S Matrix for Gravitational Field. II. Local Measure; General Relations; Elements of Renormalization Theory

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A method is suggested for obtaining the local measure in the generating functional for nonlinear gauge fields within the framework of covariant S-matrix theory. The local measure is obtained explicitly for the case of the gravitational field. The measure obtained is proved to cancel all the divergences of the type $\delta^{(4)}(0)$ which arise in the theory due to its nonlinearity. It is proved that the value of the measure that is obtained is also required by the canonical formalism for gauge fields, and thus it secures the unitarity of the S matrix. A new version of the canonical formalism for the gravitational field is given, which leads to the explicit Hamiltonian in terms of independent canonical variables. The quantization procedure in this approach to the canonical formalism is just the usual canonical quantization carried out in the Lorentz-covariant gauge. The new canonical formalism directly gives the value of the local measure in the Feynman integral. It is proved that besides securing the unitarity of the S matrix and eliminating the strongest divergences, the local measure obtained secures the gauge independence of the S matrix. This property results from the fact that the Jacobian of the gauge transformation of the field differentials is not equal to unity, contrary to the statement in previous works. The gauge transformation of the local measure exactly compensates for this Jacobian. The consequences of the gauge invariance of the theory are studied next. The complete set of generalized Ward identities for the Green's functions is obtained in the transverse gauge. The set of quantum equations of motion for the gravitational field is derived, and the problems connected with these equations are discussed. In the framework of the first-order formalism, the quantized Einstein equations are shown to take the form of the Schwinger-Dyson equations. A set of gauge relations for bare vertices is obtained. The analysis of the generalized Ward identities for the Green's functions at the threshold is given. In this connection the gravitational and fictitious-field wave-function renormalization constants Z_2 and $\tilde{Z}_2$ as well as the fictitious interaction vertex renormalization constant $\tilde{Z}_1$ and the infinite number of graviton vertex renormalization constants $Z_1, Z_1^{(2)}, \ldots, Z_1^{(n)}, \ldots$ are introduced. An infinite set of Ward relations for these renormalization constants is obtained, $Z_1 Z_2^{-1} = \tilde{Z}_1 \tilde{Z}_2^{-1}$, $(Z_1^{(n)})^{-1} = Z_1^{-n} Z_2^{n-1}$, which leaves only two independent renormalization constants Z_1 and Z_2 and secures the gauge invariance of the renormalized theory. Further, the new invariance properties of the quantum theory of the gravitational field are investigated, which are connected with peculiarities of the symmetry breaking and with the existence of the dimensional Planck length. The "scale" invariance or "homogeneity" of the theory is proved, which leads by means of Euler's theorem to a new infinite set of relations, obeyed by the Green's functions. Analysis of these relations at threshold gives a new identity for the renormalization constants: $Z_1 = Z_2$. Using the fact that the gravitational constant enters the theory only through the dimensionless space-time coordinates, the general off-mass-shell relations for the Green's functions are obtained from the "scale" identities. Possible anomalous singularities of the Green's functions are investigated. The "scale"-invariant regularization is introduced. It is also proved, using the "homogeneity" properties, that the renormalization constants do not depend on the gauge. As a result, the threshold asymptotic behavior of all Green's functions of the theory is made finite by means of only one renormalization constant—that renormalizing the gravitational constant κ.

I. INTRODUCTION

Recently considerable success has been achieved in the quantum theory of gauge-invariant fields, the gravitational field in particular. Methods have been developed by several authors in Refs. 1–4, after a well-known Feynman work of 1963, which make it possible to overcome the difficulties connected with the degeneracy of the gauge-invariant Lagrangian and to construct the unitary and gauge-invariant S matrix for the gravitational field within the framework of the functional-integration method.

The gravitational-field Lagrangian without the divergence-type term may be written as[5]

$$L = \sqrt{-g}\, g^{\mu\nu}(\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}) = D^{(\phi,\theta),(\psi,f),\alpha,\sigma}(g_{\mu\nu})(\partial_\sigma g_{\phi\theta})(\partial_\alpha g_{\psi f}) . \qquad (1.1)$$

Let us introduce the collective indices, running from 1 to 10:

$$A=(\phi,\theta),\quad B=(\psi,f),\ldots,$$

and define

$$D^{AB}=D^{(\phi,\theta),(\psi,f),\alpha,\sigma}(g_{\mu\nu})n_\alpha n_\sigma, \tag{1.2}$$

with arbitrary n_α. When $(n_\alpha)=(1,0,0,0)$,

$$\tfrac{1}{2}D^{AB}=\frac{\partial^2 L}{\partial(\partial_0 g_A)\partial(\partial_0 g_B)}\ .$$

One can check by direct computation that the Lagrangian (1.1) is degenerate because

$$\det D^{AB}=0$$

for any n_α.

The modified generating functional[6] for the gravitational field may be presented in terms of Refs. 1–4 as

$$Z[j^{\mu\nu}]=\frac{1}{Z^{\text{lin}}}\int J_\chi[g]\exp\left[i\int(L+g_{\mu\nu}j^{\mu\nu})d^4x\right]\times\prod_x\prod_\alpha\delta(\chi^\alpha)d\mu(g), \tag{1.3}$$

where L denotes the Lagrangian (1.1), χ^α the gauge conditions, and Z^{lin} the normalizing integral over paths for linearized theory:

$$L^{\text{lin}}=D^{(\phi,\theta),(\psi,f),\alpha,\sigma}(\delta_{\mu\nu})(\partial_\sigma g_{\phi\theta})(\partial_\alpha g_{\psi f}). \tag{1.4}$$

The Jacobian $J_\chi[g]$ is determined by the given gauge condition as

$$J_\chi[g]=\exp\operatorname{Tr}\ln Q_\chi^{\mu\beta}[Q_\chi^{(\text{lin})-1}]_{\beta\nu}\ , \tag{1.5}$$

where the operator $\hat{Q}_\chi^{\mu\beta}$ is determined up to a local factor according to the following recipe.

Let us perform the gauge transformation of the given gauge conditions χ^α with the aid of the general coordinate transformation function f:

$$x^\mu\to\bar{x}^\mu=x^\mu+f^\mu(x).$$

Confining ourselves to the first order in f, we obtain

$$\delta^f\chi^\alpha=\hat{Q}_\chi^{\alpha\beta}f^\beta. \tag{1.6}$$

One may use any variables $A_{\mu\nu}$, parametrizing the gravitational field. For instance, in Sec. II we shall use the variables

$$A_{\mu\nu}=g_{\mu\nu}\ .$$

In some cases the choice of variables

$$A_{\mu\nu}=\hat{g}^{\mu\nu}\equiv\sqrt{-g}\,g^{\mu\nu} \tag{1.7}$$

is more convenient.

The gauge transformation of the variables reads

$$\delta^f A_{\mu\nu}=\nabla^{\mu\nu|\sigma}(A)f^\sigma. \tag{1.8}$$

From (1.6) and (1.8) one obtains

$$\delta^f\chi^\alpha(x)=\int dy\,\frac{\delta\chi^\alpha(A|x)}{\delta A_{\mu\nu}(y)}\nabla^{\mu\nu|\sigma}(A|y)f^\sigma(y),$$

or for the kernel of $\hat{Q}$,

$$\hat{Q}^{T\alpha\beta}_\chi(x,y)=\nabla^{+\mu\nu}_\alpha(A|y)\frac{\delta\chi^\beta(A|x)}{\delta A_{\mu\nu}(y)}\ , \tag{1.9}$$

where $\nabla^{+\mu\nu}_\alpha$ is conjugated to $\nabla^{\mu\nu|\alpha}$. For instance, in terms of the variables (1.7),

$$\nabla^{\beta\gamma|\sigma}(\hat{g})=-(\partial_\sigma\hat{g}^{\beta\gamma})+\hat{g}^{\beta\rho}\delta^\gamma_\sigma\partial_\rho+\hat{g}^{\gamma\rho}\delta^\beta_\sigma\partial_\rho-\hat{g}^{\beta\gamma}\partial_\sigma\ , \tag{1.10}$$

$$\nabla^{+\mu\nu}_\alpha(\hat{g})=-(\partial_\sigma\hat{g}^{\nu\sigma})\delta^\mu_\alpha-(\partial_\sigma\hat{g}^{\mu\sigma})\delta^\nu_\alpha-\hat{g}^{\nu\sigma}\delta^\mu_\alpha\partial_\sigma-\hat{g}^{\mu\sigma}\delta^\nu_\alpha\partial_\sigma+\hat{g}^{\mu\nu}\partial_\alpha\ . \tag{1.11}$$

A "conjugate" operator is everywhere defined by equalities of the type

$$\int F_{\mu\nu}(x)\nabla^{\mu\nu|\sigma}(A|x)f^\sigma(x)dx\equiv\int f^\sigma(x)\nabla^{+\mu\nu}_\sigma(A|x)\times F_{\mu\nu}(x)dx\ .$$

It is important that $\nabla(A)$ is always linear in the variables A (which has been observed already in Ref. 2).

Now, beginning with Sec. III we shall use the linear and transverse gauge, which in terms of the variables (1.7) reads

$$\chi^\mu(\hat{g})=\partial_\nu\hat{g}^{\mu\nu}. \tag{1.12}$$

However, in Sec. II we shall need another nonlinear gauge condition (2.17). Under the gauge conditions (1.12) one obtains

$$\begin{aligned}\hat{Q}^\gamma_\sigma&=\partial_\mu\nabla^{\gamma\mu|\sigma}\\&=-(\partial^2_{\sigma\beta}\hat{g}^{\beta\gamma})+\hat{g}^{\beta\rho}\delta^\gamma_\sigma\partial^2_{\beta\rho}+(\partial_\beta\hat{g}^{\beta\rho})\delta^\gamma_\sigma\partial_\rho-(\partial_\beta\hat{g}^{\beta\gamma})\partial_\sigma\ ,\end{aligned} \tag{1.13}$$

and the conjugated operator

$$\begin{aligned}\hat{Q}^{T\mu}_\alpha&=-\nabla^{+\mu\nu}_\alpha\partial_\nu\\&=(\partial_\sigma\hat{g}^{\nu\sigma})\delta^\mu_\alpha\partial_\nu+(\partial_\sigma\hat{g}^{\mu\sigma})\partial_\alpha+\hat{g}^{\nu\sigma}\delta^\mu_\alpha\partial^2_{\nu\sigma}\ .\end{aligned} \tag{1.14}$$

The operators $\hat{Q}$ and $\hat{Q}^T$ are different from each other, but it makes no difference which of them is to be used in the expression (1.5). Calculation of the operator $\hat{Q}$ under the nonlinear gauge conditions (2.17) gives expression (2.21). It is important that if the gauge conditions are linear, the operator $\hat{Q}$ is also linear in the field variables.

As is shown in Refs. 1, 2, and 7 the Lagrange-multipliers method may be generalized, which makes it possible to obtain the S-matrix generating functional for the nondegenerate gauges $\chi^\alpha \neq 0$:

$$Z[j^{\mu\nu}] = \frac{1}{Z^{\rm lin}} \int J_\chi[g] \exp\left\{i \int \left(L + g_{\mu\nu} j^{\mu\nu} - \frac{1}{2\alpha} \chi^\mu \chi^\nu \delta_{\mu\nu}\right) d^4x\right\} d\mu(g) . \tag{1.15}$$

When $\alpha \to 0$, $Z[j^{\mu\nu}]$ reduces to expression (1.3), where the gauge is degenerate. α is a parameter fixing the gauge.

It was shown in Ref. 1 that the generating functional (1.15) determines the S matrix, which is unitary, gauge-invariant, and independent of α. The unitarity and gauge independence of the S matrix needs some more discussion, as we shall see later.

The generating functional may be also written down in the first-order formalism, where the components of the metric tensor and of the Christoffel symbols are treated as independent:

$$Z[j^{\mu\nu}] = \frac{1}{Z^{\rm lin}} \int J_\chi[g, \Gamma] \exp\left\{i \int (L(g, \Gamma) + g_{\mu\nu} j^{\mu\nu}) d^4x\right\} \times \prod_{x,\alpha} \delta(\chi^\alpha(g, \Gamma)) d\mu(g, \Gamma). \tag{1.16}$$

To make the S-matrix theory formulation complete, it is necessary to solve the problem of a local measure $d\mu(g)$ in the integral (1.15) and $d\mu(g, \Gamma)$ in the integral (1.16). In the following expression for measure, the local factor $\phi(x)$,

$$d\mu(g) = \prod_x \phi(x) \prod_{\mu \leq \nu} dg_{\mu\nu} = \exp\left\{\delta^{(4)}(0) \int \ln\phi(x) d^4x\right\} \prod_{x, \mu \leq \nu} dg_{\mu\nu} ,$$

exists due to the peculiarity of nonlinear theories.

The problem of local measure has not been solved in Refs. 1–4. However, this measure is important for unitarity, gauge invariance, and cancellation of the strongest divergences of the type $\exp[\delta^{(4)}(0) f]$, which appear in the nonlinear theories with the interaction Lagrangian quadratic in field derivatives. Specification of the measure for gravity is not straightforward because of the degeneracy of the field Lagrangian. However, for the S matrix expressed in terms of the proper integrals (1.15), (1.16) the measure may be specified. In the present paper we suggest a way of obtaining the local measure for nonlinear field theories with nondegenerate and degenerate Lagrangians and obtain the local measure for gravity.

In Sec. II of this article the local measure is obtained from the requirement of cancellation of the terms of the type $\delta^{(4)}(0)$ in the functional integral.[8] Next we show that the same value of the measure is also required by the canonical formalism for systems with constraints.

Further in Sec. II we suggest a new version of the canonical formalism for the gravitational field which possesses some advantages over the usual one developed by Dirac and others. First, it makes it possible to obtain the explicit expression for the Hamiltonian in terms of independent canonical variables. Second, the quantization procedure for such a Hamiltonian does not need any additional elements except those required by the usual canonical quantization. Third, quantization is carried out directly in a Lorentz-covariant gauge, while in the approach of Ref. 9 the use of noncovariant gauges leads to serious difficulties when the transition to a covariant gauge is performed.[10] In our approach to the canonical formalism, developed in Sec. II, the quantization is straightforward and leads to the correct expression for the S matrix in the degenerate harmonic gauge, directly giving the correct value of the local measure in the Feynman functional integral.

Next, we show that in spite of the presence of the noncovariant factor g^{00} or $g^{\mu\mu}$ in the local measure obtained (it would be better to say "due to this presence"), the local measure together with the field differentials remains invariant under the gauge transformations. This property results from the fact that the Jacobian of the gauge transformation of the field differentials is not equal to unity. Here the essential peculiarities of the gauge group of general relativity come to light. As to the local factors, it is shown that the Jacobian arising is exactly compensated for by the gauge transformation of the local measure. The latter circumstance plays an essential role in the proof of the gauge independence of the S matrix, because it is only when it is taken into account that the procedure of integration over the gauge group, developed in Ref. 3, leads to the correct proof. As a result, it is shown in Sec. II that the local measure obtained is the unique one that makes it possible to secure the unitarity and gauge independence of the S matrix, and cancellation of the strongest divergences of the type $\delta^{(4)}(0)$.

We proceed in Sec. III with a study of the consequences of gauge invariance of the theory, which results in a set of identities for the Green's functions analogous to the Nöther identities in classical theory. These relations are known as generalized Ward identities. The well-known Ward identity in electrodynamics is a particular case of these relations. The complete set of generalized Ward identities in electrodynamics was first obtained by one of the present authors in Ref. 11. The general method developed in Ref. 11 is, in principle, applicable to any gauge theory. The generalized Ward identities for the Yang-Mills field were obtained and analyzed in Ref. 12(a). The suggestion that there should be an infinity of gravitational Ward identities was spelled out in Ref. 2. A discussion of the connection between the conserved currents and the source currents in the case of gravitons may be found in Ref. 12(b). In Sec. III we shall obtain a complete set of generalized Ward identities for the Green's functions of the theory of gravity. We shall show that all these identities may be derived in the transverse gauge from the generating Ward identity in the presence of sources. [See the note in Ref. 12(c).] In its turn, the generating Ward identity is proved to be the gauge-invariant consequence of the complete set of equations of motion for the generating functional and the Green's functions. The set of quantum equations of motion for the gravitational field is derived in Sec. III and the problems connected with these equations are discussed. It is shown that in the framework of the first-order formalism, the quantized Einstein equations take directly the form of equations of the Schwinger-Dyson type in electrodynamics. Some first examples of application of the generalized Ward identities are also given in Sec. III.

Sections IV and V are devoted to the elements of renormalization theory. The threshold renormalization constants are introduced: the gravitational and fictitious-field wave-function renormalization constants Z_2 and $\tilde{Z}_2$, the fictitious interaction vertex renormalization $\tilde{Z}_1$, and the infinite number of graviton vertex renormalization constants Z_1, $Z_1^{(2)}, \ldots, Z_1^{(n)}, \ldots$, in accordance with the infinite number of the bare graviton vertices. The complete analysis of generalized Ward identities for the Green's functions at the threshold is given in Sec. IV. Such an analysis for the theory of gravity presents some technical difficulties in comparison with the similar treatment in Ref. 12(a) for the Yang-Mills field. An independent investigation is needed of the threshold asymptotics of some of the Green's functions in coinciding points, which is performed in Appendix B. The set of gauge relations of the bare vertices also needed for the analysis of Sec. IV is obtained in Appendix A. As a result, we obtain in Sec. IV an infinite set of Ward relations for the renormalization constants

$$Z_1 Z_2^{-1} = \tilde{Z}_1 \tilde{Z}_2^{-1},$$

$$(Z_1^{(n)})^{-1} = Z_1^{-n} Z_2^{n-1},$$

which leaves only two independent renormalization constants Z_1 and Z_2 and secures, as we show, the gauge invariance of the renormalized theory. Section IV ends with the construction of the renormalized Lagrangian in an arbitrary gauge and with the discussion of some peculiarities of the gravitational-field wave-function renormalization, which seem to suggest one more relation between the renormalization constants,

$$Z_1 = Z_2.$$

This relation exists, but it is not the Ward identity. We obtain this new relation between the renormalization constants in Sec. V.

In Sec. V the new invariance properties of the quantum theory of gravity are studied, which are connected mainly with the peculiarities of symmetry breaking, with conformal properties of the Einstein Lagrangian, and with existence of the fundamental constant $(\kappa\hbar)^{1/2}$ having the dimension of length. A proof is given of the invariance property, called the "scale" invariance or "homogeneity" of the theory, which states that the generating functions and Green's functions of the theory are the homogeneous functions of some of their arguments. Analytically, the homogeneity is expressed by the known Euler theorem just as gauge invariance is expressed by the Nöther identities. Application of the Euler theorem leads to a new infinite set of relations, obeyed by the Green's functions. Analysis of these relations at the threshold gives the new identity between the renormalization constants $Z_1 = Z_2$.

The "scale" identities mentioned contain the derivatives of the Green's functions with respect to the parameters of the theory. Further, using the fact that the gravitational constant κ enters the theory only through the dimensionless space-time coordinates, we exclude the derivatives with respect to κ from the "scale" identities and obtain the general off-mass-shell relations for the Green's functions.

The influence of ultraviolet and infrared divergences upon the "scale" properties is also discussed in Sec. V. The "scale"-invariant regularization is introduced and the possible anomalous singularities of the Green's functions are investigated. It is also proved, using the "homogeneity" properties, that the renormalization constants do not depend on the gauge.

As a result, the divergences are eliminated from the threshold asymptotics of all the Green's functions of the theory with the aid of only one essentially divergent constant—the gravitational-constant renormalization.

The article ends with a remark on the possible renormalizability of the theory, at least on the mass shell.

II. LOCAL MEASURE AND THE CANONICAL FORMALISM IN THE COVARIANT GAUGE

A. Appearance of the Local Measure and Cancellation of the Divergences $\delta^{(4)}(0)$ for the Theories with Nondegenerate Lagrangians

The S matrix for quantized fields coupling in the cases when the interaction Lagrangian depends nonlinearly on the field derivatives (or when the interaction Hamiltonian depends nonlinearly on the canonical momenta) cannot be presented by the usual Feynman integral over fields of the exponential of the field Lagrangian. In fact, in terms of independent fields, the unitary S matrix is presented as follows[13]:

$$S = :\exp\left\{\frac{i}{2}\int \frac{\delta}{\delta\phi(x)} D_F(x-y)\frac{\delta}{\delta\phi(y)}\, d^4x\, d^4y\right\} \times \exp\left\{i\int \tilde{L}_{\rm int}(x)\, d^4x\right\}\Big|_{\phi=\phi\,{\rm in}} :, \qquad (2.1)$$

where

$$\exp\left\{i\int \tilde{L}_{\rm int}(\phi(x))\, d^4x\right\} = \exp\left\{\frac{1}{2i}\int \frac{\delta}{\delta\pi(x)}\frac{\delta}{\delta\pi(x)}\, d^4x\right\} \times \exp\left\{-i\int H_{\rm int}(\pi,\phi)\, d^4x\right\}\Big|_{\pi=\partial L_0/\partial\dot{\phi}}. \qquad (2.2)$$

In the cases when the Lagrangian is not degenerate and the interaction Hamiltonian $H_{\rm int}(\pi,\phi)$ is expressed in terms of independent canonical variables, the generating functional of the S matrix (2.1) may be presented as the following functional integral[13]:

$$Z[j] = \frac{1}{Z^0}\int \prod_x d\phi\, \rho(\phi)\exp\left\{i\int [L(\phi)+\phi j]\, d^4x\right\}, \qquad (2.3)$$

where

$$\rho(\phi) \equiv \int \prod_x d\pi \exp\left\{i\int \left[L(\phi,\dot{\phi}+\pi) - L(\phi) - \pi\frac{\partial L(\phi,\dot{\phi}+\pi)}{\partial\pi}\right] d^4x + \delta(0)\,{\rm Tr}\ln\frac{\delta^2 L(\phi,\dot{\phi}+\pi)}{\delta\pi\delta\pi}\right\}.$$

It may be easily verified that provided $L_{\rm int}$ does not depend or depends linearly on the $\dot{\phi}$ expression, (2.3) reduces to the usual Feynman type $[\rho(\phi)=1]$. In the case of quadratic dependence of $L_{\rm int}$ on $\dot{\phi}$, (2.3) leads to the following expression for the generating functional:

$$Z[j] = \frac{1}{Z^0}\int \prod_x d\phi \exp\left\{i\int (L(\phi)+j\phi)\, d^4x + \tfrac{1}{2}\delta(0)\,{\rm Tr}\ln\frac{\partial^2 L(\phi,\dot{\phi})}{\partial\dot{\phi}\partial\dot{\phi}}\right\}$$
$$= \frac{1}{Z^0}\int \exp\left\{i\int (L(\phi)+j\phi)\, d^4x\right\} \times \prod_x \left[\det\frac{\partial^2 L}{\partial\dot{\phi}(x)\partial\dot{\phi}(x)}\right]^{1/2} d\phi. \qquad (2.4)$$

The same result can also be obtained from the continual representation of (2.2) in terms of independent canonical variables:

$$Z[j] = \frac{1}{Z^0}\int \exp\left\{i\int (\pi\dot{\phi} - H(\pi,\phi) + j\phi)\, d^4x\right\} \times \prod_x d\phi\, d\pi. \qquad (2.5)$$

Consequently the requirement of unitarity of the S matrix in the general case leads by means of the canonical formalism to the modification (2.3) of the Feynman integral. For local theories of the type mentioned, it leads to the existence of local measure, for which we have the expression

$$\prod_x \left[\det\frac{\partial^2 L}{\partial(\partial_0\phi_A)\partial(\partial_0\phi_B)}\right]^{1/2} \qquad (2.6)$$

in the integral (2.4) and the value 1 in the expression (2.5) provided the Lagrangian in (2.4) is nondegenerate and the Hamiltonian in (2.5) is expressed in terms of independent canonical variables. On the other hand, it may be proved that the local measure cancels the divergences of the type $\delta(0)$ which appear in the integral (2.4) when the integration over ϕ is performed. The latter statement is well illustrated by the stationary-phase method. Let us consider the integral (2.4) with the Lagrangian of the type (1.1)

$$L = D^{AB\mu\nu}(\phi)(\partial_\mu\phi_A)(\partial_\nu\phi_B),$$

but not degenerate, and perform the series expansion of the action functional near the extremal:

$$S = S_0 + \frac{1}{2}\int d^4x\, d^4y \frac{\delta^2 S}{\delta\phi_A(x)\delta\phi_B(y)}\Big|_{\phi_A=\phi_A^0}\bar{\phi}_A(x)\bar{\phi}_B(y) + \cdots$$
$$= S_0 + \frac{1}{2}\int d^4x\, d^4y\, \Omega^{AB}(x,y)\bar{\phi}_A(x)\bar{\phi}_B(y) + \cdots,$$

$$\bar{\phi}_A = \bar{\phi}_A - \bar{\phi}_A^0.$$

Replacing in (2.4)

$$\prod_{x,A} d\phi_A = \prod_{x,A} d\bar{\phi}_A,$$

and confining ourselves to the quadratic deviations from the extremal, we evaluate the resulting Gaussian integral:

$$Z|_{J=0}=\exp\left\{i(S_0-S_0^{\text{lin}})+\tfrac{1}{2}\delta^{(4)}(0)\int\left[\ln\det\frac{\partial^2 L}{\partial\dot\phi_A\partial\dot\phi_B}-\ln\det\frac{\partial^2 L^{\text{lin}}}{\partial\dot\phi_A\partial\dot\phi_B}\right]d^4x+\cdots\right\}\frac{[\det\Omega^{AB}_{\text{lin}}(x,y)]^{1/2}}{[\det\Omega^{AB}(x,y)]^{1/2}}\ . \quad (2.7)$$

Now, one has to single out the part $\propto\delta^{(4)}(0)$ from the term

$$\tfrac{1}{2}\text{Tr}\ln\Omega^{AB}_{\text{lin}}(x,y)-\tfrac{1}{2}\text{Tr}\ln\Omega^{AB}(x,y)$$

of (2.7). The kernel of the second variation reads

$$\Omega^{AB}(x,y)=[\Omega_1^{AB\mu\nu}(x)\partial_\mu\partial_\nu+\Omega_2^{AB\mu}(x)\partial_\mu+\Omega_3^{AB}(x)]\times\delta^{(4)}(x-y),$$

where

$$\Omega_1^{AB\mu\nu}(x)=2D^{AB\mu\nu}(\phi(x)),$$

the local term here being determined only by the matrix attached to the higher derivatives.

Indeed, one obtains the equation for the kernel $G^A_B(x,y)$ of the operator under consideration, $\hat G^A_C=\hat\Omega^{AB}_{\text{lin}}[\hat\Omega^{-1}]_{BC}$:

$$D^{AB\mu\nu}(\phi(x))\partial_\mu\partial_\nu G^C_B(x,y)+\cdots=D^{AC\mu\nu}(\delta_{\alpha\beta})\partial_\mu\partial_\nu\times\delta^{(4)}(x-y)+\cdots\ . \quad (2.8)$$

It may be seen from (2.8) that if $M^A_B(x,n)$ is chosen in the form

$$M^A_B(x,n)=D^{AC}(\delta)[D^{-1}(\phi(x))]_{CB}\,,$$

$$D^{AB}(\phi(x))\equiv D^{AB\mu\nu}(\phi(x))n_\mu n_\nu$$

one has for the local term,

$$\ln\frac{[\det\Omega^{AB}_{\text{lin}}(x,y)]^{1/2}}{[\det\Omega^{AB}(x,y)]^{1/2}}=\tfrac{1}{2}\text{Tr}\ln\hat\Omega^{AB}_{\text{lin}}[\hat\Omega^{-1}]_{BC}$$
$$=\tfrac{1}{2}\text{Tr}\ln\int M^A_B(x,n)e^{in(x-y)}d^4n+\cdots$$
$$=\tfrac{1}{2}\delta^{(4)}(0)\int\ln f(x)d^4x+\cdots. \quad (2.9)$$

There is some ambiguity in the final expression for $f(x)$ here. It is the ambiguity in the value of the local part of the coupling factors of the type

$$\partial_\mu D_F(x-y)\partial_\nu|_{x=y}=\int\frac{k_\mu k_\nu}{k^2}e^{ik(x-y)}d^4k\Big|_{x=y} \quad (2.10)$$

arising in (2.9) and depending on the way of calculating. In the framework of the Hamiltonian formalism, it follows from the definition of $D_F(x-y)$ as

$$\langle\theta(t-t')\phi(x)\phi(x')+\theta(t'-t)\phi(x')\phi(x)\rangle_0$$

that the term $\propto\delta^{(4)}(0)$ in (2.10) appears only when $\mu=\nu=0$, since it is the only case when $\delta(t_x-t_y)$ appears after the θ function is differentiated. The same conclusion follows from the representation of D_F, where the integration over k_0 is performed first. Here again, $\delta(t)$ appears only in $\partial^2_{00}D_F$. Thus in the framework of Hamiltonian technique one obtains

$$f(x)=\det M^B_A(x,n)\quad[(n)=(1,0,0,0)]$$
$$=\frac{\det D^{AB00}(\delta)}{\det D^{AB00}(\phi(x))}\ . \quad (2.11)$$

Substituting (2.11) into (2.9) and (2.7), one makes sure that the terms of the type $\delta^{(4)}(0)$ are canceled out at least on the extremal. When one chooses some other way of calculating the integrals of the type of (2.10) [for example by means of covariant regularization in k space, in which case $f(x)=\det D^{AB\mu\mu}(\delta)/\det D^{AB\mu\mu}(\phi(x))$ (see also Sec. II E)], the local measure in the integrals (2.4) and (2.7) must be also chosen not as (2.6), which the Hamiltonian method gives, but as

$$\prod_x\left[\det\frac{\partial^2 L}{\partial(\partial_\mu\phi_A)\partial(\partial_\mu\phi_B)}\right]^{1/2}, \quad (2.12)$$

the divergencies in (2.4) being compensated for again.

It is essential for the present discussion that whichever definite, but unique, way of calculating local measure and the local term in the functional integral is chosen, one will always obtain as a result the cancellation of divergent terms $\propto\delta(0)$ by the local measure.

Cancellation of the terms $\propto\delta(0)$ (for the systems concerned) by the local measure (2.6) was rigorously proved in Ref. 14 for the case of a finite number of degrees of freedom. It was shown in Ref. 14 that after the covariant finite-multiple approximation in the integral (2.4) is performed,

$$\lim_{\epsilon\to0}\frac{1}{Z^{\text{lin}}}\int\prod_{k,A}d\phi^k_A\exp\left\{i\sum_k D^{AB\mu\nu}(\phi k)\frac{1}{\epsilon^\mu}\frac{1}{\epsilon^\nu}\times\Delta\phi^k_A\Delta\phi^k_B\Big(\prod_\mu\epsilon^\mu\Big)\right\}\times\prod_k\left[\det D^{AB\mu\nu}(\phi_k)\frac{1}{\epsilon^\mu\epsilon^\nu}\right]^{1/2}, \quad (2.13)$$

the change of variables ϕ^k_A in the multiple integral (2.13) is possible, which leads to the cancellation of the terms

$$\prod_k \left[\det D^{AB\mu\nu}(\phi^k)\frac{1}{\epsilon^\mu\epsilon^\nu}\right]^{1/2} , \tag{2.14}$$

and reduces the exponential in the functional integral to the usual Gaussian type while the functional left-integrated over the usual Feynman measure turns out to be free from the terms $\exp\delta(0)$. In the functional-integration language the ambiguity in the expressions (2.9), (2.10) discussed above means the ambiguity in the limit $\epsilon_\mu \to 0$ in the expressions (2.13), (2.14). It is remarkable that the local measure (2.14) does not depend on the approximation, while the integral (2.13) does depend on the choice of it.[14] We shall write the expressions (2.6), (2.12), and (2.14) for local factors in the measure in a covariant form,

$$\lim_{n\to\infty}\left[\frac{\det D^{AB\mu\nu}(\phi(x))n_\mu n_\nu}{\det D^{AB\mu\nu}(\delta)n_\mu n_\nu}\right]^{1/2} \tag{2.15}$$

(see also Sec. II E), the Hamilton expression (2.6) being obtained from here by choosing the limit: $n_i/n_0 \to 0,\ n_0\to\infty$.

B. Calculation of the Measure for the Theories with Degenerate Lagrangians: The Case of the Gravitational Field

The main idea of our method is to use the generalized Lagrange-multipliers formalism, developed in Refs. 1, 2, and 7, which leads to expression (1.15) for the generating functional in a nondegenerate gauge. The advantage of the expression (1.15) over the expression (1.3) is that the noninvariant term

$$\Delta L(\alpha) = -\frac{1}{2\alpha}\delta_{\mu\nu}\chi^\mu(g)\chi^\nu(g), \tag{2.16}$$

quadratic in field derivatives, is added in (1.15) to the degenerate gravitational-field Lagrangian (1.1). For this reason the gauge invariance of (1.15) is broken before the transition to the mass shell and the Lagrangian degeneracy is removed. This in its turn permits us to obtain the expression for the local measure (1.17) in the integral (1.15) from the requirement of cancellation of the terms $\propto\delta^{(4)}(0)$, which are to be found in (1.15) both in the field action and in $\ln J_\chi$, the final expression for the S matrix being free from the divergences of the above-mentioned type.

