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# Inclusive Approach to the Multiplicity Distribution*

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Starting with the Mueller-Regge assumption, a multiperipheral-like integral equation for the multiplicity generating function is constructed and solved. The integral equation, therefore, involves *inclusive* parameters only. A one-to-one correspondence between our result and that of an exclusive multiperipheral model is established through a proper orthogonal matrix. Explicit expressions for the multiplicity distribution and correlations are obtained. Possible applications and generalizations are discussed within the framework of a two-component picture of hadronic productions.

## I. INTRODUCTION

We demonstrate in this paper how the multiplicity distribution for hadronic collision at high energies can be obtained if we assume the $n$-particle inclusive spectra satisfy the Mueller-Regge expansion with a *finite* number of Regge poles. A multiperipheral-like integral equation[1] for the generating function of this distribution is constructed in terms of inclusive parameters. The solution of this equation is explicitly obtained for the physically interesting case of two poles, although the general case of arbitrary number of poles can also be written down.

The Mueller-Regge approximation is plausible provided we are willing to leave out the diffractive component of the total cross section. Therefore, we are working within the framework of the standard two-component picture,[2] and the generating function obtained refers to the short-range correlation component only. For this reason, our scheme does not allow the possibility of a *complete* bootstrap, although partial bootstrap is possible if additional dynamical assumptions are made.

After an appropriate leading-particle modification, the generating function, which is written completely in terms of inclusive parameters, is compared with a similar one resulting from the exclusive multiperipheral model[3-5] (MPM). We find that they can be put in one-to-one correspondence with each other. This connection can be made by a proper orthogonal matrix, as has been found previously in Ref. 5. This equivalence allows us to express multiplicities, inclusive cross sections, and correlations in terms of parameters of either model whichever is more convenient, and provide us with a new technique of parametrizing experimental data.

We introduce notations and review the Mueller analysis of the multiplicity distribution in Sec. II. We construct and solve the integral equation for the generating function in Sec. III. In Sec. IV, we compare our results with that of a multiperipheral model. We show that they are indeed equivalent. Possible applications and generalizations are discussed in Sec. V.

For simplicity, we shall assume the existence of only one type of scalar particle and ignore the dependence on transverse momenta throughout the discussion.

## II. MULTIPLICITY GENERATING FUNCTION AND CORRELATIONS

We first review the formalism developed by Mueller[6] in relating correlation parameters of inclusive processes to the exclusive multiplicity distribution. We then discuss how the knowledge of inclusive distributions allows us to obtain the multiplicity generating function.

We denote the total cross section for the process initiated by particles $a$ and $b$ by

$$\sigma_T^{(a,b)}(Y) \equiv \sum_{N=2}^{\infty} \sigma_N(a+b \to N)$$
$$\simeq \sum_{N=2}^{\infty} \frac{e^{-Y}}{N!} \int d\Phi_N |T(a+b \to N)|^2 , \qquad (2.1)$$

and the $k$-particle inclusive density by

$$d_k(y_1, y_2, \ldots, y_k; y) \equiv \frac{1}{\sigma_T} \frac{d^k\sigma}{dy_1 \cdots dy_k}$$
$$= \frac{1}{\sigma_T} \sum_{N=0}^{\infty} \frac{e^{-Y}}{N!} \times \int d\Phi_N |T(a+b \to c_1 + \cdots + c_k + N)|^2 , \qquad (2.2)$$

where $Y = y_b - y_a > 0$ is the rapidity separation between the initial particles, $y_i$'s are rapidities of produced particles, and $d\Phi_N$ is the invariant phase space (integrated over the transverse momenta). If we integrate $d_k(y_1, \ldots, y_k; Y)$ over the $y_i$'s, we obtain, from (2.1) and (2.2), the $k$th binomial moment of the multiplicity distribution

$$D_k(Y) \equiv \langle n(n-1) \cdots (n-k+1) \rangle = \int dy_1 \cdots dy_k \, d_k(y_1 \cdots y_k; Y), \quad k \geq 1 . \qquad (2.3)$$

