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Decay Spectra of Particles and Resonances Produced in a Central Plateau

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We consider the two-body decay of the spinless resonance or particle produced in a central plateau, with an arbitrary transverse-momentum spectrum. The spectrum of the decay products is calculated exactly as an integral over the spectrum of the centrally produced resonance or particle. Special forms applicable to large and small momentum transfer are presented along with an accurate inversion formula. We show how the large-transverse-momentum behavior of the resonance production is replicated in the decay products. The decay $\pi^0 \to \gamma + \gamma$ is considered in detail.

I. INTRODUCTION

The recent verification of the existence of a central plateau in the CERN Intersecting Storage Rings (ISR) experiments[1] allows us to probe deeper into the detailed mechanism of pionization. In previous papers[2,3] we have investigated the properties of the pionization spectrum in $q_\perp{}^2$ resulting from the internal-damping structure. As originally discussed by Amati, Stanghellini, and Fubini[4] (ASF), the pions in the central plateau arise from fireballs or resonances produced in a chain of peripheral pion exchanges. In this paper we calculate the inclusive spectrum of a particle resulting from decay of a spinless two-particle resonance which is peripherally produced in a central-plateau region. The generality of the calculation allows it to be applied also to the case of a π^0 produced in the central plateau, which then decays into two photons. It can then be used to infer the π^0 spectrum from the γ spectrum.

Our work is an extension of the treatment of these problems as recently considered by others[5-10] Our formulation (1) includes an exact treatment of the kinematics and integrations; (2) is applicable to any $q_\perp{}^2$ spectrum of produced resonances or π^0's; (3) applies to both large and small $q_\perp$; (4) has the integrations performed analytically, not numerically; (5) gives a unified treatment of massive and massless final particles. The formulation includes many of the earlier results as limiting or special cases.

The calculation proceeds by considering a resonance of momentum q and mass $q^2 \equiv m^2$ being produced in a central plateau with a spectrum $\rho(q^2, q_\perp{}^2)$ independent of longitudinal momentum. This then decays into two particles of masses μ_1 and μ_2 so that $q = q_1 + q_2$. Since only one particle q_1 is observed in the single-particle spectrum, we must integrate over the momentum of q_2. It is convenient to work with

$$\begin{aligned} \eta &\equiv (q_1^\perp + q_2^\perp)^2 + m^2 , \\ \eta_1 &= (q_1^\perp)^2 + \mu_1{}^2 , \\ \eta_2 &= (q_2^\perp)^2 + \mu_2{}^2 , \\ m_0{}^2 &\equiv m^2 - \mu_2{}^2 + \mu_1{}^2 , \end{aligned} \qquad (1.1)$$

where the $\perp$ denotes two-dimensional transverse vectors. The integration over q_2 is performed by converting to integrals over η, η_2, and the rapidity $y_2 = \sinh^{-1}(q_2^\parallel/\eta_2{}^{1/2})$. The η_2 and y_2 integrations are performed exactly for infinite energy, and the integral over η, with the general function $\rho(\eta) \equiv \rho(m^2, q_\perp{}^2)$ remains.

In Sec. II we formulate the problem and calculate the decay pionization spectrum for $\eta_1 \geq m_0^4/4m^2$. In Sec. III we study the large-η_1 approximation and show how the large-η behavior of $\rho(\eta)$ replicates itself in the large-η_1 behavior. In Sec. IV the decay spectrum is calculated for $\eta_1 \leq m_0^4/4m^2$. The $\pi^0 \to 2\gamma$ decay is presented in Sec. V.

II. RESONANCE DECAY SPECTRUM

We assume that a spinless resonance or particle R is produced in the central-plateau region and calculate the transverse-momentum distribution of its decay products. For notation we call the decay products π_1, and π_2 of mass μ_1 and μ_2 and consider them as distinguishable. In Sec. V, however, they are considered as massless photons.

The inclusive R production $a+b \to R(\pi_1\pi_2)+X$

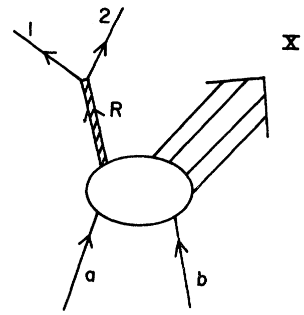

FIG. 1. Inclusive cross section for $a+b \to R+X$, where $R \to 1+2$.