So let us write the local factor $\phi(x)$ in (1.17) as

$$\phi(x) = \frac{1}{\phi_1(x)}\frac{1}{\phi_2(x)}.$$

Note that calculation of the measure by the method suggested is to be carried out for some definite choice of the gauge χ^α; the result, however, is independent of the gauge. It is convenient to carry out the calculations in the following gauge:

$$\chi_{(\beta)} = h_\mu(\beta)(-g)^{-1/4}(\partial_\sigma\sqrt{-g}\,g^{\sigma\mu}), \tag{2.17}$$

where $h_\mu(\beta)$ is a tetrad.[15]

The revised Lagrangian [from (1.1) and (2.16)] is nondegenerate; consequently we can use formula (2.15) for evaluation of the local factor $\phi_1(x)$ coming from the field action. The matrix

$$D^{AB}(\alpha) = D^{AB} + \Delta D^{AB}(\alpha)$$

[from (1.2) and (2.16)] calculated for the gauge (2.17) reads

$$D^{AB}(\alpha) = \tfrac{1}{4}\sqrt{-g}\,n^2\Big[\left(1+\frac{1}{2\alpha}\right)g^{\phi\theta}g^{\psi f} - \tfrac{1}{2}(g^{\phi\psi}g^{\theta f}+g^{\phi f}g^{\theta\psi}) + \frac{1}{2}\left(1+\frac{1}{\alpha}\right)\left(\frac{n^\phi n^f}{n^2}g^{\theta\psi}+\frac{n^\theta n^f}{n^2}g^{\phi\psi}+\frac{n^\phi n^\psi}{n^2}g^{\theta f}+\frac{n^\theta n^\psi}{n^2}g^{\phi f}\right)$$
$$-\left(1+\frac{1}{\alpha}\right)\frac{n^\phi n^\theta}{n^2}g^{\psi f} - \left(1+\frac{1}{\alpha}\right)\frac{n^\psi n^f}{n^2}g^{\phi\theta}\Big], \tag{2.18}$$

where the notation $n^\alpha = g^{\alpha\beta}n_\beta$, $n^2 = g^{\mu\nu}n_\mu n_\nu$ is introduced and symmetrization of indices (ϕ,θ), (ψ, f), and (A, B) is performed.

The direct evaluation of the determinant of the matrix (2.18) gives us

$$\det D^{AB}(\alpha) = \left(\frac{1}{4}\right)^{10}\frac{1}{\alpha^4}(g^{\mu\nu}n_\mu n_\nu)^{10}. \tag{2.19}$$

It is essential that the constant α factors out and thus cancels out after normalization. Performing normalization of (2.19) over $\det D^{AB}_{\text{lin}}(\alpha)$ and choosing the Hamiltonian way of transition $n_\mu\to\infty$, one gets from (2.15) and (2.19)

$$\frac{1}{\phi_1(x)} = (g^{00})^5. \tag{2.20}$$

The local factor $\phi_2(x)$ coming from $\ln J_\chi[g]$ in (1.15) is still to be found. The Jacobian $J_\chi[g]$ is given by (1.5) where the operator $\hat{Q}_\chi$ calculated by the method of Ref. 4 under the gauge (2.17) reads[16]

$$\hat{Q}_\mu(\gamma) = (-\partial_\mu g_{\nu\lambda} + 2\partial_\nu g_{\mu\lambda} + 2g_{\mu\lambda}\partial_\nu)\{(-g)^{1/2}(g^{\sigma\lambda}g^{\mu\nu} - \tfrac{1}{2}g^{\sigma\mu}g^{\lambda\nu})\partial_\sigma h_\mu(\gamma)(-g)^{-1/4}$$
$$+\tfrac{1}{2}(-g)^{-1/4}[\delta^\lambda_\rho h^\nu(\gamma) - \tfrac{1}{2}g^{\lambda\nu}h_\rho(\gamma)](\partial_\sigma\sqrt{-g}\,g^{\rho\sigma})\}. \tag{2.21}$$

Singling out the local part of $\ln J_\chi[g]$, which is determined by (2.21), may be carried out just as in the case of the kernel of the second variation of action. As a result one obtains

$$\phi_2(x) = (-g)^{3/2}(g^{00})^4, \quad g = \det g_{\mu\nu} . \tag{2.22}$$

Now from the requirement of cancellation of both local contributions by the local measure, we get, finally, the measure in the integrals (1.3) and (1.15):

$$d\mu(g) = \prod_x (-g)^{-3/2} g^{00} \prod_{\mu\leq\nu} dg_{\mu\nu} . \tag{2.23}$$

The measure in the integral (1.16) can be obtained using the requirement that after integration over forty variables $\Gamma^\alpha_{\beta\sigma}$ which may be performed explicitly, the integral (1.16) returns to the form (1.3). Thus we have

$$d\mu(g, \Gamma) = \prod_x (-g)^{7/2} g^{00} \prod_{\mu\leq\nu} dg_{\mu\nu} \prod_{\alpha, \beta\leq\sigma} d\Gamma^\alpha_{\beta\sigma} . \tag{2.24}$$

Expressions (2.23), (2.24) correspond to the Hamiltonian way of calculating the integrals of the type (2.10). In the manifestly Lorentz-covariant theory, the invariant regularization in k space may prove to be more convenient.

Then, the expressions for measure take the form

$$d\mu(g) = \prod_x (-g)^{-3/2} g^{\alpha\alpha} \prod_{\mu\leq\nu} dg_{\mu\nu} , \tag{2.25}$$

$$d\mu(g, \Gamma) = \prod_x (-g)^{7/2} g^{\gamma\gamma} \prod_{\mu\leq\nu} dg_{\mu\nu} \prod_{\alpha, \beta\leq\sigma} d\Gamma^\alpha_{\beta\sigma} . \tag{2.26}$$

However, as was discussed above, in any case provided that the calculations are uniform, the local measure obtained compensates for the divergences $\delta^{(4)}(0)$ in the functional integral, the S matrix being free from these divergences.

The measure naturally depends on the choice of the variables of integration. Transition to the other variables may be carried out by calculating the corresponding functional Jacobians. Let us give the values of measure for some choices of variables[17]:

$$\begin{aligned} d\mu(g) &= \prod_x (-g)^{-(20p+3)/2} g^{00} \prod_{\mu\leq\nu} d(g)^p g_{\mu\nu} \\ &= \prod_x (-g)^{-[10(2p-1)+3]/2} g^{00} \prod_{\mu\leq\nu} d(g)^p g^{\mu\nu} \\ &= \prod_x (-g)^{[5(1-2p)-6]/4} g^{00} \prod_{(\mu, (\alpha))} d(g)^p h_\mu(\alpha) \\ &= \prod_x (-g)^{[5(3-8p)-6]/4} g^{00} \prod_{(\mu, (\alpha))} d(g)^p h^\mu(\alpha) . \end{aligned} \tag{2.27}$$

The expression of the S matrix in the Dirac gauge is of special interest because of the possibility of comparing our results with the requirements of the canonical formalism.[9] It was mentioned above that in the Feynman integral over independent canonical variables, the local measure is equal to unity. Let us show that our results agree with the canonical formalism.

Let us consider the generating functional (1.16) with the measure obtained (2.24) in the Dirac gauge (this consideration follows our work[8]):

$$\chi^0 = q^{ik}\pi_{ik} , \quad \chi^i = \partial_k[(g^{(3)})^{1/3} e^{ik}], \quad i, k = 1, 2, 3 \tag{2.28}$$

where

$$\pi_{ik} = \frac{1}{\sqrt{-g}\, g^{00}} \Gamma^0_{ik} , \quad q^{ik} = g g^{00} e^{ik} ,$$

$$e^{ik} g_{ik'} = \delta^k_{k'} ,$$

$$g^{(3)} = \det g_{ik} = g g^{00} .$$

In Ref. 1 the operator $Q^{\mu\beta}(g, \Gamma)$ for the gauge conditions (2.28) was calculated. Let us confine ourselves to the quasiclassical approximation, when the classical equations of motion may be used in the expression for $Q^{\mu\beta}(g, \Gamma)$ (see Ref. 1). As was shown in Ref. 1, $Q^{\mu\beta}(g, \Gamma)$ takes the form

$$\begin{aligned} Q^i_j &= (-g^{(3)})^{1/3}(\delta^i_j e^{em}\partial^2_{em} + \tfrac{1}{3} e^{ei}\partial^2_{ej}) , \\ Q^0_j &= 0 , \quad Q^0_0 = \sqrt{-g}\, (R^{(3)} + e^{ik}\nabla_i\nabla_k) \end{aligned} \tag{2.29}$$

independent of $\Gamma^\alpha_{\beta\sigma}$. Using this circumstance, let us integrate in (1.16) over 34 variables Γ different from Γ^0_{ik}. As a result we get the additional factor

$$\prod_x [(-g)^{-11/2}(g^{00})^{-5}]$$

and the Lagrangian transforms to the canonical form[9]:

$$L(\pi, q, \lambda) = \pi_{ik}\partial_0 q^{ik} - \lambda^\mu T_\mu(\pi, q) , \tag{2.30}$$

where

$$\lambda^k = \frac{g^{0k}}{g^{00}} , \quad \lambda^0 = \frac{1}{\sqrt{-g}\, g^{00}} .$$

In the integral left a change of the variables of integration $g_{\mu\nu}$, Γ^0_{ik} to the variables q^{ik}, λ^μ, π_{ik}, leads to the Jacobian:

$$\prod_x (\sqrt{-g}\, g^{00})^5 .$$

Let us pass now from the operator $Q^{\mu\beta}$ [Eq. (2.29)] to the operator $Q^{\mu\beta}_{\text{can}}$:

$$Q^\mu_{j\,\text{can}} = Q^\mu_j , \quad Q^0_{0\,\text{can}} = \frac{1}{\lambda^0} Q^0_0 , \tag{2.31}$$

which was obtained in Ref. 9 by application of the general canonical expression for the $Q^{\mu}_{\beta\,\text{can}}$ equal to the Poisson bracket[9]:

$$Q^{\mu\beta}_{\text{can}} = \{\chi^\mu, T^\beta\} .$$

As a result of such a transition we get the additional local factor

$$\prod_x (\sqrt{-g}\, g^{00})^{-1} .$$

Finally the integral (1.16) with the measure (2.24) comes to the form

$$\int J_{\text{can}} \exp\left\{i \int L(\pi, q, \lambda) d^4x\right\} \prod_{x,\alpha} \delta(\chi^\alpha(\pi, q)) d\mu(q, \pi, \lambda) \tag{2.32}$$

equal to the Feynman integral over independent canonical variables.[9] Taking into account the local factors which arise, we have for the measure in (2.32)

$$d\mu(\pi, q, \lambda) = \prod_x \prod_{i\le k} dq^{ik} \prod_{m\le n} d\pi_{mn} \prod_\sigma d\lambda^\sigma ,$$

with the local factor equal to

$$\prod_x [(-g)^{7/2} g^{00}][(-g)^{-11/2} (g^{00})^{-5}] \times [\sqrt{-g}\, g^{00}]^5 [\sqrt{-g}\, g^{00}]^{-1} = 1 .$$

Thus the local measure obtained satisfies the requirements of the canonical formalism. Note that we have used the classical equations for $Q^{\mu\beta}$ here, but, as we have shown above, the local measure is preserved in quasiclassics.

We have used here the canonical quantization procedure for systems with constraints, which resulted in expression (2.32).

Another version of the Hamiltonian formalism may be suggested which possesses certain advantages.

C. Hamiltonian Formalism in Covariant Gauge

First let us represent the Jacobian J_χ with the aid of additional fields $\overline{C}^\alpha$, C^β, obeying Fermi statistics:

$$J_\chi = \int \exp\left[i \int \overline{C}^\alpha(x) \hat{Q}_{\alpha\beta} C^\beta(x) dx\right] \prod_{x,\alpha\beta} d\overline{C}^\alpha dC^\beta . \tag{2.33}$$

Such a representation in the functional integral (1.3) has been used in Refs. 2 and 3.

δ functionals of the gauge conditions may be represented with the aid of Lagrange multipliers:

$$\prod_{x,\mu} \delta(\chi^\mu) = \int \exp\left[-i \int \eta_\mu \chi^\mu dx\right] \prod_{x,\mu} d\eta_\mu . \tag{2.34}$$

Now let us choose the transverse harmonic gauge

$$\chi^\mu(\hat{g}) = \partial_\nu \hat{g}^{\mu\nu} . \tag{2.35}$$

Note that, provided that the gauge (2.35) is degenerate, the operator (1.13) may be written in the form

$$\hat{Q}^{\alpha\beta} = \delta^{\alpha\beta} \sqrt{-g}\, \Box , \tag{2.36}$$

where $\Box$ denotes the covariant Laplacian[18]

$$\Box = g^{\mu\nu}(\partial^2_{\mu\nu} - \Gamma^\alpha_{\mu\nu}\partial_\alpha) . \tag{2.37}$$

The idea is to consider the initial classical theory of gravity as the theory described by the following Lagrangian:

$$L(\hat{g}, \overline{C}, C, \eta) = \sqrt{-g}\, R - \eta_\mu \partial_\nu \hat{g}^{\mu\nu} + \hat{g}^{\mu\nu} \partial_\mu \overline{C}^\alpha \partial_\nu C^\alpha . \tag{2.38}$$

Here the Lagrangian of the C field is generally covariant and Hermitian due to the degeneracy of the gauge.

Let us now perform the transition from the variables $\hat{g}^{\mu\nu}$ to the variables q^{ik}, λ^μ. Then the Lagrangian of the field $\hat{g}$ takes the form (2.30) and we obtain

$$\begin{aligned} L(q^{ik}, \lambda^\mu, \eta_\mu, \overline{C}^\alpha, C^\alpha) = {}& \pi_{ik}(q, \lambda)\partial_0 q^{ik} - \lambda^\mu T_\mu(\pi(q, \lambda); q) + \frac{1}{\lambda_0^2}(\eta_0 + \eta_i \lambda^i)\partial_0\lambda^0 - \frac{\eta_i}{\lambda^0}\partial_0\lambda^i \\ & - \left[\eta_i \partial_k\left(q^{ik}\lambda_0 + \frac{\lambda^i\lambda^k}{\lambda^0}\right) + \eta_0 \partial_i \frac{\lambda^i}{\lambda^0}\right] + \frac{1}{\lambda^0}\partial_0\overline{C}^\alpha\partial_0 C^\alpha + \left(q^{ik}\lambda_0 + \frac{\lambda^i\lambda^k}{\lambda^0}\right)\partial_i\overline{C}^\alpha\partial_k C^\alpha \\ & + \frac{\lambda^i}{\lambda^0}(\partial_0\overline{C}^\alpha\partial_i C^\alpha + \partial_i\overline{C}^\alpha\partial_0 C^\alpha) . \end{aligned} \tag{2.39}$$

Let us now find the Hamiltonian formulation of the dynamics of the system represented by the Lagrangian (2.39), containing the complete set of fields.

Define the canonical momenta:

$$q^{ik} \to \pi_{ik} = \pi_{ik}(q, \lambda) ,$$

$$\lambda^i, \lambda^0 \to \pi_i = -\frac{\eta_i}{\lambda^0} , \quad \pi_0 = \frac{1}{\lambda_0^2}(\eta_0 + \eta_i\lambda^i) ,$$

$$\eta^i, \eta^0 \to P_i = 0, \quad P_0 = 0 ,$$

$$\overline{C}^\alpha \to p_\alpha = \frac{1}{\lambda^0}\partial_0 C^\alpha + \frac{\lambda^i}{\lambda^0}\partial_i C^\alpha ,$$

$$C^\alpha \to \overline{p}_\alpha = \frac{1}{\lambda^0}\partial_0\overline{C}^\alpha + \frac{\lambda^i}{\lambda^0}\partial_i\overline{C}^\alpha$$

Thus the momenta conjugate to the fields η are equal to zero, while the η fields themselves serve as momenta conjugated to the fields λ. When the latter circumstance is taken into account, the Lagrangian (2.39) may be rewritten identically in the form

$$L=\pi_{ik}\partial_0 q^{ik}+\pi_0\partial_0\lambda^0+\pi_i\partial_0\lambda^i+\bar{p}_\alpha\partial_0 C^\alpha+\partial_0\bar{C}^\alpha p_\alpha - H(q,\pi^{ik},\lambda,\pi^\mu,\bar{C},p,C,\bar{p})\,. \qquad (2.40)$$

The Hamiltonian takes the form

$$H=\lambda^\mu T_\mu(\pi^{ik},q)+\lambda^0\bar{p}_\alpha p_\alpha-\bar{p}_\alpha\lambda^k\partial_k C_\alpha-(\partial_i\bar{C}_\alpha)\lambda^i p_\alpha + \pi^0\left(\lambda_0{}^2\partial_k\frac{\lambda^k}{\lambda^0}\right)-\pi_i[\lambda^k\partial_k\lambda^i+\lambda^0\partial_k(\lambda^0 q^{ik})] - q^{ik}\lambda_0\partial_i\bar{C}^\alpha\partial_k C^\alpha\,. \qquad (2.41)$$

Note that this Hamiltonian has no constraints.

Thus the formulas (2.40), (2.41) present the Hamilton formulation of the theory of gravity with a Hermitian Hamiltonian, containing eighteen pairs of independent fields, some of these fields—$\bar{C}$, C and their conjugated momenta—being the elements of Grassman algebra.[19] The last property is unusual from the point of view of classical theory, but not contradictory, however. As to the quantum theory, use of the Hamiltonian formalism developed here directly leads, as we shall show, to the covariant S-matrix theory formulation, described in Sec. I, with the local measure coinciding with the value obtained above.

The S matrix thus obtained is pseudounitary in complete Hilbert space, that is,

$$\sum_m \langle n|S|m\rangle\langle m|S^*|n'\rangle=\delta_{nn'}\,. \qquad (2.42)$$

It may be shown, however, with the aid of Ward identities which will be obtained in Sec. III, that in the cases when n and n' are physical states, the contribution to the unitarity condition (2.42) from the nonphyiscal degrees of freedom among the variables $\hat{g}^{\mu\nu}$ and from the fictitious degrees of freedom $\bar{C}$, C are exactly compensated in the sum over m in (2.42). It means that the theory, which is pseudounitary in the complete space, is also effectively unitary in the physical subspace and thus is identical to the original theory with two independent degrees of freedom when all the equations of constraints are solved. It must be stressed that all the matrix elements of all the transitions generally are nonzero, but some of the degrees of freedom being quantized with indefinite metrics, the compensation mentioned above takes place in the matrix elements squared.

It must be noted that contrary to the generalized Hamiltonian formalism with Lagrange multipliers, the version presented here is the usual Hamiltonian formalism with the explicit Hamiltonian (2.41) expressed in terms of independent canonical variables.

This is the reason why, in order to quantize the system (2.41), we do not need the special procedure of Ref. 9 but the usual canonical quantization will do.

Thus, the functional integral has the usual form:

$$\int\exp\left\{i\int[\pi_{ik}\partial_0 q^{ik}+\pi_\alpha\partial_0\lambda^\alpha+\bar{p}_\alpha\partial_0 C^\alpha+\partial_0\bar{C}^\alpha p_\alpha-H]dx\right\} \times\prod_x d\bar{p}^\alpha dp^\alpha d\pi_{ik}dq^{ik}d\lambda^\alpha d\pi_\alpha d\bar{C}^\alpha dC_\gamma\,. \qquad (2.43)$$

Let us perform here the integration over all the momenta. After integration over π_{ik} we get the local factor

$$\prod_x\lambda_0{}^{-3}|q^{ik}|^{-2}\,. \qquad (2.44)$$

The integration over $\bar{p}^\alpha$ and p^α, taking into account the statistics, gives the local factor

$$\prod_x\lambda_0{}^4\,, \qquad (2.45)$$

while the Lagrangian returns to the form

$$L=\sqrt{-g}\,R+\hat{g}^{\mu\nu}\partial_\mu\bar{C}^\alpha\partial_\nu C^\alpha\,,$$

which differs from (2.38) by the absence of gauge conditions with Lagrange multipliers. However, the integration over π^0, π^i gives the δ functionals

$$\prod_x\delta^{(1)}(\lambda_0{}^2\partial_\mu\hat{g}^{0\mu})\delta^{(3)}[\lambda^0(\partial_\mu\hat{g}^{i\mu}-\lambda^i\partial_\mu\hat{g}^{0\mu})] = \prod_x\frac{1}{\lambda_0{}^5}\prod_x\delta^{(4)}(\partial_\mu\hat{g}^{\mu\nu})\,. \qquad (2.46)$$

Finally the functional integral (2.43) takes exactly the form (1.3):

$$\int\exp\left\{i\int[L(g)+\bar{C}\hat{Q}_\chi C]dx\right\}\prod_x\delta(\chi)d\bar{C}dC\,d\mu(g) = \int\exp\left[i\int L(g)dx+\mathrm{Tr}\ln Q\right]\prod_x\delta(\chi(g))d\mu(g)\,,$$

with the local measure having, by virtue of (2.44)–(2.46), the value in $g_{\mu\nu}$ variables

$$d\mu(g)=\prod_x(-g)^{-3/2}g^{00}\prod_{\mu\le\nu}dg_{\mu\nu}$$

coinciding with that of Ref. 8.

Thus the canonical formalism (both versions), which secures the unitarity of the S matrix, leads to the same value of the local measure as the requirement of cancellation of the divergences $\propto\delta^{(4)}(0)$ does.

Most fascinating is however the statement that in spite of its noncovariant form (presence of the term g^{00} or $g^{\mu\mu}$), this measure also secures the gauge independence of the S matrix.

D. The Gauge Invariance of the Measure

Let us choose for definiteness the $g^{\mu\nu}$ parametrization and denote by ϕ the set of fields

$$\phi^A(x)=\{g^{\mu\nu}(x),\ \bar{C}^\alpha(x),\ C^\alpha(x)\}.$$

Then the functional integral (1.15) in the nondegenerate gauge takes the form

$$Z=\int e^{iS[\phi(x)]}M[g(x)]\prod_{x,A}d\phi^A(x). \qquad (2.47)$$

It will be sufficient to take into account only the terms containing the higher derivatives in the action here:

$$S[\phi(x)]=\int d^4x\,D^{AB\mu\nu}(\phi(x))\frac{\partial}{\partial x^\mu}\phi_A(x)\frac{\partial}{\partial x^\nu}\phi_B(x). \qquad (2.48)$$

As we have shown, the local measure is equal to

$$M[g(x)]=\prod_x g^p(x)g^{00}(x)$$
$$=\prod_x[\det\mathfrak{D}^{AB00}(\phi(x))]^{1/2}, \qquad (2.49)$$

where

$$\det\mathfrak{D}=\frac{\det D_1}{(\det D_2)^2} \qquad (2.50)$$

and D_1, D_2 are the cells of the matrix D^{AB00} related to the fields g and $\bar{C}, C$, respectively. We shall not fix the value of p, which depends on the choice of variables (2.27).

Let us find the gauge transformation of the local measure. Perform the general coordinate transformation:

$$x^\mu\to\bar{x}^\mu(x)=x^\mu+f^\mu(x), \qquad (2.51)$$

$$\bar{g}^{\mu\nu}(\bar{x})=\frac{\partial\bar{x}^\mu(x)}{\partial x^\alpha}\frac{\partial\bar{x}^\nu(x)}{\partial x^\beta}g^{\alpha\beta}(x). \qquad (2.52)$$

To the first order in f,

$$\bar{g}^{\mu\nu}(\bar{x}(x))=g^{\mu\nu}(x)+g^{\mu\beta}(x)\partial_\beta f^\nu(x)+g^{\nu\beta}(x)\partial_\beta f^\mu(x). \qquad (2.53)$$

The transformed measure reads

$$M^f\equiv M[\bar{g}(\bar{x})]$$
$$=\exp\left\{\delta^{(4)}(0)\left[\int\ln[\bar{g}^p(\bar{x})\bar{g}^{00}(\bar{x})]d^4\bar{x}\right]\right\}$$
$$=\exp\left\{\delta^{(4)}(0)\left[\int\ln[g^p(x)g^{00}(x)]d^4x+2p\int\partial_\alpha f^\alpha(x)d^4x\right.\right.$$
$$+\int[\partial_\alpha f^\alpha(x)]\ln[g^p(x)g^{00}(x)]d^4x$$
$$\left.\left.-2\int\frac{g^{0\mu}(x)}{g^{00}(x)}\partial_\mu f^0(x)d^4x\right]\right\}$$

to the first order in f. Note that the second term in the exponential here vanishes because it has the form of a surface integral and the function f tends to zero at infinity. (We consider the gauge transformations which preserve the flat asymptotics of the gravitational field. This is assumed in all considerations concerning S-matrix theory.)

As a result we obtain

$$M[\bar{g}(\bar{x})]=M[g(x)]\Delta_1[g(x),f], \qquad (2.54)$$

where

$$\Delta_1[g(x),f]=\exp\left\{\delta^{(4)}(0)\left[\int[\partial_\mu f^\mu(x)]\times\ln[g^p(x)g^{00}(x)]d^4x-2\int\frac{g^{0\mu}(x)}{g^{00}(x)}\partial_\mu f^0(x)d^4x\right]\right\}. \qquad (2.55)$$

Let us calculate now the gauge transformation of the field differentials in the functional integral (2.47). We shall provide the $\bar{C}, C$ fields with the transformation properties of scalars:

$$\bar{C}^\alpha(x)\to\bar{C}^\alpha(\bar{x}),\quad C^\alpha(x)\to C^\alpha(\bar{x}). \qquad (2.56)$$

We impose such transformation properties following the requirement of general covariance of the Lagrangian (2.38) in the degenerate harmonic gauge. Nevertheless, it is no more than a matter of convenience, because these fields do not enter the final expression for the functional integral.

We start with the definition

$$\left(\prod_x d\phi(x)\right)^f\equiv\prod_{\bar{x}}d\bar{\phi}(\bar{x}). \qquad (2.57)$$

Considering the $\bar{x}$ to be functions of x let us calculate first the transformation Jacobian from the differentials $\prod_x d\bar{\phi}(\bar{x}(x))$ to $\prod_x d\phi(x)$. One has from (2.53)

$$\det\frac{\delta\bar{g}^{\mu\nu}(\bar{x}(x))}{\delta g^{\alpha\beta}(x)}=1+5\delta^{(4)}(0)\int\partial_\mu f^\mu(x)d^4x$$
$$=1,$$

since the integral of the divergence vanishes. This conclusion does not depend on the choice of the variables of integration. In regard to (2.56) one finds

$$\prod_x d\bar{\phi}(\bar{x}(x))=\prod_x d\phi(x). \qquad (2.58)$$

The Jacobian of the transformation

$$\prod_{\bar{x}}d\bar{\phi}(\bar{x})=\Delta_2\prod_x d\bar{\phi}(\bar{x}(x)) \qquad (2.59)$$

still remains to be found.

It seems at first glance that, by virtue of the one-to-one correspondence $\bar{x} \leftrightarrow x$, a transformation such as (2.59) means only a change in the order of placing the differentials in their product over points. These formal arguments were the reason for the wrong statement found in the literature (see Refs. 1 and 10) that the "permutation of points" Jacobian Δ_2 is equal to unity. We shall show that contrary to this statement, the field differentials of Eq. (2.59) are not equal to each other; to be more precise, the functional integrals over them do not coincide.

This phenomenon no longer seems paradoxical if it is taken into consideration that the gauge group of general relativity has the fundamental property of being the coordinate transformation group. Except for the form of the field functions the gauge transformation affects the space-time argument, generally mixing up the time variable with the space variables, the latter being just the indices enumerating the degrees of freedom. It is easily understood from this remark that the coordinate transformation in the integrand of the functional integral, which formally looks like the identical permutation of points, is in fact a nontrivial operation, for it leads to the change of T ordering in the S matrix.

This fundamental aspect of quantum gravity has not yet received proper accounting in the literature. (See also Sec. II E.)

It follows already from the present discussion, restricted to consideration of local measure, that while the Jacobian of the form transformation (2.58) turns out to be unity, the Jacobian of the point transformation (2.59) does not.

In order to calculate the Jacobian Δ_2 let us perform a trivial change of notation of integration variables in the integral (2.47) to present it in the form

$$Z = \int e^{iS[\bar{\phi}(\bar{x})]} M[\bar{g}(\bar{x})] \prod_{\bar{x}} d\bar{\phi}(\bar{x})$$
$$= \int e^{iS[\bar{\phi}(\bar{x})]} M[\bar{g}(\bar{x})] \Delta_2 \prod_{x} d\bar{\phi}(\bar{x}(x)) \qquad (2.60)$$

and use the following trick: The quantity

$$N = M[\bar{g}(\bar{x})]\Delta_2 \qquad (2.61)$$

is the local measure in the last integral of (2.60).

Since the general method of obtaining the local measure is available (found as a result of canonical quantization), let us carry out the independent calculation of the local measure N in the integral (2.60) over

$$\prod_x d\bar{\phi}(\bar{x}(x)) \; . \qquad (2.62)$$

In order to do that it is necessary to pass from functions of $\bar{x}$ to functions of x in the expression (2.60), for the integration in (2.60) is carried out over the functions of x. One obtains

$$S[\bar{\phi}(\bar{x})] = \int d^4\bar{x} D^{AB\mu\nu}(\bar{\phi}(\bar{x})) \frac{\partial}{\partial \bar{x}^\mu} \bar{\phi}_A(\bar{x}) \frac{\partial}{\partial \bar{x}^\nu} \bar{\phi}_B(\bar{x})$$
$$= \int d^4 x E^{AB\mu\nu}(\bar{\phi}(x)) \frac{\partial}{\partial x^\mu} \bar{\phi}_A(x) \frac{\partial}{\partial x^\nu} \bar{\phi}_B(x) \, , \qquad (2.63)$$

where the notation

$$\bar{\phi}_A(x) \equiv \bar{\phi}_A(\bar{x}(x))$$

and

$$E^{AB\mu\nu}(\bar{\phi}(x)) = \left| \frac{\partial \bar{x}^\alpha}{\partial x^\beta} \right| D^{AB\gamma\sigma}(\bar{\phi}) \frac{\partial x^\mu}{\partial \bar{x}^\gamma} \frac{\partial x^\nu}{\partial \bar{x}^\sigma} \qquad (2.64)$$

is introduced. Then according to our general recipe, the local measure N in the integral (2.60) is determined by an expression analogous to (2.49):

$$N = \prod_x [\det \mathfrak{E}^{AB00}(\bar{\phi}(x))]^{1/2} \, . \qquad (2.65)$$

Note that it is $\prod_x$ that stands in (2.65), because the integration in (2.60) is carried out with respect to the functions of x.

In parallel to (2.50),

$$\det \mathfrak{E} = \frac{\det E_1}{(\det E_2)^2} \; . \qquad (2.66)$$

Remember now that the determinant $|D^{AB\mu\nu} n_\mu n_\nu|$ was calculated earlier for arbitrary n_μ, not only for $(n) = (1, 0, 0, 0)$. Using the expression for this determinant, one obtains from (2.64)

$$[\det \mathfrak{E}^{AB00}(\bar{\phi}(x))]^{1/2} = \left| \frac{\partial \bar{x}}{\partial x} \right|^K \bar{g}^p(x) \bar{g}^{\mu\nu}(x) \frac{\partial x^0}{\partial \bar{x}^\mu} \frac{\partial x^0}{\partial \bar{x}^\nu} \, . \qquad (2.67)$$

Here the value of the power K is of no importance, for to the first order in f,

$$\prod_x \left| \frac{\partial \bar{x}}{\partial x} \right|^K = \prod_x [1 + \partial_\alpha f^\alpha(x)]^K$$
$$= \exp\left\{ \delta^{(4)}(0) \left[-K \int \partial_\alpha f^\alpha(x) d^4 x \right] \right\}$$
$$= 1 \, .$$

As a result one obtains from (2.65) and (2.67), to the first order in f,

$$N = \prod_x \bar{g}^p(\bar{x}(x)) \bar{g}^{00}(\bar{x}(x)) \left[1 - 2 \frac{\bar{g}^{0\mu}(\bar{x}(x))}{\bar{g}^{00}(\bar{x}(x))} \partial_\mu f^0(x) \right] . \qquad (2.68)$$

The next task is to single out the factor $M[\bar{g}(\bar{x})]$ from the quantity N determined by Eqs. (2.61), (2.68). In order to do that it is necessary to pass from $\prod_x$ in (2.68) to $\prod_{\bar{x}}$. One has, to the first order in f,

$$N=\exp\left\{\delta^{(4)}(0)\left[\int \ln[\bar{g}^{p}(\bar{x}(x))\bar{g}^{00}(\bar{x}(x))]\,d^4x + 2\int \frac{\bar{g}^{0\mu}(\bar{x}(x))}{\bar{g}^{00}(\bar{x}(x))}\partial_\mu f^0(x)d^4x\right]\right\}. \tag{2.69}$$

Substituting here

$$d^4x=d^4\bar{x}[1-\partial_\mu f^\mu(x)], \tag{2.70}$$

we shall once more accomplish the transition to the infinitesimal expansion, but contrary to (2.68), it is the expansion $\prod_{\bar{x}}$. As a result, we obtain

$$N=\left[\prod_{\bar{x}}\bar{g}^{p}(\bar{x})\bar{g}^{00}(\bar{x})\right]\times\exp\left\{\delta^{(4)}(0)\left[-\int \ln[\bar{g}^{p}(\bar{x})\bar{g}^{00}(\bar{x})][\partial_\mu f^\mu(x)]d^4\bar{x} + 2\int \frac{\bar{g}^{0\mu}(\bar{x})}{\bar{g}^{00}(\bar{x})}\partial_\mu f^0(x)d^4\bar{x}\right]\right\}. \tag{2.71}$$

However,

$$\left[\prod_{\bar{x}}\bar{g}^{p}(\bar{x})\bar{g}^{00}(\bar{x})\right]\equiv M[\bar{g}(\bar{x})].$$

Finally from (2.71) and (2.61) one obtains the following expression for the "permutation of points" Jacobian Δ_2:

$$\Delta_2[\bar{g}(\bar{x}),f]=\exp\left\{\delta^{(4)}(0)\left[-\int[\partial_\mu f^\mu(x)]\ln[\bar{g}^{p}(\bar{x})\bar{g}^{00}(\bar{x})]d^4\bar{x} + 2\int\frac{\bar{g}^{0\mu}(\bar{x})}{\bar{g}^{00}(\bar{x})}\partial_\mu f^0(x)d^4\bar{x}\right]\right\}. \tag{2.72}$$

To the first order in f it is evident that

$$\Delta_2[\bar{g}(\bar{x}),f]=\Delta_2[g(x),f]=\frac{1}{\Delta_2[g(x),-f]}\ .$$

Comparing now Eqs. (2.72) and (2.55) one verifies that

$$\Delta_2[g(x),-f]=\Delta_1[g(x),f]\,. \tag{2.73}$$

Now making use of (2.57), (2.58), (2.59), and (2.73) one finds that

$$\left[\prod_x d\phi(x)\right]^f=\frac{1}{\Delta_1}\left[\prod_x d\phi(x)\right],$$

and Eq. (2.54) states that

$$(M[g])^f=\Delta_1 M[g],$$

to the first order in f.

Thus it is proved that

$$\left[M[g]\prod_x d\phi(x)\right]^f=M[g]\prod_x d\phi(x)\,. \tag{2.74}$$

The local measure

$$M[g]=\prod_x(-g)^{-3/2}g^{00},$$

together with the field differentials is gauge-invariant.

