$$\epsilon(y,x)=\epsilon_0(y,x)+\sum_1^N \delta_m(x)g_m(y). \tag{A11}$$

It then follows that

$$\|\epsilon(\cdot,x)\|^2=\|\epsilon_0(\cdot,x)\|^2+\|\sum_1^N \delta_m(x)g_m(\cdots)\|^2 +2\ \text{Real}\sum_1^N \delta_m(x)(\epsilon_0(\cdots,x),g_m(\cdots)). \tag{A12}$$

However, one can easily see from Eq. (A9) that

$$(\epsilon_0(\cdots,x),g_m(\cdots))=0. \tag{A13}$$

Therefore

$$\epsilon(\cdots,x)\geq\epsilon_0(\cdots,x). \quad \text{Q.E.D.} \tag{A14}$$

Actually, condition (A13) suggests that the optimal $\mathcal{L}_N(y,x)$ should be the reproducing kernel of the sub-Hilbert space spanned by $g_m(x)$, $m=1,2,\ldots,N$, which, according to a theorem[5] on the completeness of a Hilbert space, must be $g_m(Y)(K^{-1})_{mn}g_n(x)$.

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Deuteron Spin Alignment in High-Energy Elastic Proton-Deuteron Backscattering*

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The spin alignment $\langle T_{20}\rangle$ of recoil deuterons in high-energy p-d elastic scattering at 180° is calculated in two models. One is the simple nucleon-transfer or pickup mechanism. The other is the extended pickup mechanism of Kerman and Kisslinger that includes, in addition to nucleon exchange, the exchange of $N^*(1688)$ resonance which is taken to be present in the deuteron with a small (1–2%) probability. For proton lab energies around 1 GeV, the Kerman-Kisslinger model gives alignments that are characteristically different from the predictions of the simple exchange model, showing that the presence of the N^* in the deuteron has a significant effect at these energies.

I. INTRODUCTION

It has been known for some time that deuteron spin-polarization effects are present in high-energy elastic proton-deuteron scattering. The Glauber multiple scattering theory shows that the cross section is strongly dependent on deuteron polarization, due to the D-state admixture in the deuteron wave function.[1] It has been also suggested[2] that for the same reason elastic π-d or p-d scattering can be used to produce aligned deuteron beams. On the experimental side, both vector and tensor polarization of deuterons in foward p-d scattering have been observed by means of double scattering techniques.[3] The Glauber theory, with a suitable extension to include nucleon spin-orbit effects, is known to account fairly well for the observed polarization.[4] In contrast, not very much is known about deuteron polarization effects in backward p-d scattering at high energy, a domain in which the differential cross section shows a striking peak. It is of interest to investigate these effects at back angles in view of the possibility that such an investigation might help distinguish the different models that have been proposed to explain the structure of the backward cross section.

In the present work an attempt is made to study deuteron polarization[5] effects in high-energy backward p-d elastic scattering. The point of view is taken that the mechanism for backward scattering is a nucleon-transfer (pickup) process.[6] Two mod-

els are considered, the simple nucleon-transfer mechanism and an extension of it by Kerman and Kisslinger[7] in which they propose a new structure for the deuteron that includes a component consisting of the $N^*(1688)$ and nucleon. For reasons of simplicity, consideration is limited to polarization effects at 180°. At this angle the recoil deuterons have a nonvanishing alignment. Results are presented for the typical proton energy of 1 GeV, the energy at which the effects of the N^* resonance are expected to be important in the Kerman-Kisslinger model.

II. NUCLEON-TRANSFER MECHANISM

Elastic proton-deuteron backscattering is envisioned to take place via the nucleon-transfer mechanism shown diagrammatically in Fig. 1. The neutron originally in the deuteron combines with the initial proton to form the deuteron in the final state. Taking the wave function of the deuteron to be

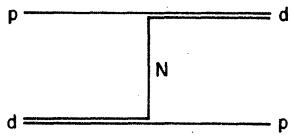

FIG. 1. Simple nucleon-transfer mechanism.