(Fig. 1) contributes to the single-particle spectrum for π_1 in the central plateau[11]:

$$E_1 \frac{d\sigma}{d^3q_1} = \frac{d\sigma^{a+b \to X+1+2}}{d^2q_1^\perp dy_1} \equiv \rho_1(\eta_1), \tag{2.1}$$

$$\rho_1(\eta_1) = \frac{1}{2s} \frac{1}{2(2\pi)^3} \sum_X \int d\varphi_2 \int d\Phi_X (2\pi)^4 \delta^4(q_1+q_2+q_X-p_a-p_b) \left| \frac{T(ab \to RX)\, T(R \to \pi_1\pi_2)}{(q_1+q_2)^2 - m_R^2 + i\Gamma m_R} \right|^2, \tag{2.2}$$

where

$$d\varphi_2 = \frac{d^2q_2^\perp dy_2}{2(2\pi)^3}$$

and

$$T(R \to \pi_1\pi_2) = (16\pi\Gamma m_R)^{1/2}.$$

We assume that the inclusive integration and summation over X produces the spinless resonances R in a central-plateau region constant in the R's rapidity:

$$\frac{d\sigma^{ab \to X+R}}{d^2q_\perp dy} \equiv \rho(q^2, q_\perp^2). \tag{2.3}$$

This gives

$$\rho_1(\eta_1) = \int \frac{d^2q_2^\perp dy_2}{2(2\pi)^3} \frac{(16\pi\Gamma m_R)\rho((q_1+q_2)^2, (q_1+q_2)_\perp^2)}{|(q_1+q_2)^2 - m_R^2 + i\Gamma m_R|^2}. \tag{2.4}$$

We can make this into a superposition of spectra for various masses $m^2 = (q_1+q_2)^2$ of the virtual resonance by introducing $\int dm^2 \sigma(m^2 - (q_1+q_2)^2) = 1$:

$$\rho_1(\eta_1) = \left[\frac{1}{\pi} \int dm^2 \frac{\Gamma m_R}{|m^2 - m_R^2 + i\Gamma m_R|^2}\right] \frac{1}{\pi} \int d^2q_2^\perp\, dy_2\, \delta((q_1+q_2)^2 - m^2)\, \rho(m^2, (q_1+q_2)_\perp^2). \tag{2.5}$$

In the narrow-width limit $\Gamma \ll m_R$ the integral in brackets becomes 1, and we evaluate the remaining integral at $m^2 = m_R^2$. For the remainder of this paper we will suppress this integral over the resonance virtual mass as well as the dependence of ρ on m^2.

We now proceed to convert the integral over $d^2q_2^\perp\, dy_2$ to $d\eta\, d\eta_2\, dy_2$ and perform the integrations over $d\eta_2\, dy_2$, leaving only the $d\eta$ integral over the unspecified resonance spectrum

$$\rho(\eta) \equiv \rho(m^2, (q_1+q_2)_\perp^2). \tag{2.6}$$

Using the variables:

$$q_1^0 = \eta_1^{1/2} \cosh y_1, \quad q_1^z = \eta_1^{1/2} \sinh y_1,$$
$$q_2^0 = \eta_2^{1/2} \cosh y_2, \quad q_2^z = \eta_2^{1/2} \sinh y_2, \tag{2.7}$$
$$\eta_1 = (q_1^\perp)^2 + \mu_1^2, \quad \eta_2 = (q_2^\perp)^2 + \mu_2^2,$$

we compute the argument of the δ function:

$$(q_1+q_2)^2 - m^2 = \mu_1^2 + \mu_2^2 + (q_1^\perp)^2 + (q_2^\perp)^2 - m^2 - (q_1^\perp + q_2^\perp)^2 + 2(\eta_1\eta_2)^{1/2} \cosh(y_1 - y_2). \tag{2.8}$$