E. Discussion of the Previous Result

The essence of the above considerations lies in the formal demonstration of the fact that it is not immaterial which coordinate system is used in the definition of the integration variables of the functional integral. This conclusion deserves to be explained in more detail.

The problem is that the expression for the amplitude [for example, Eq. (2.47)] with the invariant classical action involved is, in fact, too formal. It implies, in reality, the noncovariant (under general coordinate transformations) procedure of the functional-integral (or S-matrix) construction, in the course of which the infinite expressions appear.

In any approach to S-matrix theory, such a procedure makes use of an ordering parameter. In the framework of the functional-integration technique, the ordering parameter is involved through a limiting transition from the integral defined on a lattice of points. If the points of the lattice are defined as intersections of the curves $x^i=\text{constant}$ with the hypersurfaces $x^0=\text{constant}$, then x^0 is the ordering parameter. To evaluate the infinitesimal amplitude, the classical action S must be computed between successive hypersurfaces $x^0=\tau^j$ and $x^0=\tau^{j+1}$, $\tau^{j+1}-\tau^j=\epsilon$. Then in the limit $\epsilon\to 0$ the quantity $S_{\tau_j}^{\tau_{j+1}}$ necessarily contains a term divergent as $\delta(0)$ and this term is noncovariant. Indeed, the action functional is homogeneous in the field derivatives $\partial_\mu g$. Clearly if we specify $g^{(j)}$ and $g^{(j+1)}$ on τ^j and τ^{j+1} arbitrarily, the time derivative of the classical trajectory $g(x)$ that connects these boundary values will go to infinity as $\epsilon\to 0$, while derivatives with respect to x^i will tend to some average of those on τ^j and τ^{j+1} and thus remain finite[20]:

$$\partial_i g(x)=O(1),\quad \partial_0 g(x)=O(1/\epsilon).$$

Thus the divergent term in the action

$$S_{\tau_j}^{\tau_{j+1}}\propto\int_{\tau_j}^{\tau_{j+1}}dx^0\int d^3x\,D(g)g^{00}\partial_0 g\partial_0 g$$

is noncovariant. It carries the information on the choice of the ordering parameter. It is also clear that such a divergent term does not appear in the functional integral over independent canonical variables, because the canonical formalism does not contain derivatives with respect to the ordering parameter nonlinearly, these derivatives being included in the canonical momenta.

If the infinitesimal amplitude is evaluated in some other coordinates $\bar{x}$, then the above-mentioned mechanism will lead to a divergent term containing derivatives $\partial g/\partial \bar{x}^0$, and thus different from the previous one. Here the role of the coordinate system comes to light: Although the action functional is covariant in the notation of the integrand of the functional integral, it transforms nontrivially under the general coordinate transformations.

The notation $\prod_x dg$ in the functional integral indicates which particular coordinate system has been used to define a lattice and an ordering parameter $\tau = x^0$. This notation and the noncovariant local measure are the only reminders in the formal expression (2.47) of the actual procedure of its evaluation. According to our notations we ascribe the change in the action under coordinate transformations to the change of the field differentials

$$\prod_x dg \to \prod_{\bar{x}} dg ,$$

which gives rise to the "permutation of points" Jacobian Δ_2.

What we have proved is that transformations of the local measure and the noncovariant divergent term in the action agree with each other in such a way that they compensate each other in any coordinate system. According to our notations we say that the symbol $\prod_x dg$ is noncovariant, but the measure

$$d\mu(g) = M[g(x)] \prod_x dg$$

is covariant as a whole.

What we have not proved is that the Jacobian Δ_2 possesses only a local part, i.e., that the term $\propto \delta^{(4)}(0)$ is the only noncovariant term finally coming from the action. We believe that the proof may be given by means of the canonical formalism developed in the present paper. If so, then finally the functional integral does not depend on the choice of the ordering parameter, provided that the boundaries of variation of this parameter are $\pm\infty$. This statement is essential for gauge independence of the S matrix.

In the framework of the canonical formalism, the ordering parameter is involved through the definition of the canonical momenta

$$\pi_\tau = \frac{\partial L}{\partial(\partial q/\partial \tau)} .$$

Here again the notation $\prod_x d\pi\, dq$ indicates the particular choice of this definition. This notation prescribes also the rule of singling the local term out of expressions of the type of (2.10); the argument of the $\theta(\tau)$ function in the definition of $D_F(x-y)$ will be $\tau = x^0$ or $\tau = \bar{x}^0$, subject to the choice of the ordering parameter. Expression (2.15) corresponds to the choice of the ordering parameter as (quantization on a spacelike hypersurface)

$$\tau = x^\mu n_\mu .$$

Choosing the ordering parameter as (Lorentz-invariant quantization of the theory in Euclidean variables [6])

$$\tau = x^\mu x^\mu = x^2 ,$$

we shall obtain expressions (2.12), (2.25), and (2.26), provided that calculation of the corresponding integral (2.10) is carried out in a Euclidean domain.

F. Independence of S Matrix from the Gauge Condition

It is not difficult to show now that the property (2.74) of the local measure secures the gauge independence of the S matrix (independence on the choice of the gauge condition χ). Indeed, in order to give the proof of gauge independence, the gauge displacement of the variables of integration $g \to g^f$ is to be accomplished in the integral

$$\int e^{iS[g]} J_\chi[g] \prod_x \delta(\chi(g))\, d\mu(g), \tag{2.75}$$

and the procedure of invariant integration over the gauge group is to be applied.[3] Note that the Jacobian J_χ itself is given by the group integral[3]:

$$J_\chi^{-1}[g] = \int \prod_x \delta(\chi(g^f))\, d\tilde{\mu}(f). \tag{2.76}$$

It is essential for the proof given in Ref. 3 that after the change of variables in the integrand of (2.75) is accomplished, the dependence on the function f should remain only in the δ functional $\delta(\chi(g^f))$.

But by virtue of the group integral invariance,

$$J_\chi[g^f] = J_\chi[g] .$$

Also,

$$S[g^f] = S[g] .$$

Consequently only the equality

$$d\mu(g^f) = d\mu(g)$$

is required, which is true, as we have shown, provided the value of the measure is

$$d\mu(g) = \prod_x (-g)^{-3/2} g^{00} \prod_{\mu\le\nu} dg_{\mu\nu}\ .$$

It is easily understood that the proof of gauge independence of the S matrix given in Ref. 3 becomes incorrect if the property (2.74) of the measure is not taken into account.

One may notice that the "permutation-of-points" Jacobian will not appear if one performs the gauge displacement of variables in the functional integral directly, according to Eq. (1.8). However, it is a specific feature of coordinate transformations that the right-hand side of Eq. (1.8) contains the so-called transport term with the time derivative of the field functions. Because of this term such a naive displacement in the functional integral would be singular as it follows from the discussion of Sec. II E. In order to perform a correct displacement one must first make a trivial change of notation in the functional integral [as in Eq. (2.60)] and then use Eq. (2.53), where the transport term is absent in the right-hand side. The result will be the same as that of the naive procedure, but the Jacobian Δ_2 will appear.

It must yet be shown that the local factor does not appear in the invariant measure over the group of general coordinate transformations. Let us present the measure in the integral (2.76) as

$$d\tilde{\mu}(f) = \Sigma[f(x)] \prod_{x,\mu} df^\mu(x)\,, \tag{2.77}$$

where

$$\Sigma[f(x)] = \exp\left[\delta^{(4)}(0)\int \ln\sigma(f(x))d^4x\right] \tag{2.78}$$

is the local measure.

Let us require left invariance of the measure (2.77), under the infinitesimal group transformation with the function f_0.

It is not difficult to show that

$$\begin{aligned} f_3(x) &\equiv (f_0 \times f) \\ &= f(x) + f_0(x + f(x))\ . \end{aligned} \tag{2.79}$$

Now

$$\Sigma[f(x)] \prod_{x,\mu} df^\mu(x) = \Sigma[f_3(x)] \prod_{x,\mu} df_3^\mu(x)\ . \tag{2.80}$$

It is easily seen that contrary to the previous case, the "permutation of points" effect does not arise in (2.80). Provided that the function f_0 is small and slowly varying, we obtain

$$f_0^\sigma(x+f(x)) = f_0^\sigma(x) + [\partial_\gamma f_0^\sigma(x)]\,f^\gamma(x)$$

and

$$\det\frac{\delta f_3^\mu(x)}{\delta f^\nu(y)} = \exp\left\{\delta^{(4)}(0)\left[-\int \partial_\nu f_0^\nu(x)d^4x\right]\right\} = 1\ .$$

Hence it follows that

$$\Sigma[f(x)] = \Sigma[f_3(x)]\ . \tag{2.81}$$

The same conclusion follows also from the right-invariance requirement.

To the first order in the gauge function f_0, one obtains from (2.81)

$$\Sigma[f+f_0] = \Sigma[f]\ .$$

Now it follows directly from the general view of local measure (2.78) that to the first order,

$$\Sigma[f+f_0] = \Sigma[f]\Sigma[f_0]\ .$$

As a result we obtain

$$\Sigma[f] = 1 + O(f^2)\ .$$

Thus, in the first-order neighborhood, the local measure in the invariant group integral is trivial. This is sufficient because the group integrals of the type (2.76) are concentrated in the neighborhood of the first order.[3]

There have been attempts to specify the measure in the Feynman integral for gravity (see Refs. 20 and 21), and what is more, the measure (2.23) obtained in the present paper coincides with the result of Ref. 20. It must be emphasized, however, that all the early attempts to obtain the measure cannot be considered as correct ones because they deal with a nonunitary S matrix [additional diagrams in the expansion of $\mathrm{Tr}\ln Q(Q^{(0)})^{-1}$ were not taken into account].

As to Refs. 22 and 10 dealing with the correct integrals (1.3), (1.16), the value of the measure in the harmonic gauge given in these references,

$$d\mu(g) = \prod_x g^{-5/2} \prod_{\mu\le\nu} dg_{\mu\nu}\,, \tag{2.82}$$

differs from the value (2.23), which we obtained in Ref. 8, by the factor

$$g^{(3)} = gg^{00}\ .$$

The mistake in the arguments of Ref. 10 consists of the following: The authors obtain expression (2.82) using some special gauge in which

$$g^{(3)} = \text{const.} \tag{2.83}$$

Their next conclusion, however, that the same value of measure holds also for the harmonic gauge, unfortunately ignores the fact that the transition from the gauge (2.83) with the measure (2.82) to the harmonic gauge cannot be performed identically. Indeed, in order to perform such a transition in the functional integral, the procedure of gauge displacement of variables and the subsequent integration over the group is needed.[3] As was already noted, such a procedure leads to the same integral in another gauge, only provided that the local measure together with the field differentials is gauge-invariant. We have shown that the required property is possessed by the measure (2.23), not the measure (2.82). In order to pass from the gauge (2.83) to the harmonic gauge, one has to bring the factor $g^{(3)}$ into the expression (2.82), which by virtue of (2.83) changes nothing, but after the transition to the harmonic gauge is accomplished this factor cannot be discarded.

III. THE GENERALIZED WARD IDENTITIES AND DYNAMICAL EQUATIONS FOR THE GREEN'S FUNCTIONS

The local measure obtained in Sec. II completes the covariant and unitary S-matrix theory formulation for the gravitational field.

The next task consists of the construction of the renormalization theory. We shall begin the struggle against the infinities with the derivation of a number of consequences of the presence of a non-Abelian gauge group in the theory.

The gauge invariance of the classical theory is analytically expressed by means of Nöther identities:

$$\nabla^{+\mu\nu}_{\alpha}(\hat{g})\frac{\delta S[g]}{\delta \hat{g}^{\mu\nu}(x)}=0\,. \tag{3.1}$$

Here, $S[g]$ is a gauge-invariant action functional.

The Nöther identities in the quantum domain are known as generalized Ward identities.[11,23]

Here we shall derive the complete set of such relations for the Green's functions of the theory of gravity.

From this section on, we shall use the variables $\hat{g}^{\mu\nu}$ and the linear gauge conditions

$$\chi^{\mu}(\hat{g})=\partial_{\nu}\hat{g}^{\mu\nu}\,. \tag{3.2}$$

A. The Generating Ward Identity in Presence of Sources

Let us follow the alterations in the integrand of the functional integral under the change of the variables

$$\hat{g}^{\mu\nu}\rightarrow(\hat{g}^{\mu\nu})^{f}\,. \tag{3.3}$$

First, let us perform the identical change of notation of the integration variables in the generating functional (1.15) with the gauge conditions (3.2), local measure (2.23), and $\alpha\neq 0$. One obtains

$$Z[j^{\mu\nu}]=\frac{1}{Z^{\text{lin}}}\int J_{\chi}[\hat{g}^{f}]\exp\left\{iS[\hat{g}^{f}]-\frac{i}{2\alpha}\int\chi^{\mu}(\hat{g}^{f})\chi^{\mu}(\hat{g}^{f})d^{4}x+i\int(\hat{g}^{\mu\nu})^{f}j_{\mu\nu}d^{4}x\right\}d\mu(\hat{g}^{f})\,. \tag{3.4}$$

The infinitesimal displacement (3.3) reads

$$[\hat{g}^{\mu\nu}(x)]^{f}=\nabla^{\mu\nu|\sigma}(\hat{g}|x)f^{\sigma}(x)\,. \tag{3.5}$$

The efficiency of such a choice of the displacement is due first of all to

$$S[\hat{g}^{f}]=S[\hat{g}]\,.$$

Further,

$$\delta^{f}\int\chi^{\mu}(\hat{g})\chi^{\mu}(\hat{g})d^{4}x =2\int\chi^{\mu}(\hat{g}(x))\nabla^{+\alpha\beta}_{\sigma}(y)\frac{\delta\chi^{\mu}(\hat{g}(x))}{\delta\hat{g}^{\alpha\beta}(y)}f^{\sigma}(y)d^{4}x\,d^{4}y =2\int f^{\sigma}(y)\hat{Q}^{T\mu}_{\sigma}\chi^{\mu}(\hat{g}(y))d^{4}y$$

according to the definition of the operator $\hat{Q}^{T}$ by (1.9) and (1.14).

Finally, let us use the gauge invariance of the local measure, together with the field differentials proved in Sec. II:

$$d\mu(\hat{g}^{f})=d\mu(\hat{g})\,.$$

Then one finds that to the zeroth order in f, the integral (3.4) has exactly the form (1.15), which differs from (3.4) by the trivial change of notations. Hence one obtains to the first order in f

$$\int\exp\left\{iS[\hat{g}]-\frac{i}{2\alpha}\int\chi^{\mu}\chi^{\mu}d^{4}x+i\int\hat{g}^{\mu\nu}j_{\mu\nu}d^{4}x+\ln J_{\chi}[\hat{g}]\right\}\times\int d^{4}y\,f^{\sigma}(y)\left[-\frac{1}{\alpha}\hat{Q}^{T\mu}_{\sigma}\chi^{\mu}(y)+\frac{1}{i}\nabla^{+\mu\nu}_{\sigma}\frac{\delta\ln J_{\chi}[\hat{g}]}{\delta\hat{g}^{\mu\nu}(y)}+\nabla^{+\mu\nu}_{\sigma}j_{\mu\nu}(y)\right]d\mu(\hat{g})=0\,,$$

which must be true for an arbitrary function f. Let us present the functional at the point y, integrated here, with the aid of variations with respect to external sources.

As a result, we obtain the following equation for the generating functional:

$$\left[-\frac{1}{\alpha}\hat{Q}^{T\mu}_{\sigma}\chi^{\mu}(x)+\frac{1}{i}\nabla^{+\mu\nu}_{\sigma}\frac{\delta}{\delta\hat{g}^{\mu\nu}(x)}\ln J[\hat{g}]+\nabla^{+\mu\nu}_{\sigma}j_{\mu\nu}(x)\right]\Bigg|_{\hat{g}=i^{-1}\delta/\delta j}Z[j]=0\,, \tag{3.6}$$

which is the consequence of gauge invariance of the action and the local measure.

Let us operate from the left with

$$\hat{Q}^{-1}{}^{\sigma}_{\gamma}\Big|_{\hat{g}=i^{-1}\delta/\delta j}\,.$$

We obtain

$$\left[-\frac{1}{\alpha}\chi^{\gamma}(x)+\mathcal{L}^{1}{}_{\gamma}(x)+\hat{Q}^{-1}{}^{\sigma}_{\gamma}\nabla^{+\mu\nu}_{\sigma}j_{\mu\nu}(x)\right]\Big|_{\hat{g}=i^{-1}\delta/\delta j}Z[j]=0, \tag{3.7}$$

$$\mathcal{L}^{1}{}_{\gamma}(x)=\frac{1}{i}\hat{Q}^{-1}{}^{\alpha}_{\gamma}\nabla^{+\mu\nu}_{\alpha}(x)\frac{\delta}{\delta\hat{g}^{\mu\nu}(x)}\operatorname{Tr}\ln Q^{T\rho}_{\sigma}(z,y)\,. \tag{3.8}$$

Now Eq. (3.7) may be presented as

$$\left[-\frac{1}{\alpha}\chi^{\gamma}(x)+\mathcal{L}^{1}{}_{\gamma}(x)+\mathcal{L}^{2}{}_{\gamma}(x)+:\hat{Q}^{-1}{}^{\sigma}_{\gamma}\nabla^{+\mu\nu}_{\sigma}j_{\mu\nu}(x):\right]\Big|_{\hat{g}=i^{-1}\delta/\delta j}Z[j]=0, \tag{3.9}$$

where the normal-product notation : : means that sources stand to the left of the operator $\hat{Q}^{-1}\nabla^{+}$ when the functional differentiation is meant. $\mathcal{L}^{2}{}_{\gamma}(x)$ denotes the result of commutation of this operator with the sources:

$$\mathcal{L}^{2}{}_{\gamma}(x)=\hat{Q}^{-1\alpha}_{\gamma}\nabla^{+\mu\nu}_{\alpha}(x)j_{\mu\nu}(x)-:\hat{Q}^{-1\alpha}_{\gamma}\nabla^{+\mu\nu}_{\alpha}(x)j_{\mu\nu}(x):$$
$$=\frac{1}{i}\int\frac{\delta}{\delta\hat{g}^{\mu\nu}(x')}\hat{Q}^{-1\alpha}_{\gamma}(x)\nabla^{+\mu\nu}_{\alpha}(x)\delta(x-x')dx'\,. \tag{3.10}$$

Let us show no hat the quantities $\mathcal{L}^{1}{}_{\gamma}(x)$ and $\mathcal{L}^{2}{}_{\gamma}(x)$ in Eq. (3.9) exactly compensate for each other. Such a compensation was mentioned already in Refs. 1 and 2, but it was verified earlier only up to the local factors. We shall present the computation to make sure that these quantities are compensated for exactly. Let us show first of all that the operator $\nabla^{+\mu\nu}_{\alpha}$ in (3.9) commutes with the sources. Really, one has directly from (1.11)

$$\frac{\delta}{\delta\hat{g}^{\mu\nu}(x')}\nabla^{+\mu\nu}_{\alpha}(x)\delta(x-x')=\{-5[\partial_{\alpha}\delta(x-x')]+5\delta(x-x')\overleftarrow{\partial}_{\alpha}\}\delta(x-x')$$
$$\equiv 0\,.$$

Next we introduce the function which obeys the following equations:

$$\hat{Q}^{T\sigma}_{\alpha}(x)F^{\gamma}_{\sigma}(x,y)=\delta^{\gamma}_{\alpha}\delta(x-y)\,, \tag{3.11}$$

$$\hat{Q}^{\sigma}_{\alpha}(y)F^{\alpha}_{\beta}(x,y)=\delta^{\sigma}_{\beta}\delta(x-y)\,. \tag{3.12}$$

It is important that the operators $\hat{Q}$ and $\hat{Q}^{T}$ defined by Eqs. (1.13), (1.14) are not identical. The quantities (3.8), (3.10) are presented as

$$\mathcal{L}^{1}{}_{\gamma}(x)=\frac{1}{i}\hat{Q}^{-1\rho}_{\gamma}\mathfrak{N}^{1}_{\rho}(x)\,, \tag{3.13}$$

$$\mathcal{L}^{2}{}_{\gamma}(x)=-\frac{1}{i}\hat{Q}^{-1\rho}_{\gamma}\mathfrak{N}^{2}_{\rho}(x)\,, \tag{3.14}$$

where

$$\mathfrak{N}^{1}{}_{\rho}(x)=\nabla^{+\mu\nu}_{\rho}(x)\int\left[\frac{\delta}{\delta\hat{g}^{\mu\nu}(x)}(\hat{Q}^{\alpha}_{\beta}(z)\right]F^{\beta}_{\alpha}(y,z))_{z=y}dy\,, \tag{3.15}$$

$$\mathfrak{N}^{2}{}_{\rho}(x)=\int\left[\frac{\delta}{\delta\hat{g}^{\mu\nu}(y)}\hat{Q}^{T\sigma}_{\rho}(x)\right]\nabla^{\mu\nu|\alpha}(y)F^{\alpha}_{\sigma}(x,y)dy\,. \tag{3.16}$$

Using directly Eqs. (1.10)–(1.14), one obtains for (3.16)

$$\mathfrak{N}^{2}{}_{\rho}(x)=[\partial_{\mu\nu}{}^{2}(x)\nabla^{\mu\nu|\alpha}(y)F^{\alpha}_{\rho}(x,y)]_{y=x}$$
$$+[\partial_{\mu}(x)\hat{Q}^{\mu}_{\alpha}(y)F^{\alpha}_{\rho}(x,y)]_{y=x}$$
$$+[\partial_{\rho}(x)\hat{Q}^{\mu}_{\alpha}(y)F^{\alpha}_{\mu}(x,y)]_{y=x}\,. \tag{3.17}$$

Here the last two terms have the form

$$\partial_{\rho}\delta(x-y)|_{x=y}=0$$

according to Eq. (3.12). Performing similar calculations, one obtains for (3.15)

$$\mathfrak{N}^{1}{}_{\rho}(x)=-\nabla^{+\mu\nu}_{\rho}(x)\{[\delta^{\alpha}_{\beta}\partial_{\mu}(x)+\delta^{\alpha}_{\mu}\partial_{\beta}(x)]\partial_{\nu}(y)F^{\beta}_{\alpha}(x,y)+\partial_{\beta\mu}{}^{2}(x)F^{\beta}_{\nu}(x,y)\}_{y=x}\,. \tag{3.18}$$

Here the operator $\nabla^{+}(x)$ acts upon the complete dependence on x of the expression in curly brackets, that is, upon both arguments of the function $F(x,x)$. By means of explicit calculation of the action of the operator $\nabla^{+}(x)$ upon both arguments, Eq. (3.18) transforms identically, after cumbersome calculations, to the form

$$\mathfrak{N}^{1}{}_{\rho}(x)=[\partial_{\mu\nu}{}^{2}(x)\nabla^{\mu\nu|\alpha}(y)F^{\alpha}_{\rho}(x,y)]_{y=x}+[\partial_{\beta}(y)\hat{Q}^{T\mu}_{\rho}(x)F^{\beta}_{\mu}(x,y)]_{y=x}+[\partial_{\beta}(x)\hat{Q}^{T\nu}_{\rho}(x)F^{\beta}_{\nu}(x,y)]_{y=x}$$
$$+[\partial_{\rho}(x)\hat{Q}^{\alpha}_{\beta}(y)F^{\beta}_{\alpha}(x,y)]_{y=x}+[\partial_{\alpha}(x)\hat{Q}^{\alpha}_{\beta}(y)F^{\beta}_{\rho}(x,y)]_{y=x}+[\partial_{\rho}(y)\hat{Q}^{T\beta}_{\alpha}(x)F^{\alpha}_{\beta}(x,y)]_{y=x}\,. \tag{3.19}$$

By virtue of Eqs. (3.11), (3.12), all the terms in (3.19) except for the first one vanish.

Thus we obtain

$$\mathfrak{N}^{1}{}_{\rho}(x)=[\partial_{\mu\nu}{}^{2}(x)\nabla^{\mu\nu|\alpha}(y)F^{\alpha}_{\rho}(x,y)]_{y=x}$$
$$=\mathfrak{N}^{2}{}_{\rho}(x)\,, \tag{3.20}$$

and finally,

$$\mathcal{L}^1{}_\gamma(x)+\mathcal{L}^2{}_\gamma(x)=0\,. \tag{3.21}$$

Equation (3.9), with (3.21) taken into account, takes the form

$$\left[-\frac{1}{\alpha}\chi^\gamma(x)+:\hat{Q}^{-1}{}^\sigma_\gamma\nabla^{+\mu\nu}_\sigma j_{\mu\nu}(x):\right]\Big|_{\hat{g}=i^{-1}\delta/\delta j} Z[j]=0\,. \tag{3.22}$$

The last equation is the generating Ward identity in the presence of sources, which holds due to gauge invariance of the theory. [See Ref. 12(c).]

B. Generalized Ward Identities for Green's Functions

Let us operate from the left with

$$\Phi\left[\frac{1}{i}\frac{\delta}{\delta j}\right],$$

where $\Phi[\hat{g}]$ is an arbitrary functional. After commutation with the sources, the latter must be set equal to zero. Then the generating Ward identity (3.22) may be translated to Green's-function language according to the formula[6]

$$\frac{1}{Z[j]}\Phi\left[\frac{1}{i}\frac{\delta}{\delta j}\right]Z[j]\Big|_{j=0}=\langle 0|T\Phi[\hat{g}]|0\rangle\,. \tag{3.23}$$

As a result, we obtain the equation

$$\frac{i}{\alpha}\langle 0|T\chi^\rho(x)\Phi[\hat{g}]|0\rangle=\left\langle 0\left|T\hat{Q}^{-1}{}^\sigma_\rho\nabla^{+\mu\nu}_\sigma(x)\frac{\delta}{\delta\hat{g}^{\mu\nu}(x)}\Phi[\hat{g}]\right|0\right\rangle, \tag{3.24}$$

which is the generating expression for all the generalized Ward identities of the theory of gravity.

By means of substitution of a functional

$$\Phi_{nm}[\hat{g}]=\chi^{\alpha_1}(x_1)\cdots\chi^{\alpha_n}(x_n)\hat{g}^{\beta_1\gamma_1}(y_1)\cdots\hat{g}^{\beta_m\gamma_m}(y_m) \tag{3.25}$$

into (3.24) we shall obtain an infinite set of relations for the Green's functions.

The generalized Ward identities thus obtained necessarily contain information on the longitudinal components of the Green's functions. However, when $m\neq 0$, they also contain information on the transverse components. It is sufficient for the present discussion to confine ourselves to the first two sets of these relations, which correspond to the values $m=0$ and $m=1$ in (3.25).

The first set may be easily obtained by the consecutive substitution of $\Phi_{n0}[\hat{g}]$, $n=1,2,\ldots$ into (3.24). One finds

$$\langle 0|T\chi^{\alpha_1}(x_1)|0\rangle=0\,, \tag{3.26}$$

$$\langle 0|T\chi^{\alpha_1}(x_1)\chi^{\alpha_2}(x_2)|0\rangle=\frac{\alpha}{i}\delta^{\alpha_1\alpha_2}\delta(x_1-x_2)\,, \tag{3.27}$$

$\ldots,$

$$\langle 0|T\chi^{\alpha_1}(x_1)\cdots\chi^{\alpha_n}(x_n)|0\rangle=\begin{cases}0, & n \text{ odd}\\ \left(\dfrac{\alpha}{i}\right)^{n/2}\sum\delta^{\alpha_{k(1)}\alpha_{i(1)}}\cdots\delta^{\alpha_{k(n/2)}\alpha_{i(n/2)}}\times\delta(x_{k(1)}-x_{i(1)})\cdots\delta(x_{k(n/2)}-x_{i(n/2)}), & n \text{ even}.\end{cases} \tag{3.28}$$

Here $\sum$ denotes the sum over all the decompositions of n to $n/2$ pairs $(k(1),i(1)),\ldots,(k(n/2),i(n/2))$.

The second set of generalized Ward identities is given by the generating expression (3.24) in the form of a recurrence formula, which reduces the order n by two:

$$\frac{i}{\alpha}\langle 0|T\chi^{\alpha_1}(x_1)\chi^{\alpha_2}(x_2)\cdots\chi^{\alpha_n}(x_n)\hat{g}^{\beta\gamma}(y)|0\rangle=\sum_{j=2}^{n}\langle 0|T\chi^{\alpha_2}(x_2)\cdots\chi^{\alpha_{j-1}}(x_{j-1})\chi^{\alpha_{j+1}}(x_{j+1})\cdots\chi^{\alpha_n}(x_n)\hat{g}^{\beta\gamma}(y)|0\rangle$$
$$\times\delta^{\alpha_1\alpha_j}\delta(x_1-x_j)+\langle 0|T\chi^{\alpha_2}(x_2)\cdots\chi^{\alpha_n}(x_n)E^{\alpha_1|\beta\gamma}(x_1,y)|0\rangle\,, \tag{3.29}$$

where the following notation is introduced:

$$E^{\alpha|\beta\gamma}(x,y)\overset{\text{def}}{=}\hat{Q}^{-1}{}^\sigma_\alpha(x)\nabla^{+\beta\gamma}_\sigma(x)\delta(x-y)\equiv\nabla^{\beta\gamma|\sigma}(y)Q^{-1}{}^\sigma_\alpha(x,y)\,. \tag{3.30}$$

One obtains consecutively from (3.29) that

$$\langle 0|T\hat{g}^{\beta\gamma}(y)\chi^\alpha(z)|0\rangle=\frac{\alpha}{i}\langle 0|TE^{\alpha|\beta\gamma}(z,y)|0\rangle\,, \tag{3.31}$$

$$\langle 0|T\hat{g}^{\beta\gamma}(y)\chi^{\alpha_1}(z_1)\chi^{\alpha_2}(z_2)|0\rangle=\frac{\alpha}{i}\langle 0|TE^{\alpha_1|\beta\gamma}(z_1,y)\chi^{\alpha_2}(z_2)|0\rangle+\frac{\alpha}{i}\langle 0|\hat{g}^{\beta\gamma}(y)|0\rangle\delta^{\alpha_1\alpha_2}\delta(z_1-z_2)\,, \tag{3.32}$$

$$\langle 0|T\hat{g}^{\beta\gamma}(y)\chi^{\alpha_1}(z_1)\chi^{\alpha_2}(z_2)\chi^{\alpha_3}(z_3)|0\rangle$$
$$=\frac{\alpha}{i}\langle 0|TE^{\alpha_1|\beta\gamma}(z_1,y)\chi^{\alpha_2}(z_2)\chi^{\alpha_3}(z_3)|0\rangle$$
$$+\left(\frac{\alpha}{i}\right)^2\langle 0|TE^{\alpha_2|\beta\gamma}(z_2,y)|0\rangle\delta^{\alpha_1\alpha_3}\delta(z_1-z_3)$$
$$+\left(\frac{\alpha}{i}\right)^2\langle 0|TE^{\alpha_3|\beta\gamma}(z_3,y)|0\rangle\delta^{\alpha_1\alpha_2}\delta(z_1-z_2),$$
$$\cdots. \qquad (3.33)$$

Now, by virtue of the translation invariance,

$$\langle 0|\hat{g}^{\alpha\beta}(x)|0\rangle=A^{(0)}\delta^{\alpha\beta},$$

where the value of a constant $A^{(0)}$ is to be determined independently.[24] Later on, we shall use the arbitrariness in the choice of the value of this constant to choose it in the initial theory in such a way as to secure the average field in the renormalized theory to be equal to $\delta^{\alpha\beta}$.

Since we consider the quantum theory of asymptotically flat gravitational fields which possess the same relativity principle as the flat space does,[24] the translation-invariance requirement remains.

Let us define the field operator

$$\phi^{\alpha\beta}(x)\overset{\text{def}}{=}\hat{g}^{\alpha\beta}(x)-\langle 0|\hat{g}^{\alpha\beta}(x)|0\rangle,$$
$$\langle 0|\phi^{\alpha\beta}(x)|0\rangle=0.$$

Further, we shall deal with the Green's functions of the field ϕ, the latter being the usual field operator with the vacuum expectation equal to zero.