$$\psi_{\text{deut}} = R_S(r)\, Y_{00}\, \chi^1 + R_D(r)[\, Y_2\, \chi^1]^1 , \tag{1}$$

one obtains the following expression for the transition amplitude for going from the initial state (M_i, m_i) to the final state (M_f, m_f) (see Refs. 8, 9):

$$T(M_f, m_f; M_i, m_i) = \delta(\vec{p}_f + \vec{d}_f, \vec{p}_i + \vec{d}_i) \left(-\frac{\kappa^2 + \Delta^2}{M_N}\right)$$

$$\times \sum_{L, L'=0,2} R_L(\Delta) R_{L'}(\Delta') \sum_m Y^*_{L', z'-z+m}(\hat{\Delta}') Y_{Lm}(\hat{\Delta})$$

$$\times \{C(\tfrac{1}{2}\tfrac{1}{2}1; m_i, M_i - m - m_f) C(L'11; z'-z+m,\ z-m_f-m)$$

$$\times C(\tfrac{1}{2}\tfrac{1}{2}1; m_f, M_i - m_f - m) C(L11; m, M_i - m)\}, \tag{2}$$

where $\vec{p}_f, \vec{d}_f; \vec{p}_i, \vec{d}_i$ are the final and initial momenta of the proton and deuteron with $\vec{\Delta} = \frac{1}{2}\vec{d}_i - \vec{p}_f$, $\vec{\Delta}' = \frac{1}{2}\vec{d}_f - \vec{p}_i$; Δ, Δ' are the magnitudes of $\vec{\Delta}$ and $\vec{\Delta}'$. The binding energy B of the deuteron is given by $B = \kappa^2/M_N$, M_N being the nucleon mass. M_i, m_i (M_f, m_f) are deuteron and proton spin projections in the initial (final) state, with $z = M_i + m_i$ and $z' = M_f + m_f$. R_L and $R_{L'}$ are Fourier transforms of deuteron wave functions

$$R_S(\Delta) = R_{L=0}(\Delta) = \left(\frac{2}{\pi}\right)^{1/2} \int dr\, r^2 j_0(\Delta r) R_S(r),$$

$$R_D(\Delta) = R_{L=2}(\Delta) = -\left(\frac{2}{\pi}\right)^{1/2} \int dr\, r^2 j_2(\Delta r) R_D(r). \tag{3}$$

The Y's are spherical harmonics and the C's are Clebsch-Gordan coefficients.

In the center-of-mass (c.m.) frame we have $\Delta = \Delta'$ for elastic scattering. For the special case of 180° scattering in the c.m. frame we have in addition $\vec{\Delta} = -\vec{\Delta}' = \frac{1}{2}\vec{p}_i$. As only backscattering will be considered here, it is natural to choose the z axis to be along $\vec{\Delta}$. Having made this choice, one sees from Eq. (2) that the backscattering amplitude vanishes unless $m = 0$ *and* $z' - z + m = 0$ which implies that $z' = z$. Hence, with the condition $M_i + m_i = M_f + m_f$, one obtains, omitting the δ function,

$$T(M_f, m_f; M_i, m_i) = -\left(\frac{\kappa^2 + \Delta^2}{M_N}\right) \sum_{L, L'=0,2} R_L(\Delta) R_{L'}(\Delta) Y^*_{L'0}(-\hat{z}) Y_{L0}(\hat{z})$$

$$\times [C(\tfrac{1}{2}\tfrac{1}{2}1; m_i, M_i - m_f) C(L'11; 0, M_f) C(\tfrac{1}{2}\tfrac{1}{2}1; m_f, M_i - m_f) C(L11; 0, M_i)]. \tag{4}$$

Of the $6 \times 6 = 36$ possible combinations of initial and final spin states, only some give rise to a nonzero T because of the condition $M_i + m_i = M_f + m_f$.

The amplitude T is sufficient for one to study polarization effects in backscattering. For unpolarized initial particles it turns out that only one of the eight parameters necessary to specify fully the polarization of the final deuterons is nonvanishing at 180°, if one assumes rotation and parity invariance. This is discussed below.