The δ function is then satisfied at two values of y_2 given by

$$\eta = \eta_1 + \eta_2 + 2(\eta_1\eta_2)^{1/2}\cosh(y_1 - y_2)\,. \tag{2.9}$$

Integrating over all y_2 for infinite energy will then give a θ function when (2.9) can be satisfied:

$$\rho_1(\eta_1) = \frac{2}{\pi}\int d^2q_2^\perp \frac{\rho(\eta)\,\theta(\eta^{1/2} - \eta_1^{1/2} - \eta_2^{1/2})}{2\eta_1^{1/2}\eta_2^{1/2}\,|\sinh(y_1 - y_2)|}\,. \tag{2.10}$$

Calculating the denominator from (2.9) gives

$$\rho_1(\eta_1) = \frac{2}{\pi}\int d^2q_2^\perp \frac{\rho(\eta)\,\theta(\eta^{1/2} - \eta_1^{1/2} - \eta_2^{1/2})}{\Delta^{1/2}(\eta, \eta_1, \eta_2)}\,, \tag{2.11}$$

where

$$\Delta(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx\,. \tag{2.12}$$

We now convert

$$d^2q_2^\perp = \tfrac{1}{2}\,d(q_2^\perp)^2\,d\,\theta_{12}$$
$$= -\tfrac{1}{2}\,d\eta_2\,\frac{d(\cos\theta_{12})}{\sin\theta_{12}}\,. \tag{2.13}$$

Using

$$\eta = (q_1^\perp)^2 + (q_2^\perp)^2 + 2\,|q_1^\perp|\,|q_2^\perp|\,\cos\theta_{12} + m^2\,, \tag{2.14}$$

we have

$$d^2q_2^\perp = (2)\tfrac{1}{2}\,\frac{d\eta_2\,d\eta}{[-\Delta((\eta - m^2), (q_1^\perp)^2, (q_2^\perp)^2)]^{1/2}}\,, \tag{2.15}$$

where the extra (2) is included for the double-valued mapping $\eta \to \pm\,\theta_{12}$. Rewriting this in terms of η_1, η_2 we have from (2.11)

$$\rho_1(\eta_1) = \frac{2}{\pi}\int d\eta\int d\eta_2\,\frac{\rho(\eta)\theta(\eta^{1/2} - \eta_1^{1/2} - \eta_2^{1/2})}{\Delta^{1/2}(\eta, \eta_1, \eta_2)}\;\frac{\theta(-\Delta(\eta - m^2, \eta_1 - \mu_1^{\,2}, \eta_2 - \mu_2^{\,2}))}{[-\Delta(\eta - m^2, \eta_1 - \mu_1^{\,2}, \eta_2 - \mu_2^{\,2})]^{1/2}}\,, \tag{2.16}$$

where the latter θ function guarantees a physical angle between $q_1^\perp$ and $q_2^\perp$.

We define the four roots in η_2 of the denominator:

$$\begin{aligned} a &= (\eta^{1/2} + \eta_1^{1/2})^2\,,\\ b &= \mu_2^{\,2} + [\,(\eta - m^2)^{1/2} + (\eta_1 - \mu_1^{\,2})^{1/2}]^2\,,\\ c &= (\eta^{1/2} - \eta_1^{1/2})^2\,,\\ d &= \mu_2^{\,2} + [(\eta - m^2)^{1/2} - (\eta_1 - \mu_1^{\,2})^{1/2}]^2\,. \end{aligned} \tag{2.17}$$

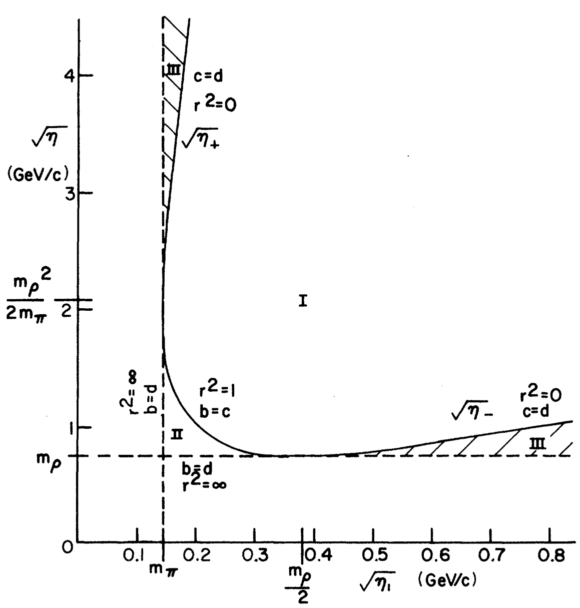

FIG. 2. Kinematic boundaries for $\rho \to \pi + \pi$. In region I: $b > c > d$; in region II: $c > b > d$; regions III are not allowed kinematically.