The resulting generalized Ward identities are easily translated to the Green's functions of the field ϕ language. The form of the first set of relations (3.26)–(3.28) remains unchanged by virtue of transversality of the gauge. As to the second set, one can make sure that the terms containing the average field cancel out and that the recurrence formula

$$\frac{i}{\alpha}\langle 0|T\chi^{\alpha_1}(x_1)\chi^{\alpha_2}(x_2)\cdots\chi^{\alpha_n}(x_n)\phi^{\beta\gamma}(y)|0\rangle$$
$$=\sum_{j=2}^{n}\langle 0|T\chi^{\alpha_2}(x_2)\cdots\chi^{\alpha_{j-1}}(x_{j-1})\chi^{\alpha_{j+1}}(x_{j+1})\cdots$$
$$\times\chi^{\alpha_n}(x_n)\phi^{\beta\gamma}(y)|0\rangle\delta^{\alpha_1\alpha_j}\delta(x_1-x_j)$$
$$+\langle 0|T\chi^{\alpha_2}(x_2)\cdots\chi^{\alpha_n}(x_n)E^{\alpha_1|\beta\gamma}(x_1,y)|0\rangle \qquad (3.34)$$

has just the form of that for the $\hat{g}$ field (3.29). The difference in the values of these Green's functions for n even results from the difference in their values for $n=0$. In particular, Eq. (3.32) takes the form

$$\langle 0|T\phi^{\beta\gamma}(y)\chi^{\alpha_1}(z_1)\chi^{\alpha_2}(z_2)|0\rangle$$
$$=\frac{\alpha}{i}\langle 0|TE^{\alpha_1|\beta\gamma}(z_1,y)\chi^{\alpha_2}(z_2)|0\rangle. \qquad (3.35)$$

C. Equations Defining the Dependence of the Green's Functions on the Gauge

For the first example of application of the generalized Ward identities in the exact theory we shall obtain the equations for the dependence of the Green's functions

$$\langle 0|T\phi^{\beta\gamma}(y)\chi^{\alpha_1}(x_1)\cdots\chi^{\alpha_n}(x_n)|0\rangle \qquad (3.36)$$

on the parameter α fixing the gauge. The function (3.36) is presented by the ratio of functional integrals:

$$\langle 0|T\phi^{\beta\gamma}(y)\chi^{\alpha_1}(x_1)\cdots\chi^{\alpha_n}(x_n)|0\rangle=\frac{\int(\exp i\{S[\hat{g},C,\overline{C}]-1/(2\alpha)\int\chi^{\mu}\chi^{\mu}dz\})\chi^{\alpha_1}(x_1)\cdots\chi^{\alpha_n}(x_n)\phi^{\beta\gamma}(y)d\mu(\hat{g},C,\overline{C})}{\int(\exp i\{S[\hat{g},C,\overline{C}]-(1/2\alpha)\int\chi^{\mu}\chi^{\mu}dz)d\mu(\hat{g},C,\overline{C})}.$$

Differentiating the numerator and denominator with respect to α, one obtains

$$\frac{d}{d\alpha}\langle 0|T\phi^{\beta\gamma}(y)\chi^{\alpha_1}(x_1)\cdots\chi^{\alpha_n}(x_n)|0\rangle=\frac{i}{2\alpha^2}\left[\left\langle 0\left|T\phi^{\beta\gamma}(y)\chi^{\alpha_1}(x_1)\cdots\chi^{\alpha_n}(x_n)\int\chi^{\mu}(z)\chi^{\mu}(z)dz\right|0\right\rangle\right.$$
$$\left.-\langle 0|T\phi^{\beta\gamma}(y)\chi^{\alpha_1}(x_1)\cdots\chi^{\alpha_n}(x_n)|0\rangle\left\langle 0\left|T\int\chi^{\mu}(z)\chi^{\mu}(z)dz\right|0\right\rangle\right]$$
$$=\frac{i}{2\alpha^2}\left[\left\langle 0\left|T\phi^{\beta\gamma}(y)\chi^{\alpha_1}(x_1)\cdots\chi^{\alpha_n}(x_n)\int\chi^{\mu}(z)\chi^{\mu}(z)dz\right|0\right\rangle\right.$$
$$\left.-\frac{\alpha}{i}4[\delta^{(4)}(0)]^2\langle 0|T\phi^{\beta\gamma}(y)\chi^{\alpha_1}(x_1)\cdots\chi^{\alpha_n}(x_n)|0\rangle\right], \qquad (3.37)$$

where the Ward identity (3.27) was used. The generalized Ward identities make it possible now to carry out the explicit subtraction of the divergence $[\delta^{(4)}(0)]^2$ in the last expression. Indeed, taking the trace over two of $(n+3)$ points in the recurrence formula (3.34) one obtains for the function on the right-hand side of (3.37)

$$\frac{i}{\alpha}\left\langle 0\left|T\phi^{\beta\gamma}(y)\chi^{\alpha_1}(x_1)\cdots\chi^{\alpha_n}(x_n)\int\chi^{\mu}(z)\chi^{\mu}(z)dz\right|0\right\rangle=4[\delta^{(4)}(0)]^2\langle 0|T\phi^{\beta\gamma}(y)\chi^{\alpha_1}(x_1)\cdots\chi^{\alpha_n}(x_n)|0\rangle$$
$$+n\langle 0|T\phi^{\beta\gamma}(y)\chi^{\alpha_1}(x_1)\cdots\chi^{\alpha_n}(x_n)|0\rangle$$
$$+\left\langle 0\left|T\chi^{\alpha_1}(x_1)\cdots\chi^{\alpha_n}(x_n)\int E^{\nu|\beta\gamma}(z,y)\chi^{\nu}(z)dz\right|0\right\rangle. \qquad (3.38)$$

When this representation is substituted into Eq. (3.37) the divergent term cancels out and one obtains

$$\frac{d}{d\alpha}\langle 0|T\phi^{\beta\gamma}(y)\chi^{\alpha_1}(x_1)\cdots\chi^{\alpha_n}(x_n)|0\rangle=\frac{1}{2\alpha}\Big[n\langle 0|T\phi^{\beta\gamma}(y)\chi^{\alpha_1}(x_1)\cdots\chi^{\alpha_n}(x_n)|0\rangle$$
$$+\left\langle 0\left|T\chi^{\alpha_1}(x_1)\cdots\chi^{\alpha_n}(x_n)\int E^{\nu|\beta\gamma}(z,y)\chi^{\nu}(z)dz\right|0\right\rangle\Big]. \qquad (3.39)$$

Another way of dealing with Eq. (3.37) may prove to be more convenient because it does not lead to the appearance of the fictitious-particle Green's functions. The method consists of singling out the disconnected parts of the $(n+3)$-point function integrated in Eq. (3.37), and the subsequent use of the Ward identities of the first set only.

For example, for the two-point function one has

$$\frac{d}{d\alpha}\langle 0|T\phi^{\alpha\beta}(x)\phi^{\mu\nu}(y)|0\rangle$$
$$=\frac{i}{2\alpha^2}\Big[2\int dz\langle 0|T\phi^{\alpha\beta}(x)\chi^{\sigma}(z)|0\rangle\langle 0|T\phi^{\mu\nu}(y)\chi^{\sigma}(z)|0\rangle$$
$$+\left\langle 0\left|T\phi^{\alpha\beta}(x)\phi^{\mu\nu}(y)\int\chi^{\sigma}(z)\chi^{\sigma}(z)dz\right|0\right\rangle_{\text{c.p.}}\Big]. \qquad (3.40)$$

Here c.p. denotes the connected part of the four-point function.

The first term on the right-hand side of (3.40) may be represented in momentum space as the product of propagators. As for the second term, the integral of the connected part of the four-point function, its analysis is very difficult even if only the threshold asymptotic behavior is considered. It may be shown that this integral is proportional to the propagator on the mass shell, but the coefficient remains unknown. The gauge-invariance properties are not sufficient to carry out completely the calculation of the derivatives with respect to α. In Sec. V we shall investigate some other properties of the theory which will permit us to find the mass-shell asymptotic behavior of the integrals of the type (3.40). As we shall show, they may be expressed directly in terms of the vertex Green's functions with one of the momenta equal to zero.

The final calculation of the dependence on α being deferred until Sec. V, let us return to the general properties of the Green's functions.

D. The Quantized Einstein Equations

The generalized Ward identities which were obtained here do not determine the Green's functions completely. Relation (3.6) is the gauge-invariance consequence of the complete set of equations of motion for the generating functional.

The derivation of these equations may be performed by the well-known general method of integration by parts in the functional integral.

Since the integrand of the functional integral vanishes at the boundaries of the integration region, let us write

$$\int\frac{\delta}{\delta\hat{g}^{\mu\nu}(x)}\Big[\exp\Big\{iS[\hat{g}]-\frac{i}{2\alpha}\int\chi^{\beta}\chi^{\beta}d^4x+i\int\hat{g}^{\alpha\beta}j_{\alpha\beta}d^4x$$
$$+\ln J_{\chi}[\hat{g}]+\delta^{(4)}(0)\int\ln M(\hat{g}(x))d^4x\Big\}\Big]$$
$$\times\prod_{x,\,\alpha\le\beta}d\hat{g}^{\alpha\beta}(x)=0.$$

Passing to variations with respect to external sources, one obtains from here the equations of motion for the generating functional:

$$\left\{\frac{\delta S[\hat{g}]}{\delta\hat{g}^{\mu\nu}(x)}+\frac{1}{\alpha}\partial_{\mu\sigma}{}^2\hat{g}^{\sigma\nu}(x)+\frac{1}{i}\frac{\delta}{\delta\hat{g}^{\mu\nu}(x)}\ln J[\hat{g}]+\frac{1}{i}\delta^{(4)}(0)\frac{1}{M(\hat{g}(x))}\frac{\partial M(\hat{g}(x))}{\partial\hat{g}^{\mu\nu}(x)}+j_{\mu\nu}(x)\right\}\Bigg|_{\hat{g}=i^{-1}\delta/\delta j}Z[j]=0. \qquad (3.41)$$

When variations of these equations are performed and the sources are put equal to zero, the infinite set of equations of motion for the Green's functions follows, which determines the dynamics completely.

The question of the boundary conditions to these equations, as well as some of their properties, will be taken up in Sec. V.

It must only be noted here that the equations of motion (3.41) necessarily contain terms of the type $\delta^{(4)}(0)$. We have seen that the corresponding term in the functional integral exactly compensates the divergences $\delta^{(4)}(0)$ arising in the action $S[\hat{g}]$ and in $\ln J_\chi[\hat{g}]$ when the exponential of these quantities is integrated.

We have seen also that the divergences mentioned arise due to nonlinearity of the terms with the higher time derivatives. The local terms in the equations of motion play the same role. Indeed, the extremal

$$\frac{\delta S[\hat{g}]}{\delta \hat{g}^{\mu\nu}(x)}$$

contains nonlinear terms with the second derivatives and the squares of the first time derivatives of the field functions in coinciding points. It may be shown that functional integrals of the types

$$\langle f(\hat{g})\partial_{00}{}^2\hat{g}(x)\rangle$$

and

$$\langle f(\hat{g})\partial_0\hat{g}(x)\partial_0\hat{g}(x)\rangle$$

diverge as $\delta(0)$. In terms of operators these divergences appear because we deal with the T products of fields in Eqs. (3.41): Commutation of the time derivative with the T-ordering operation yields $\delta(0)$ when the θ function is differentiated in coinciding points.

The local term in Eq. (3.41) cancels these divergences and the equations of motion are free of them. It would be interesting to carry out explicitly the subtraction of these divergences.

This question as well as the application of generalized Ward identities for the proof of S-matrix unitarity in the physical subspace, which was mentioned in Sec. II when the canonical formalism in a covariant gauge was considered, will be studied in future work.

The only thing to be noted here is that Eq. (3.6), obtained earlier, may be also derived as a consequence of the general equations of motion (3.41). To perform the derivation, it is sufficient to operate from the left with

$$\nabla^{+\mu\nu}_{\gamma}(\hat{g})|_{\hat{g}=i^{-1}\delta/\delta j}\,.$$

It must be taken into account, however, that the mentioned property

$$\frac{\delta S[\hat{g}]}{\delta\hat{g}^{\mu\nu}(x)}\bigg|_{\hat{g}=i^{-1}\delta/\delta j} Z[j]\propto\delta^{(4)}(0)$$

causes the classical Nöther identities (3.1) in application to the generating-functional variations to take the form

$$\left\{\nabla^{+\mu\nu}_{\sigma}(\hat{g})\left[\frac{\delta S[\hat{g}]}{\delta\hat{g}^{\mu\nu}(x)}+\frac{1}{i}\delta^{(4)}(0)\frac{1}{M(\hat{g}(x))}\frac{\partial M(\hat{g}(x))}{\partial\hat{g}^{\mu\nu}(x)}\right]\right\}\bigg|_{\hat{g}=i^{-1}\delta/\delta j} Z[j]=0\,. \tag{3.42}$$

Taking into account (3.42) one obtains Eq. (3.6) as a result of application of the Nöther identities in the quantum domain.

Equations (3.41) are the general dynamical equations of Schwinger's type in the second-order formalism. There is another version of the theory: the first-order formalism, which in some respect possesses advantages over the second-order formalism. The advantages are: (a) There is only one bare vertex in the classical Lagrangian of the first-order formalism; (b) under some conditions the quantum equations for the Green's functions in this formalism take directly the form of equations of Schwinger-Dyson type in electrodynamics.

These conditions are concerned with the local measure and consist of the obligatory use of a regularization in which all the terms of the type $\delta(0)$ vanish; for example, the one suggested in Ref. 7. The problem is that the quantum equations in functional derivatives in the first-order formalism are purely quadratic in functional derivatives with the exception of the terms coming from the local measure. In the regularization of Ref. 7 these terms vanish. The terms $\propto\delta(0)$ which come from the action functional also vanish in the regularization of Ref. 7, so that the compensation of these divergences is automatically guaranteed.

In the first-order formalism the expressions (1.16) and (2.24) for the generating functional must be used, and the source of the field Γ must be introduced by adding a term $J^{\mu\nu}_{\sigma}(x)\Gamma^{\sigma}_{\mu\nu}(x)$ to the Lagrangian in the integrand of (1.16). We shall consider the $\hat{g}^{\mu\nu}$ parametrization, the nondegenerate harmonic gauge (1.12), and the operator (1.14) which corresponds to this gauge in the first-order formalism as well as in the second-order one. The Jacobian of this operator in Eq. (1.16) must be represented with the aid of additional Fermi fields $\overline{C}$, C as in Eq. (2.33), and the sources of these fields must be introduced by adding a term $\overline{\eta}_\alpha C^\alpha+\overline{C}^\alpha\eta_\alpha$ to the Lagrangian. Then the generating functional takes the form

$$Z=Z[j_{\mu\nu},J^{\alpha\beta}_{\sigma},\overline{\eta}_\alpha,\eta_\beta]\,.$$

In order to write down the Dyson equations, the usual definitions must be introduced for the Green's functions in the presence of sources:

$$\frac{1}{Z}\frac{\delta Z}{\delta i j_{\mu\nu}(x)}=\langle\hat{g}^{\mu\nu}(x)\rangle\,,$$
$$\frac{1}{Z}\frac{\delta Z}{\delta i J^{\mu\nu}_{\sigma}(x)}=\langle\Gamma^{\sigma}_{\mu\nu}(x)\rangle\,; \tag{3.43}$$

$$\frac{\delta^2 \ln Z}{\delta ij_{\alpha\beta}(x)\delta ij_{\mu\nu}(y)} = \frac{1}{i} G^{(\alpha\beta)(\mu\nu)}(x,y)\,, \tag{3.44}$$

$$\frac{\delta^2 \ln Z}{\delta ij_{\alpha\beta}(x)\delta i J^{\mu\nu}_{\gamma}(y)} = \frac{1}{i} E^{(\alpha\beta)(\gamma|\mu\nu)}(x,y)\,, \tag{3.45}$$

$$\frac{\delta^2 \ln Z}{\delta i J^{\mu\nu}_{\gamma}(x)\delta i J^{\alpha\beta}_{\sigma}(y)} = \frac{1}{i} D^{(\gamma|\mu\nu)(\sigma|\alpha\beta)}(x,y)\,, \tag{3.46}$$

$$\frac{1}{Z}\frac{\delta^2 Z}{\delta i\eta^{\alpha}(x)\delta i\bar{\eta}^{\beta}(y)}\bigg|_{\eta=\bar{\eta}=0} = M^{\alpha\beta}(x,y)\,. \tag{3.47}$$

Let us consider the matrices

$$\mathfrak{g}^k(x) = \begin{pmatrix} j \\ J \end{pmatrix}, \quad \mathfrak{G}^{mn}(x,y) = \begin{pmatrix} G & E \\ E & D \end{pmatrix}, \quad m,n,k = 1, \ldots, 50 \tag{3.48}$$

where m, n, k will be replaced by $(\alpha\beta)$ or $(\gamma|\alpha\beta)$, and introduce the polarization $(\Pi, \tilde{\Pi})$ and the mass (Σ) operators according to the formulas

$$-\frac{\delta \mathfrak{G}^{mn}(x,x)}{\delta \mathfrak{g}^k(y)} = \int \Pi^{mn}_{r}(x,z)\mathfrak{G}^{rk}(z,y)dz\,, \tag{3.49}$$

$$\frac{\delta M^{\alpha\beta}(x,x')}{\delta ij_{\gamma\sigma}(y)} = \int \tilde{\Pi}^{\alpha\beta}_{(\mu\nu)}(x,x'|z)G^{(\mu\nu)(\gamma\sigma)}(z,y)dz\,, \tag{3.50}$$

$$\frac{\delta M^{\alpha\beta}(x,x')}{\delta i J^{\mu\nu}_{\sigma}(y)} = \int \tilde{\Pi}^{\alpha\beta}_{(\gamma|\phi\lambda)}(x,x'|z)D^{(\gamma|\phi\lambda)(\sigma|\mu\nu)}(z,y)dz\,, \tag{3.51}$$

$$-\frac{\delta M^{\mu\gamma}(x,y)}{\delta j_{\nu\sigma}(x')} = \int \Sigma^{\mu\lambda}_{(\nu\sigma)}(x,x'|z)M^{\lambda\gamma}(z,y)dz\,. \tag{3.52}$$

The equations of motion of the quantized Einstein theory in the first-order formalism may be derived using the standard technique.[6] Provided that the regularization of Ref. 7 will be used everywhere, these equations are of Schwinger-Dyson type and have the following form:

$$\begin{aligned}
&\int dz\,\Big\{\Big[\partial^x_\nu\delta^{\alpha}_{\gamma}\tfrac{1}{2}(\delta^{\phi}_{\mu}\delta^{\lambda}_{\alpha}+\delta^{\lambda}_{\mu}\delta^{\phi}_{\alpha})-\partial^x_\alpha\delta^{\alpha}_{\gamma}\tfrac{1}{2}(\delta^{\phi}_{\mu}\delta^{\lambda}_{\nu}+\delta^{\lambda}_{\mu}\delta^{\phi}_{\nu})+\langle\Gamma^{\alpha}_{\rho\nu}(x)\rangle\delta^{\rho}_{\gamma}\tfrac{1}{2}(\delta^{\phi}_{\mu}\delta^{\lambda}_{\alpha}+\delta^{\lambda}_{\mu}\delta^{\phi}_{\alpha})+\langle\Gamma^{\rho}_{\mu\alpha}(x)\rangle\delta^{\alpha}_{\gamma}\tfrac{1}{2}(\delta^{\phi}_{\rho}\delta^{\lambda}_{\nu}+\delta^{\lambda}_{\rho}\delta^{\phi}_{\nu})\\
&\quad-\langle\Gamma^{\alpha}_{\rho\alpha}(x)\rangle\delta^{\rho}_{\gamma}\tfrac{1}{2}(\delta^{\phi}_{\mu}\delta^{\lambda}_{\nu}+\delta^{\lambda}_{\mu}\delta^{\phi}_{\nu})-\langle\Gamma^{\rho}_{\mu\nu}(x)\rangle\delta^{\alpha}_{\gamma}\tfrac{1}{2}(\delta^{\phi}_{\rho}\delta^{\lambda}_{\alpha}+\delta^{\lambda}_{\rho}\delta^{\phi}_{\alpha})\Big]\delta(x-z)\\
&\quad+i\Pi^{(\rho|\mu\alpha)(\alpha|\rho\nu)}_{(\gamma|\phi\lambda)}(x,z)-i\Pi^{(\rho|\mu\nu)(\alpha|\rho\alpha)}_{(\gamma|\phi\lambda)}(x,z)\Big\}E^{(\gamma|\phi\lambda)(\theta\zeta)}(z,y)\\
&\qquad+\int dz\,\Big\{\Big[\frac{1}{\alpha}\partial^x_{[\mu}\partial^x_{\sigma}\tfrac{1}{2}(\delta^{\sigma}_{\phi}\delta^{\nu]}_{\lambda}+\delta^{\nu]}_{\phi}\delta^{\sigma}_{\lambda})\Big]\delta(x-z)+i\Pi^{(\rho|\mu\alpha)(\alpha|\rho\nu)}_{(\phi\lambda)}(x,z)-i\Pi^{(\rho|\mu\nu)(\alpha|\rho\alpha)}_{(\phi\lambda)}(x,z)\\
&\qquad\quad+i\Big[-\partial^x_{[\nu}\partial^{x'}_{\mu]}\tilde{\Pi}^{\alpha\alpha}_{(\phi\lambda)}(x,x'|z)-\partial^x_{[\nu}\partial^{x'}_{\gamma}\tilde{\Pi}^{\mu]\gamma}_{(\phi\lambda)}(x,x'|z)-\partial^{x'}_{[\nu}\partial^{x'}_{\gamma}\tilde{\Pi}^{\mu]\gamma}_{(\phi\lambda)}(x,x'|z)\Big]_{x=x'}\Big\}G^{(\phi\lambda)(\theta\zeta)}(z,y)\\
&\qquad\qquad=-\tfrac{1}{2}(\delta^{\theta}_{\mu}\delta^{\zeta}_{\nu}+\delta^{\theta}_{\nu}\delta^{\zeta}_{\mu})\delta(x-y)\,,
\end{aligned}\tag{3.53}$$

$$\begin{aligned}
&\int dz\,\Big\{\Big[\partial^x_\alpha\tfrac{1}{2}(\delta^{\sigma}_{\phi}\delta^{\beta}_{\lambda}+\delta^{\beta}_{\phi}\delta^{\sigma}_{\lambda})+\langle\Gamma^{\beta}_{\mu\alpha}(x)\rangle\tfrac{1}{2}(\delta^{\mu}_{\phi}\delta^{\sigma}_{\lambda}+\delta^{\sigma}_{\phi}\delta^{\mu}_{\lambda})+\langle\Gamma^{\sigma}_{\mu\alpha}(x)\rangle\tfrac{1}{2}(\delta^{\mu}_{\phi}\delta^{\beta}_{\lambda}+\delta^{\beta}_{\phi}\delta^{\mu}_{\lambda})-\langle\Gamma^{\nu}_{\alpha\nu}(x)\rangle\tfrac{1}{2}(\delta^{\sigma}_{\phi}\delta^{\beta}_{\lambda}+\delta^{\beta}_{\phi}\delta^{\sigma}_{\lambda})\Big]\delta(x-z)\\
&\quad+i\Pi^{(\mu\sigma)(\beta|\mu\alpha)}_{(\phi\lambda)}(x,z)+i\Pi^{(\mu\beta)(\sigma|\mu\alpha)}_{(\phi\lambda)}(x,z)-i\Pi^{(\sigma\beta)(\nu|\alpha\nu)}_{(\phi\lambda)}(x,z)\Big\}E^{(\phi\lambda)(\eta|\theta\zeta)}(z,y)\\
&\qquad+\int dz\,\Big\{\Big[\langle\hat{g}^{\mu\sigma}(x)\rangle\delta^{\beta}_{\gamma}\tfrac{1}{2}(\delta^{\phi}_{\mu}\delta^{\lambda}_{\alpha}+\delta^{\phi}_{\alpha}\delta^{\lambda}_{\mu})+\langle\hat{g}^{\mu\beta}(x)\rangle\delta^{\sigma}_{\gamma}\tfrac{1}{2}(\delta^{\phi}_{\mu}\delta^{\lambda}_{\alpha}+\delta^{\phi}_{\alpha}\delta^{\lambda}_{\mu})-\langle\hat{g}^{\sigma\beta}(x)\rangle\delta^{\nu}_{\gamma}\tfrac{1}{2}(\delta^{\phi}_{\alpha}\delta^{\lambda}_{\nu}+\delta^{\phi}_{\nu}\delta^{\lambda}_{\alpha})\Big]\delta(x-z)\\
&\qquad\quad+i\Pi^{(\mu\sigma)(\beta|\mu\alpha)}_{(\gamma|\phi\lambda)}(x,z)+i\Pi^{(\mu\beta)(\sigma|\mu\alpha)}_{(\gamma|\phi\lambda)}(x,z)-i\Pi^{(\sigma\beta)(\nu|\alpha\nu)}_{(\gamma|\phi\lambda)}(x,z)\Big\}D^{(\gamma|\phi\lambda)(\eta|\theta\zeta)}(z,y)\\
&\qquad\qquad=-\tfrac{1}{2}\Big[\delta^{\eta}_{\alpha}(\delta^{\sigma}_{\theta}\delta^{\beta}_{\zeta}+\delta^{\beta}_{\theta}\delta^{\sigma}_{\zeta})-\tfrac{1}{3}\delta^{\beta}_{\alpha}(\delta^{\eta}_{\theta}\delta^{\sigma}_{\zeta}+\delta^{\sigma}_{\theta}\delta^{\eta}_{\zeta})-\tfrac{1}{3}\delta^{\sigma}_{\alpha}(\delta^{\eta}_{\theta}\delta^{\beta}_{\zeta}+\delta^{\beta}_{\theta}\delta^{\eta}_{\zeta})\Big]\delta(x-y)\,,
\end{aligned}\tag{3.54}$$

$$\begin{aligned}
&\int dz\,\Big\{\Big[\partial^x_\nu\delta^{\alpha}_{\gamma}\tfrac{1}{2}(\delta^{\phi}_{\mu}\delta^{\lambda}_{\alpha}+\delta^{\phi}_{\alpha}\delta^{\lambda}_{\mu})-\partial^x_\alpha\delta^{\alpha}_{\gamma}\tfrac{1}{2}(\delta^{\phi}_{\mu}\delta^{\lambda}_{\nu}+\delta^{\phi}_{\nu}\delta^{\lambda}_{\mu})+\langle\Gamma^{\alpha}_{\rho\nu}(x)\rangle\delta^{\rho}_{\gamma}\tfrac{1}{2}(\delta^{\phi}_{\mu}\delta^{\lambda}_{\alpha}+\delta^{\rho}_{\alpha}\delta^{\lambda}_{\mu})+\langle\Gamma^{\rho}_{\mu\alpha}(x)\rangle\delta^{\alpha}_{\gamma}\tfrac{1}{2}(\delta^{\phi}_{\rho}\delta^{\lambda}_{\nu}+\delta^{\phi}_{\nu}\delta^{\lambda}_{\rho})\\
&\quad-\langle\Gamma^{\alpha}_{\rho\alpha}(x)\rangle\delta^{\rho}_{\gamma}\tfrac{1}{2}(\delta^{\phi}_{\mu}\delta^{\lambda}_{\nu}+\delta^{\phi}_{\nu}\delta^{\lambda}_{\mu})-\langle\Gamma^{\rho}_{\mu\nu}(x)\rangle\delta^{\alpha}_{\gamma}\tfrac{1}{2}(\delta^{\phi}_{\rho}\delta^{\lambda}_{\alpha}+\delta^{\phi}_{\alpha}\delta^{\lambda}_{\rho})\Big]\delta(x-z)+i\Pi^{(\rho|\mu\alpha)(\alpha|\rho\nu)}_{(\gamma|\phi\lambda)}(x,z)-i\Pi^{(\rho|\mu\nu)(\alpha|\rho\alpha)}_{(\gamma|\phi\lambda)}(x,z)\\
&\quad+i\Big[-\partial^x_{[\nu}\partial^{x'}_{\mu]}\tilde{\Pi}^{\alpha\alpha}_{(\gamma|\phi\lambda)}(x,x'|z)-\partial^x_{[\nu}\partial^{x'}_{\gamma}\tilde{\Pi}^{\mu]\gamma}_{(\gamma|\phi\lambda)}(x,x'|z)-\partial^{x'}_{[\nu}\partial^{x'}_{\gamma}\tilde{\Pi}^{\mu]\gamma}_{(\gamma|\phi\lambda)}(x,x'|z)\Big]_{x=x'}\Big\}D^{(\gamma|\phi\lambda)(\eta|\theta\zeta)}(z,y)\\
&\qquad+\int dz\,\Big\{\Big[\frac{1}{\alpha}\partial^x_{[\mu}\partial^x_{\sigma}\tfrac{1}{2}(\delta^{\sigma}_{\phi}\delta^{\nu]}_{\lambda}+\delta^{\nu]}_{\phi}\delta^{\sigma}_{\lambda})\Big]\delta(x-z)+i\Pi^{(\rho|\mu\alpha)(\alpha|\rho\nu)}_{(\phi\lambda)}(x,z)-i\Pi^{(\rho|\mu\nu)(\alpha|\rho\alpha)}_{(\phi\lambda)}(x,z)\Big\}E^{(\phi\lambda)(\eta|\theta\zeta)}(z,y)=0\,,
\end{aligned}\tag{3.55}$$

$$\begin{aligned}
&\int dz\,\Big\{\Big[\partial^x_\alpha\tfrac{1}{2}(\delta^{\sigma}_{\phi}\delta^{\beta}_{\lambda}+\delta^{\beta}_{\phi}\delta^{\sigma}_{\lambda})+\langle\Gamma^{\beta}_{\mu\alpha}(x)\rangle\tfrac{1}{2}(\delta^{\mu}_{\phi}\delta^{\sigma}_{\lambda}+\delta^{\sigma}_{\phi}\delta^{\mu}_{\lambda})+\langle\Gamma^{\sigma}_{\mu\alpha}(x)\rangle\tfrac{1}{2}(\delta^{\mu}_{\phi}\delta^{\beta}_{\lambda}+\delta^{\beta}_{\phi}\delta^{\mu}_{\lambda})-\langle\Gamma^{\nu}_{\alpha\nu}(x)\rangle\tfrac{1}{2}(\delta^{\sigma}_{\phi}\delta^{\beta}_{\lambda}+\delta^{\beta}_{\phi}\delta^{\sigma}_{\lambda})\Big]\delta(x-z)\\
&\quad+i\Pi^{(\mu\sigma)(\beta|\mu\alpha)}_{(\phi\lambda)}(x,z)+i\Pi^{(\mu\beta)(\sigma|\mu\alpha)}_{(\phi\lambda)}(x,z)-i\Pi^{(\sigma\beta)(\nu|\alpha\nu)}_{(\phi\lambda)}(x,z)\Big\}G^{(\phi\lambda)(\theta\zeta)}(z,y)\\
&\qquad+\int dz\,\Big\{\Big[\langle\hat{g}^{\mu\sigma}(x)\rangle\delta^{\beta}_{\gamma}\tfrac{1}{2}(\delta^{\phi}_{\mu}\delta^{\lambda}_{\alpha}+\delta^{\phi}_{\alpha}\delta^{\lambda}_{\mu})+\langle\hat{g}^{\mu\beta}(x)\rangle\delta^{\sigma}_{\gamma}\tfrac{1}{2}(\delta^{\phi}_{\mu}\delta^{\lambda}_{\alpha}+\delta^{\phi}_{\alpha}\delta^{\lambda}_{\mu})-\langle\hat{g}^{\sigma\beta}(x)\rangle\delta^{\nu}_{\gamma}\tfrac{1}{2}(\delta^{\phi}_{\alpha}\delta^{\lambda}_{\nu}+\delta^{\phi}_{\nu}\delta^{\lambda}_{\alpha})\Big]\delta(x-z)\\
&\qquad\quad+i\Pi^{(\mu\sigma)(\beta|\mu\alpha)}_{(\gamma|\phi\lambda)}(x,z)+i\Pi^{(\mu\beta)(\sigma|\mu\alpha)}_{(\gamma|\phi\lambda)}(x,z)-i\Pi^{(\sigma\beta)(\nu|\alpha\nu)}_{(\gamma|\phi\lambda)}(x,z)\Big\}E^{(\gamma|\phi\lambda)(\theta\zeta)}(z,y)=0\,,
\end{aligned}\tag{3.56}$$

$$\int dz\,\{[\langle\partial_\sigma\hat{g}^{\nu\sigma}(x)\rangle\delta^\mu_\lambda\partial^x_\nu+\langle\partial_\sigma\hat{g}^{\mu\sigma}(x)\rangle\delta^{\alpha}_{\lambda}\partial^x_\alpha+\langle\hat{g}^{\nu\sigma}(x)\rangle\delta^\mu_\lambda\partial^x_\nu\partial^x_\sigma]\delta(x-z)$$
$$+i[\partial^{x'}_\sigma\partial^x_\nu\Sigma^{\mu\lambda}_{(\nu\sigma)}(x,x'|z)+\partial^{x'}_\sigma\partial^x_\alpha\Sigma^{\alpha\lambda}_{(\mu\sigma)}(x,x'|z)+\partial^x_\nu\partial^x_\sigma\Sigma^{\mu\lambda}_{(\nu\sigma)}(x,x'|z)]_{x=x'}\}M^{\lambda\gamma}(z,y)=i\delta^{\mu\gamma}\delta(x-y),$$
$$\eta=\overline{\eta}=0\,. \quad (3.57)$$

In these equations the notation $[\mu\nu]$ means that symmetrization with respect to the indices μ,ν is supposed. The α appearing once in Eq. (3.53) and once in Eq. (3.55) denotes the parameter fixing the gauge.

When the sources are switched off in these equations, one must put

$$\langle\hat{g}^{\mu\nu}(x)\rangle=A^{(0)}\delta^{\mu\nu},\quad \langle\Gamma^{\alpha}_{\beta\sigma}(x)\rangle=0,$$

while the polarization and mass operators are connected with the three-particle vertex functions in the usual way. The connection between these vertex functions and the higher-order vertex functions may be found with the aid of standard technique.[6]

In Secs. IV and V we shall analyze the divergences which appear in the threshold asymptotic behavior of the Green's functions. One might think that this analysis is much simpler in the framework of the first-order formalism than in the second-order one, because there is only one bare vertex in the classical Lagrangian of the first-order formalism. However, because of the formal nonrenormalizability of the theory, one has to analyze not only the lowest-order Green's functions, but all the exact Green's functions in both versions of the theory. Besides that, a part of the Green's functions of the first-order theory which does not contribute to the scattering amplitudes may contain the additional divergences which one does not need to analyze. On the other hand, the second-order formalism is more symmetrical and the Ward identities are of a rather simple form in this theory. That is why we prefer to present the subsequent analysis in the framework of the second-order formalism.

