{T}_R e^{ikr^*}, & r^*\to+\infty\\ (k_{-\infty})^{-1/2}(e^{ikr^*}+\mathcal{S}_R e^{-ikr^*}), & r^*\to-\infty.\end{cases} \quad \text{(A4)}$$

The incoming waves are normalized to unit incident flux.

The constant Wronskian $W(R,L)=RL'-LR'$, when evaluated at the two boundaries, provides a relation between the transmission coefficients:

$$W(-\infty)=-2i\mathcal{T}_L=W(+\infty)=-2i\mathcal{T}_R\,. \quad \text{(A5)}$$

Obviously $\mathcal{T}_L=\mathcal{T}_R\equiv\mathcal{T}$. Similarly the constancy of the Wronskian $W(R,\bar{L})$ gives a relation involving reflection coefficients,

$$\mathcal{S}_R/\mathcal{T}=-\bar{\mathcal{S}}_L/\bar{\mathcal{T}}\,, \quad \text{(A6)}$$

where the overbar denotes complex conjugation.

To solve Eq. (A3), where the δ-function source represents a radiating particle at r_0^*, we require that $G\propto R$ for $r^*>r_0^*$ and $G\propto L$ for $r^*<r_0^*$. We also require the matching conditions $[G]=0$, $[G']=-1$ at $r^*=r_0^*$ as dictated by the δ-function source. The appropriate solution is

$$G(r^*,r_0^*)=\begin{cases}\dfrac{i}{2\mathcal{T}}L(r_0^*)R(r^*), & r^*>r_0^*\\ \dfrac{i}{2\mathcal{T}}R(r_0^*)L(r^*), & r^*<r_0^*\end{cases} \quad \text{(A7)}$$

which has the asymptotic behavior

$$G(r^*,r_0^*)\sim\begin{cases}\tfrac{1}{2}i\,L(r_0^*)(k_\infty)^{-1/2}e^{ikr^*}, & r^*\to+\infty\\ \tfrac{1}{2}i\,R(r_0^*)(k_{-\infty})^{-1/2}e^{-ikr^*}, & r^*\to-\infty\,.\end{cases} \quad \text{(A8)}$$

The derivative of the Green's function is required to solve the general equation (A1). That derivative is

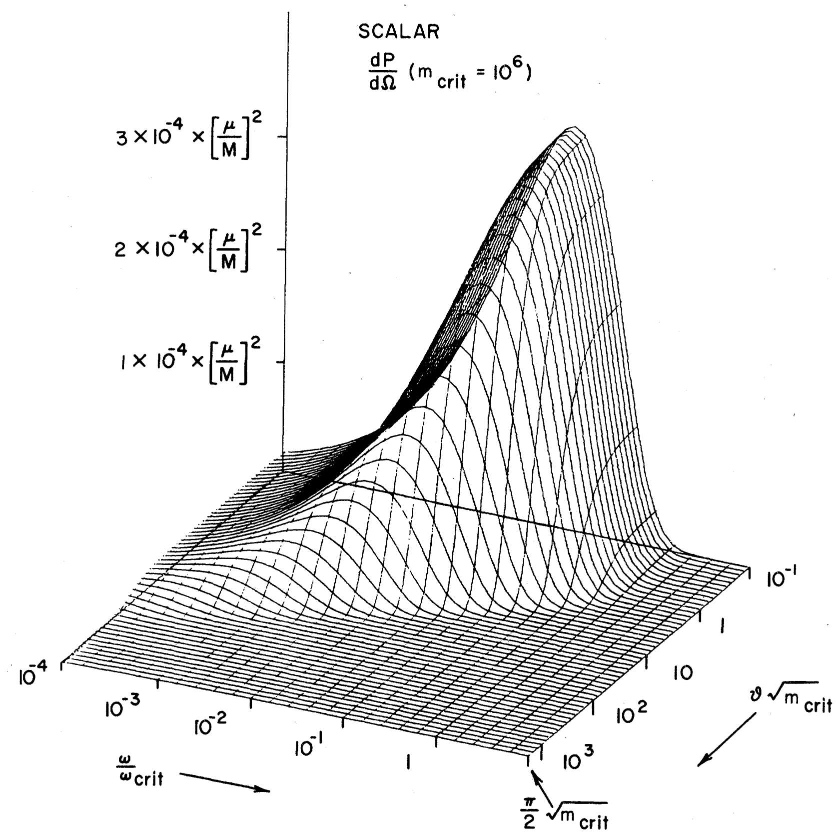

FIG. 2. Power per unit solid angle as a function of angle $\vartheta=\pi/2-\theta$ (θ is the polar angle) and frequency. The scaling of the ϑ axis is chosen so that the half-width of the beam is at $\vartheta\sqrt{m_{\rm crit}}=1$ for $\omega/\omega_{\rm crit}=1$.