This gives

$$\rho_1(\eta_1) = \frac{2}{\pi}\int d\eta\int d\eta_2 \times \frac{\rho(\eta)\theta((a - \eta_2)(c - \eta_2))\,\theta((b - \eta_2)(\eta_2 - d))}{[-(\eta_2 - a)(\eta_2 - b)(\eta_2 - c)(\eta_2 - d)]^{1/2}}\,. \tag{2.18}$$

The θ functions then give the limits

$$\begin{aligned} &\eta_2 \le c \le a\,,\\ &d \le \eta_2 \le b\,. \end{aligned} \tag{2.19}$$

Therefore, for the upper limit on η_2 we must know when b is greater or less than c. The solutions in η to the equations $b = c$ and $c = d$ are formally

$$\eta_\pm \equiv \frac{m_0^{\,4}}{4\mu_1^{\,4}}\left[\eta_1^{1/2} \pm (\eta_1 - \mu_1^{\,2})^{1/2}\left(1 - \frac{4\mu_1^{\,2}m^2}{m_0^{\,4}}\right)^{1/2}\right]^2\,. \tag{2.20}$$

By substituting $\eta = \eta_+$ into (2.17), we find that η_+ corresponds to $c = d$ but not to $b = c$. It is therefore the upper boundary of the η integration region (see Fig. 2).

In substituting $\eta = \eta_-$ in (2.17), we find that for $\eta_1 > m_0^4/4m^2$

$$(\eta_- - m^2)^{1/2} = \frac{m_0^{\,2}}{2\mu_1^{\,2}}\left[(\eta_1 - \mu_1^{\,2})^{1/2} - \eta_1^{1/2}\left(1 - \frac{4\mu_1^{\,2}m^2}{m_0^{\,4}}\right)^{1/2}\right]\,, \tag{2.21}$$

while for $\eta_1 < m_0^{\,4}/4m^2$, the right-hand side has the opposite sign. Then for $\eta_1 > m_0^{\,4}/4m^2$, the substitution $\eta = \eta_-$ corresponds to $c = d$ but not to

$b=c$ and is the lower limit of the η integration. However, for $\eta_1 < m_0^4/4m^2$ $\eta=\eta_-$ corresponds to $b=c$ (Fig. 2).

In this section we treat the case $\eta_1 > m_0^4/4m^2$ which for a ρ resonance is $|q_1^\perp| > 0.36$ GeV/c and for a π^0 decay is $|q_1^\perp| > 0.07$ GeV/c. For $\eta_1 > m_0^4/4m^2$ we find from (2.17) that $b > c$ always and, therefore, the limits are

$$d \le \eta_2 \le c \,. \tag{2.22}$$

Also, for $\eta_1 > m_0^4/4m^2$ we have shown that

$$\eta_- \le \eta \le \eta_+ \,. \tag{2.23}$$

with $a > b > c > d$ we define

$$r^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)}\,,$$

$$r^2 = \frac{m_0^4 - 4[(\eta\eta_1)^{1/2} - (\eta - m^2)^{1/2}(\eta_1 - \mu_1^2)^{1/2}]^2}{16[\eta\eta_1(\eta - m^2)(\eta_1 - \mu_1^2)]^{1/2}} \,. \tag{2.24}$$

Then the exact result for the single-particle spectrum from (2.18) is,[12] for $\eta_1 > m_0^4/4m^2$,

$$\rho_1(\eta_1) = \int_{\eta_-}^{\eta_+} d\eta \, \frac{F(\frac{1}{2}, \frac{1}{2}; 1; r^2)\rho(\eta)}{2[\eta\eta_1(\eta - m^2)(\eta_1 - \mu_1^2)]^{1/4}} \,. \tag{2.25}$$

F is a hypergeometric function which is 1 at $r^2 = 0$. It is related to elliptic integrals[12]:

$$F(\tfrac{1}{2}, \tfrac{1}{2}; 1; r^2) = \frac{2}{\pi} F(\tfrac{1}{2}\pi, r) = \frac{2}{\pi} K(r) \,. \tag{2.26}$$

III. SPECTRA FOR η_1 LARGE COMPARED WITH m^2

The relation between the resonance spectrum $\rho(\eta)$ and the decay spectrum $\rho_1(\eta_1)$ becomes even simpler for large η_1.