IV. WARD RELATIONS FOR RENORMALIZATION CONSTANTS

A. Propagators, Vertices, and Renormalization Constants

Let us introduce some necessary notations.[25] Instead of the n-point function of the field $\hat{g}$,

$$\langle 0|T\hat{g}^{a_1}\cdots\hat{g}^{a_n}|0\rangle=\frac{1}{Z}\frac{\delta^n Z[j]}{\delta ij^{a_1}\cdots\delta ij^{a_n}}\bigg|_{j=0}, \quad (4.1)$$

we shall deal with the n-point function of the field ϕ [see Fig. 1(a)],

$$\langle 0|T\phi^{a_1}\cdots\phi^{a_n}|0\rangle, \quad (4.2)$$

as was discussed above. It is convenient to single out the connected part of (4.2) [see Fig. 1(b)]:

$$\frac{\delta^n \ln Z[j]}{\delta ij^{a_1}\cdots\delta ij^{a_n}}\bigg|_{j=0}. \quad (4.3)$$

In addition to the propagator of the field ϕ [see Fig. 1(c)],

$$\frac{1}{i}G^{ab}=\frac{\delta^2\ln Z[j]}{\delta ij^a\delta ij^b}\bigg|_{j=0}, \quad (4.4)$$

we introduce also the propagator of the fictitious particles $\overline{C}^\alpha(x)$, $C^\beta(x)$ [see Fig. 1(d)],

$$M^{\alpha\beta}(x,y)=\langle 0|T\overline{C}^\alpha(x)C^\beta(y)|0\rangle$$
$$\equiv\langle 0|TQ^{-1\alpha\beta}(x,y)|0\rangle\,. \quad (4.5)$$

Now, according to the well-known definition, we introduce the proper vertex functions of the field ϕ [see Fig. 1(e)],

$$\Gamma^{(n+2)}_{a_1a_2\cdots a_{n+2}}=-\frac{\delta^n G^{-1}{}_{a_1a_2}[j]}{\delta i\langle\hat{g}^{a_3}\rangle\cdots\delta i\langle\hat{g}^{a_{n+2}}\rangle}\bigg|_{j=0},\quad n\geq 1\,. \quad (4.6)$$

Let us introduce also the proper vertex functions for the interaction of the fictitious particles with gravitons [see Fig. 1(f)],

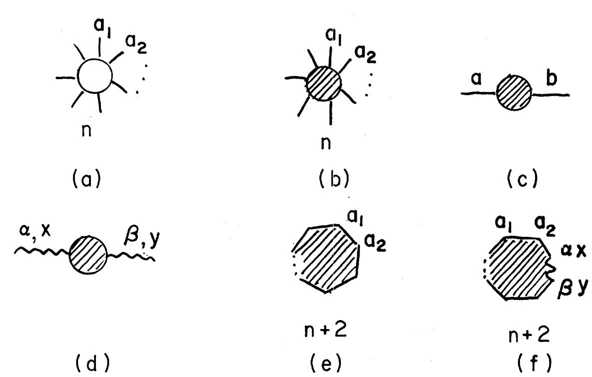


FIG. 1. Graphical notation of the exact Green's functions of the theory: (a) n-point function of the field ϕ, (b) its connected part, (c) exact graviton propagator, (d) exact propagator of fictitious particles, (e) proper vertex functions of graviton interaction, and (f) proper vertex functions of graviton interaction with the fictitious particles.

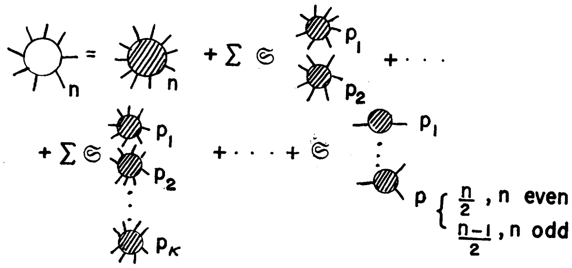


FIG. 2. Representation of the n-point function in terms of the connected parts of the lower-order p-point functions. Decomposition is carried out in the number k of disconnected pieces. Σ denotes the sum over $p_1, p_2, \ldots, p_k$ with the condition $\sum_{i=1}^{k} p_i = n$ in every term, while $\mathfrak{S}$ denotes symmetrization with respect to the indices $a_1, \ldots, a_n$ in such a way that every particular arrangement of indices should enter the expansion only once.

$$\gamma^{(n+2)}_{\alpha\beta|\mu_1\nu_1,\ldots,\mu_n\nu_n}(x,y|z_1,\ldots,z_n) \equiv \gamma^{(n+2)}_{\alpha\beta}(x,y|a_1\cdots a_n) = -\frac{\delta^n M^{-1}{}_{\alpha\beta}(x,y|j)}{\delta i\langle\hat{g}^{a_1}\rangle\cdots\delta i\langle\hat{g}^{a_n}\rangle}\bigg|_{j=0}. \tag{4.7}$$

Expansion of the n-point function in the connected parts of the lower-order p-point functions is given in Fig. 2. The connected part of the n-point function is expanded in the proper vertex functions as shown in Fig. 3.

The $(n+2)$-point function

$$\Theta^{\alpha\beta}(x,y|a_1\cdots a_n) = \langle 0|TQ^{-1\alpha\beta}(x,y)\phi^{a_1}\cdots\phi^{a_n}|0\rangle \tag{4.8}$$

may be expanded in the proper vertex functions (4.6) and (4.7).

The bare propagators and the bare vertices are determined by the expansion of the action functional

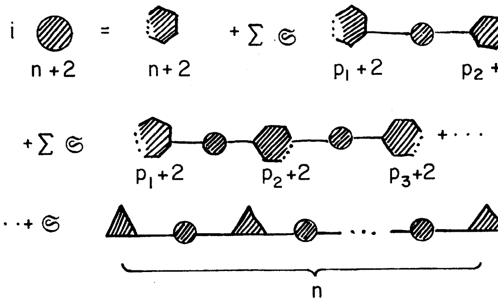


FIG. 3. Expansion of the connected part of the n-point function in the proper vertex functions with the same notations as in Fig. 2.

$$\frac{1}{\kappa}S[\hat{g},\alpha] + \int \overline{C}_\alpha(x)\hat{Q}^{\alpha\beta}(\hat{g}|x)C_\beta(x)dx,$$
$$S[\hat{g},\alpha] \equiv \int\sqrt{-g}\,R\,dx - \frac{1}{2\alpha}\int\chi^\mu(x)\chi^\mu(x)dx \tag{4.9}$$

in terms of the variations

$$\frac{\delta}{\delta\hat{g}^a}\bigg|_{\hat{g}=\langle 0|\hat{g}|0\rangle}$$

Let us denote

$$i\left[\frac{\delta^2 S[\hat{g},\alpha]}{\delta\hat{g}^a\delta\hat{g}^b}\bigg|_{\hat{g}=\delta}\right]^{-1} = G^{(0)ab}(\alpha),$$

$$-\frac{\delta^{(n+2)}S[\hat{g}]}{\delta\hat{g}^{a_1}\cdots\delta\hat{g}^{a_{n+2}}}\bigg|_{\hat{g}=\delta} = \Gamma^{(0)(n+2)}_{a_1\cdots a_{n+2}}, \qquad n\geq 1$$

$$[\hat{Q}^{\alpha\beta}|_{\hat{g}=\delta}\delta(x-y)]^{-1} = M^{(0)\alpha\beta}(x,y),$$

$$-\frac{\delta^n\hat{Q}^{\alpha\beta}\delta(x-y)}{\delta\hat{g}^{a_1}\cdots\delta\hat{g}^{a_n}}\bigg|_{\hat{g}=\delta} = \gamma^{(0)(n+2)}_{\alpha\beta}(x,y|a_1\cdots a_n).$$

By virtue of

$$\langle 0|\hat{g}^{\mu\nu}(x)|0\rangle = A^{(0)}\delta^{\mu\nu},$$

the bare propagators and the infinite set of the bare graviton vertices are represented as

$$\kappa A^{(0)}G^{(0)ab}\left(\frac{\alpha}{A^{(0)}}\right), \quad (A^{(0)})^{-1}M^{(0)\alpha\beta}(x,y),$$
$$\frac{1}{\kappa}(A^{(0)})^{-(n+1)}\Gamma^{(0)(n+2)}_{a_1\cdots a_{n+2}}. \tag{4.10}$$

It may be easily verified that the average field $A^{(0)}$ enters expressions (4.10) with just the powers mentioned.

Now the gauge conditions being linear, only the first bare vertex

$$\gamma^{(0)(3)}_{\alpha\beta}(x,y|a) \tag{4.11}$$

does not vanish:

$$\gamma^{(0)(n+2)}_{\alpha\beta}(x,y|a_1\cdots a_n) = 0, \quad n>1. \tag{4.12}$$

The bare vertices are not independent of each other. The gauge invariance of the gravitational field action functional imposes a set of constraints on them which is analogous to the Ward identities for the exact Green's functions. The gauge relations of the bare vertices are obtained in Appendix A. Another set of ("scale") relations of the bare vertices is obtained in Sec. V C.

In the present section we shall examine the divergences arising in the threshold asymptotic behavior of the Green's functions. Since each new bare vertex in the Lagrangian generally brings in a new type of overlapping divergences, the renormalization procedure as prescribed by perturbation theory generally requires an infinite number of renormalization constants. We shall formulate the renormalization conditions as the requirement that the threshold asymptotic behavior of the renormalized propagators and vertex functions should have the same structure as in the lowest order of perturbation theory. For convenience of further consideration we shall deviate slightly from the usual rules of introducing the Z multipliers, and introduce the latter as the factors by which the threshold asymptotic behavior of unrenormalized Green's functions differs from the values of the bare Green's functions (4.10)–(4.12):

$$\Gamma^{(n+2)}_{\mu_1\nu_1;\ldots;\mu_{n+2}\nu_{n+2}}(p_1,\ldots,p_{n+2}) \xrightarrow[\substack{\Sigma p_i=0\\ p_i\to 0}]{} (Z_1^{(n)})^{-1}\frac{1}{\kappa}(A^{(0)})^{-(n+1)} \times\Gamma^{(0)(n+2)}_{\mu_1\nu_1;\ldots;\mu_{n+2}\nu_{n+2}}(p_1,\ldots,p_{n+2}),$$

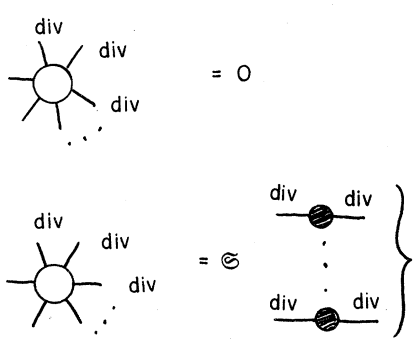
$$\gamma^{(3)}_{\alpha\beta|\mu\nu}(k,p|q) \xrightarrow[\substack{k+p+q=0\\ k,p,q\to 0}]{} \tilde{Z}_1^{-1}\gamma^{(0)(3)}_{\alpha\beta|\mu\nu}(k,p|q), \tag{4.13}$$

$$\gamma^{(n+2)}_{\alpha\beta|\mu_1\nu_1;\ldots;\mu_n\nu_n}(k,p|q_1\cdots q_n) \xrightarrow[\substack{k+p+\Sigma q_i=0\\ k,p,q_i\to 0}]{} 0, \quad n>1.$$

Now the graviton propagator depends on the parameter α, fixing the gauge. But the S matrix does not depend on α on the mass shell. That is why we shall consider first the case of degenerate gauge ($\alpha=0$), while the generalization to the case of arbitrary (nondegenerate) gauge will be given at the end of the present section. The renormalization conditions for the propagators in degenerate gauge have the form

$$G^{\mu_1\nu_1;\mu_2\nu_2}(p,\alpha)\xrightarrow[p^2\to 0]{} Z_2\kappa A^{(0)}G^{(0)\mu_1\nu_1;\mu_2\nu_2}\left(p,\frac{\alpha}{A^{(0)}}\right), \quad \alpha\to 0$$

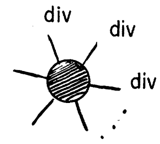
$$M^{\alpha\beta}(p)\xrightarrow[p^2\to 0]{}\tilde{Z}_2\frac{1}{A^{(0)}}M^{(0)\alpha\beta}(p). \tag{4.14}$$



(4.15) (a) n odd

$\frac{n}{2}$ (4.16) (b) n even

FIG. 4. The generalized Ward identities, obtained in Sec. III.



= 0 (4.17)

n ≥ 3

FIG. 5. The first set of generalized Ward identities for the Green's functions.

We shall show that as a consequence of the gauge invariance of the theory, the renormalization constants Z_2, $\tilde{Z}_2$, $Z_1\equiv Z_1^{(1)}$ are the only independent ones in the whole infinite set.

B. Ward Identities for Propagators and Vertex Functions

In order to obtain the relations of the renormalization constants, let us represent the generalized Ward identities of Sec. III in terms of propagators and proper vertices.

Let us begin with the first set of generalized Ward identities (3.26)–(3.28), which is given by Figs. 4(a) and 4(b). It is not difficult to make sure that Eqs. (4.15), (4.16) may be uniformly put into the form of Fig. 5 and

$$p_\mu G^{\mu\nu;\alpha\beta}(p,\alpha)p_\alpha = i\alpha\delta^{\nu\beta}. \tag{4.18}$$

Let us use now the expansion presented by Fig. 3. Then the first set of generalized Ward identities takes the form

$$\mathrm{div}G^{aa'}\mathrm{div}G^{bb'}\mathrm{div}G^{cc'}\Gamma^{(3)}_{a'b'c'}=0, \tag{4.19}$$

$$(\mathrm{div}G)\cdots(\mathrm{div}G)[\Gamma^{(n+2)}+\Sigma\mathfrak{S}\Gamma^{(p_1+2)}G\Gamma^{(p_2+2)}+\cdots+\mathfrak{S}\Gamma^{(3)}G\Gamma^{(3)}G\cdots G\Gamma^{(3)}]=0, \tag{4.20}$$

where $(\mathrm{div}G)\cdots(\mathrm{div}G)$ consists of $n+2$ factors of $\mathrm{div}G$.

As to the second set of generalized Ward identities, it must be noticed that the Green's functions entering the right-hand sides of Eqs. (3.29)–(3.35) may be represented in terms of the Green's functions Θ [Eq. (4.8)], the identity of order n containing not only the Θ function of order n, but also the Θ functions of order $(n+1)$ taken at coinciding points. The higher-order Green's function, taken at coinciding points, generally has the pole asymptotic behavior of the lower-order function, so that both contributions are to be analyzed.

Correspondingly, let us represent the function from the right-hand side of (3.34) in the form

$$\langle 0|TE^{\alpha_1|\beta\gamma}(x_1,y)\chi^{\alpha_2}(x_2)\cdots\chi^{\alpha_n}(x_n)|0\rangle$$
$$=A^{(0)}\nabla^{(0)\beta\gamma|\sigma}(y)\partial^{x_2}_{\alpha_2'}\cdots$$
$$\times\partial^{x_n}_{\alpha_n'}\Theta^{\sigma\alpha_1|\alpha_2'\alpha_2;\cdots;\alpha_n'\alpha_n}(x_1,y|x_2,\ldots,x_n)$$
$$+\mathcal{E}^{\alpha_1|\beta\gamma|\alpha_2\cdots\alpha_n}(x_1,y|x_2,\ldots,x_n), \tag{4.21}$$

where the linearity of the operator ∇ in $\hat{g}$ is used,

$$\nabla^{\beta\gamma|\sigma}(\hat{g}|y)=A^{(0)}\nabla^{(0)\beta\gamma|\sigma}(y)+\nabla^{\beta\gamma|\sigma}(\phi|y), \tag{4.22}$$

$$\nabla^{(0)\beta\gamma|\sigma}\equiv\nabla^{\beta\gamma|\sigma}(\delta|y)$$
$$=\delta^\beta_\sigma\partial^y_\gamma+\delta^\gamma_\sigma\partial^y_\beta-\delta^{\beta\gamma}\partial^y_\sigma, \tag{4.23}$$

and the following notation is introduced:

$$\mathcal{E}^{\alpha_1|\beta\gamma|\alpha_2,\cdots\alpha_n}(x_1,y|x_2,\ldots,x_n)\equiv\langle 0|T\nabla^{\beta\gamma|\sigma}(\phi|y)Q^{-1\sigma}_{\alpha_1}(x_1,y)\,\chi^{\alpha_2}(x_2)\cdots\chi^{\alpha_n}(x_n)|0\rangle. \tag{4.24}$$

As a result we obtain the generalized Ward identities of the second set (3.31), (3.35) in momentum space in the form

$$\frac{1}{\alpha}ip_\nu G^{\beta\gamma;\alpha\nu}(p,\alpha)=iA^{(0)}(\delta^\beta_\sigma p_\gamma+\delta^\gamma_\sigma p_\beta-\delta^{\beta\gamma}p_\sigma)M^{\sigma\alpha}(p)+\mathcal{E}^{\alpha|\beta\gamma}(-p,p), \tag{4.25}$$

$$\frac{1}{\alpha}ik_{\alpha_1'}G^{\alpha_1\alpha_1';\sigma_1\sigma_1'}(k)iq_{\alpha_2'}G^{\alpha_2\alpha_2';\sigma_2\sigma_2'}(q)G^{\beta\gamma;\sigma_3\sigma_3'}(p)\Gamma^{(3)}_{\sigma_1\sigma_1';\sigma_3\sigma_3';\sigma_2\sigma_2'}(k,p,q)$$
$$=A^{(0)}(\delta^\beta_\sigma p_\gamma+\delta^\gamma_\sigma p_\beta-\delta^{\beta\gamma}p_\sigma)M^{\alpha_1\alpha_1'}(k)M^{\sigma\sigma'}(p)iq_{\alpha_2'}G^{\alpha_2\alpha_2';\rho\rho'}(q)\gamma^{(3)}_{\alpha_1'\sigma'|\rho\rho'}(k,p|q)+\mathcal{E}^{\alpha_1|\beta\gamma|\alpha_2}(k|p|q),$$
$$k+p+q=0. \tag{4.26}$$

C. The Analysis of Threshold Asymptotic Behavior of the Green's Functions and Generalized Ward Identities

The asymptotic behavior of the Green's functions $\mathcal{E}$ is obtained separately in Appendix B.

Let us consider first the exact graviton propagator. The highest possible Lorentz-invariant structure of the propagator with respect to the symmetry properties contains five independent scalar functions. However, with respect to the lowest-order Ward identity of the first set (4.18), only three of these functions remain independent. In terms of these three functions, which we denote $G^{\parallel}(p^2,\alpha)$, $G^{LT}(p^2,\alpha)$, $G^{TT}(p^2,\alpha)$, the exact graviton propagator has the form

$$\frac{1}{i}G^{\mu\nu;\beta\gamma}(p,\alpha)=\frac{p_\mu p_\gamma p_\beta p_\nu}{(p^2)^2}G^{\parallel}+\frac{1}{p^2}(\delta_{\beta\gamma}p_\mu p_\nu+\delta_{\mu\nu}p_\beta p_\gamma)\left(\alpha G^{LT}-G^{\parallel}-2G^{TT}-\frac{\alpha}{p^2}\right)$$
$$+\frac{1}{p^2}(\delta_{\nu\beta}p_\gamma p_\mu+\delta_{\nu\gamma}p_\beta p_\mu+\delta_{\mu\beta}p_\gamma p_\nu+\delta_{\mu\gamma}p_\beta p_\nu)\left(\frac{\alpha}{2p^2}+G^{TT}\right)$$
$$+\delta_{\mu\nu}\delta_{\beta\gamma}(G^{\parallel}+2G^{TT}-2\alpha G^{LT})+(\delta_{\mu\beta}\delta_{\nu\gamma}+\delta_{\mu\gamma}\delta_{\nu\beta})\left(\frac{\alpha}{2p^2}-G^{TT}\right), \tag{4.27}$$

$$\frac{1}{i}G^{\mu\nu;\beta\gamma}(p,\alpha)p_\beta=p_\gamma\left(\frac{p_\mu p_\nu}{p^2}-\delta_{\mu\nu}\right)\alpha G^{LT}+\frac{\alpha}{p^2}(\delta_{\nu\gamma}p_\mu+\delta_{\mu\gamma}p_\nu-\delta_{\mu\nu}p_\gamma), \tag{4.28}$$

$$\frac{1}{i}p_\mu G^{\mu\nu;\beta\gamma}(p,\alpha)p_\beta=\alpha\delta_{\nu\gamma}. \tag{4.29}$$

The bare propagator, which may be easily obtained from the Lagrangian, is given by the formula (4.27) with the values

$$G^{(0)\parallel}=0,\quad G^{(0)LT}=0,$$
$$\kappa A^{(0)}G^{(0)TT}\left(p^2,\frac{\alpha}{A^{(0)}}\right)=\frac{1}{p^2}(A^{(0)}+\tfrac{1}{2}\alpha)\kappa. \tag{4.30}$$

Since the exact propagator in the pole term must have the structure of the bare propagator, we obtain

$$G^{\parallel}(p^2,\alpha)\xrightarrow[p^2\to 0]{}0,$$
$$G^{LT}(p^2,\alpha)\xrightarrow[p^2\to 0]{}0. \tag{4.31}$$

Thus, the pole asymptotic behavior of the exact graviton propagator has the form

$$\frac{1}{i}G^{\mu\nu;\beta\gamma}(p,\alpha)=\Big[2\delta_{\mu\nu}\delta_{\beta\gamma}-\delta_{\mu\beta}\delta_{\nu\gamma}-\delta_{\mu\gamma}\delta_{\nu\beta}-\frac{2}{p^2}(\delta_{\beta\gamma}p_\mu p_\nu+\delta_{\mu\nu}p_\beta p_\gamma)+\frac{1}{p^2}(\delta_{\nu\beta}p_\gamma p_\mu+\delta_{\nu\gamma}p_\beta p_\mu+\delta_{\mu\beta}p_\gamma p_\nu+\delta_{\mu\gamma}p_\beta p_\nu)\Big]$$
$$\times G^{TT}(p^2,\alpha)$$
$$+\Big[\tfrac{1}{2}(\delta_{\mu\beta}\delta_{\nu\gamma}+\delta_{\mu\gamma}\delta_{\nu\beta})+\frac{1}{2p^2}(\delta_{\nu\beta}p_\gamma p_\mu+\delta_{\nu\gamma}p_\beta p_\mu+\delta_{\mu\beta}p_\gamma p_\nu+\delta_{\mu\gamma}p_\beta p_\nu)-\frac{1}{p^2}(\delta_{\beta\gamma}p_\mu p_\nu+\delta_{\mu\nu}p_\beta p_\gamma)\Big]\frac{\alpha}{p^2},$$
$$p^2\to 0. \quad (4.32)$$

The propagator (4.32) in the degenerate gauge ($\alpha=0$) is transverse with respect to any of the indices.

Now the renormalization condition (4.14) takes the form

$$G^{TT}(p^2,\alpha)\xrightarrow[\substack{p^2\to 0\\ \alpha\to 0}]{}Z_2\kappa A^{(0)}\frac{1}{p^2}. \quad (4.33)$$

As to the propagator of fictitious quanta, its highest possible Lorentz-invariant structure is

$$M^{\alpha\beta}(p)=\left(\delta^{\alpha\beta}-\frac{p^\alpha p^\beta}{p^2}\right)M^{\perp}(p^2)+\frac{p^\alpha p^\beta}{p^2}M^{\parallel}(p^2). \quad (4.34)$$

The corresponding bare propagator is equal to

$$\delta^{\alpha\beta}\frac{1}{A^{(0)}}\frac{1}{p^2}.$$

Thus, we obtain

$$[M^{\perp}(p^2)-M^{\parallel}(p^2)]\xrightarrow[p^2\to 0]{}0,$$

and the renormalization condition (4.14) takes the form

$$M^{\perp}_{p^2\to 0}(p^2)=M^{\parallel}_{p^2\to 0}(p^2)$$
$$=\tilde{Z}_2\frac{1}{A^{(0)}}\frac{1}{p^2}. \quad (4.35)$$

We shall begin with the lowest-order generalized Ward identities (4.25) and (4.19).

A relation of the bare propagators, analogous to the generalized Ward identity (4.25), is obtained in Appendix A, Eq. (A12), but we also possess rather simple explicit representations for the functions occurring in Eq. (4.25). Taking into account Eqs. (4.28), (4.31), (4.34), and (4.35), one obtains from (4.25)

$$\frac{i}{p^2}(\delta^{\alpha\beta}p^\gamma+\delta^{\alpha\gamma}p^\beta-\delta^{\beta\gamma}p^\alpha)=i(\delta^{\alpha\beta}p^\gamma+\delta^{\alpha\gamma}p^\beta-\delta^{\beta\gamma}p^\alpha)\frac{1}{p^2}\tilde{Z}_2$$
$$+\mathcal{E}^{\alpha|\beta\gamma}_{p^2\to 0}(-p,p). \quad (4.36)$$

The following expression has been obtained in Appendix B, Eq. (B9):

$$\mathcal{E}^{\alpha|\beta\gamma}_{p^2\to 0}(-p,p)=[c_1p^\alpha\delta^{\beta\gamma}+c_2(p^\gamma\delta^{\alpha\beta}+p^\beta\delta^{\alpha\gamma})]M^{\perp}(p^2). \quad (4.37)$$

Substituting Eq. (4.37) into Eq. (4.36) with regard for (4.35), one obtains

$$\frac{1}{p^2}(\delta^{\alpha\beta}p^\gamma+\delta^{\alpha\gamma}p^\beta)\left(i-i\tilde{Z}_2-c_2\frac{1}{A^{(0)}}\tilde{Z}_2\right)$$
$$+\frac{1}{p^2}\delta^{\beta\gamma}p^\alpha\left(-i+i\tilde{Z}_2-c_1\frac{1}{A^{(0)}}\tilde{Z}_2\right)=0. \quad (4.38)$$

Hence,

$$c_1=-iA^{(0)}(\tilde{Z}_2^{-1}-1), \quad (4.39)$$
$$c_2=iA^{(0)}(\tilde{Z}_2^{-1}-1). \quad (4.40)$$

Thus the lowest-order Ward identity (4.25) permits us to define the constants c_1 and c_2.

Let us consider now the generalized Ward identity (4.19). After the transition to the mass shell with regard for (4.13), Eq. (4.19) takes the form

$$Z_1^{-1}(A^{(0)})^{-2}\mathrm{div}G^{(0)aa'}\mathrm{div}G^{(0)bb'}\mathrm{div}G^{(0)cc'}\Gamma^{(0)(3)}_{a'b'c'}=0. \quad (4.41)$$

Equation (A14) of Appendix A shows, however, that (4.41) is satisfied identically and thus it does not lead to relations for the renormalization constants.

Let us consider therefore the higher-order generalized Ward identity of the second set (4.26). With regard to Eqs. (4.28), (4.31), (4.34), (4.35), and (4.13), one may represent Eq. (4.26) as follows:

$$\frac{\alpha}{q^2k^2}(A^{(0)})^{-2}Z_1^{-1}(\delta^{\sigma_1\alpha_1}k^{\sigma_2}+\delta^{\sigma_2\alpha_1}k^{\sigma_1}-\delta^{\sigma_1\sigma_2}k^{\alpha_1})(\delta^{f_1\alpha_2}q^{f_2}+\delta^{f_2\alpha_2}q^{f_1}-\delta^{f_1f_2}q^{\alpha_2})G^{\beta\gamma;\phi\theta}(p,\alpha)\Gamma^{(0)(3)}_{\sigma_1\sigma_2;f_1f_2;\phi\theta}(k,q,p)$$
$$\underset{\text{pole}}{=}\frac{\alpha}{q^2k^2p^2}(A^{(0)})^{-1}\tilde{Z}_2^{\,2}\tilde{Z}_1^{-1}(\delta^{\gamma\theta}p^\beta+\delta^{\theta\beta}p^\gamma-\delta^{\beta\gamma}p^\theta)(\delta^{f_1\alpha_2}q^{f_2}+\delta^{f_2\alpha_2}q^{f_1}-\delta^{f_1f_2}q^{\alpha_2})\gamma^{(0)(3)}_{\alpha_1\theta|f_1f_2}(k,p|q)$$
$$+\mathcal{E}^{\alpha_1|\beta\gamma|\alpha_2}_{k^2\to 0,p^2\to 0,q^2\to 0;k+p+q=0}(k|p|q). \quad (4.42)$$

An expression has been obtained in Appendix B, Eq. (B16), for the asymptotic behavior of the function $\mathcal{E}^{\alpha_1|\beta\gamma|\alpha_2}$. Putting (B16) into the form symmetric with respect to k, p, q, and taking into account (4.35) and (4.13), one obtains

$$\mathcal{E}^{\alpha_1|\beta\gamma|\alpha_2}(k|p|q) \underset{\substack{k^2\to 0, p^2\to 0, q^2\to 0\\ k+p+q=0}}{=} -\frac{\alpha}{q^2k^2p^2}(A^{(0)})^{-2}\tilde{Z}_2{}^2\tilde{Z}_1{}^{-1}[\tilde{c}_2(\delta^{\gamma\theta}p^\beta+\delta^{\theta\beta}p^\gamma)+\tilde{c}_1\delta^{\beta\gamma}p^\theta]$$
$$\times(\delta^{f_1\alpha_2}q^{f_2}+\delta^{f_2\alpha_2}q^{f_1}-\delta^{f_1f_2}q^{\alpha_2})\gamma^{(0)(3)}_{\alpha_1\theta|f_1f_2}(k,p|q). \tag{4.43}$$

It was also proved in Appendix B, Eq. (B17), that

$$\tilde{c}_1=ic_1, \quad \tilde{c}_2=ic_2. \tag{4.44}$$

However, the constants c_1 and c_2 were obtained from the previous Ward identity: (4.39)–(4.40). Thus, one obtains with the aid of (4.44)

$$\tilde{c}_1=A^{(0)}(\tilde{Z}_2{}^{-1}-1), \quad \tilde{c}_2=-A^{(0)}(\tilde{Z}_2{}^{-1}-1). \tag{4.45}$$

It must be noted that the exact propagator $G^{\beta\gamma;\phi\theta}(p,\alpha)$, containing the constant α, still remains on the left-hand side of (4.42). Let us cancel α entering (4.42) as a multiplier and then pass to the limit $\alpha\to 0$. Then, for the resulting factor

$$G^{\beta\gamma;\phi\theta}_{\alpha\to 0, p^2\to 0}(p,\alpha),$$

one may use the renormalization condition (4.14). Substituting (4.45) into (4.43) and (4.42), one verifies that the addition of the term $\mathcal{E}^{\alpha_1|\beta\gamma|\alpha_2}$ in (4.42) is equivalent to multiplication of the right-hand side of (4.42) by the factor $\tilde{Z}_2{}^{-1}$.

Finally one obtains

$$\frac{1}{q^2k^2}(A^{(0)})^{-1}Z_1{}^{-1}Z_2(\delta^{\sigma_1\alpha_1}k^{\sigma_2}+\delta^{\sigma_2\alpha_1}k^{\sigma_1}-\delta^{\sigma_1\sigma_2}k^{\alpha_1})(\delta^{f_1\alpha_2}q^{f_2}+\delta^{f_2\alpha_2}q^{f_1}-\delta^{f_1f_2}q^{\alpha_2})G^{(0)\beta\gamma;\phi\theta}_{\alpha\to 0}(p,\alpha)\Gamma^{(0)(3)}_{\sigma_1\sigma_2;f_1f_2;\phi\theta}(k,q,p)$$
$$\underset{\text{pole}}{=}\frac{1}{q^2k^2p^2}(A^{(0)})^{-1}\tilde{Z}_1{}^{-1}\tilde{Z}_2(\delta^{\gamma\theta}p^\beta+\delta^{\theta\beta}p^\gamma-\delta^{\beta\gamma}p^\theta)(\delta^{f_1\alpha_2}q^{f_2}+\delta^{f_2\alpha_2}q^{f_1}-\delta^{f_1f_2}q^{\alpha_2})\gamma^{(0)(3)}_{\alpha_1\theta|f_1f_2}(k,p|q). \tag{4.46}$$

The corresponding relation of bare vertices has been obtained in Appendix A, Eq. (A18). Comparing (4.46) with (A18), one finds that

$$Z_1Z_2{}^{-1}=\tilde{Z}_1\tilde{Z}_2{}^{-1} \tag{4.47}$$

—the first Ward relation for renormalization constants.

In order to obtain the other relations, let us consider the whole first set of generalized Ward identities (4.20). The corresponding relations of bare vertices have been obtained in Appendix A, Eq. (A23). Comparing the exact identities (4.20) with their analogs for bare vertices (A23) one may obtain successively the infinite set of Ward relations for the threshold renormalization constants, determined by Eqs. (4.13) in the degenerate gauge:

$$(Z_1^{(n)})^{-1}=Z_1{}^{-n}Z_2{}^{n-1}, \quad n\geq 1. \tag{4.48}$$

It must be noted that the first set of generalized Ward identities at the threshold is exhausted by the relations (4.48). As to the second set, only the first two relations have been examined. It is obvious, however, that by virtue of Eqs. (4.12), (4.13), the higher-order generalized Ward identities for the fictitious-interaction Green's functions cannot produce new relations for the threshold renormalization constants.

Thus, Eqs. (4.47), (4.48) represent the complete set of Ward relations for the renormalization constants. Let us proceed with the consequences of these relations.

D. Lagrangian of the Renormalized Theory (Renormalizations at the Threshold)

The action functional of initial theory in the degenerate transverse gauge reads

$$S[\phi,\overline{C},C;A^{(0)},\kappa]$$
$$=\frac{(A^{(0)})^{-1}}{\kappa}\left[\frac{1}{2!}\frac{\delta^2S}{\delta\hat{g}^a\delta\hat{g}^b}\bigg|_{\hat{g}=\delta}\phi^a\phi^b+\sum_{n=1}^{\infty}(A^{(0)})^{-n}\frac{1}{(n+2)!}\frac{\delta^{(n+2)}S}{\delta\hat{g}^{a_1}\cdots\delta\hat{g}^{a_{n+2}}}\bigg|_{\hat{g}=\delta}\times\phi^{a_1}\cdots\phi^{a_{n+2}}\right]$$
$$+A^0\left[\int\overline{C}_\alpha\hat{Q}^{(0)\alpha\beta}C_\beta dx+(A^{(0)})^{-1}\int\overline{C}_\alpha\frac{\delta\hat{Q}^{\alpha\beta}}{\delta\hat{g}^a}\bigg|_{\hat{g}=\delta}C_\beta\phi^a dx\right]. \tag{4.49}$$

Let us perform the following change of the field variables in (4.49):

$$\phi^a_R=Z_2{}^{-1/2}\phi^a, \quad \overline{C}^\alpha_R=\tilde{Z}_2{}^{-1/2}\overline{C}^\alpha, \quad C^\alpha_R=\tilde{Z}_2{}^{-1/2}C^\alpha. \tag{4.50}$$

Then the terms

$$(A^{(0)})^{-n}\phi^{a_1}\cdots\phi^{a_{n+2}}=(A^{(0)})^{-n}Z_2{}^{(n+2)/2}\phi^{a_1}_R\cdots\phi^{a_{n+2}}_R, \quad n\geq 1$$

$$(A^{(0)})^{-1}\overline{C}^\alpha C^\beta\phi^a=(A^{(0)})^{-1}\tilde{Z}_2Z_2{}^{1/2}\overline{C}^\alpha_R C^\beta_R\phi^a_R,$$

by virtue of the Ward identities

$$1 = Z_1^{(n)}(Z_1^{-n} Z_2^{n-1}),$$
$$1 = \tilde{Z}_1 \tilde{Z}_2^{-1} Z_1^{-1} Z_2,$$

take the form

$$(A^{(0)})^{-n}\phi^{a_1}\cdots\phi^{a_{n+2}} = Z_1^{(n)}\phi_R^{a_1}\cdots\phi_R^{a_{n+2}} \times[(A^{(0)})^{-1}Z_1^{-1}Z_2^{3/2}]^n,$$
$$(A^{(0)})^{-1}\overline{C}^\alpha C^\beta \phi^a = \tilde{Z}_1 \overline{C}_R^\alpha C_R^\beta \phi_R^a[(A^{(0)})^{-1}Z_1^{-1}Z_2^{3/2}].$$

Now the constant $A^{(0)}$ plays the role of a coupling constant in this theory. Let us define the renormalized value of this constant as

$$(A_R^{(0)})^{-1} = (A^{(0)})^{-1}[Z_1^{-1}Z_2^{3/2}] \tag{4.51}$$

and choose the initial value $A^{(0)}$ in such a way that the renormalized constant $A_R^{(0)}$ should be finite.