III. ALIGNMENT OF RECOIL DEUTERONS

The spin state of a deuteron taking part in a scattering is in general a statistical mixture of pure states, which is conveniently specified by giving the density matrix

$$\rho=\tfrac{1}{3}\sum_{J,M}\langle T_{JM}\rangle T^{\dagger}_{JM}, \quad M=-J,\ldots,+J;\ J=0,1,2 \tag{5}$$

where the brackets denote statistical expectation values.[10] The T_{JM} are irreducible tensor operators constructed out of the components of the spin operator $\vec{S}$ of the deuteron. Explicitly

$$\begin{aligned}
&T_{00}=1 \quad \text{(identity operator)},\\
&T_{11}=-\tfrac{1}{2}\sqrt{3}\,(S_x+iS_y),\\
&T_{10}=(\tfrac{3}{2})^{1/2}S_z,\\
&T_{22}=\tfrac{1}{2}\sqrt{3}\,(S_x+iS_y)^2,\\
&T_{21}=-\tfrac{1}{2}\sqrt{3}\,[(S_x+iS_y)S_z+S_z(S_x+iS_y)],\\
&T_{20}=(\tfrac{1}{2})^{1/2}(3S_z^{\,2}-2),\\
&T_{J,-M}=(-)^M T^{\dagger}_{JM}, \qquad \mathrm{Tr}[T_{JM}T^{\dagger}_{J'M'}]=3\delta_{JM,J'M'}.
\end{aligned} \tag{6}$$

The specification of the parameters $\langle T_{JM}\rangle$ completely fixes the spin state of the deuteron.

In the elastic scattering $p+d\to p+d$, the polarization $\langle T_{JM}\rangle$ of the final deuteron for the case of unpolarized initial particles is given by

$$I_0\langle T_{JM}\rangle=\tfrac{1}{6}\mathrm{Tr}[MM^{\dagger}T_{JM}] \tag{7}$$

when the final proton spin is not observed.[10] Here I_0 is the unpolarized differential cross section. M is the transition amplitude matrix in spin space.

The parameters $\langle T_{JM}\rangle$ specifying the polarization may be looked at in two ways in the case of elastic scattering. These specify the polarization of final state deuterons in scattering of unpolarized protons off unpolarized deuterons. A second scattering is necessary to analyze the polarization so produced. Thus, in a double-scattering experiment the first scattering acts as the "polarizer" and the second as the "analyzer." Alternatively, if one considers the scattering of protons off a polarized deuteron target whose polarization is now given by $\langle T_{JM}\rangle$, then these parameters give the dependence of the differential cross section on the initial polarization. This relationship between the two cases follows from time reversal invariance.[10] In recent years low-energy experiments involving polarized deuterons have also been done.[11]

For the case of exact backscattering all but one of the $\langle T_{JM}\rangle$ in Eq. (7) vanish because of invariance reasons. If one assumes that M is invariant under rotations and reflections, one can easily show[12] that only $\langle T_{20}\rangle$ is nonzero at 180° in the c.m. frame for the choice of z axis described in Sec. II.

It is now a simple matter to calculate $\langle T_{20}\rangle$. Defining a related quantity called alignment by the relation $A\equiv\langle 3S_z^{\,2}-2\rangle=\sqrt{2}\,\langle T_{20}\rangle$, if N_+, N_0, N_- are the numbers of outgoing deuterons with spin projections 1, 0, −1, respectively, along the z axis, then

$$\begin{aligned}
A&=3\left(\frac{N_++N_-}{N_++N_-+N_0}\right)-2\\
&=\frac{N_++N_--2N_0}{N_++N_-+N_0}.
\end{aligned} \tag{8}$$

Bounds for A are readily obtained:

$$-2\leq A\leq 1. \tag{9}$$

To calculate the alignment we note that

$$N_+\propto\sum_{i,f}|T(1,m_f;M_i,m_i)|^2, \tag{10}$$

with similar relations for N_0 and N_-. Evaluating the nonzero amplitudes in Eq. (4) and using those to find N_+, N_0, and N_- one finally obtains for the alignment for 180° scattering the simple expression

$$A(180°)=\frac{2\sqrt{2}R_S(\Delta)R_D(\Delta)-R_D^{\,2}(\Delta)}{R_S^{\,2}(\Delta)+R_D^{\,2}(\Delta)}. \tag{11}$$

The alignment vanishes if there is no D state. It is equal to −1 when the S-state wave function is zero.