Relations among partial cross sections $\sigma_N$, binomial moments $D_k$, and inclusive densities $d_k$ can best be summarized by the multiplicity generating function

$$I(z, Y) \equiv \frac{1}{\sigma_T} \sum_{N=2}^{\infty} \sigma_N z^N = \sum_{k=0}^{\infty} D_k(Y) \frac{h^k}{k!} \equiv \exp\left[ \sum_{k=1}^{\infty} C_k(Y) \frac{h^k}{k!} \right], \qquad (2.4)$$

where either $z$ or $h \equiv z - 1$ can be used as the generating parameter. In (2.4), $D_0(Y) \equiv 1$, the functions $C_k(Y)$ are correlation parameters, and the last equality is also known as the cluster expansion. One can show that $C_k(Y)$ is related to the $k$th correlation function

$$C_k(Y) = \int dy_1 \cdots dy_k \, c_k(y_1, \ldots, y_k; Y) , \qquad (2.5)$$

where $c_k(y_1, \ldots, y_k; Y)$ can be obtained from inclusive densities by[6]

$$\begin{aligned} d_1(y_1) &\equiv c_1(y_1), \\ d_2(y_1, y_2) &\equiv c_1(y_1)c_1(y_2) + c_2(y_1, y_2), \\ d_3(y_1, y_2, y_3) &\equiv c_1(y_1)c_1(y_2)c_1(y_3) + c_1(y_1)c_2(y_2, y_3) \\ &\quad + c_1(y_2)c_2(y_1, y_3) + c_1(y_3)c_2(y_1, y_2) \\ &\quad + c_3(y_1, y_2, y_3) , \\ &\ldots . \end{aligned} \qquad (2.6)$$

Although the above construction is general, it is particularly useful if one works within the framework of the short-range correlation picture. In this case, all $c_k(y_1, \ldots, y_k; Y)$'s vanish whenever the rapidity separation $|y_i - y_j|$ of any pair of particles becomes large compared to a characteristic length $L$, called the correlation length. The effective integration volume in (2.5) is then reduced from $Y^k$ to $L^{k-1}Y$, and we find, asymptotically,[6] that

$$C_k(Y) \simeq a_k Y + b_k , \quad k = 1, 2, \ldots . \qquad (2.7)$$

A particular realization of the short-range correlation picture is the Mueller-Regge model where the leading $J$-plane singularity is a factorizable pole. It then follows that all inclusive densities $d_k(y_1, \ldots, y_k; Y)$ satisfy the scaling condition at high energies, and (2.7) is automatically guaranteed. Furthermore, the correlation length is given by $L = (\alpha_P - \alpha_M)^{-1}$, where $\alpha_P$ and $\alpha_M$ are the two leading $J$-plane singularities.

Our primary aim is to calculate $I(z, Y)$, assuming the inclusive densities are given, either from the Mueller-Regge analysis, or from experiments. Technically, once the $d_k$'s are known, we can immediately calculate the $c_k$'s, which, upon integrating over the $y_i$'s, yield all the correlation parameters $C_k(Y)$. Substituting them into (2.4), $I(z, Y)$ is then obtained. However, this procedure, though straightforward, is very complicated, except possibly in the "weak"-coupling-limit analysis of Ref. 2. Therefore, we shall adopt a different procedure for calculating $I(z, Y)$ in Sec. III by directly utilizing the "recurrence" property of the $D_k$'s. Once $I(z, Y)$ is found, the correlation parameters $C_k$ can then be obtained by a direct differentiation.