Then we obtain

$$(A^{(0)})^{-n}\phi^{a_1}\cdots\phi^{a_{n+2}} = Z_1^{(n)}(A_R^{(0)})^{-n}\phi_R^{a_1}\cdots\phi_R^{a_{n+2}}, \tag{4.52}$$

$$(A^{(0)})^{-1}\overline{C}^\alpha C^\beta \phi^a = \tilde{Z}_1 (A_R^{(0)})^{-1}\overline{C}_R^\alpha C_R^\beta \phi_R^a, \tag{4.53}$$

and

$$\phi^a\phi^b = Z_2 \phi_R^a \phi_R^b, \tag{4.54}$$
$$\overline{C}^\alpha C^\beta = \tilde{Z}_2 \overline{C}_R^\alpha C_R^\beta. \tag{4.55}$$

Thus, as a result of Ward identities the "coupling constants" of all bare graviton and fictitious-particle interaction vertices are renormalized with the aid of a single quantity $Z_1^{-1}Z_2^{3/2}$.

Let us introduce the renormalized gravitational constant κ_R,

$$\frac{(A^{(0)})^{-1}}{\kappa} = \frac{(A_R^{(0)})^{-1}}{\kappa_R}, \quad \kappa_R = \kappa(Z_1^{-1}Z_2^{3/2}), \tag{4.56}$$

and choose the initial value of the constant κ in such a way that the renormalized constant κ_R should be finite.

Substituting Eqs. (4.51)–(4.56) into Eq. (4.49), we obtain the expression for the action functional in terms of renormalized quantities:

$$S[\phi, \overline{C}, C; A^{(0)}, \kappa] \equiv S_R[\phi_R, \overline{C}'_R, C'_R; A_R^{(0)}, \kappa_R]$$
$$= \frac{(A_R^{(0)})^{-1}}{\kappa_R}\left[Z_2 \frac{1}{2!}\frac{\delta^2 S}{\delta\hat{g}^a\delta\hat{g}^b}\bigg|_{\hat{g}=\delta}\phi_R^a\phi_R^b + \sum_{n=1}^{\infty} Z_1^{(n)}(A_R^{(0)})^{-n}\frac{1}{(n+2)!}\frac{\delta^{(n+2)}S}{\delta\hat{g}^{a_1}\cdots\delta\hat{g}^{a_{n+2}}}\bigg|_{\hat{g}=\delta}\phi_R^{a_1}\cdots\phi_R^{a_{n+2}}\right]$$
$$+A_R^{(0)}\left[\tilde{Z}_2\int dx \overline{C}_\alpha'^R \hat{Q}^{(0)\alpha\beta}C_\beta'^R + \tilde{Z}_1(A_R^{(0)})^{-1}\int \overline{C}_\alpha'^R \frac{\delta\hat{Q}^{\alpha\beta}}{\delta\hat{g}^a}\bigg|_{\hat{g}=\delta}C_\beta'^R\phi_R^a dx\right], \tag{4.57}$$

where

$$C'^R = \left(\frac{A^{(0)}}{A_R^{(0)}}\right)^{1/2} C^R,$$
$$\overline{C}'^R = \left(\frac{A^{(0)}}{A_R^{(0)}}\right)^{1/2} \overline{C}^R$$

are the actual renormalized fictitious fields.

The renormalization constant $\tilde{Z}_2$ may be eliminated from the expression for the S matrix by means of a trivial change of the local measure.[26] The constant $A_R^{(0)}$ may be set equal to unity.

Thus, as a consequence of generalized Ward identities, only two essentially divergent quantities, the wave-function ϕ renormalization constant Z_2 and the gravitational-constant renormalization constant $Z_1^{-1}Z_2^{3/2}$, survive in the theory.

It is seen from (4.57) that the generalized Ward identities permit us to transform identically the initial Lagrangian to such a form that after the infinite factors are included in the parameters of the theory, all the Green's functions possess the finite threshold asymptotic behavior. This does not mean, however, that divergences have been removed from off-threshold amplitudes.

E. Gauge Invariance of the Renormalized Theory

The renormalized gravitational field operator reads

$$\hat{g}_R = A_R^{(0)}\delta + \phi_R, \tag{4.58}$$

and the asymptotic behavior of all the Green's functions of the field $\hat{g}_R$ is finite by virtue of the finiteness of $A_R^{(0)}$ and the Green's functions of the field ϕ_R. It is a surprise, however, that the constant and the variable parts of the gravitational field operator are generally renormalized with the aid of different quantities:

$$\hat{g}_R = Z_2^{-1/2}[(Z_1 Z_2^{-1})A^{(0)}\delta + \phi]. \tag{4.59}$$

The renormalized gravitational field operator $\hat{g}_R$ may be expressed directly in terms of the unrenormalized operator $\hat{g}$ only, provided that one extra relation takes place:

$$Z_1 = Z_2. \tag{4.60}$$

However, the generalized Ward identities, which were analyzed here, do not yield this relation. As we shall show in Sec. V, relation (4.60) is true, but it is not a Ward identity. In other words, the gauge invariance of the theory does not require this relation, but some other properties do. These new properties of quantum gravity will be studied in Sec. V.

The main consequence of generalized Ward identities for renormalization constants is the statement that the counterterms, renormalizing the threshold asymptotic behavior of the Green's functions, do not break the gauge invariance of the theory.

This statement is not quite evident because if Eq. (4.60) is not considered, then by virtue of Eq. (4.59) the renormalized Lagrangian as a whole does not represent the invariant expression in terms of the field $\hat{g}_R$. The renormalized Lagrangian of gravitational field has the form of a curvature scalar up to the constant factor in terms of another metric field:

$$\begin{aligned}\tilde{g} &= \phi_R + Z_1^{-1} Z_2 A_R^{(0)} \delta \\ &= Z_2^{-1/2}(\phi + A^{(0)}\delta) \\ &= Z_2^{-1/2}\hat{g}\,. \end{aligned} \tag{4.61}$$

Contrary to (4.59), (4.61) is proportional to the unrenormalized $\hat{g}$ but is generally not finite. Nevertheless, the gauge invariance may be proved.

Indeed, the gauge invariance of the initial theory was true by virtue of the action $S[\phi, A^{(0)}]$ invariance in terms of the integration variables $\hat{g} = A^{(0)}\delta + \phi$. In the framework of renormalized theory, one may transform the differentials

$$\prod d\hat{g}_R = \prod d\phi_R = \prod d\tilde{g}\,,$$

and consider $\tilde{g}$ as the integration variables of renormalized theory. It remains to be noted that by virtue of (4.57),

$$S_R[\phi_R, A_R^{(0)}] \equiv S[\phi, A^{(0)}]\,,$$

and thus provided that S is invariant under the transformations of $\hat{g}$, S_R is also invariant under the same transformations of $\hat{g}$ and consequently it it is invariant under the transformations of $\tilde{g}$, because $\tilde{g}$ is proportional to $\hat{g}$ and the gauge transformations are homogeneous.

Finally, the action S_R is invariant under the transformations of the integration variables of renormalized theory. Q.E.D.

It must be emphasized that the variables $\tilde{g}$ and g_R, determined by the formulas (4.61) and (4.58), coincide only by virtue of the extra relation (4.60).

F. Renormalizations in Arbitrary Gauge and Finiteness of Longitudinal Parts of the Threshold Amplitudes

The Ward relations (4.47) and (4.48) have been derived in the degenerate gauge ($\alpha = 0$). However, exact Green's functions generally depend on the gauge, which results in their dependence on the parameter α. Consequently, the constants Z_1 and Z_2 also generally depend on α. Let us consider now Green's-function renormalization in the arbitrary (nondegenerate) gauge.

Renormalization constants $Z_1^{(n)}(\alpha)$, $\tilde{Z}_1(\alpha)$, $\tilde{Z}_2(\alpha)$ may be introduced with the aid of the same formulas as Eqs. (4.13), (4.14), while introduction of $Z_2(\alpha)$ needs a certain discussion. When $\alpha \neq 0$, the exact propagator (4.32) contains a free term proportional to α and a term with the function $G^{TT}(p^2, \alpha)$ also dependent on α. The bare function (4.30) reads

$$A^{(0)} G^{(0)TT}\left(p^2, \frac{\alpha}{A^{(0)}}\right) = (A^{(0)} + \tfrac{1}{2}\alpha)\frac{1}{p^2}\,.$$

Only the function $G^{TT}(p^2, \alpha)|_{p^2\to 0}$ is to be renormalized. The renormalization constant $Z_2(\alpha)$ must be introduced in such a way that the Ward relations in the case $\alpha \neq 0$ should remain in a form of (4.47) and (4.48).

Let us show that this requirement leads to the following definition of $Z_2(\alpha)$:

$$G^{TT}(p^2, \alpha)|_{p^2\to 0} = [A^{(0)} Z_2(\alpha) + \tfrac{1}{2}\alpha]\frac{1}{p^2}\,. \tag{4.62}$$

The requirement that the form of the Ward identities corresponding to $\alpha \neq 0$ should remain the same as in the degenerate case is a nontrivial one, because the propagator possesses the free term dependent on α. Indeed, let us return to Eq. (4.42). When $\alpha \neq 0$, there is the exact propagator with the coefficient of the form

$$(A^{(0)})^{-2} Z_1^{-1}(\alpha) G^{\beta\gamma;\mu\nu}(p, \alpha, A^{(0)})|_{p^2\to 0}$$

on the left-hand side of (4.42), while on the right-hand side there is a factor

$$(A^{(0)})^{-1} \tilde{Z}_2(\alpha) \tilde{Z}_1^{-1}(\alpha)\,.$$

Equation (4.42) holds for any value of the parameter $A^{(0)}$; in particular, it remains unchanged under the following substitution:

$$A^{(0)} \to A'^{(0)} = \frac{1}{Z_2(\alpha)}\,. \tag{4.63}$$

Now, using the definition (4.62) and the representation (4.32) one finds that

$$G^{\beta\gamma;\mu\nu}(p, \alpha, A'^{(0)}) = G^{(0)\beta\gamma;\mu\nu}(p, \alpha)\,.$$

With the aid of the analog of (4.42) for the bare functions, Eq. (A18), one may cancel the tensor factors in Eq. (4.42) with the value $A'^{(0)}$ (4.63). Then it follows that

$$(A'^{(0)})^{-2} Z_1^{-1}(\alpha) = (A'^{(0)})^{-1} \tilde{Z}_2(\alpha) \tilde{Z}_1^{-1}(\alpha)\,,$$

and thus

$$Z_2(\alpha)Z_1^{-1}(\alpha)=\tilde{Z}_2(\alpha)\tilde{Z}_1^{-1}(\alpha) \tag{4.64}$$

by virtue of (4.63).

Using the same trick, one may easily prove that provided the renormalization constant $Z_2(\alpha)$ is determined by Eq. (4.62), the Ward identity

$$[Z_1^{(n)}(\alpha)]^{-1}=Z_1^{-n}(\alpha)Z_2^{n-1}(\alpha) \tag{4.65}$$

holds.

It is evident that if the definition of $Z_2(\alpha)$ differs from Eq. (4.62), the Ward identities cannot have the form of (4.64) and (4.65).

In Sec. V the functions $Z_1(\alpha)$ and $Z_2(\alpha)$ will be found. It will be shown that not only the charge renormalization

$$Z_1^{-1}(\alpha)Z_2^{3/2}(\alpha)=Z_1^{-1}(0)Z_2^{3/2}(0),$$

but the constants Z_1 and Z_2 themselves do not depend on the gauge:

$$Z_1(\alpha)=Z_1(0), \quad Z_2(\alpha)=Z_2(0). \tag{4.66}$$

It means that in the expansion of the exact Green's functions in powers of α, only the terms of zeroth order become infinite at the threshold. That is why only the transverse part of the wave function in the action functional

$$S[g,\alpha]=S[g]-\frac{1}{2\alpha}\int\chi^\mu(x)\chi^\mu(x)dx$$

is to be renormalized:

$$\phi_R^{T\mu\nu}(x)=Z_2^{1/2}\phi^{T\mu\nu}(x),$$

which is equivalent to (4.50) in the case of the degenerate gauge. Then in the case of an arbitrary (nondegenerate) gauge, all considerations are valid which were carried out earlier for the case of degenerate gauge. As a result, the asymptotic behavior of the transverse parts of the Green's functions is finite in view of renormalization, while the asymptotic behavior of the longitudinal parts is finite by virtue of relations (4.66).

V. THE SCALE PROPERTIES OF THE THEORY AND THE RELATION $Z_1=Z_2$

In the previous sections we have analyzed the consequences of invariance of the theory of gravity under the gauge group of general coordinate transformations. It will be shown in the present section that the theory under consideration possesses additional properties of invariance connected with the conformal transformations of the Einstein Lagrangian, with the pecularities of symmetry breaking, and with existence of a fundamental constant having the dimension of length.

Considering first the generating functions of the theory as functionals of sources, we shall prove a theorem of invariance (homogeneity) which they satisfy and deduce a new set of relations for the Green's functions. Further, we shall consider the properties of the equations of motion for the generating G function as the functional of the average field and prove a second theorem of homogeneity. As a result, we shall obtain the second set of relations for the Green's functions. Both new sets of "scale" relations contain the derivatives of the Green's functions with respect to the parameters of the theory.

Combination of these relations gives the remarkable identities which state that the derivative of the Green's function with respect to the parameter $A^{(0)}$ is equivalent to the derivative of the corresponding generating function with respect to the "average field" argument, i.e., to the higher-order Green's function, with one of the momenta equal to zero. The threshold analysis of these identities will produce a fundamental identity for the renormalization constants:

$$Z_1=Z_2.$$

Further, using the fact that gravitational constant κ enters the theory only through the dimensionless space-time coordinates, we shall exclude the derivatives with respect to κ from the "scale" identities and obtain the general off-mass-shell relations for the Green's functions.

Next, we discuss the introduction of the "scale"-invariant regularization and possible anomalous singularities of the Green's functions. At the end of Sec. V the independence of renormalization constants on the gauge will be proved.

A. The First Homogeneity Theorem

Let us present the generating functional (1.15) as

$$Z\left[J\left|\frac{1}{\kappa},\frac{1}{A^{(0)}},\frac{1}{\alpha}\right.\right]=\frac{\int\exp\left\{\frac{i}{\kappa}\mathcal{S}_1[\hat{g}]+\mathcal{S}_2[\hat{g},\bar{C},C]+i\hat{g}^aJ_a\right\}\prod_a d\phi^a\prod_{x,\mu}d\bar{C}^\mu(x)\prod_{x,\nu}dC^\nu(x)}{\int\exp\left\{\frac{i}{\kappa}\mathcal{S}_1[\hat{g}]+\mathcal{S}_2[\hat{g},\bar{C},C]\right\}\prod_a d\phi^a\prod_{x,\mu}d\bar{C}^\mu(x)\prod_{x,\nu}dC^\nu(x)}, \tag{5.1}$$

$$\hat{g}^{\mu\nu}(x)=A^{(0)}\delta^{\mu\nu}+\phi^{\mu\nu}(x).$$

Here the parameters entering the theory are explicitly designated as arguments of the generating functional, and the following notation is introduced:

$$\mathcal{S}_1[\hat{g}] = \int \sqrt{-g}\, R\, dx - \frac{1}{\alpha}\int \chi^\mu \chi^\mu\, dx, \tag{5.2}$$

$$\mathcal{S}_2[\hat{g}, \bar{C}, C] = \int \bar{C}^\mu(x)\hat{Q}^{\mu\nu}C^\nu(x)\,dx + \delta(0)\int \ln M[\hat{g}]\,dx\,. \tag{5.3}$$

The expression of these functionals in terms of the field ϕ is given by the formula (4.49), where the parameter $A^{(0)}$ enters explicitly.

The integration in (5.1) is carried out with respect to functions vanishing at infinity:

$$\phi\,(t=\pm\infty) = 0\,. \tag{5.4}$$

In addition to this, according to the definition we have

$$\langle 0|\phi\,(x)|0\rangle = 0\,.$$

Let us perform the following change of variables in the integrand of (5.1):

$$\phi' = \frac{\phi}{A^{(0)}}\,, \quad \bar{C}' = (A^{(0)})^{1/2}\bar{C}, \quad C' = (A^{(0)})^{1/2}C. \tag{5.5}$$

Then with respect to (5.2) and (4.49) one obtains

$$\mathcal{S}_1[\phi, A^{(0)}, \alpha] = A^{(0)}\mathcal{S}_1\left[\phi', 1, \frac{\alpha}{A^{(0)}}\right]\,.$$

Also,

$$\bar{C}QC \equiv A^{(0)}\bar{C}Q^{(0)}C + \gamma^{(0)}\phi\bar{C}C$$
$$= \bar{C}'Q^{(0)}C' + \gamma^{(0)}\phi'\bar{C}'C'\,.$$

Further, the local measure transforms as

$$\prod_x M[\hat{g}(x)] = \prod_x M[A^{(0)}\delta + \phi\,(x)]$$
$$= C\prod_x M[\delta + \phi'(x)]\,,$$

where C is the constant factor which cancels by the identical factors in the normalizing integral of (5.1), as well as the Jacobian of the transformation (5.5).

The transformation of the term with sources reads

$$\hat{g}^a J_a = (A^{(0)}\delta + \phi^a)J_a$$
$$= (\delta + \phi'^a)(A^{(0)}J_a)\,.$$

It must be noted that the change of the variables (5.5) does not alter the asymptotic behavior of the functions at infinity (5.4).

Finally, as a result of such a change of integration variables, the generating functional returns to the initial form (5.1), but with the arguments altered, so that the following theorem is proved:

$$Z\left[J\left|\frac{1}{\kappa}, \frac{1}{A^{(0)}}, \frac{1}{\alpha}\right.\right] = Z\left[A^{(0)}J\left|\frac{A^{(0)}}{\kappa}, 1, \frac{A^{(0)}}{\alpha}\right.\right]\,. \tag{5.6}$$

Equation (5.6) states that the generating functional is a homogeneous function of order 0 with respect to all of its arguments.[27]

The invariance (homogeneity) property (5.6) of the theory explicitly states that the average field also plays the role of a coupling constant in this theory. It must also be noted that in the course of derivation of Eq. (5.6) the properties of conformal transformations of the Einstein Lagrangian were exploited, which yield, in particular, the result that the constant average field satisfies the classical equations of motion identically. The fact that the constant $A^{(0)}$ cannot be defined by the equations of motion is specific for the symmetry breaking in the quantum theory of gravity. Later on, we shall once more exploit this property when the quantum equations of motion are considered.

Such peculiarities of a theory with symmetry breaking mean that the average field enters the theory only through a boundary condition. Indeed, one may represent the generating functional (5.1) completely in terms of the field $\hat{g}$ as the integration variables. Then the integrand of (5.1) will not depend on the parameter $A^{(0)}$ at all, but the dependence of the integral on this parameter will remain the same as before, because the new variables of integration possess the property

$$\hat{g}^{\mu\nu}(t=\pm\infty) = A^{(0)}\delta^{\mu\nu} \tag{5.7}$$

and

$$\langle 0|\hat{g}^{\mu\nu}(x)|0\rangle = A^{(0)}\delta^{\mu\nu}\,.$$

Considering integration with respect to the field ϕ, we make explicit the dependence of (5.1) on the parameter $A^{(0)}$.[28]

Performing the differentiations of (5.6) with respect to sources,

$$\frac{\delta^n}{\delta i J^{a_1}\cdots\delta i J^{a_n}} = (A^{(0)})^n\frac{\delta^n}{\delta i A^{(0)}J^{a_1}\cdots\delta i A^{(0)}J^{a_n}}\,,$$

we shall easily obtain the homogeneity theorem concerning the higher-order generating functions as functionals of sources: The "average field" functional

$$\langle\hat{g}^a\rangle\left(J\left|\frac{1}{\kappa}, \frac{1}{A^{(0)}}, \frac{1}{\alpha}\right.\right) \overset{\text{def}}{=} \frac{1}{Z}\delta Z\left[J\left|\frac{1}{\kappa}, \frac{1}{A^{(0)}}, \frac{1}{\alpha}\right.\right]\Big/\delta i J^a \tag{5.8}$$

is a homogeneous function of order -1 of its four arguments. The generating function

$$G^{ab}\left(J\left|\frac{1}{\kappa}, \frac{1}{A^{(0)}}, \frac{1}{\alpha}\right.\right) \tag{5.9}$$

is a homogeneous function of order -2, and so on.

Let us now apply the Euler theorem[29] to these functions. We obtain

$$\frac{\delta\langle\hat{g}^a\rangle}{\delta J^b}J^b-\left(\kappa\frac{\partial}{\partial\kappa}+A^{(0)}\frac{\partial}{\partial A^{(0)}}+\alpha\frac{\partial}{\partial\alpha}\right)\langle\hat{g}^a\rangle=-\langle\hat{g}^a\rangle,$$

$$\frac{\delta G^{ab}}{\delta J^c}J^c-\left(\kappa\frac{\partial}{\partial\kappa}+A^{(0)}\frac{\partial}{\partial A^{(0)}}+\alpha\frac{\partial}{\partial\alpha}\right)G^{ab}=-2G^{ab},$$

and so on.

Let us now put the sources here equal to zero. Then the first infinite set of relations for the Green's functions follows as

$$\left(\kappa\frac{\partial}{\partial\kappa}+A^{(0)}\frac{\partial}{\partial A^{(0)}}+\alpha\frac{\partial}{\partial\alpha}\right)A^{(0)}=A^{(0)}, \tag{5.10}$$

$$\left(\kappa\frac{\partial}{\partial\kappa}+A^{(0)}\frac{\partial}{\partial A^{(0)}}+\alpha\frac{\partial}{\partial\alpha}\right)G^{ab}(A^{(0)};\kappa;\alpha)=2G^{ab}(A^{(0)};\kappa;\alpha), \tag{5.11}$$

$$\left(\kappa\frac{\partial}{\partial\kappa}+A^{(0)}\frac{\partial}{\partial A^{(0)}}+\alpha\frac{\partial}{\partial\alpha}\right)\langle 0|T\phi^a\phi^b\phi^c|0\rangle=3\langle 0|T\phi^a\phi^b\phi^c|0\rangle, \tag{5.12}$$

generally (see Fig. 6). The first relation (5.10) is satisfied identically, while the rest of them state that derivatives of n-point functions with respect to the parameters $A^{(0)}$, κ, and α are not independent.

From (5.12) and (5.11), one easily obtains

$$\left(\kappa\frac{\partial}{\partial\kappa}+A^{(0)}\frac{\partial}{\partial A^{(0)}}+\alpha\frac{\partial}{\partial\alpha}\right)\Gamma^{(3)}_{abc}=-3\Gamma^{(3)}_{abc},$$

$$\left(\kappa\frac{\partial}{\partial\kappa}+A^{(0)}\frac{\partial}{\partial A^{(0)}}+\alpha\frac{\partial}{\partial\alpha}\right)\Gamma^{(n+1)}_{a_1\cdots a_{n+1}}=-(n+1)\times\Gamma^{(n+1)}_{a_1\cdots a_{n+1}}, \tag{5.14}$$

and from (5.11) it follows that

$$\left(\kappa\frac{\partial}{\partial\kappa}+A^{(0)}\frac{\partial}{\partial A^{(0)}}+\alpha\frac{\partial}{\partial\alpha}\right)G_{ab}{}^{-1}(A^{(0)};\kappa;\alpha)=-2G_{ab}{}^{-1}(A^{(0)};\kappa;\alpha), \tag{5.15}$$

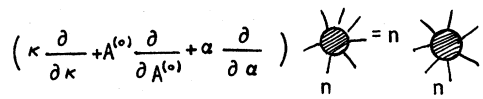
which will be of further use.

B. The Second Homogeneity Theorem

The generating $\mathcal{G}$ function as the functional of the average field,

$$G^{ab}(\langle g\rangle|A^{(0)},\kappa,\alpha), \tag{5.16}$$

is of the most interest because it generates directly the vertex Green's functions according to the formula (4.6). The generating G function (5.16) may be obtained from (5.9) by means of inverting the function (5.8),



(5.13)

FIG. 6. The first set of "scale" identities for the Green's functions.

$$J^a=J^a\left(\langle g\rangle\left|\frac{1}{\kappa},\frac{1}{A^{(0)}},\frac{1}{\alpha}\right.\right), \tag{5.17}$$

and substituting the expression obtained into (5.9). Hence, one may see that the dependences of the functions (5.9) and (5.16) on the parameters are generally quite different from each other.

In order to analyze the generating function (5.16), let us consider the set of quantum equations of motion obtained in Sec. III. Let us denote

$$\frac{\delta\mathcal{S}_1[\hat{g}]}{\delta\hat{g}^a}=T_1^a[\hat{g}]=T_1^{\mu\nu}(\hat{g}(x)),$$

$$\frac{1}{i}\frac{\delta\mathcal{S}_2[\hat{g}]}{\delta\hat{g}^a}=T_2^a[\hat{g}]=T_2^{\mu\nu}(\hat{g}(x)),$$

where

$$\mathcal{S}_2[\hat{g}]=\mathrm{Tr}\ln\hat{Q}[\hat{g}]\hat{Q}^{(0)-1}+\delta(0)\int\ln M[\hat{g}]dx. \tag{5.18}$$

It must be emphasized that the functionals T_1 and T_2, being defined in terms of the field $\hat{g}$, depend on only one argument.

Equations (3.41) for the generating functional take the form

$$\left\{\frac{1}{\kappa}T_1^a\left[\frac{\delta}{i\delta J}\right]+T_2^a\left[\frac{\delta}{i\delta J}\right]+J^a\right\}Z[J]=0. \tag{5.19}$$

Multiplying Eq. (5.19) by $Z^{-1}[J]$ from the left, we obtain

$$-J^a[\langle g\rangle]=\frac{1}{\kappa}T_1^a\left[\langle g\rangle^b+G^{bc}(\langle g\rangle)\frac{\delta}{\delta\langle g\rangle^c}\right]1+T_2^a\left[\langle g\rangle^b+G^{bc}(\langle g\rangle)\frac{\delta}{\delta\langle g\rangle^c}\right]1. \tag{5.20}$$

At last, let us perform one more variation with respect to the average field. Finally we get the equation

$$-\delta^{af}=G^{fe}(\langle g\rangle)\frac{\delta}{\delta\langle g\rangle^e}\left\{\frac{1}{\kappa}T_1^a\left[\langle g\rangle^b+G^{bc}(\langle g\rangle)\frac{\delta}{\delta\langle g\rangle^c}\right]+T_2^a\left[\langle g\rangle^b+G^{bc}(\langle g\rangle)\frac{\delta}{\delta\langle g\rangle^c}\right]\right\}1 \tag{5.21}$$

for the function

$$G^{ab}(\langle g\rangle | A^{(0)}, \kappa, \alpha)$$

of interest.

The boundary conditions read

$$\langle g^{\mu\nu}(x)\rangle(J)|_{J=0} = A^{(0)}\delta^{\mu\nu}, \tag{5.22}$$

$$G^{ab}(J)|_{J=0} = G^{ab}(\langle g\rangle)|_{\langle g^{\mu\nu}\rangle = A^{(0)}\delta^{\mu\nu}} . \tag{5.23}$$

It must be noted first of all that the parameter $A^{(0)}$ does not appear in the equation (5.21) for the G function. It appears only in the boundary conditions (5.22) and (5.23), which reflects the peculiarities of symmetry breaking, mentioned earlier. Further, Eqs. (5.22), (5.23) serve as the boundary conditions to the equations for the functions of sources, but not to the Eq. (5.21), where the average field is the argument of the G function, so that these conditions are applied to the function $G(\langle g\rangle)$ only as the conditions of transition to the vertex Green's functions, which we are finally interested in. Thus we have

$$\frac{\partial}{\partial A^{(0)}} G^{ab}(\langle g\rangle | A^{(0)}, \kappa, \alpha) = 0 . \tag{5.24}$$

It follows from (5.24) that the G function as the function of sources (5.9) and the Green's functions depend parametrically on $A^{(0)}$, as a consequence of the boundary condition (5.23) only.

We have proved earlier the homogeneity theorem for the generating functions as the functionals of sources. Let us show that the G function as the functional of the average field possesses a similar property.

Let us find how the functionals $T_1[\zeta]$ and $T_2[\zeta]$ transform under multiplication of the argument ζ by a constant.

From the known conformal-transformation properties of the curvature, one finds

$$\mathcal{S}_1[A\zeta;\alpha] = A\mathcal{S}_1\left[\zeta;\frac{\alpha}{A}\right] .$$

Hence

$$T_1[A\zeta;\alpha] = T_1\left[\zeta;\frac{\alpha}{A}\right] . \tag{5.25}$$

Now, as the operator $\hat{Q}[\zeta]$ and the local measure $M[\zeta]$ are homogeneous in ζ and the functional $\mathcal{S}_2[\zeta]$ has the form of Tr ln, one finds

$$T_2[A\zeta] = \frac{1}{A} T_2[\zeta] . \tag{5.26}$$

Using Eqs. (5.25), (5.26) in Eq. (5.21) and omitting the indices, one obtains

$$-\delta = A^2 G(\langle g\rangle | \kappa;\alpha)\frac{\delta}{\delta A\langle g\rangle}\left\{\frac{1}{\kappa A}T_1\left[A\langle g\rangle + A^2G(\langle g\rangle|\kappa;\alpha)\frac{\delta}{\delta A\langle g\rangle}\middle| A\alpha\right] + T_2\left[A\langle g\rangle + A^2 G(\langle g\rangle|\kappa;\alpha)\frac{\delta}{\delta A\langle g\rangle}\right]\right\} 1 . \tag{5.27}$$

Let us now denote

$$\langle\bar{g}\rangle = A\langle g\rangle, \quad \bar{\kappa} = A\kappa, \quad \bar{\alpha} = A\alpha,$$

$$\bar{G}(\langle\bar{g}\rangle|\bar{\kappa};\bar{\alpha}) = A^2G(\langle g\rangle|\kappa, \alpha) .$$

One may notice that Eq. (5.27) for the function $\bar{G}$ has just the form of Eq. (5.21) for the function G written down in terms of the new arguments. Thus we conclude

$$\bar{G}(\langle\bar{g}\rangle|\bar{\kappa};\bar{\alpha}) = G(\langle\bar{g}\rangle|\bar{\kappa};\bar{\alpha}),$$

or

$$G(A\langle g\rangle| A\kappa; A\alpha) = A^2G(\langle g\rangle|\kappa;\alpha), \tag{5.28}$$

which holds for arbitrary A.

Equation (5.28) represents the second homogeneity theorem, concerning the generating function as a functional of the average field. Note that by virtue of (5.24), for the generating function (5.16) the number of arguments is less by one in comparison with the function (5.9).

Since, by virtue of Eq. (5.28), the G function is homogeneous of the order 2, the function

$$G_{ab}^{-1}(\langle g\rangle|\kappa;\alpha)$$

is homogeneous of the order −2.

Let us apply the Euler theorem to this function. Then one finds that

$$\frac{\delta G_{ab}^{-1}(\langle g\rangle|\kappa;\alpha)}{\delta\langle g\rangle^c}\langle g\rangle^c + \frac{\partial G_{ab}^{-1}(\langle g\rangle|\kappa;\alpha)}{\partial\kappa}\kappa + \frac{\partial G_{ab}^{-1}(\langle g\rangle|\kappa;\alpha)}{\partial\alpha}\alpha = -2G_{ab}^{-1}(\langle g\rangle|\kappa;\alpha) . \tag{5.29}$$

Performing functional differentiations of (5.29) with respect to $\langle g\rangle$ and then putting $\langle g^{\mu\nu}\rangle = A^{(0)}\delta^{\mu\nu}$ we obtain the second infinite set of relations for the Green's functions, independent of Ward identities.

One obtains consecutively in the momentum space

$$-i\Gamma^{(3)}_{\alpha\beta;\gamma\sigma;\mu\nu}(p,-p,0)A^{(0)}\delta^{\mu\nu} + \kappa\frac{\partial}{\partial\kappa}G_{\alpha\beta;\gamma\sigma}^{-1}(p|A^{(0)};\kappa;\alpha) + \alpha\frac{\partial}{\partial\alpha}G_{\alpha\beta;\gamma\sigma}^{-1}(p|A^{(0)};\kappa;\alpha) = -2G_{\alpha\beta;\gamma\sigma}^{-1}(p|A^{(0)};\kappa;\alpha), \tag{5.30}$$

$$i\Gamma^{(n+2)}_{\alpha_1\beta_1,\ldots,\alpha_{n+1}\beta_{n+1},\mu\nu}\Big(p_1,\ldots,p_n,-\sum_{i=1}^{n}p_i,0\Big)\delta^{\mu\nu}A^{(0)}+\Big(\kappa\frac{\partial}{\partial\kappa}+\alpha\frac{\partial}{\partial\alpha}\Big)\Gamma^{(n+1)}_{\alpha_1\beta_1,\ldots,\alpha_{n+1}\beta_{n+1}}\Big(p_1,\ldots,p_n-\sum_{i=1}^{n}p_i\Big)$$
$$=-(n+1)\Gamma^{(n+1)}_{\alpha_1\beta_1,\ldots,\alpha_{n+1}\beta_{n+1}}\Big(p_1,\ldots,p_n,-\sum_{i=1}^{n}p_i\Big),$$
$$n\geq 2. \quad (5.31)$$

In the case in which the gauge is degenerate, the terms $\alpha[\partial/(\partial\alpha)]$ may be omitted in the relations (5.31)

C. The Relation $Z_1 = Z_2$ in an Arbitrary Gauge: Scale Relations of the Bare Vertices

Comparing two sets of "scale" identities, (5.10)–(5.15) and (5.29)–(5.31), we obtain the relations mentioned in the beginning of this section:

$$\frac{\partial}{\partial A^{(0)}}G_{\alpha\beta;\gamma\sigma}{}^{-1}(p|A^{(0)};\kappa;\alpha)=-i\Gamma^{(3)}_{\alpha\beta;\gamma\sigma;\mu\nu}(p,-p,0)\delta^{\mu\nu}, \quad (5.32)$$

$$\frac{\partial}{\partial A^{(0)}}\Gamma^{(3)}_{\alpha\beta;\gamma\sigma;\phi\theta}(p,k,-p-k|A^{(0)};\kappa;\alpha)=i\Gamma^{(4)}_{\alpha\beta;\gamma\sigma;\phi\theta;\mu\nu}(p,k,-p-k,0)\delta^{\mu\nu}, \quad (5.33)$$

$$\frac{\partial}{\partial A^{(0)}}\Gamma^{(4)}_{\alpha\beta;\gamma\sigma;\phi\theta;\rho\zeta}(p,k,q,-p-k-q|A^{(0)};\kappa;\alpha)=i\Gamma^{(5)}_{\alpha\beta;\gamma\sigma;\phi\theta;\rho\zeta;\mu\nu}(p,k,q,-p-k-q,0)\delta^{\mu\nu}, \quad (5.34)$$

and so on.