For $\eta_1 \gg \mu_1^2$, the limits in (2.20) become

$$\eta_+ \to \frac{m_0^4}{4\mu_1^4}\eta_1\left[1 + \left(1 - \frac{4\mu_1^2 m^2}{m_0^4}\right)^{1/2}\right]^2 \equiv \eta_1 c_+ \,, \tag{3.1}$$

$$\eta_+ \xrightarrow[m^2 \gg \mu_1^2, \mu_2^2]{} \frac{m^4}{\mu_1^4}\eta_1 \,,$$

$$\eta_- \to \frac{m_0^4}{4\mu_1^4}\eta_1\left[1 - \left(1 - \frac{4\mu_1^2 m^2}{m_0^4}\right)^{1/2}\right]^2 \equiv \eta_1 c_- \,. \tag{3.2}$$

For $\eta_1 \gg m^2$, r^2 approaches zero as

$$r^2 \approx \frac{m_0^4}{16\eta_1\eta} < \frac{m_0^4}{16\eta_1^2} \,. \tag{3.3}$$

In fact, for $\eta_1 \ge 1.5m^2$ we have to better than 1% accuracy

$$F(\tfrac{1}{2}, \tfrac{1}{2}; 1; r^2) \cong 1 \,.$$

In this region

$$\rho_1(\eta_1) \cong \frac{1}{2\eta_1^{1/2}} \int_{\eta_1 c_-}^{\eta_1 c_+} d\eta \, \frac{\rho(\eta)}{[\eta(\eta - m^2)]^{1/4}} \,. \tag{3.4}$$

We may invert this by differentiation and ignore ρ at the upper limit if it is rapidly falling to obtain

$$\rho(\eta_1 c_-) = -\frac{2}{c_-^{1/2}}\left[\eta_1\left(\eta_1 - \frac{m^2}{c_-}\right)\right]^{1/4} \frac{d}{d\eta_1}[\eta_1^{1/2}\rho_1(\eta_1)] \,. \tag{3.5}$$

We are particularly interested in the connection between the asymptotic behavior of $\rho(\eta)$ and that of $\rho_1(\eta_1)$. For the case $\eta \gg m^2 \gg \mu_1^2$ we take the upper limit effectively infinite:

$$\rho_1(\eta_1) \to \frac{1}{2\eta_1^{1/2}} \int_{\eta_1 c_-}^{\infty} d\eta \frac{\rho(\eta)}{\eta^{1/2}} \,. \tag{3.6}$$

The three cases for large-η behavior,

$$\rho(\eta) \to [\eta^{-\alpha}, e^{-a\eta}, e^{-b\eta^{1/2}}] \,, \tag{3.7}$$

become after integration at large η_1 ($\alpha > 0$)

$$\rho_1(\eta_1) \to \left[\eta_1^{-\alpha}, \frac{e^{-a\eta_1}}{a\eta_1}, \frac{e^{-b\eta_1^{1/2}}}{b\eta_1^{1/2}}\right] \,. \tag{3.8}$$

A simple ASF model for resonance production with peripheral pion exchange gives $\rho(\eta) \propto \eta^{-1}$ due to the pion propagators alone.[3] The resulting η_1^{-1} spectrum is inconsistent with data. In fact, internal form factors must also be included[3] to give $\rho(\eta) \propto \eta^{-4}$ and therefore $\rho_1(\eta_1) \propto \eta_1^{-4}$ to fit the data.