$$\frac{\partial G(r^*, r_0^*)}{\partial r_0^*} = \begin{cases} \frac{i}{2\Upsilon} L'(r_0^*) R(r^*), & r^* > r_0^* \\ \frac{i}{2\Upsilon} R'(r_0^*) L(r^*), & r^* < r_0^*. \end{cases} \tag{A9}$$

APPENDIX B: THE WKB METHOD NEAR A PARABOLIC POTENTIAL MAXIMUM

We wish to solve the Schrödinger equation

$$-\frac{d^2\psi}{dx^2} + (V-E)\psi = 0 \tag{B1}$$

for a potential $V(x)$ which is adequately represented near its maximum at $x=0$ by a parabola:

$$V(x) \approx \tfrac{1}{2} V''(0) x^2, \tag{B2}$$

where we have selected the energy scale so that $V(0)=0$. For a barrier maximum, we have

$$V''(0) < 0. \tag{B3}$$

For x within the parabolic region, where Eq. (B2) holds, Eq. (B1) becomes

$$\frac{d^2\psi}{dx^2} + [E + \tfrac{1}{2} |V''(0)| x^2] \psi = 0. \tag{B4}$$

By defining

$$w = |\tfrac{1}{2} V''(0)|^{1/2}, \quad \xi = w^{1/2} x, \quad \epsilon = -E/w, \tag{B5}$$

we may put Eq. (B4) into dimensionless form:

$$\frac{d^2\psi}{d\xi^2} + (\xi^2 - \epsilon)\psi = 0. \tag{B6}$$

Finally we define

$$z = (1-i)\xi, \quad \nu = -\tfrac{1}{2}(1+i\epsilon), \tag{B7}$$

so that Eq. (B6) takes the well-known form[24]

$$\frac{d^2\psi}{dz^2} + (\nu + \tfrac{1}{2} - \tfrac{1}{4} z^2)\psi = 0. \tag{B8}$$

Since ν is never an integer, two linearly independent solutions of (B8) are

$$\psi = D_\nu(z), \quad \psi = D_\nu(-z), \tag{B9}$$

where $D_\nu(\pm z)$ are parabolic cylinder functions.

In order to obtain WKB connection formulas, we must know the asymptotic behavior of $D_\nu(\pm z)$. For $x>0$, we have $\arg z = -\pi/4$, and the asymptotic for D_ν for large $|z|$ is

$$D_\nu(z) \sim z^\nu \exp(-\tfrac{1}{4} z^2). \tag{B10}$$

On the other hand, if $x<0$, then $\arg z = 3\pi/4$, and the asymptotic form for D_ν is

$$D_\nu(z) \sim z^\nu \exp(-\tfrac{1}{4} z^2) - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{\nu\pi i} z^{-\nu-1} \exp(\tfrac{1}{4} z^2). \tag{B11}$$

These expressions may be rewritten in terms of x as follows:

$$D_\nu(\eta x) \sim (2w)^{-1/4} x^{-1/2} \exp(-\tfrac{1}{8}\pi\epsilon + \tfrac{1}{2} i w x^2 - \tfrac{1}{2} i\epsilon \ln x - \tfrac{1}{4} i\epsilon \ln 2w + \tfrac{1}{8} i\pi), \qquad x \gg 0, \tag{B12}$$

$$D_\nu(\eta x) \sim (2w)^{-1/4} |x|^{-1/2} \exp(\tfrac{3}{8}\pi\epsilon + \tfrac{1}{2} i w x^2 - \tfrac{1}{2} i\epsilon \ln|x| - \tfrac{1}{4} i\epsilon \ln 2w - \tfrac{3}{8} i\pi) + \frac{\sqrt{2\pi}}{\Gamma(-\nu)} (2w)^{-1/4} |x|^{-1/2} \exp(\tfrac{1}{8}\pi\epsilon - \tfrac{1}{2} i w x^2 + \tfrac{1}{2} i\epsilon \ln|x| + \tfrac{1}{4} i\epsilon \ln 2w + \tfrac{1}{8} i\pi), \quad x \ll 0 \tag{B13}$$

where $\eta = (1-i) w^{1/2}$.