Let us now consider the mass-shell asymptotic behavior in the relations obtained with the aid of the formulas of Sec. IV:

$$G^{\mu\nu;\alpha\beta}(p|A^{(0)};\kappa;\alpha)\xrightarrow[p^2\to 0]{}\kappa\alpha G_2^{(0)\mu\nu;\alpha\beta}(p)+\kappa[A^{(0)}Z_2(A^{(0)};\kappa;\alpha)+\tfrac{1}{2}\alpha]G_1^{(0)\mu\nu;\alpha\beta}(p,\ \alpha=0), \quad (5.35)$$

$$\Gamma^{(3)}_{\mu\nu;\alpha\beta;\gamma\sigma}(p,k,-p-k)\xrightarrow[\substack{p^2\to 0,\,k^2\to 0,\\ (p+k)^2\to 0}]{}\frac{1}{\kappa(A^{(0)})^2}Z_1{}^{-1}(A^{(0)};\kappa;\alpha)\,\Gamma^{(0)(3)}_{\mu\nu;\alpha\beta;\gamma\sigma}(p,k,-p-k). \quad (5.36)$$

Further, at the threshold,

$$\Gamma^{(4)}(p,k,q,-p-k-q)\to\frac{1}{\kappa(A^{(0)})^3}(Z_1^{(2)}(A^{(0)};\kappa;\alpha))^{-1}\,\Gamma^{(0)(4)}(p,k,q,-p-k-q), \quad (5.37)$$

and so on.

Only Eq. (5.35) needs some explanations. As was discussed above, the exact propagator (4.32) possesses a free term proportional to α, which is denoted here as

$$\kappa\alpha G_2^{(0)\mu\nu;\alpha\beta},$$

and a completely transverse term, which is denoted as

$$\kappa[A^{(0)}Z_2+\tfrac{1}{2}\alpha]G_1^{(0)\mu\nu;\alpha\beta}.$$

Besides this, it was shown in Sec. IV that the renormalization constant $Z_2(\alpha)$ must be introduced as in Eq. (5.35) in order that the Ward identities should remain in a form of (4.64) and (4.65). In order to obtain the relations for renormalization constants from the threshold asymptotic behavior of Eqs. (5.32)–(5.34), we shall need one extra set of relations among the bare vertices, different from the gauge relations found in Appendix A.

The gravitational field Lagrangian possesses a "homogeneity" property:

$$S[A\hat{g}]=AS[\hat{g}],\quad \alpha=\infty \quad (5.38)$$

$$\frac{\delta^n S[A\hat{g}]}{\delta A\hat{g}^{a_1}\cdots\delta A\hat{g}^{a_n}}=\frac{\delta^n S[\hat{g}]}{\delta\hat{g}^{a_1}\cdots\delta\hat{g}^{a_n}}A^{-n+1}. \quad (5.39)$$

Application of Euler's theorem gives

$$\frac{\delta^{n+1}S[\hat{g}]}{\delta\hat{g}^{a_1}\cdots\delta\hat{g}^{a_n}\delta\hat{g}^{a_{n+1}}}\hat{g}^{a_{n+1}}=(-n+1)\frac{\delta^n S[\hat{g}]}{\delta\hat{g}^{a_1}\cdots\delta\hat{g}^{a_n}}. \quad (5.40)$$

Equations (5.40) taken at

$$\hat{g}^{\mu\nu}=\delta^{\mu\nu}$$

represent an infinite set of new ("scale") relations among the bare vertices.

One of the momenta is always equal to zero on the left-hand side of Eq. (5.40) taken at $\hat{g}=\delta$; however, a generalization to the case of nonzero momenta may also be given. For example one may prove that

$$\frac{1}{i}\delta^{\mu\nu}\Gamma^{(0)(3)}_{\mu\nu;\alpha\beta;\rho\phi}(q,p,k)$$
$$=\tfrac{1}{2}[G^{(0)}_{\alpha\beta;\rho\phi}(q)]^{-1}-\tfrac{1}{2}[G^{(0)}_{\alpha\beta;\rho\phi}(p)]^{-1}-\tfrac{1}{2}[G^{(0)}_{\alpha\beta;\rho\phi}(k)]^{-1}$$
$$+\tfrac{1}{4}[\delta^{\alpha\beta}(q_\rho p_\phi)+q_\phi p_\rho)+\delta^{\rho\phi}(q_\alpha k_\beta+q_\beta k_\alpha)$$
$$+(\delta_{\alpha\rho}\delta_{\beta\phi}+\delta_{\alpha\phi}\delta_{\beta\rho})q^2]$$
$$+\tfrac{1}{8}[\delta^{\alpha}_{\phi}(p_\rho k_\beta-k_\beta p_\rho)+\delta^{\alpha}_{\rho}(p_\phi k_\beta-k_\beta p_\phi)$$
$$+\delta^{\beta}_{\phi}(p_\rho k_\alpha-k_\alpha p_\rho)+\delta^{\beta}_{\rho}(p_\phi k_\alpha-k_\alpha p_\phi)]\,,$$
$$k+p+q=0,\quad \alpha=\infty\,. \tag{5.41}$$

The relations (5.40), (5.41) may be very efficiently used in perturbation theory calculations, because in some cases they give a possibility of avoiding the highly complicated explicit expressions for bare vertices.

When $q=0$, Eq. (5.41) states that

$$i\Gamma^{(0)(3)}_{\alpha\beta;\gamma\sigma;\mu\nu}(p,-p,0)\delta^{\mu\nu}=[G^{(0)}_{\alpha\beta;\gamma\sigma}(p,\alpha)]^{-1}|_{\alpha=\infty}\,. \tag{5.42}$$

When $n\geq 3$, we obtain from (5.40) the relations for the higher-order bare vertices:

$$i\Gamma^{(0)(n+2)}_{\alpha_1\beta_1,\ldots,\alpha_{n+1}\beta_{n+1},\mu\nu}\left(p_1,\ldots,p_n,-\sum_{i=1}^{n}p_i,0\right)\delta^{\mu\nu}$$
$$=-n\Gamma^{(0)(n+1)}_{\alpha_1\beta_1,\ldots,\alpha_{n+1}\beta_{n+1}}\left(p_1,\ldots,p_n,-\sum_{i=1}^{n}p_i\right),$$
$$n\geq 2\,. \tag{5.43}$$

Using relations (5.42), (5.43) to cancel the tensor factors in the threshold asymptotic behavior of relations (5.32)–(5.34), we obtain the new infinite set of identities for renormalization constants $Z_2(A^{(0)},\kappa)$, $Z_1^{(n)}(A^{(0)},\kappa)$ in an arbitrary gauge:

$$\frac{\partial}{\partial A^{(0)}}\left(\frac{1}{A^{(0)}}Z_2^{-1}\right)=-\frac{1}{(A^{(0)})^2}Z_1^{-1} \tag{5.44}$$

$$\frac{\partial}{\partial A^{(0)}}\left(\frac{1}{(A^{(0)})^{n+1}}(Z_1^{(n)})^{-1}\right)=-\frac{n+1}{(A^{(0)})^{n+2}}(Z_1^{(n+1)})^{-1}\,,$$
$$n\geq 1\,. \tag{5.45}$$

Next, Ward relations

$$(Z_1^{(n)})^{-1}=Z_1^{-n}Z_2^{n-1}$$

must be used. It then follows that

$$\frac{\partial}{\partial A^{(0)}}\left(\frac{1}{A^{(0)}}Z_2^{-1}\right)=-\frac{1}{(A^{(0)})^2}Z_1^{-1}\,, \tag{5.46}$$

$$\frac{\partial}{\partial A^{(0)}}\left(\frac{1}{(A^{(0)})^2}Z_1^{-1}\right)=-\frac{2}{(A^{(0)})^3}Z_1^{-2}Z_2\,, \tag{5.47}$$

$$\frac{\partial}{\partial A^{(0)}}\left(\frac{1}{(A^{(0)})^3}Z_1^{-2}Z_2\right)=-\frac{3}{(A^{(0)})^4}Z_1^{-3}Z_2^{2}\,, \tag{5.48}$$

$\ldots\,.$

Only the first two equations of this infinite set are independent. The third one is consistent with the previous two. As a condition of consistency, one obtains from (5.46)–(5.48), after some simple arithmetic, a new fundamental relation for renormalization constants:

$$Z_1=Z_2 \tag{5.49}$$

in the arbitrary gauge. The role of this relation was discussed at the end of Sec. IV.

One can show that (5.49) is the only consistency condition following from the whole infinite set of relations (5.46)–(5.48).

It must be emphasized that, just as the Ward identities (4.64) and (4.65) were obtained from the requirement that the renormalized theory should preserve the property of gauge invariance of the initial theory, in an analogous manner relation (5.49) expresses the requirement that the renormalized theory should preserve other properties of invariance of the initial theory: the homogeneity properties.

Solving Eqs. (5.46)–(5.49) for renormalization constants as functions of $A^{(0)}$ one easily finds that

$$\frac{\partial Z_1}{\partial A^{(0)}}=\frac{\partial Z_2}{\partial A^{(0)}}=0\,. \tag{5.50}$$

D. The Dependence of the Green's Functions on κ and the General Off-Mass-Shell Relations

The homogeneity properties of the theory which resulted in Eqs. (5.49), (5.50) do not embrace the whole variety of "scale" properties of the theory.

There is one more, almost obvious, property, which is the consequence of the essential feature of quantum gravity: the existence of a fundamental constant $\sqrt{\kappa}$, having the dimension of length ($\hbar=1$) (Planck's length[30]). This property states that κ enters the dimensionless generating G function only through its space-time arguments. This statement becomes apparent if one works in units for which $\hbar=c=\kappa=1$. It may also be easily proved using Eq. (5.21) for the generating G function:

$$G^{ab}(\langle g\rangle|\kappa)=G(x,y\|\langle g\rangle;\kappa) \tag{5.51}$$

(the discrete indices are omitted).

Let us denote

$$x'=\frac{x}{\sqrt{\kappa}}\,,\quad y'=\frac{y}{\sqrt{\kappa}}$$

and define a new function G':

$$G'(x',y'\|\langle g\rangle(z'))\overset{\text{def}}{=}G(x,y\|\langle g\rangle(z);\kappa). \tag{5.52}$$

The equation for the new function G' may be shown to have the same form as Eq. (5.21) for the function (5.51) of the new arguments and the value

$$\kappa=1.$$

So it follows that

$$G'(x', y' \| \langle g\rangle (z')) = G(x', y' \| \langle g\rangle (z'); 1) . \quad (5.53)$$

Comparing Eq. (5.53) with the definition (5.52) of the function G', one finds

$$G(x, y \| \langle g\rangle (z);\kappa) = G\left(\frac{x}{\sqrt{\kappa}}, \frac{y}{\sqrt{\kappa}} \Big\| \langle g\rangle \left(\frac{z}{\sqrt{\kappa}}\right); 1\right), \quad (5.54)$$

which gives for the Green's functions in momentum space

$$G(p|\kappa) = \kappa^2 G(\sqrt{\kappa}\, p|1), \quad (5.55)$$

$$\Gamma^{(n+2)}(p_1, \ldots, p_{n+2}|\kappa) = \kappa^{-2}\Gamma^{(n+2)}(\sqrt{\kappa}\, p_1, \ldots, \sqrt{\kappa}\, p_{n+2}|1),$$

$$\sum p_i = 0 . \quad (5.56)$$

One can see that Eqs. (5.55), (5.56) again have the form of "homogeneity property," this time concerning the Green's functions as the functions of momenta and $1/\sqrt{\kappa}$. Application of the Euler theorem immediately gives

$$p^{\sigma}\frac{\partial}{\partial p^{\sigma}} G^{\mu\nu;\,\alpha\beta}(p) - 2\kappa \frac{\partial}{\partial \kappa} G^{\mu\nu;\,\alpha\beta}(p) = -4G^{\mu\nu;\,\alpha\beta}(p), \quad (5.57)$$

$$-p^{\sigma}\frac{\partial}{\partial p^{\sigma}} G_{\mu\nu;\alpha\beta}{}^{-1}(p) + 2\kappa \frac{\partial}{\partial \kappa} G_{\mu\nu;\,\alpha\beta}{}^{-1}(p) = -4G_{\mu\nu;\,\alpha\beta}{}^{-1}(p), \quad (5.58)$$

$$\sum_{i=1}^{n+2} p_i^{\sigma}\frac{\partial}{\partial p_i^{\sigma}} \Gamma^{(n+2)}_{\alpha_1\beta_1,\ldots,\alpha_{n+2}\beta_{n+2}}(p_1, \ldots, p_{n+2}) - 2\kappa\frac{\partial}{\partial\kappa}\Gamma^{(n+2)}_{\alpha_1\beta_1,\ldots,\alpha_{n+2}\beta_{n+2}}(p_1, \ldots, p_{n+2}) = 4\Gamma^{(n+2)}_{\alpha_1\beta_1,\ldots,\alpha_{n+2}\beta_{n+2}}(p_1, \ldots, p_{n+2}),$$

$$\sum_{i=1}^{n+2} p_i = 0 . \quad (5.59)$$

Let us now consider the second set of scale identities (5.30)–(5.31). The derivatives with respect to κ entering these identities may be excluded now with the aid of Eqs. (5.57)–(5.59). Then one obtains the set of off-mass-shell relations for the Green's functions:

$$p^{\sigma}\frac{\partial}{\partial p^{\sigma}} G_{\alpha\beta;\rho\gamma}{}^{-1}(p) + 2\alpha\frac{\partial}{\partial\alpha} G_{\alpha\beta;\rho\gamma}{}^{-1}(p) = 2i\Gamma^{(3)}_{\alpha\beta;\rho\gamma;\mu\nu}(p, -p, 0)A^{(0)}\delta^{\mu\nu}, \quad (5.60)$$

$$-p^{\sigma}\frac{\partial}{\partial p^{\sigma}}\Gamma^{(3)}_{\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3}(p, k, -p-k) - k^{\sigma}\frac{\partial}{\partial k^{\sigma}}\Gamma^{(3)}_{\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3}(p, k, -p-k) - 2\Gamma^{(3)}_{\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3}(p, k, -p-k) = 2i\Gamma^{(4)}_{\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3,\mu\nu}(p, k, -p-k, 0)A^{(0)}\delta^{\mu\nu},$$

$$\alpha = 0 \quad (5.61)$$

$$-\sum_{i=1}^{n+1} p_i^{\sigma}\frac{\partial}{\partial p_i^{\sigma}}\Gamma^{(n+2)}_{\alpha_1\beta_1,\ldots,\alpha_{n+2}\beta_{n+2}}\left(p_1, \ldots, p_{n+1}, -\sum_{j=1}^{n+1} p_j\right) - 2n\Gamma^{(n+2)}_{\alpha_1\beta_1,\ldots,\alpha_{n+2}\beta_{n+2}}\left(p_1, \ldots, p_{n+1}, -\sum_{j=1}^{n+1} p_j\right) = 2i\Gamma^{(n+3)}_{\alpha_1\beta_1,\ldots,\alpha_{n+2}\beta_{n+2},\mu\nu}\left(p_1, \ldots, p_{n+1}, -\sum_{j=1}^{n+1} p_j, 0\right)A^{(0)}\delta^{\mu\nu},$$

$$\alpha = 0. \quad (5.62)$$

Equations (5.60)–(5.62), obeyed by the Green's functions, are the "scale-invariance" relations which may be regarded on a level with the generalized Ward identities as the exact consequences of the dynamical equations.

As to the renormalization constants Z_1 and Z_2, the threshold analysis of Eqs. (5.55), (5.56) gives the evident result that

$$\frac{\partial}{\partial\kappa} Z_2(\kappa, \alpha) = \frac{\partial}{\partial\kappa} Z_1(\kappa, \alpha) = 0 . \quad (5.63)$$

E. Scale-Invariant Regularization

Equations (5.50), (5.63) state that threshold renormalization constants do not depend either on $A^{(0)}$ or on κ. This means that radiation corrections to the threshold asymptotic behavior of the Green's functions are either absent or coupling-constant-independent. Both of these possibilities seem paradoxical; nevertheless we shall show that the conclusion is correct and the details depend on the way of dealing with the divergent integrals.

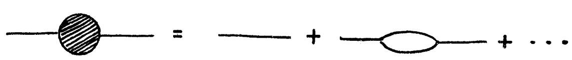


FIG. 7. Graphical representation of the integrals of perturbation theory.

In order to analyze what happens one must first make sure that the integrals of perturbation theory formally possess the same invariance properties as the exact functions do, according to the considerations of the present section. Let us consider, for example, the perturbation expansion of the graviton propagator (see Fig. 7). Performing formally the change of integration variables in the integrals in momentum space

$$k \to \left(\frac{\kappa}{A^{(0)}}\right)^{1/2} k\ ,$$

one obtains, provided that the boundaries of integration region are infinite,

$$G(p|\kappa;A^{(0)}) = \kappa^2 G\left(\left(\frac{\kappa}{A^{(0)}}\right)^{1/2} p \,\middle|\, 1;1\right)$$

in any order of perturbation theory, just as is required by Eq. (5.55).

However, the case is somewhat more complicated in view of ultraviolet divergences. The calculation of the integrals of the type presented by Fig. 7 needs the introduction of a regularization. For instance, the introduction of a cutoff in the momentum space would immediately break our present conclusions, for the Green's functions would depend on the additional dimensional parameter. The latter would enter the Euler theorem as well as invalidate Eq. (5.55) and all the consequences. However, to secure the invariance of the theory, the regularization must also be introduced in an invariant way. In the present case, except for gauge invariance, regularization must also secure the homogeneity of the theory. While introduction of a gauge-invariant regularization presented a certain problem,[7,31] the introduction of a "scale"-invariant regularization is straightforward: Perform a formal change of integration variables

$$k \to \left(\frac{\kappa}{A^{(0)}}\right)^{1/2} k$$

in the divergent integrals and introduce the upper limit Λ or any other type of regularization in the integrals with respect to dimensionless variables $(\kappa/A^{(0)})^{1/2}k$. It is evident that the invariance requirements of the type (5.55) are then satisfied manifestly and the dependence of the Green's functions on the dimensionless parameter Λ does not affect the "scale" properties of the Green's functions in any case.

It must be emphasized that the choice of dimensionless units does not yet guarantee invariant regularization because Λ would become dimensional again when the gravitational constant κ is reintroduced. Another necessary step consists in proclaiming Λ (which is numerical in the celebrated system of units) purely numerical. Then changes of system of units will touch the momenta, but not Λ.

Reference 7 gives an example of a regularization which is both gauge-invariant and scale-invariant. Thus, provided that the invariant regularization is introduced, all the conclusions of the present section are also true when the ultraviolet divergences are taken into account, but perturbation theory is no longer an expansion in the "coupling constant" $(\kappa/A^{(0)})^{1/2}$. The quantity $(\kappa/A^{(0)})^{1/2}p$ serves as the true external parameter for the scattering amplitudes. Working in absolute units, one makes it obvious that all orders of perturbation theory become important at the same time because there is no small pure number in the theory to distinguish the various orders.

Perturbation theory acquires its literal meaning only in the framework of such an invariant regularization technique which annihilates all the power divergences in terms of the upper limit Λ, so that only the logarithmic divergences remain. The regularization of Ref. 7 possesses this property too. The perturbation expansion regularized in such a manner becomes at the same time an expansion in the powers of external momenta, each distinct order being of a unique power in momenta. In this case there will be no radiation corrections to Z multipliers: $Z_1 = Z_2 = 1$. However, it must be stressed that this is not the property of the theory, but the property of a particular regularization technique (for example that of Ref. 7), since generally neither the gauge invariance nor the scale invariance require the obligatory annihilation of power divergences.

So generally the constants Z_1 and Z_2 as functions of the infinite parameter of scale-invariant regularization, Λ, would possess radiative corrections from all orders of perturbation theory, notwithstanding that they are coupling-constant-independent. In the general case $Z_2 \geqslant 1$, contrary to electrodynamics, because in gravitational theory there are charges of only one sign and thus radiative corrections make the experimental charge larger than (or equal to) the initial one. It must be noted that the peculiarity of the theory of gravity which is the nonlinear gauge theory causes not only the renormalized three-graviton-vertex function, but all the renormalized vertex functions of graviton scattering at the threshold, to be equal to corresponding bare vertices with the renormalized values of κ and $A^{(0)}$, all infinite radiative corrections being included in Z multipliers.

Now, we considered a particular form of pole asymptotic behavior of the Green's functions throughout the present paper. It must be noticed, however, that if the infrared divergences exist in the theory (see Ref. 32), they may lead to the occurrence of anomalous singularities. The additional powers in momenta may be formally included into Z multipliers:

$$Z_2 \to \mathfrak{z}_2(p^2),$$

$$Z_1 \to \mathfrak{z}_1(p^2, k^2, pk),$$

and so on.

Although Ward identities in a simple form of (4.47), (4.48) would not hold for $\mathfrak{z}$ quantities, the scale properties of the theory will yield certain relations among them, provided that the invariant regularization is introduced. Using the exact relations (5.60)–(5.62), one obtains in the degenerate gauge

$$\mathfrak{z}_2^{-1}(p^2) + p^2\frac{\partial}{\partial p^2}\,\mathfrak{z}_2^{-1}(p^2) = \mathfrak{z}_1^{-1}(p^2, p^2, -p^2) \tag{5.64}$$

and the analogous relations for the higher-order anomalous factors $\mathfrak{z}_1^{(n)}$. If the anomalous singularity has the form

$$\mathfrak{z}_2(p^2) = \frac{\text{const}}{(p^2)^\gamma},$$

Eq. (5.64) gives

$$(1+\gamma)\,\mathfrak{z}_2^{-1}(p^2) = \mathfrak{z}_1^{-1}(p^2, p^2, -p^2). \tag{5.65}$$

Equation (5.65) replaces the $Z_1 = Z_2$ relation, while the higher-order relations for $\mathfrak{z}_1^{(n)}$ with respect to Eq. (5.65) replace the Ward identities for renormalization constants.

F. Gauge Independence of Renormalization Constants

We shall end the present section with the calculation of the dependence of Z_1 and Z_2 on the constant α, fixing the gauge.

Let us use Eqs. (5.58)–(5.59) to exclude the derivatives with respect to κ from Eqs. (5.30)–(5.31). [The threshold asymptotic behavior of the Green's functions being known, the derivatives with respect to momenta may be calculated explicitly in Eqs. (5.58)–(5.59).] Thus we obtain on the mass shell (m.s.)

$$\begin{aligned}\alpha\frac{\partial}{\partial\alpha}G^{\alpha\beta;\gamma\sigma}(p|A^{(0)};\kappa;\alpha) \underset{\text{m.s.}}{=}\; & G^{\alpha\beta;\gamma\sigma}(p|A^{(0)};\kappa;\alpha) \\ & -iG^{\alpha\beta;\alpha'\beta'}(p)G^{\gamma\sigma;\gamma'\sigma'}(p) \\ & \times\Gamma^{(3)}_{\alpha'\beta';\gamma'\sigma';\mu\nu}(p,-p,0)A^{(0)}\delta^{\mu\nu},\end{aligned} \tag{5.66}$$

$$\begin{aligned}\alpha\frac{\partial}{\partial\alpha}\Gamma^{(3)}_{\alpha\beta;\gamma\sigma;\phi\theta}(p,k,-p-k|A^{(0)};\kappa;\alpha) \underset{\text{m.s.}}{=}\; & -2\Gamma^{(3)}_{\alpha\beta;\gamma\sigma;\phi\theta}(p,k,-p-k|A^{(0)};\kappa;\alpha) \\ & -i\Gamma^{(4)}_{\alpha\beta;\gamma\sigma;\phi\theta;\mu\nu}(p,k,-p-k,0|A^{(0)};\kappa;\alpha)A^{(0)}\delta^{\mu\nu}.\end{aligned} \tag{5.67}$$

Relations (5.66), (5.67) together with the Ward identity represent the complete set of equations for renormalization constants as the functions of α.

Substitution of the asymptotic expressions (5.35)–(5.37) into the set of equations (5.66), (5.67) is not straightforward, because the propagator (5.35) possesses the free term, dependent on α. It will be sufficient to consider only one structure in the propagator, of the type

$$(\delta^{\mu\alpha}\delta^{\nu\beta} + \delta^{\mu\beta}\delta^{\nu\alpha}),$$

which does not mix up with other structures when multiplication with summation over collective indices is performed.

Confining ourselves to this structure, we find from (4.32) that

$$G_1^{(0)\mu\nu;\alpha\beta}(p, \alpha=0) = -\frac{1}{p^2}(\delta^{\mu\alpha}\delta^{\nu\beta} + \delta^{\mu\beta}\delta^{\nu\alpha}),$$

$$G_2^{(0)\mu\nu;\alpha\beta}(p, \alpha=0) = \frac{1}{2p^2}(\delta^{\mu\alpha}\delta^{\nu\beta} + \delta^{\mu\beta}\delta^{\nu\alpha}).$$

Then the free parameter α cancels in the expression (5.35) for the exact propagator, this being the consequence of the correct introduction of $Z_2(\alpha)$, and we obtain

$$\begin{aligned}G^{\mu\nu;\alpha\beta}(p|A^{(0)};\kappa;\alpha)|_{p^2\to 0} = \;& \frac{1}{p^2}(\delta^{\mu\alpha}\delta^{\nu\beta} + \delta^{\nu\alpha}\delta^{\mu\beta}) \\ & \times[-\kappa A^{(0)}Z_2(\alpha)].\end{aligned} \tag{5.68}$$

Further, with the aid of relation (5.42) for bare vertices we find that

$$\begin{aligned}iG^{\alpha\beta;\alpha'\beta'}(p)G^{\gamma\sigma;\gamma'\sigma'}(p)\Gamma^{(3)}_{\alpha'\beta';\gamma'\sigma';\mu\nu}(p,-p,0)A^{(0)}\delta^{\mu\nu} \\ = \frac{1}{p^2}(\delta^{\mu\alpha}\delta^{\nu\beta} + \delta^{\nu\alpha}\delta^{\mu\beta})[-\kappa A^{(0)}Z_1^{-1}(\alpha)Z_2^{\,2}(\alpha)].\end{aligned} \tag{5.69}$$

Substitution of (5.68) and (5.69) into Eq. (5.66) gives a differential equation for $Z_2(\alpha)$ and $Z_1(\alpha)$. In order to obtain the second equation, let us substitute the pole asymptotic behavior of Γ functions (5.36) and (5.37) into the second equation (5.67) and use the fact that the function $Z_2(\alpha)$ is introduced in such a way that the Ward identity

$$[Z_1^{(2)}(\alpha)]^{-1}=Z_1^{-2}(\alpha)Z_2(\alpha)$$

holds. Using Eq. (5.43) to cancel the bare vertices in Eq. (5.67), one finds the second differential equation. The complete set of equations reads

$$\alpha\frac{d}{d\alpha}Z_2=Z_2(1-Z_1^{-1}Z_2), \tag{5.70}$$

$$\alpha\frac{d}{d\alpha}Z_1^{-1}=2Z_1^{-1}(Z_1^{-1}Z_2-1). \tag{5.71}$$

In order to solve this nonlinear set of equations, let us remember that we have obtained

$$Z_1(\alpha)=Z_2(\alpha).$$

The solution of Eqs. (5.70), (5.71) immediately follows:

$$\frac{dZ_1(\alpha)}{d\alpha}=\frac{dZ_2(\alpha)}{d\alpha}=0.$$

Thus not only charge renormalization, but Z_1 and Z_2 themselves do not depend on the gauge; and all the relations for renormalization constants, which were obtained earlier, hold in the arbitrary gauge. Construction of the renormalized Lagrangian in the arbitrary gauge was discussed at the end of Sec. IV.

We shall get an idea about the information carried by the "scale" properties of the theory if we compare the equation (5.60), which was obtained here, with the similar equation (3.40) of Sec. III. One finds that the integral of the connected part of the four-point function, arising in Eq. (3.40), is unambiguously expressed in terms of the $\Gamma^{(3)}$ function with one of the momenta equal to zero. Similarly, calculation of the derivatives with respect to α of the higher-order Green's functions by means of the method of Sec. III leads to the integrals of the connected parts of the Green's functions of the field ϕ analogous to (3.40) or to the integrals of the Green's functions of fictitious fields, analogous to (3.39). Using the relations obtained in the present section, one easily finds explicit expressions for these integrals in terms of higher-order vertex functions with one of the momenta equal to zero.

VI. CONCLUSION

The relation obtained,

$$Z_1=Z_2,$$

and the gauge independence of the renormalization constants are specific distinctions of the theory of gravity from the theory of the Yang-Mills field.

In particular, it is possible in the Yang-Mills theory to make the renormalization $Z_2(\alpha)$ finite by means of an appropriate choice of the value of α.[12(a)] Then, only one essentially divergent constant remains in this theory: the charge-renormalization constant. We see that similar considerations cannot be applied to the theory of gravity, because in this case Z_2 does not depend on α. The relation $Z_1=Z_2$ serves the equivalent of these additional considerations for the theory of gravity. By virtue of this relation, there remains in the theory of gravity, as well as in the Yang-Mills theory and in electrodynamics, only one essentially divergent constant, renormalizing the threshold asymptotic behavior of all the Green's functions.

It goes without saying that the main question of whether the theory is renormalizable or not remains unsolved. The Green's functions are unrenormalizable from the viewpoint of the perturbation theory, but we have seen that the latter does not exist in the usual sense. The true parameter for the scattering amplitudes is the quantity $\sqrt{\kappa}\,p$, so that the next problem is the renormalization of the terms following the threshold terms in the expansion of the Green's functions in $\sqrt{\kappa}\,p$. There are reasons to think that this task will not require the introduction of additional higher-order invariant structures in the Lagrangian, because structures of this type are present already in the covariant and unitary S matrix.[18] In this case, the theory would prove to be renormalizable, at least on the mass shell.

APPENDIX A: THE GAUGE RELATIONS OF THE BARE VERTICES [33]

The starting point for the derivation of gauge relations among the bare vertices is the Nöther identity[34]:

$$\nabla^{+\mu\nu}_{\ \beta}(x)\frac{\delta S}{\delta\hat{g}^{\mu\nu}(x)}\equiv\nabla^{+b}_{\ \beta}\frac{\delta S}{\delta\hat{g}^{b}}=0, \tag{A1}$$

where

$$S=\int\sqrt{-g}\,R\,dx.$$

Performing the consecutive differentiation of (A1) with respect to $\hat{g}^{a_i}$ and after that putting $\hat{g}=\delta$, we shall obtain the infinite set of relations which the bare vertices obey.

First we obtain a set of exact relations, where $\hat{g}$ is arbitrary:

$$\frac{\delta}{\delta\hat{g}^{a_1}}(\nabla^{+b}_{\ \beta})\frac{\delta S}{\delta\hat{g}^{b}}+\nabla^{+b}_{\ \beta}\frac{\delta^2 S}{\delta\hat{g}^{b}\delta\hat{g}^{a_1}}=0, \tag{A2}$$

$$\frac{\delta}{\delta\hat{g}^{a_2}}(\nabla^{+b}_{\ \beta})\frac{\delta^2 S}{\delta\hat{g}^b\delta\hat{g}^{a_1}}+\frac{\delta}{\delta\hat{g}^{a_1}}(\nabla^{+b}_{\ \beta})\frac{\delta^2 S}{\delta\hat{g}^b\delta\hat{g}^{a_2}}+\nabla^{+b}_{\ \beta}\frac{\delta^3 S}{\delta\hat{g}^b\delta\hat{g}^{a_1}\delta\hat{g}^{a_2}}=0, \quad \text{(A3)}$$

$\cdots,$

$$\sum_{i=1}^{n}\frac{\delta}{\delta\hat{g}^{a_i}}(\nabla^{+b}_{\ \beta})\frac{\delta^n S}{\delta\hat{g}^b\delta\hat{g}^{a_1}\cdots\delta\hat{g}^{a_{i-1}}\delta\hat{g}^{a_{i+1}}\cdots\delta\hat{g}^{a_n}}+\nabla^{+b}_{\ \beta}\frac{\delta^{n+1}S}{\delta\hat{g}^b\delta\hat{g}^{a_1}\cdots\delta\hat{g}^{a_n}}=0. \quad \text{(A4)}$$

We used here the linearity of the operator ∇^+ in $\hat{g}$.