IV. DECAY SPECTRUM FOR $\eta_1 < m_0^4/4m^2$

For $\eta_1 < m_0^4/4m^2$, we find from Fig. 2 two regions of η, η_1 space which give different η_2 limits. η_1 is fixed and for region I (Fig. 2)

$$\eta_- < \eta < \eta_+ ; \; b > c \text{ so } d \le \eta_2 \le c \,. \tag{4.1}$$

But for region II

$$m^2 \le \eta < \eta_- ; \; d < b < c \text{ so } d \le \eta_2 \le b \,. \tag{4.2}$$

From (2.16) we now include the additional contributions of region II:

$$\rho_1(\eta_1)=\frac{2}{\pi}\int_{\eta_-}^{\eta_+}d\eta\,\rho(\eta)\int_d^c d\eta_2\,[(a-\eta_2)(b-\eta_2)(c-\eta_2)(\eta_2-d)]^{-1/2}$$
$$+\frac{2}{\pi}\int_{m^2}^{\eta_-}d\eta\,\rho(n)\int_d^b d\eta_2\,[(a-\eta_2)(c-\eta_2)(b-\eta_2)(\eta_2-d)]^{-1/2}\,. \qquad (4.3)$$

In the second integral $a>c>b>d$, giving[12] the result

$$\rho_1(\eta_1)=\int_{\eta_-}^{\eta_+}d\eta\,\frac{F(\frac{1}{2},\frac{1}{2};1;r^2)\rho(\eta)}{2[\eta\eta_1(\eta-m^2)(\eta_1-\mu_1^2)]^{1/4}}+2\int_{m^2}^{\eta_-}d\eta\,\frac{F(\frac{1}{2},\frac{1}{2};1;r^{-2})\rho(\eta)}{\{m_0^4-4[\eta^{1/2}\eta_1^{1/2}-(\eta-m^2)^{1/2}(\eta_1-\mu_1^2)^{1/2}]^2\}^{1/2}}\,. \qquad (4.4)$$

The $1/r^2$ in the second integral resulted from the interchange of b and c in the ordering and in (2.24). In the first integral $r^2\leq 1$, and in the second $1/r^2\leq 1$.

We note that for $\eta_1<m_0^4/4m^2$ the curve $r^2=1$ or $b=c$ always occurs in the integration region, Fig. 2, and it is no longer possible to approximate the hypergeometric function by 1 throughout the entire region.

We take the limiting case $\eta_1\to\mu_1^2$ or $(q_1^\perp)^2\to 0$. In this limit[13]

$$r^2\propto\frac{1}{(\eta_1-\mu_1^2)^{1/2}}\to\infty\,, \qquad (4.5)$$
$$F(\tfrac{1}{2},\tfrac{1}{2};1;r^2)\to\frac{\ln r}{r}$$
$$\propto(\eta_1-\mu_1^2)^{1/4}\ln(\eta_1-\mu_1^2)\,,$$
$$(\eta_+-\eta_-)\propto(\eta_1-\mu_1^2)^{1/2}\,.$$

Combining the above we find that the first integral in (4.4) vanishes as $\eta_1\to\mu^2$, and in the second integral the hypergeometric function approaches 1:

$$\rho_1(\eta_1=\mu_1^2)=2\int_{m^2}^{m_0^4/4\mu_1^2}d\eta\,\frac{\rho(\eta)}{(m_0^4-4\eta\mu_1^2)^{1/2}}\,. \qquad (4.6)$$

If $m^2\gg 4\mu_1^2$ and $\rho(\eta)$ is rapidly falling, we find that

$$\rho_1(\eta_1=\mu_1^2)\cong\frac{2}{m_0^2}\int_{m^2}^{\infty}d\eta\,\rho(\eta)\,. \qquad (4.7)$$

V. $\pi^0\to 2\gamma$ DECAY SPECTRUM

The decay $\pi^0\to 2\gamma$ is of course characterized by a spinless decaying particle with no width, $\mu=0$, $\eta_1=(q_1^\perp)^2$, $m=m_{\pi^0}$, and $\rho_{\pi^0}(\eta)$ is the π^0 production spectrum. The kinematics and results are obtained from the general case above by taking the $\mu\to 0$ limit. In this limit we have from (2.20) (denoting variables for $\mu=0$ with primes)

$$\eta'_+=+\infty\,,$$
$$\eta'_-=\left(\eta_1^{1/2}+\frac{m^2}{4\eta_1^{1/2}}\right)^2\,. \qquad (5.1)$$

The resulting kinematic region is shown in Fig. 3. Again there are two regions of integration for $\eta_1<m^2/4$.