The asymptotic forms for $D_\nu(-\eta x)$ follow from letting $x \to -x$ in (B12). These forms must be compared with the WKB solutions in the regions far from the potential maximum.

In the WKB region, the solutions have the form

$$\psi \sim k^{-1/2} e^{\pm iy}, \tag{B14}$$

where k and y, as functions of x, are defined by

$$k = (E-V)^{1/2}, \quad y = \int^x k\, dx. \tag{B15}$$

We now specialize to the case $E<0$ (i.e., $\epsilon>0$), so that there are classical turning points (where $V=E$) at $x=a$ and at $x=-a$. We also assume that the potential does not depart significantly from parabolic form before x reaches the asymptotic region, where the WKB approximation is valid.

In other words, we assume the existence of regions of overlap in which the parabolic approximation (B2) and the WKB approximation (B14) are simultaneously valid. This situation is illustrated in Fig. 3.

For definiteness, we take

$$y=\int_a^x k\,dx \quad \text{for } x>a\,,$$
$$y=-\int_x^{-a} k\,dx \quad \text{for } x<-a\,, \qquad \text{(B16)}$$

so that y is positive on the right of the potential barrier and negative on the left. Within the parabolic region, we have explicitly

$$k=(w^2x^2-|E|)^{1/2} \quad \text{for } |x|>a\,. \qquad \text{(B17)}$$

By integrating we find

$$|y|=\tfrac{1}{2}\epsilon\{(w/\epsilon)^{1/2}|x|(wx^2/\epsilon-1)^{1/2} -\ln[(w/\epsilon)^{1/2}|x|+(wx^2/\epsilon-1)^{1/2}]\} \quad \text{for } |x|>a\,. \qquad \text{(B18)}$$

In the asymptotic region $|x|>>a$, Eqs. (B17) and (B18) take on simpler form:

$$|y|\sim\tfrac{1}{2}wx^2-\tfrac{1}{2}\epsilon\ln|x|-\tfrac{1}{4}\epsilon\ln(4w/\epsilon)\,,$$
$$k\sim w|x|\,,\quad |x|>>a\,. \qquad \text{(B19)}$$

By comparing Eqs. (B12) and (B19) we find that

$$D_\nu(\eta x)=\begin{cases}Ak^{-1/2}e^{iy}\,, & x>>a\\ Bk^{-1/2}e^{iy}+Ck^{-1/2}e^{-iy}, & x<<-a\end{cases}$$
$$D_\nu(-\eta x)=\begin{cases}Bk^{-1/2}e^{-iy}+Ck^{-1/2}e^{iy}, & x>>a\\ Ak^{-1/2}e^{-iy}\,, & x<<-a\end{cases} \qquad \text{(B20)}$$

with

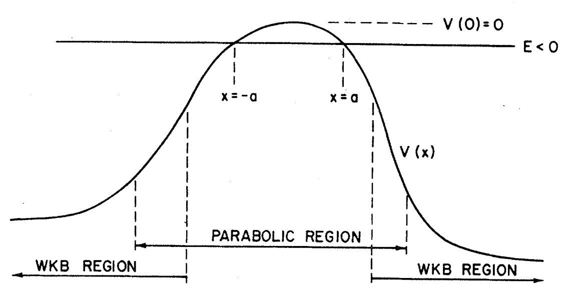

FIG. 3. Parabolic WKB potential. The potential for the Schrödinger wave equation (B1), showing the parabolic region, the WKB region, and the regions of overlap, in which both approximations hold simultaneously.

$$A=(\tfrac{1}{2}w)^{1/4}\exp(-\tfrac{1}{8}\pi\epsilon+\tfrac{1}{8}i\pi-\tfrac{1}{4}i\epsilon\ln\tfrac{1}{2}\epsilon)\,,$$
$$B=(\tfrac{1}{2}w)^{1/4}\exp(\tfrac{1}{8}\pi\epsilon+\tfrac{1}{8}i\pi+\tfrac{1}{4}i\epsilon\ln\tfrac{1}{2}\epsilon)\,\frac{\sqrt{2\pi}}{\Gamma(-\nu)}\,,$$
$$C=(\tfrac{1}{2}w)^{1/4}\exp(\tfrac{3}{8}\pi\epsilon-\tfrac{3}{8}i\pi-\tfrac{1}{4}i\epsilon\ln\tfrac{1}{2}\epsilon)\,. \qquad \text{(B21)}$$