IV. EXTENDED NUCLEON-TRANSFER MECHANISM OF KERMAN AND KISSLINGER

A generalization of the nucleon-transfer mechanism was proposed by Kerman and Kisslinger (henceforth denoted KK) to explain the pronounced back peak seen in high-energy p-d elastic scattering.[13] At 1 GeV they take the exchange of the spin-$\frac{5}{2}$ $N^*(1688)$ nucleon resonance to contribute in addition to the nucleon exchange in the pickup mechanism for the backscattering. Diagrammatically the KK extension of the simple pickup model is the one shown in Fig. 2. On the basis of a nonrelativistic Lagrangian for πNN^* coupling, chosen in analogy to the static Chew-Low Lagrangian for πNN interaction, KK estimate that the deuteron can contain 1-2% of N^* components. The new baryonic structure of the deuteron manifests itself

in the wave function, which is now written as

$$\psi_{\text{deut}}=R_S(r)Y_{00}\chi^1+R_D(r)[Y_2\chi^1]^1 + \sum_{S=2,3} a_S R_{D*}(r)[Y_2\chi^S]^1 . \quad (12)$$

The first two terms are the $L=0$ and $L=2$ components of the conventional wave function. χ^S is the spin wave function corresponding to N^*N, so that $S=2$ or 3. $R_{D*}(r)$ is the radial wave function of the D^* component. The $L=4$ component that can combine with $S=3$ to give $J=1$ is neglected, as is also any dependence of the radial form $R_{D*}(r)$ on S, so that in this model the effect of including the N^* is to introduce a new $L=2$ component in the wave function. The constant a_S is calculated in perturbation theory. To do this, however, the wave function

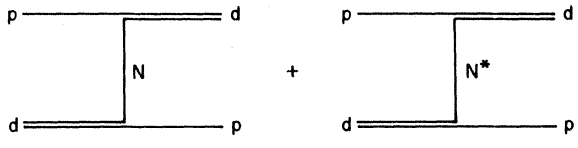

FIG. 2. Nucleon-transfer mechanism extended to include N^* resonance.

R_{D*} must be known. KK have used two different models for $R_{D*}(r)$ in their work.

With the choice of z axis along $\vec{\Delta}$, the KK-model Born approximation amplitude for exact backscattering in the c.m. frame is found to be

$$T(M_f, m_f; M_i, m_i)= T_{NN}+T_{NN*}, \quad (13)$$

where T_{NN} is the amplitude in Eq. (4) and

$$T_{NN*}=-\left(\frac{\kappa^{*2}+\Delta^2}{M_{N*}}\right)\sum_{S,S'=2,3} R_{D*,S}(\Delta)R_{D*,S'}(\Delta)Y_{20}^*(-\hat{z})Y_{20}(\hat{z}) \times[C(\tfrac{1}{2}\tfrac{5}{2}S';m_i,M_i-m_f)C(2S'1;0M_f)C(\tfrac{1}{2}\tfrac{5}{2}S;m_f,M_i-m_f)C(2S1;0M_i)], \quad (14)$$

where[14] $\kappa^{*2}=1.286\,M_N(B+M_{N*}-M_N)$ and $R_{D*,S}(\Delta)\equiv a_S R_{D*}(\Delta)$, with

$$R_{D*}(\Delta)=-\left(\frac{2}{\pi}\right)^{1/2}\int dr\, r^2 j_2(\Delta r)R_{D*}(r) . \quad (15)$$

The amplitudes T_{NN} and T_{NN*} for backscattering are zero unless $M_f+m_f=M_i+m_i$.

As in the simple nucleon-transfer case, the alignment A can be calculated once the nonzero amplitudes in Eq. (13) are computed. The resulting expression for A is, however, quite long. Introducing the following notation to simplify matters somewhat:

$$E=-\left(\frac{\kappa^2+\Delta^2}{4\pi M_N}\right), \quad E^*=-\left(\frac{\kappa^{*2}+\Delta^2}{4\pi M_{N*}}\right)$$
$$S=R_S(\Delta), \quad D=R_D(\Delta), \quad D_2=a_2R_{D*}(\Delta), \quad D_3=a_3R_{D*}(\Delta), \quad (16)$$

the alignment for 180° scattering is given by

$$A(180°)=N/D , \quad (17)$$

where

$$N=E^2(6.363\,S^3D-2.25\,S^2D^2+6.363\,S^3D-2.25\,D^4) + E^{*2}(1.25\,D_2^4-1.069\,D_2^3D_3+3.107\,D_2^2D_3^2-0.07633\,D_2D_3^3-0.7857\,D_3^4) + 2EE^*S^2(0.5\,D_2^2+0.6682\,D_2D_3-0.5\,D_3^2) + 2EE^*SD(0.7071\,D_2^2+1.795\,D_2D_3+2.929\,D_3^2)+2EE^*D^2(0.25\,D_2^2+0.9356\,D_2D_3-0.5714\,D_3^2) ,$$

$$D=E^2(2.25S^4+4.5S^2D^2+2.25\,D^4)+E^{*2}(1.25\,D_2^4-1.069\,D_2^3D_3+4.071\,D_2^2D_3^3+1.985\,D_2D_3^3+1.556\,D_3^4) + 2EE^*S^2(0.5\,D_2^2+1.871D_2D_3+1.75\,D_3^2)+2EE^*SD(0.7071\,D_2^2+0.9448D_2D_3-0.7071\,D_3^2) + 2EE^*D^2(0.25\,D_2^2-0.2673D_2D_3+0.07143\,D_3^2) .$$