From (2.24) we now have

$$r'^2=\frac{m^4-4\eta_1[\eta^{1/2}-(\eta-m^2)^{1/2}]^2}{16\eta_1\eta^{1/2}(\eta-m^2)^{1/2}}\,. \qquad (5.2)$$

For the photon spectrum we include an extra factor of two for identical photons. For $\eta_1>m^2/4$ or $|q_1^\perp|>\frac{1}{2}m_{\pi^0}=0.07$ GeV we obtain from (2.25)[14]

$$\rho_\gamma(\eta_1)=2\int_{\eta'_-}^{\infty}d\eta\,\frac{F(\frac{1}{2},\frac{1}{2};1;r'^2)\rho_{\pi^0}(\eta)}{2\eta_1^{1/2}[\eta(\eta-m^2)]^{1/4}}\,. \qquad (5.3)$$

Again, for $\eta_1>1.5\,m^2$ or $|q_1^\perp|>0.1$ GeV we can approximate the hypergeometric function by 1 and obtain[7] to 1% accuracy

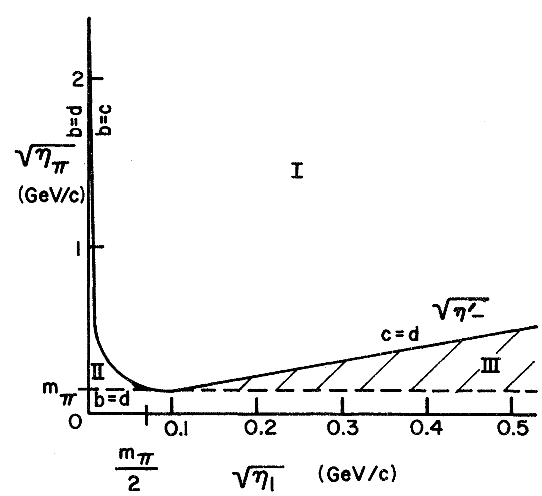

FIG. 3. Same as Fig. 2 but for $\pi^0\to\gamma+\gamma$.

$$\rho_\gamma(\eta_1)=\frac{1}{\eta_1^{1/2}}\int_{\eta'_-}^{\infty}d\eta\,\frac{\rho_{\pi^0}(\eta)}{[\eta(\eta-m^2)]^{1/4}}\,. \tag{5.4}$$

The inverse is obtained by differentiation:

$$\rho_{\pi^0}(\eta'_-)=-\left(1-\frac{m^4}{16\eta_1^2}\right)^{3/2}\eta_1^{1/2}\frac{d}{d\eta_1}[\eta_1^{1/2}\rho_\gamma(\eta_1)]\,. \tag{5.5}$$

For $\eta_1 \gg m^2$ this becomes the approximation of Sternheimer[5]:

$$\rho_{\pi^0}(q_\perp^2)\approx-\tfrac{1}{2}\frac{d}{dq_\perp}[(q_\perp)\rho_\gamma(q_\perp^2)]\,. \tag{5.6}$$

For $\eta_1 < m^2/4$, or $|q_1^\perp| < \frac{1}{2}m_{\pi^0}$ we have from (4.4)

$$\rho_\gamma(\eta_1)=\int_{\eta'_-}^{\infty}d\eta\,\frac{F(\frac{1}{2},\frac{1}{2};1;r'^2)\rho_{\pi^0}(\eta)}{\eta_1^{1/2}[\eta(\eta-m^2)]^{1/4}}+4\int_{m^2}^{\eta'_-}d\eta\,\frac{F\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{r'^2}\right)\rho_{\pi^0}(\eta)}{[m^4-4\eta_1(\eta^{1/2}-(\eta-m^2)^{1/2})^2]^{1/2}}\,. \tag{5.7}$$

The point $|q_1^\perp|=0$ obtained from (4.6) is[7]

$$\rho_\gamma(\eta_1=0)=\frac{4}{m^2}\int_{m^2}^{\infty}d\eta\,\rho_{\pi^0}(\eta)\,. \tag{5.8}$$

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[14] This result also follows from Ref. 8 [Eq. (30)–Eq. 32)] by completing an integral contained there to obtain the $F(\frac{1}{2},\frac{1}{2};1;r'^2)$ function.