It follows from (A2) when $\hat{g}=\delta$ that

$$\nabla^{+(0)b}_{\ \beta}\frac{\delta^{2(0)}S}{\delta\hat{g}^b\delta\hat{g}^a}=0. \quad \text{(A5)}$$

Let us now act upon Eq. (A3) with the operators $\nabla^{+a_1}_{\ \alpha_1}$, $\nabla^{+a_2}_{\ \alpha_2}$ and then put $\hat{g}=\delta$. With regard to (A5), one obtains

$$\nabla^{+(0)a}_{\ \alpha}\nabla^{+(0)b}_{\ \beta}\nabla^{+(0)c}_{\ \gamma}\frac{\delta^{3(0)}S}{\delta\hat{g}^a\delta\hat{g}^b\delta\hat{g}^c}=0. \quad \text{(A6)}$$

Let us now consider the functional

$$S(\alpha)=S-\frac{1}{2\alpha}\int\chi^\mu\chi^\mu dx. \quad \text{(A7)}$$

In terms of (A7), Eqs. (A1)–(A3) take the form

$$\nabla^{+b}_{\ \beta}\frac{\delta S(\alpha)}{\delta\hat{g}^b}=-\frac{1}{\alpha}\hat{Q}_{\sigma\beta}\chi^\sigma, \quad \text{(A8)}$$

$$\frac{\delta}{\delta\hat{g}^a}(\nabla^{+b}_{\ \beta})\frac{\delta S(\alpha)}{\delta\hat{g}^b}+\nabla^{+b}_{\ \beta}\frac{\delta^2 S(\alpha)}{\delta\hat{g}^b\delta\hat{g}^a}=-\frac{1}{\alpha}\left(\frac{\delta}{\delta\hat{g}^a}\hat{Q}_{\sigma\beta}\right)\chi^\sigma-\frac{1}{\alpha}\hat{Q}_{\sigma\beta}\left(\frac{\delta}{\delta\hat{g}^a}\chi^\sigma\right), \quad \text{(A9)}$$

$$\left(\frac{\delta}{\delta\hat{g}^a}\nabla^{+b}_{\ \beta}\right)\frac{\delta^2 S(\alpha)}{\delta\hat{g}^b\delta\hat{g}^c}+\left(\frac{\delta}{\delta\hat{g}^c}\nabla^{+b}_{\ \beta}\right)\frac{\delta^2 S(\alpha)}{\delta\hat{g}^b\delta\hat{g}^a}+\nabla^{+b}_{\ \beta}\frac{\delta^3 S(\alpha)}{\delta\hat{g}^b\delta\hat{g}^a\delta\hat{g}^c}=-\frac{1}{\alpha}\left(\frac{\delta}{\delta\hat{g}^a}\hat{Q}_{\sigma\beta}\right)\left(\frac{\delta}{\delta\hat{g}^c}\chi^\sigma\right)-\frac{1}{\alpha}\left(\frac{\delta}{\delta\hat{g}^c}\hat{Q}_{\sigma\beta}\right)\left(\frac{\delta}{\delta\hat{g}^a}\chi^\sigma\right). \quad \text{(A10)}$$

We used here the linearity of the operator $\hat{Q}$ and the gauge condition χ in $\hat{g}$.

Equations (A8)–(A10) present the nontrivial generalization of the previous relations because the second variations of action may be inverted in these formulas. It would not be possible to perform this inversion in Eqs. (A2), (A3) because of degeneracy of the matrix

$$\frac{\delta^2 S}{\delta\hat{g}^a\delta\hat{g}^b}.$$

Denote

$$D^{ab}[\hat{g}]=\left[\frac{\delta^2 S(\alpha)}{\delta\hat{g}^a\delta\hat{g}^b}\right]^{-1}. \quad \text{(A11)}$$

When $\hat{g}=\delta$ one obtains from (A9)

$$\frac{1}{\alpha}\partial^y_\gamma G^{(0)\rho\gamma;\sigma\alpha}(y,z)=\nabla^{(0)\sigma\alpha|\beta}(y)M^{(0)\rho\beta}(z,y) \quad \text{(A12)}$$

or

$$-\frac{1}{\alpha}\partial^y_\gamma G^{(0)\rho\gamma;\sigma\alpha}(y,z)=\nabla^{+(0)\sigma\alpha}_{\ \rho}(y)\frac{1}{\Box_z}\delta(y-z), \quad \text{(A13)}$$

$$\Box_z\equiv\partial^z_\mu\partial^z_\mu.$$

Here the transversality of the gauge was used. With regard to (A13), Eq. (A6) takes the form

$$(\text{div}G^{(0)aa'})(\text{div}G^{(0)bb'})(\text{div}G^{(0)cc'})\Gamma^{(0)(3)}_{a'b'c'}=0, \quad \text{(A14)}$$

which is the analog of the Ward identity (4.41). Equation (A12) is the analog of the Ward identity (4.25).

Inverting the second variations in (A10), one obtains, for $a=(\mu,\nu;x)$, $b=(\rho,\gamma;y)$, $c=(\phi,\theta;z)$,

$$D^{bb'}\left(\frac{\delta}{\delta\hat{g}^{b'}}\nabla^{+c}_{\ \beta}\right)\delta(z-x)+D^{cc'}\left(\frac{\delta}{\delta\hat{g}^{c'}}\nabla^{+b}_{\ \beta}\right)\delta(y-x)+D^{bb'}D^{cc'}\nabla^{+a}_{\ \beta}\frac{\delta^3 S}{\delta\hat{g}^a\delta\hat{g}^{b'}\delta\hat{g}^{c'}}$$
$$=-\frac{1}{\alpha}D^{bb'}\left(\frac{\delta}{\delta\hat{g}^{b'}}\hat{Q}_{\sigma\beta}(x)\right)[\partial^x_\alpha D^{c;\sigma\alpha}(z,x)]-\frac{1}{\alpha}D^{cc'}\left(\frac{\delta}{\delta\hat{g}^{c'}}\hat{Q}_{\sigma\beta}(x)\right)[\partial^x_\alpha D^{b;\sigma\alpha}(y,x)], \quad \text{(A15)}$$

which is a rigorous relation. Let us now put $\hat{g}=\delta$, then find the explicit form of variation of the operator ∇^+ in the first two terms of (A15), and then go to Fourier components,

$$x\leftrightarrow k,\ y\leftrightarrow p,\ z\leftrightarrow q,$$

keeping in mind that $k+p+q=0$. One notices that the terms which come from the first two terms of (A15) have the one-particle pole $1/p^2$ or $1/q^2$, while the other terms have the pole $1/p^2q^2$. Thus, when the pole asymptotic behavior is considered, one obtains

$$(\delta^\mu_\beta k^\nu+\delta^\nu_\beta k^\mu-\delta^{\mu\nu}k^\beta)G^{(0)\phi\theta;\phi'\theta'}(q)G^{(0)\rho\gamma;\rho'\gamma'}(p)\times\Gamma^{(0)(3)}_{\mu\nu;\rho'\gamma';\phi'\theta'}(k,p,q)$$
$$\underset{\text{pole}}{=}\frac{1}{\alpha}G^{(0)\rho\gamma;\rho'\gamma'}(p)q_{\theta'}G^{(0)\phi\theta;\phi'\theta'}(q)\gamma^{(0)(3)}_{\phi'\beta|\rho'\gamma'}(k,q|p)+\frac{1}{\alpha}G^{(0)\phi\theta;\phi'\theta'}(q)p_{\gamma'}G^{(0)\rho\gamma;\rho'\gamma'}(p)\gamma^{(0)(3)}_{\rho'\beta|\phi'\theta'}(k,p|q). \quad \text{(A16)}$$

Multiply this equality by q^θ and use relation (A13). Then it follows that

$$\frac{\alpha}{q^2}(\delta^\mu_\beta k^\nu+\delta^\nu_\beta k^\mu-\delta^{\mu\nu}k^\beta)(\delta^{\phi\phi'}q^{\theta'}+\delta^{\phi\theta'}q^{\phi'}-\delta^{\phi'\theta'}q^\phi)$$
$$\times G^{(0)\rho\gamma;\rho'\gamma'}(p)\Gamma^{(0)(3)}_{\mu\nu;\rho'\gamma';\phi'\theta'}(k,p,q)$$
$$\underset{\text{pole}}{=}G^{(0)\rho\gamma;\rho'\gamma'}(p)\gamma^{(0)(3)}_{\phi\beta|\rho'\gamma'}(k,q|p)$$
$$+\frac{1}{q^2}(\delta^{\phi\phi'}q^{\theta'}+\delta^{\phi\theta'}q^{\phi'}-\delta^{\phi'\theta'}q^\phi)$$
$$\times\frac{\alpha}{p^2}(\delta^{\rho'\rho}p^\gamma+\delta^{\rho'\gamma}p^\rho-\delta^{\rho\gamma}p^{\rho'})\gamma^{(0)(3)}_{\rho'\beta|\phi'\theta'}(k,p|q). \quad \text{(A17)}$$

Note that the first term on the right-hand side of (A17) possesses the one-particle pole $1/p^2$, while the other terms have the pole $1/p^2q^2$. Omitting the term with the weaker pole asymptotic behavior, one obtains finally

$$\frac{\alpha}{q^2}(\delta^\mu_\beta k^\nu+\delta^\nu_\beta k^\mu-\delta^{\mu\nu}k^\beta)(\delta^{\phi\phi'}q^{\theta'}+\delta^{\phi\theta'}q^{\phi'}-\delta^{\phi'\theta'}q^\phi)$$
$$\times G^{(0)\rho\gamma;\rho'\gamma'}(p,\alpha)\Gamma^{(0)(3)}_{\mu\nu;\rho'\gamma';\phi'\theta'}(k,p,q)$$
$$\underset{\text{pole}}{=}\frac{\alpha}{p^2q^2}(\delta^{\phi\phi'}q^{\theta'}+\delta^{\phi\theta'}q^{\phi'}-\delta^{\phi'\theta'}q^\phi)$$
$$\times(\delta^{\rho'\rho}p^\gamma+\delta^{\rho'\gamma}p^\rho-\delta^{\rho\gamma}p^{\rho'})\gamma^{(0)(3)}_{\rho'\beta|\phi'\theta'}(k,p|q), \quad \text{(A18)}$$

which is the analog of the Ward identity (4.46).

Let us return to the rigorous equality (A15) and take two divergences there: with respect to one index from the pair b and another one from the pair c. The equation thus obtained is to be varied n times with respect to $\hat{g}$; then $\hat{g}$ is to be put equal to δ and the pole asymptotic behavior is to be taken. It is not difficult to see that the first two terms on the left-hand side of (A15) will always produce the weaker pole asymptotic behavior than the other terms.

Thus we can write

$$\partial_{b_1}D^{bb'}\partial_{c_1}D^{cc'}\nabla^{+a}_\alpha\frac{\delta^3 S}{\delta\hat{g}^a\delta\hat{g}^{b'}\delta\hat{g}^{c'}}$$
$$\underset{\substack{\delta^n,\hat{g}=\delta\\ \text{pole}}}{=}\frac{1}{\alpha}\partial_{b_1}D^{bb'}\hat{\gamma}^{(0)}_{a_2\alpha|b'}(\partial_{c_1}D^{c;a}\partial_{a_1})$$
$$+\frac{1}{\alpha}\partial_{c_1}D^{cc'}\hat{\gamma}^{(0)}_{a_2\alpha|c'}(\partial_{b_1}D^{b;a}\partial_{a_1}). \quad \text{(A19)}$$

Let us now apply the operator

$$\mathfrak{F}^{f_1\cdots f_n}=\text{div}D^{f_1f_1'}\frac{\delta}{\delta\hat{g}^{f_1'}}\cdots\text{div}D^{f_nf_n'}\frac{\delta}{\delta\hat{g}^{f_n'}} \quad \text{(A20)}$$

to Eq. (A19) and define the action of this operator upon the quantity

$$\mathfrak{F}^{f_1\cdots f_n}[\partial_{c_2}D^{c;a}\partial_{a_2}]\equiv\mathcal{P}^{f_1\cdots f_n ca}_{(n+2)} . \quad \text{(A21)}$$

Analyzing the action of the operator (A20) upon both sides of Eq. (A19) and using Eqs. (A4) and (A13), one obtains

$$\mathcal{P}^{f_1\cdots f_{n+3}}_{(n+3)}|_{\hat{g}=\delta}=0, \quad n\geq 0 \quad \text{(A22)}$$

as a consequence of Eq. (A19). The equality (A22) may be written down as

$$(\text{div}G^{(0)})\cdots(\text{div}G^{(0)})$$
$$\times[\Gamma^{(0)(n+2)}+\textstyle\sum\mathfrak{S}\Gamma^{(0)(p_1+2)}G^{(0)}\Gamma^{(0)(p_2+2)}$$
$$+\cdots+\mathfrak{S}\Gamma^{(0)(3)}G^{(0)}\Gamma^{(0)(3)}G^{(0)}\cdots G^{(0)}\Gamma^{(0)(3)}]=0,$$
$$n\geq 1 \quad \text{(A23)}$$

where $(\text{div}G^{(0)})\cdots(\text{div}G^{(0)})$ consists of $(n+2)$ factors of $\text{div}G^{(0)}$. Equation (A23) is the analog of the first set of Ward identities (4.20).

APPENDIX B: THE THRESHOLD ASYMPTOTIC BEHAVIOR OF THE GREEN'S FUNCTIONS Θ IN COINCIDING POINTS

For the analysis of the Ward identities (4.25) and (4.26), it is necessary to find the threshold asymptotic behavior of the functions

$$\mathcal{E}^{\alpha|\beta\gamma}(k,p)|_{k+p=0},$$
$$\mathcal{E}^{\alpha_1|\beta\gamma|\alpha_2}(k|p|q)|_{k+p+q=0}, \quad \text{(B1)}$$

which are determined by the formula (4.24). According to (4.8), these functions are expressed in terms of the Green's functions $\Theta^{\alpha\beta}(x,y|a)$ and $\Theta^{\alpha\beta}(x,y|a_1,a_2)$ at coinciding points.

Let us consider the first of the functions (B1). One has

$$\mathcal{E}^{\alpha|\beta\gamma}(x,y)\equiv\langle 0|T\nabla^{\beta\gamma|\sigma}(\phi|y)Q^{-1\,\alpha\sigma}(x,y)|0\rangle$$
$$=(\partial^y_\rho\delta^\gamma_\mu\delta^\beta_\nu+\partial^y_\rho\delta^\beta_\mu\delta^\gamma_\nu-\partial^y_\nu\delta^\beta_\mu\delta^\gamma_\rho)\Theta^{\alpha\nu|\mu\rho}(x,y|y)$$
$$-\int dz\,\delta(y-z)(\delta^\gamma_\mu\delta^\beta_\nu+\delta^\beta_\mu\delta^\gamma_\nu)$$
$$\times\partial^z_\rho\Theta^{\alpha\nu|\mu\rho}(x,y|z). \quad \text{(B2)}$$

Going over to Fourier components and taking into account the translation invariance, one obtains

$$i\mathcal{E}^{\alpha|\beta\gamma}(-p,p)=\int dt[t^\rho(\delta^\gamma_\mu\delta^\beta_\nu+\delta^\beta_\mu\delta^\gamma_\nu)+p^\nu\delta^\beta_\mu\delta^\gamma_\rho]$$
$$\times\Theta^{\alpha\nu|\mu\rho}(t,p|-p-t). \quad \text{(B3)}$$

For the Green's function Θ one has a representation

$$i\Theta^{\alpha\nu|\mu\rho}(t,p|-p-t)=M^{\alpha\alpha'}(t)M^{\nu\nu'}(p)G^{\mu\rho;\mu'\rho'}(-t-p)$$
$$\times\gamma_{\alpha'\nu'|\mu'\rho'}(t,p|-t-p), \quad \text{(B4)}$$

corresponding to Fig. 8. We are interested in (B3) when $p\to 0$. With respect to Eq. (B4), one

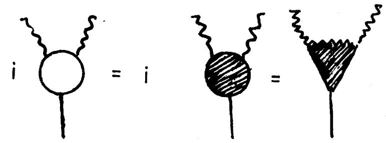


FIG. 8. The three-point function Θ.

represents the right-hand side of (B3) in terms of the following two integrals, in which we may directly put $p=0$:

$$\int M^{\alpha\alpha'}(t)G^{\mu\rho;\mu'\rho'}(-t)\gamma_{\alpha'\nu'|\mu'\rho'}(t,0|-t)dt \equiv f_1^{\alpha\mu\rho\nu'}, \tag{B5}$$

$$\int t^{\rho}M^{\alpha\alpha'}(t)G^{\mu\rho;\mu'\rho'}(-t)\gamma_{\alpha'\nu'|\mu'\rho'}(t,0|-t)dt \equiv f_2^{\alpha\mu\nu'}. \tag{B6}$$

Here, the constants $f_1^{\alpha\mu\rho\nu'}$ and $f_2^{\alpha\mu\nu'}$ are the relativistic tensors composed of $\delta^{\mu\nu}$ only. Thus

$$f_2^{\alpha\mu\nu'}=0 \tag{B7}$$

and

$$f_1^{\alpha\mu\rho\nu'}=c_1\delta^{\mu\rho}\delta^{\alpha\nu'}+c_2(\delta^{\alpha\mu}\delta^{\rho\nu'}+\delta^{\alpha\rho}\delta^{\mu\nu'}), \tag{B8}$$

where the symmetry properties are taken into account. Finally, using

$$M^{\nu\nu'}(p) \underset{p^2\to 0}{\to} \delta^{\nu\nu'}M^{\perp}(p^2),$$

one obtains from (B3) the asymptotic behavior needed:

$$\mathcal{E}^{\alpha|\beta\gamma}(-p,p) \underset{p\to 0}{=} [c_1p^{\alpha}\delta^{\beta\gamma}+c_2(p^{\gamma}\delta^{\alpha\beta}+p^{\beta}\delta^{\alpha\gamma})]M^{\perp}(p^2), \tag{B9}$$

which is determined by (B9) up to two unknown constants c_1 and c_2.

Consider now the second of the functions (B1):

$$\mathcal{E}^{\alpha_1|\beta\gamma|\alpha_2}(x|y|z)\equiv\langle 0|T\nabla^{\beta\gamma|\sigma}(\phi|y)Q^{-1\,\alpha_1\sigma}(x,y)\chi^{\alpha_2}(z)|0\rangle . \tag{B10}$$

Similarly to the previous case, one expresses (B10) with the aid of the explicit performing of the operator ∇ action in terms of the higher-order Green's function

$$\Theta^{\mu\rho|\alpha_1\sigma|\alpha_2\zeta}(t|x,y|z)=\langle 0|T\phi^{\mu\rho}(t)Q^{-1\,\alpha_1\sigma}(x,y)\phi^{\alpha_2\zeta}(z)|0\rangle \tag{B11}$$

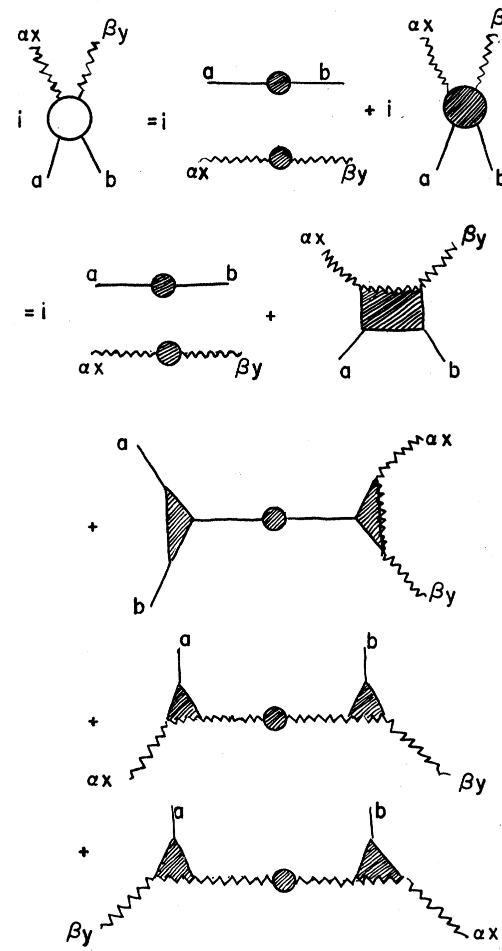


FIG. 9. The four-point function Θ.

taken at coinciding points. This expression reads in momentum space $(t\leftrightarrow s)$

$$i\mathcal{E}^{\alpha_1|\beta\gamma|\alpha_2}(k|-k-q|q)$$
$$=\int q^{\zeta}[p^{\rho}(\delta_{\mu}^{\gamma}\delta_{\nu}^{\beta}+\delta_{\mu}^{\beta}\delta_{\nu}^{\gamma})+(k+q)^{\nu}\delta_{\mu}^{\beta}\delta_{\rho}^{\gamma}]$$
$$\times\delta(s+k+p+q)\Theta^{\mu\rho|\alpha_1\nu|\alpha_2\zeta}(s|k,p|q)dp\,ds . \tag{B12}$$

For the Fourier transform of the function (B11) we use the representation of Fig. 9. We are interested in (B12) when $k,q\to 0$. As before in the integrals which arise we put $k=0$, $q=0$ directly. Then (B12) takes the form

$$\mathcal{E}^{\alpha_1|\beta\gamma|\alpha_2}(k|-k-q|q) \underset{\substack{k\to 0\\ q\to 0}}{=} -iq^{\zeta}[-k^{\rho}(\delta_{\mu}^{\gamma}\delta_{\nu}^{\beta}+\delta_{\mu}^{\beta}\delta_{\nu}^{\gamma})+(k+q)^{\nu}\delta_{\mu}^{\beta}\delta_{\rho}^{\gamma}]G^{\mu\rho;\alpha_2\zeta}(q)M^{\alpha_1\nu}(k)$$
$$+iq^{\zeta}(k+q)^{\nu}\delta_{\mu}^{\beta}\delta_{\rho}^{\gamma}M^{\alpha_1\phi}(k)G^{\alpha_2\zeta;f_1f_2}(q)[\tilde{f}_1^{\nu\mu\rho\phi f_1f_2}+\gamma^{(3)}_{\phi\theta|f_1f_2}(k,-k-q|q)M^{\theta\xi}(k+q)\tilde{f}_2^{\nu\mu\rho\xi}], \tag{B13}$$

where $\tilde{f}_1^{\nu\mu\rho\phi f_1 f_2}$ and $\tilde{f}_2^{\nu\mu\rho\xi}$ are the constant tensors.

Now note that the first term on the right-hand side of (B13) (which is the contribution from the disconnected part of the four-point function), as well as the term proportional to $\tilde{f}_1$, possesses the pole $1/q^2k^2$, while the third term, proportional to $\tilde{f}_2$, has the pole $1/[q^2k^2(k+q)^2]$. Omitting the terms with the weaker pole asymptotic behavior one obtains

$$\mathcal{E}^{\alpha_1|\beta\gamma|\alpha_2}(k|-k-q|q) \underset{\substack{k\to 0\\ q\to 0}}{=} iq^{\xi}(k+q)^{\nu}\delta^{\beta}_{\mu}\delta^{\gamma}_{\rho}M^{\alpha_1\phi}(k)G^{\alpha_2\xi;f_1f_2}(q)\gamma^{(3)}_{\phi\theta|f_1f_2}(k,-k-q|q)M^{\theta\psi}(k+q)\tilde{f}_2^{\nu\mu\rho\psi}. \quad \text{(B14)}$$

Thus only one constant tensor $\tilde{f}_2$ contributes to the final expression (B14). This means that only one of the last two diagrams of Fig. 9 contributes to the pole asymptotic behavior of the Green's function at coinciding points. This contribution reads

$$\tilde{f}_2^{\nu\mu\rho\zeta} = i\int dp M^{\nu\sigma}(p)G^{\mu\rho;c_1c_2}(-p)\gamma^{(3)}_{\zeta\sigma|c_1c_2}(0,p|-p)$$
$$= \tilde{c}_1\delta^{\mu\rho}\delta^{\nu\zeta} + \tilde{c}_2(\delta^{\mu\nu}\delta^{\rho\zeta} + \delta^{\mu\zeta}\delta^{\nu\rho}). \quad \text{(B15)}$$

Thus it follows finally from (B14) that

$$\mathcal{E}^{\alpha_1|\beta\gamma|\alpha_2}(k|-k-q|q) \underset{\substack{k\to 0\\ q\to 0}}{=} -\frac{\alpha}{q^2}M^{\perp}(k^2)M^{\perp}((k+q)^2)(\delta^{f_1\alpha_2}q^{f_2} + \delta^{f_2\alpha_2}q^{f_1} - \delta^{f_1f_2}q^{\alpha_2})$$
$$\times\gamma^{(3)}_{\alpha_1\theta|f_1f_2}(k,-k-q|q)\{\tilde{c}_1\delta^{\beta\gamma}(k+q)^{\theta} + \tilde{c}_2[\delta^{\gamma\theta}(k+q)^{\beta} + \delta^{\beta\theta}(k+q)^{\gamma}]\}, \quad \text{(B16)}$$

which gives the asymptotic behavior needed up to two unknown constants $\tilde{c}_1$ and $\tilde{c}_2$.

It is of special importance that the constants $\tilde{c}_1$ and $\tilde{c}_2$ are not the new ones. One can make sure comparing the integrals (B15) and (B5) that those integrals are exactly equal. Thus,

$$\tilde{f}_2^{\nu\mu\rho\zeta} = if_1^{\nu\mu\rho\zeta}$$

and from (B15) and (B8),

$$\tilde{c}_1 = ic_1, \quad \tilde{c}_2 = ic_2. \quad \text{(B17)}$$

In Sec. IV the constants c_1 and c_2 are determined with the aid of the lowest-order Ward identity. Then formulas (B16) and (B17) permit us to substitute in the next Ward identity the completely known asymptotic behavior of the Green's functions (B1).

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[1]We refer to the previous paper by E. S. Fradkin and I. V. Tyutin [Phys. Rev. D 2, 2841 (1970)] as I.

[2]B. S. DeWitt, Phys. Rev. 162, 1195 (1967); 162, 1239 (1967).

[3]L. D. Faddeev and V. N. Popov, Phys. Lett. 25B, 30 (1967); V. N. Popov and L. D. Faddeev, Kiev Report No. ITP-67-36, 1967 (unpublished).

[4]S. Mandelstam, Phys. Rev. 175, 1580 (1968); 175, 1604 (1968).

[5]Up to Sec. IV we use the units $\kappa = \hbar = c = 1$.

[6]E. S. Fradkin, Dokl. Akad. Nauk SSSR 98, 47 (1954); Zh. Eksp. Teor. Fiz. 29, 121 (1955) [Sov. Phys.—JETP 2, 148 (1956)]; in *Quantum Field Theory and Hydrodynamics*, Proceedings of the P. N. Lebedev Physics Institute, Vol. 29 (Nauka, Moscow, 1965).

[7]G. 't Hooft, Nucl. Phys. B33, 173 (1971). The gauge-invariant regularization was suggested in G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972).

[8]E. S. Fradkin and G. A. Vilkovisky, P. N. Lebedev Physical Institute Report No. 137, 1971 (unpublished).

[9]L. D. Faddeev, Theor. Math. Phys. USSR 1, 3 (1969); L. D. Faddeev and V. N. Popov, in proceedings of the Fifth International Conference on the Theory of Gravitation and Relativity, Tbilisi report, 1968 (unpublished).

[10]N. P. Konopleva and V. N. Popov, *The Gauge Fields* (Atomizdat, Moscow, 1972).

[11]E. S. Fradkin, Zh. Eksp. Teor. Fiz. 29, 258 (1955) [Sov. Phys.—JETP 2, 361 (1956)].

[12](a) A. A. Slavnov, Kiev Report No. ITP-71-83E, 1971 (unpublished); Theor. Math. Phys. USSR 10, 153 (1972); J. C. Taylor, University of Oxford report, 1971 (unpublished). (b) There was an attempt to derive the general form of Ward identities for the gravitational field: Abdus Salam and J. Strathdee, in *Problems of Theoretical Physics* (Nauka, Moscow, 1972). The authors did not make use of a conservation law in the Nöther form, therefore they obtained only a highly complicated expression of the conserved currents (the gravitational energy-momentum tensor) in terms of the sources. (c) We derive the generalized Ward identities performing explicit calculations in the transverse gauge (3.2) in order to show that the compensation (3.21) takes place. However, this compensation may be proved for an arbitrary gauge condition χ and thus the generalized Ward identities in a form of Eq. (3.22) and Eqs. (3.26)–(3.33) are true for an arbitrary χ. Indeed, it may be shown that independently of the gauge condition,

$$[\mathcal{L}^1_{\gamma}(x) + \mathcal{L}^2_{\gamma}(x)] \propto \int C^{(\gamma x)(\alpha y)}_{(\alpha y)}dy,$$

where the trace of the gauge-group structure constants $C^{(\gamma x)(\alpha y)}_{(\beta z)}$ appears on the right-hand side. Such a trace is zero for the coordinate group of general relativity as well as for the gauge group of the Yang-Mills theory. Thus the generalized Ward identities in the Yang-Mills theory derived for the case of the transverse gauge in Ref. 12(a) may be also derived in an arbitrary gauge.

[13]E. S. Fradkin, in *Problems of Theoretical Physics* (Nauka, Moscow, 1969).

[14]G. A. Vilkovisky, Institute for Optical-Physical Measurements Report No. 70-6, 1970 (unpublished); Theor. Math. Phys. USSR 8, 359 (1971); Theor. Math. Phys. USSR 16, 90 (1973).

[15]The choice of the gauge conditions (2.17) in their local components means that the Lagrange multipliers (*B* field) in the formulas of Ref. 1 are introduced also by their components in the orthogonal *vierbein*. This does not affect, however, the results of Ref. 1.

[16]Note that the determinant $J_\chi[g]$ is specified by the arguments of Ref. 1 only up to the local factor. However, each particular way of calculating the expression for $J_\chi[g]$ determines this factor uniquely. In particular, expression (2.21) is obtained exactly following the procedure described in Ref. 1. It is important, that since the local factor of $\ln J_\chi$ must be compensated for by the corresponding term in the local measure, any other choice of the local factor of (2.21) must be accompanied by the change of the measure, the generating functional being unaffected. Thus the measure must be calculated for at least one choice of this factor in $J_\chi[g]$.

[17]In Ref. 1 the local measure was omitted, but in the expressions (4.26)–(4.29) of Ref. 1 it is necessary to use expressions (2.23), (2.24) or (2.25), (2.26) of the present paper.

[18]It is noteworthy that the operator $\hat{Q}$ becomes in the degenerate harmonic gauge proportional to the generally covariant Laplace operator $\Box$, while the $\Box^{-1}$ operator may be presented by means of a manifestly covariant path integral in Riemannian space (Ref. 14). Use of this representation in the generating functional may give the opportunity to overstep the limits of perturbation theory.

[19]F. A. Beresin, *Secondary Quantization Method* (Nauka, Moscow, 1965).

[20]We cite here the arguments from the paper by H. Leutwyler, Phys. Rev. 134, B1155 (1964).

[21]C. W. Misner, Rev. Mod. Phys. 29, 497 (1957); B. Laurent, Ark. Fys. 16, 279 (1959); J. R. Klauder, Nuovo Cimento 19, 1059 (1961); B. S. DeWitt, J. Math. Phys. 3, 1073 (1962).

[22]L. D. Faddeev and V. N. Popov, in proceedings of the International Seminar on the Functional Methods in Quantum Field Theory and Statistics, Part II, P. N. Lebedev Physical Institute Report No. 141, 1971 (unpublished).

[23]Y. Takahashi, Nuovo Cimento 6, 370 (1957).

[24]In virtue of translation and Lorentz invariance this vacuum average must be a constant relativistic tensor. Because of our boundary conditions, we dispose of only one constant tensor, which leaves the arbitrariness only in the constant factor A^0. These are the usual consequences of a relativity principle. Since we do not deal with quantum cosmology and only the *S*-matrix theory is considered, where the curved space is "the interaction region," the physical relativity principle holds in its usual form, while the general covariance has the meaning of a gauge invariance: V. A. Fock, *The Theory of Space-Time and Gravitation* (Pergamon, New York, 1959).

[25]The first letters from the lower-case Latin alphabet will be used to denote the collective indices $a=(\mu,\nu;x)$, so $\hat{g}^{\mu\nu}(x)=\hat{g}^a$, and except for the cases mentioned specially, the indices $a, b, c, \ldots,$ repeated indicate summation over the discrete and integration with respect to continuous indices. Such condensed notation is used throughout in Ref. 2.

[26]We did not consider the vertices generated by the term $\propto \delta^{(4)}(0)$ in the modified action functional. However, the local measure itself serves as the counterterm, as we have shown, securing the cancellation of the divergences of the type $\delta^{(4)}(0)$. The wave-function renormalization in the local term results in the trivial change of a constant local measure.

[27]The function $f(x_1,\ldots,x_n)$ is called homogeneous of order m if the relation

$$f(Ax_1,\ldots,Ax_n)=A^m f(x_1,\ldots,x_n)$$

is true for arbitrary A.

[28]It must be noted that the change of integration variables $\hat{g}'=\hat{g}/A^{(0)}$, contrary to (5.5), would not be straightforward, because it changes the behavior of the functions (5.7) at infinity.

[29]If the function $f(x_1,\ldots,x_n)$ is homogeneous of order m, the following Euler theorem is true:

$$x_1\frac{\partial f}{\partial x_1}+\ldots+x_n\frac{\partial f}{\partial x_n}=mf(x_1,\ldots,x_n).$$

[30]A thorough discussion of the possible rôle of Planck's length in particle physics may be found in K. P. Stanjukovich, *Gravitational Field and Elementary Particles* (Nauka, Moscow, 1965).

[31]A. A. Slavnov, Kiev Report No. ITP-71-131E, 1971 (unpublished).

[32]It is claimed in the paper by P. P. Kulish [in proceedings of International Seminar on the Functional Methods in Quantum Field Theory and Statistics, Part II (Ref. 22)], that infrared divergences exist in the theory of the pure gravitational field. However, we do not believe this is so, because the infrared divergences always occur in the theories in which the Born amplitudes of the radiation of massless quanta do not vanish at the threshold, while in the case of the gravitational field, the bare vertex functions are quadratic in momenta. Perturbation-theory calculations in the regularization of Ref. 7 show the absence of infrared divergences on the mass shell.

[33]The idea of derivation of gauge relations of the bare vertices was given in Ref. 2, and the set of Eqs. (A1)–(A4) was obtained, but these equations do not have the form of Ward identities yet. In order to put these equations into the form of Ward identities, the investigation of Appendix A is needed. Its essence lies in the use of additional terms with the constant α in order to invert the second variation of action.

[34]Here and below, the Latin index repeated, when it stands by the operator ∇^+, does not mean the integration over the continuous index, while in other cases the summation rules (Ref. 25) remain.