Before proceeding to the analysis indicated above, we would like to discuss some general properties of $I(z, Y)$ under the assumption of short-range correlations. From (2.7), we obtain an asymptotic expression

$$I(z, Y) \simeq \sum_{k=0}^{\infty} \tilde{D}_k(Y) \frac{h^k}{k!} = \exp[A(h)Y + B(h)], \qquad (2.8)$$

where the $\tilde{D}_k(Y)$'s are the asymptotic behavior of the $D_k$'s, and

$$A(h)=\sum_{k=0}^{\infty} a_k \frac{h^k}{k!}\ , \tag{2.9}$$

$$B(h)=\sum_{k=0}^{\infty} b_k \frac{h^k}{k!}\ . \tag{2.10}$$

From the scaling property of the $d_k$'s, it follows from (2.3) that $\tilde{D}_k(Y)$ is a $k$th-order polynomial in $Y$. Therefore, $I(z, Y)$ in (2.8) can also be written as a power series about $Y=0$,

$$I(z, Y)\simeq \sum_{n=0}^{\infty} e_n(h)\frac{Y^n}{n!}=\exp[A(h)Y+B(h)]\ , \tag{2.11}$$

where the coefficient $e_n(h)$ depends on $\tilde{D}_k(Y)$, $k\geqslant n$. In particular, $e_0(h)$ only depends on the constant term of $\tilde{D}_k(Y)$, and $e_1(h)$ on the *linear* term of the polynomial $\tilde{D}_k(Y)$. However, it also follows from (2.11) that

$$e_n(h)=A(h)^n e^{B(h)}\ , \quad n=0, 1, \ldots\ . \tag{2.12}$$

Therefore, in order to find $A(h)$ and $B(h)$, it is sufficient to know only $e_0(h)$ and $e_1(h)$, i.e., the constant and the linear terms of $\tilde{D}_k(Y)$, $k=0, 1, \ldots$ . More generally, if the $\tilde{D}_k(Y)$'s are known, (2.12) is a necessary and sufficient condition for guaranteeing the short-range correlation constraints.

## III. THE INTEGRAL EQUATION FOR THE GENERATING FUNCTION

The $N$-particle inclusive density, $d_N$, is related through the generalized optical theorem[7] to an appropriate forward $(N+2)$-to-$(N+2)$ absorptive amplitude, $F_N$, by

$$d_N(y_1, \ldots, y_N; Y)=(e^Y\sigma_T)^{-1}F_N(y_1, \ldots, y_N; Y)\ . \tag{3.1}$$

The Mueller-Regge expansion is a statement on the dependence of $F_N$ on the $y_i$'s. By keeping a finite number of Regge poles in each Mueller complex angular momentum plane, a recurrence relation in $N$ can be obtained. When this is substituted into (2.3), and (2.4) is used, an integral equation for a quantity related to the multiplicity generating function $I(z, Y)$ follows.

To simplify the discussion, we shall first ignore the leading particle effect. This amounts to neglecting the triple-Regge behavior and we shall demonstrate that this will not affect the asymptotic behavior of our correlation parameters. We consider a world with only two trajectories, a leading pole with intercept $\alpha_P(0)\approx 1$ and a secondary pole with $\alpha_M(0)\approx \frac{1}{2}$. Generalization to more than two poles can be made in a straightforward manner.

The Mueller diagrams that contribute to the asymptotic limit of $F_N^{(a,b)}$ are those shown in Fig. 1(a):

$$F_0^{(a,b)}(Y)=\sum_i G_i^{(a)}G_i^{(b)}e^{\alpha_i Y}\ ,$$

$$\begin{aligned}F_N^{(a,b)}(y_1, \ldots, y_N; Y)=&\sum_{i_l} G_{i_0}^{(a)}g_{i_0,i_1}\cdots g_{i_{N-1},i_N}G_{i_N}^{(b)}\exp[\alpha_{i_0}(y_1-y_0)+\cdots+\alpha_{i_N}(y_{N+1}-y_N)]\\ &\times\theta(y_1)\theta(y_2-y_1)\cdots\theta(Y-y_N)\\ &+\text{other permutations over } (1, 2, \ldots, N) \text{ for } N\geqslant 1\ ,\end{aligned} \tag{3.2}$$