Far to the left of the potential barrier $D_\nu(-\eta x)$ behaves like a pure outgoing wave, whereas $D_\nu(\eta x)$ is a combination of an ingoing and an outgoing wave for $x<<-a$. Thus, the WKB scattering solution that represents an incident wave moving purely toward the left, which we call $\psi=L(x)$, matches to $D_\nu(-\eta x)$ only. Any admixture of $D_\nu(\eta x)$ would force the WKB solution to include an incident wave moving toward the right. By examining the $x>>a$ asymptotic forms of $D_\nu(\pm\eta x)$, the same arguments may be used to conclude that the scattering solution with an incident wave moving right only, denoted by $\psi=R(x)$, behaves like $D_\nu(\eta x)$ near the peak of the potential.

Specifically, $R(x)$ and $L(x)$ may be defined with the normalization

$$R(x)\simeq\begin{cases}\mathcal{T}k^{-1/2}e^{iy}\,, & x>>a\\ k^{-1/2}e^{iy}+\mathcal{S}k^{-1/2}e^{-iy}, & x<<-a\end{cases}$$
$$L(x)\simeq\begin{cases}k^{-1/2}e^{-iy}+\mathcal{S}k^{-1/2}e^{iy}, & x>>a\\ \mathcal{T}k^{-1/2}e^{-iy}\,, & x<<-a\,.\end{cases} \qquad \text{(B22)}$$

Here $\mathcal{S}$ and $\mathcal{T}$ are the scattering and transmission coefficients. Using the above arguments and Eqs. (B20)–(B22), in the region near the peak of the potential barrier,

$$L(x)=(\tfrac{1}{2}w)^{-1/4}\,\frac{\Gamma(-\nu)}{\sqrt{2\pi}} \times\exp(-\tfrac{1}{8}\pi\epsilon-\tfrac{1}{8}i\pi-\tfrac{1}{4}i\epsilon\ln\tfrac{1}{2}\epsilon)\,D_\nu(-\eta x)\,,$$
$$R(x)=(\tfrac{1}{2}w)^{-1/4}\,\frac{\Gamma(-\nu)}{\sqrt{2\pi}} \times\exp(-\tfrac{1}{8}\pi\epsilon-\tfrac{1}{8}i\pi-\tfrac{1}{4}i\epsilon\ln\tfrac{1}{2}\epsilon)\,D(\eta x)\,. \qquad \text{(B23)}$$

The normalizations adopted in Eqs. (B22) coincide with those specified in Eqs. (A4). This is a consequence of the fact that Eqs. (B22) provide WKB solutions whenever Eq. (B15) holds. It is therefore possible to drop the assumption that V is parabolic once the WKB approximation becomes valid [thus drop Eq. (B18) and continue using Eqs. (B22) away from the potential peak and on out to the asymptotic region envisioned in Eqs. (A4) where V becomes constant and $y\sim kx+$ const]. See Fig. 3.

In order to calculate the scalar power radiated at the peak intensities, we must evaluate $L(r^*)$ and $R(r^*)$ at $r=r_0=(3+\delta)M$ with $\delta<<1$ when, from Eq. (3.7),

$$V(r=3M)=\left(1-\frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right]\simeq\frac{l(l+1)}{27M^2},$$

$$E=m^2\omega_0{}^2\simeq\frac{m^2}{27M^2}(1-\delta). \tag{B24}$$

The parameters defined by (B5), (B7), and (B13) may be evaluated as follows:

$$\begin{aligned} w &\simeq \frac{[l(l+1)]^{1/2}}{27M^2}\simeq\frac{m}{27M^2},\\ \epsilon &= \frac{V-E}{w}\simeq 1+4q+m\delta = 1+4q+\frac{4}{\pi}\frac{m}{m_{\text{crit}}},\\ \nu &= -\tfrac{1}{2}(1+i\epsilon),\\ \eta &= (1-i)\frac{m^{1/2}}{27^{1/2}M}. \end{aligned} \tag{B25}$$