As in the simple exchange model, the alignment is zero if only the S state is present in the deuteron wave function. If one puts $D_2=D_3=0$ in Eq. (17), one recovers Eq. (11).

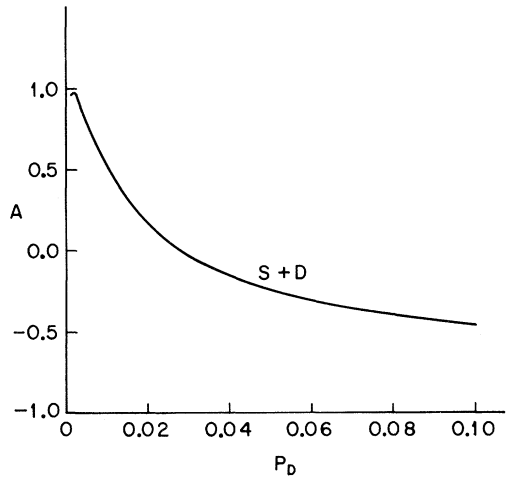

FIG. 3. Alignment of recoil deuterons in elastic p-d backscattering in the c.m. system in the simple nucleon-transfer model. The alignment is plotted against P_D, the D-state probability. The curve shown is for 1 GeV proton lab energy. The Bressel wave function for the deuteron has been used, its normalization being adjusted to give various values of P_D. "$S+D$" denotes that the curve is given by the simple exchange model, i.e., there is no D^*.

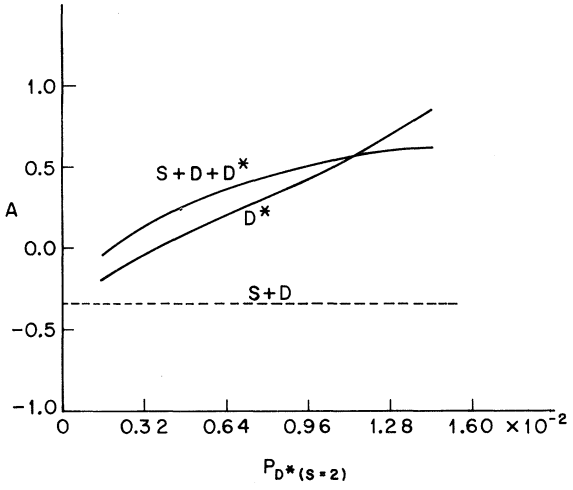

FIG. 4. Alignment of recoil deuterons in the extended nucleon-transfer model. The curves shown are all for proton lab energy of 1 GeV. The alignment is plotted against the percentage of the $S=2$ component of the D^* state. The curve labeled $S+D+D^*$ has been calculated with probabilities 6.5% and 1.6% for the D and D^* states. The curve labeled D^* is obtained by setting both S and D states to be zero in Eq. (17). The dashed line gives the position of the value of A in the absence of D^*, i.e., in the simple exchange model, at 1 GeV with 6.5% D state. The curves with a D^* in the label have been calculated with KK model I D^* wave function. Results using model II are very much similar to those shown.

V. RESULTS

The Bressel wave functions for the S and D states of the deuteron[15] have been used in the calculation. Model I and II functions of KK (see Ref. 7) have been used for the radial function R_{D^*}. Protons are assumed to be incident on stationary deuterons in the laboratory. Lab-to-c.m. kinematics has been done relativistically.