Here $2q=l-m\ll m$ and $m_{\text{crit}}=4/\pi\delta$. Using the above relations, (B23) may be simplified by noting that $x\simeq 0$ when $\delta\ll 1$:

$$|L(x)|^2\simeq|L(0)|^2\simeq|R(0)|^2 \simeq\frac{3\sqrt{6}\,M}{2\pi m^{1/2}}e^{-\pi\epsilon/4}|\Gamma(\tfrac{1}{2}+\tfrac{1}{2}i\epsilon)D_\nu(0)|^2. \tag{B26}$$

Since

$$D_\nu(0)=\frac{\pi^{1/2}}{2^{1/4}\Gamma(\frac{3}{4}+\frac{1}{4}i\epsilon)},$$

we find that

$$|L(0)|^2=|R(0)|^2 =\frac{3\sqrt{3M}}{4\pi m^{1/2}}e^{-\pi\epsilon/4}|\Gamma(\tfrac{1}{4}+\tfrac{1}{4}i\epsilon)|^2, \tag{B27}$$

where ϵ is given by (B25).

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[1]C. W. Misner, R. A. Breuer, D. R. Brill, P. L. Chrzanowski, H. G. Hughes, III, and C. M. Pereira, Phys. Rev. Lett. 28, 998 (1972).

[2]R. A. Breuer, J. Tiomno, and C. V. Vishveshwara, Nuovo Cimento Lett. 4, 857 (1972).

[3]R. A. Breuer, J. Tiomno, and C. V. Vishveshwara (unpublished).

[4]R. A. Breuer, R. Ruffini, J. Tiomno, and C. V. Vishveshwara, Phys. Rev. D 7, 1002 (1973).

[5]R. A. Breuer and C. V. Vishveshwara, Phys. Rev. D 7, 1008 (1973).

[6]M. Davis, R. Ruffini, J. Tiomno, and F. Zerilli, Phys. Rev. Lett. 28, 1352 (1972).

[7]A. Einstein, Sitzber. Kgl. Preuss. Akad. Wiss. 688 (1916); 154 (1918); see also L. Landau and E. Lifschitz, *The Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1962), p. 363.

[8]K. S. Thorne, Astrophys. J. 158, 1 (1969).

[9]P. C. Peters, Phys. Rev. D 1, 1559 (1970).

[10]M. Davis, R. Ruffini, W. H. Press, and R. H. Price, Phys. Rev. Lett. 27, 1466 (1971); R. Ruffini, Phys. Rev. D 7, 972 (1973).

[11]A. Zee, senior thesis, Dept of Physics, Princeton University, 1963 (unpublished).

[12]For flat space-time treatments of this question ("will a charge sitting on a table radiate?") see H. Bondi and T. Gold, Proc. R. Soc. A229, 416 (1955); T. Fulton and F. Rohrlich, Ann. Phys. (N.Y.) 9, 499 (1960); F. Rohrlich, *ibid.* 22, 169 (1963).

[13]H. G. Hughes, III, Ann. Phys. (N.Y.) (to be published).

[14]C. W. Misner, Phys. Rev. Lett. 28, 994 (1972).

[15]C. W. Misner, lecture to the VI Texas Symposium on Relativistic Astrophysics, New York, 1972 (unpublished); paper in *Gravitational Radiation and Collapse, I.A.U. Symposium No. 64,* edited by C. DeWitt (to be published).

[16]J. M. Bardeen, W. H. Press, and S. Teukolsky, Astrophys. J. 178, 347 (1972).

[17]P. L. Chrzanowski and C. W. Misner, 1973 (unpublished); for a summary see C. W. Misner in *Ondes et Radiations Gravitationelles,* Actes du Collogne International C.N.R.S. No. 220 (to be published).

[18]P. L. Chrzanowski, Ph. D. thesis, Dept. of Physics and Astronomy, University of Maryland, 1973 (unpublished).

[19]R. A. Matzner, J. Math. Phys. 9, 163 (1968).

[20]R. H. Price, Phys. Rev. D 5, 2419 (1972); 5, 2439 (1972).

[21]R. Ruffini, J. Tiomno, and C. V. Vishveshwara, Nuovo Cimento Lett. 3, 211 (1971); see also Ref. 5.

[22]F. Zerilli, Phys. Rev. D 2, 2141 (1970); see also Refs. 3, 4.

[23]D. W. Ford *et al.*, Ann. Phys. (N.Y.) 7, 239 (1959).

[24]*Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. II.