Results are presented for the typical proton laboratory kinetic energy of 1 GeV. In the KK model the N^* resonance has a pronounced effect at this energy on the differential cross section. It is therefore of interest to see the effect of the presence of this resonance in the deuteron on the alignment at 1 GeV. The predictions of the simple nucleon-transfer mechanism at 1 GeV are shown in Fig. 3. Choosing the values 6.5% and 1.6% for the probabilities of D and D^* components in the wave function, one obtains the results presented in Fig. 4 for KK-model calculations. It is seen that at 1

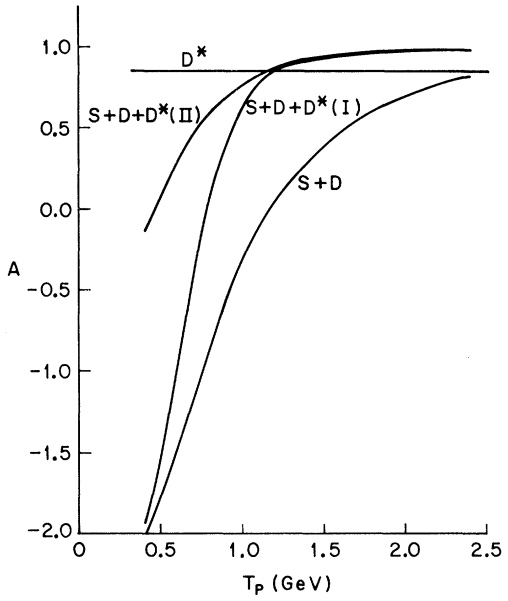

FIG. 5. Energy dependence of alignment at 180°. T_p is the proton kinetic energy in the lab. The curve labeled $S+D$ is calculated in the simple exchange model with 6.5% D state. The curves marked $S+D+D^*$ are based on KK model calculations. D^*(I) involves model I wave function for D^* with 1.6% probability. D^*(II) is model II with 2.7% probability. For both curves D^*(I) and D^*(II) the ratio a_2/a_3 in Eq. (12) is chosen to be 3, which is about the value used by KK. The D-state probability is 6.5% in both curves $S+D+D^*$. The horizontal line marked D^* is obtained by setting both S and D to be zero in Eq. (17). This turns out to be energy-independent because of the simplifying assumption that both $S=2$ and $S=3$ components of the D^* state have the same radial wave function.

GeV the D^* component has a significant effect, especially if it is predominantly in the $S=2$ configuration.[16]

The energy dependence of the alignment at 180° is shown in Fig. 5, where it is plotted as a function of the lab energy of incident protons. Results given by both models I and II for R_{D^*} are presented. The alignment varies quite markedly with energy in the range 0.5–1.5 GeV. Around 1 GeV, the simple exchange mechanism and the KK mechanism give rather different values for the alignment. In the KK model the alignment depends on the specific choice of the D^* wave function. However, model I and II functions give appreciably different values only below 1 GeV.

To sum up, at 1 GeV and slightly higher energies, the KK extension of the simple nucleon-transfer mechanism gives alignment effects at 180° that are characteristically different from the simple-exchange-model predictions. The alignments predicted are large fractions of the maximum possible value. It must be stated, however, that these conclusions are subject to the following limitations: The entire calculation has been based on a nonrelativistic formalism. The adequacy of such a procedure at the energies considered is open to question. Even otherwise the KK model is admittedly crude in view of the uncertainties regarding the πNN^* coupling and the lack of knowledge concerning the D^* wave function. Since in this work p-d scattering has been viewed as a first-order exchange process, the question also arises as to the importance of nonexchange processes and also of second-order exchange effects. Fukushima[9] estimates that nonexchange processes give negligible contribution to the cross section at 180°. He also finds that second-order corrections are not negligible. It may be that significant corrections to the foregoing results might come from more detailed considerations. The present work is therefore to be considered as an exploratory analysis of the possible deuteron polarization effects arising from its baryonic structure in high-energy p-d scattering.

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[12]The argument is quite similar to the one given in Sec. II of Ref. 10.

[13]See Ref. 7. The simple nucleon-transfer mechanism gives too low a value for the cross section at back angles.

[14]The definition of κ^* as given in Ref. 7 is used. Incidentally, the following errors in that paper may be noted: In Ref. 19, E_d should be replaced by ϵ_d. The factor 120 in Eq. (18) should read 60. A factor $a_{S_1} a_{S_2}$ should be inserted immediately after the summation sign in Eq. (18′). The Racah coefficient $W(\ell 22S; 12)$ in Eq. (18″) should read $W(\ell 12S; 12)$.

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[16]Kerman and Kisslinger use the value 3.7 for the ratio a_2/a_3 in their calculations of the cross section. See Ref. 9.