where we have put $y_a=y_0=0$ and $y_b=y_{N+1}=Y$. The $G_i$'s and $g_{ij}$'s are, respectively, external and internal Mueller-Regge couplings, and $\alpha_i$ runs over $\alpha_P$ and $\alpha_M$. To construct an integral equation, let us define an "internal Mueller-Regge amplitude," $B_{ij}^N(h, Y)$, by first removing from (3.2) those external couplings $G_i$ and then multiplying a generating parameter $h$ for each internal coupling, i.e., replace $g_{ij}$ by $hg_{ij}$. Upon integrating over each $y_i$, we obtain, for fixed $N$,

$$\begin{aligned}(\sigma_T^{(a,b)}e^Y)D_N^{(a,b)}\frac{h^N}{N!}&=\sum_{i,j}G_i^{(a)}B_{ij}^N(h, Y)G_j^{(b)}\\ &=\frac{h^N}{N!}\int dy_1\cdots dy_N F_N^{(a,b)}\ .\end{aligned} \tag{3.3}$$

It is easy to see that the permutation over 1 to $N$ in Eq. (3.2), after performing the phase-space integration, just cancels the factor $1/N!$ in Eq. (3.3). The resulting $B_{ij}^N(h, Y)$ satisfies the multiperipheral-like recurrence relation [see Fig. 1(b)]

$$B_{ij}^N(h, Y)=\sum_k\int_0^Y dy\, B_{ik}^{(N-1)}(h, y)e^{\alpha_j(Y-y)}hg_{kj}\ .$$

Defining

$$B_{ij}(h, Y)=\sum_{N=0}^{\infty}B_{ij}^N(h, Y)\ ,$$

we find that

$$\sigma_T I(z, Y)=\sum_{i,j}G_i^{(a)}B_{ij}(z-1, Y)G_j^{(b)} \tag{3.4}$$

and it satisfies

$$B_{ij}(h, Y) = \delta_{ij} e^{\alpha_j Y} + \sum_k \int_0^Y dy\, B_{ik}(h, y) e^{\alpha_j (Y-y)} h g_{kj} . \quad (3.5)$$

The above integral equation can further be simplified by performing Laplace transforms so that the solution can be obtained algebraically. (See Ref. 5 for details.) However, we shall adopt a more direct approach here by relying on our past experience with the multiperipheral model.

The linear integral equation (3.5) has a typical solution of the Regge type. Because of our two-pole input, the solution will be of the form

$$B_{ij}(h, Y) = \mu_{ij}(h) e^{\alpha_P(h) Y} + \nu_{ij}(h) e^{\alpha_M(h) Y} . \quad (3.6)$$

Substituting (3.6) into (3.5), after solving two eigenvalue conditions, we obtain

$$\alpha_P(h) + \alpha_M(h) = \gamma_P(h) + \gamma_M(h), \quad (3.7a)$$

$$\alpha_P(h) - \alpha_M(h) = \{[\gamma_P(h) - \gamma_M(h)]^2 + 4 g_{MP}^2 h^2\}^{1/2} , \quad (3.7b)$$

$$\begin{bmatrix} \mu_{PP} & \mu_{PM} \\ \mu_{MP} & \mu_{MM} \end{bmatrix} = \begin{bmatrix} \dfrac{\tau + \tau_0}{2\tau} & \dfrac{g_{MP}}{\tau} h \\ \dfrac{g_{MP}}{\tau} h & \dfrac{\tau - \tau_0}{2\tau} \end{bmatrix} , \quad (3.8)$$

$$\begin{bmatrix} \nu_{PP} & \nu_{PM} \\ \nu_{MP} & \nu_{MM} \end{bmatrix} = \begin{bmatrix} \dfrac{\tau - \tau_0}{2\tau} & -\dfrac{g_{MP}}{\tau} h \\ -\dfrac{g_{MP}}{\tau} h & \dfrac{\tau + \tau_0}{2\tau} \end{bmatrix} , \quad (3.9)$$

where $\gamma_P$, $\gamma_M$, $\tau_0$, and $\tau$ are functions of $h$, given by

$$\gamma_P(h) = \alpha_P + g_{PP} h ,$$
$$\gamma_M(h) = \alpha_M + g_{MM} h , \quad (3.10)$$
$$\tau_0(h) = \gamma_P(h) - \gamma_M(h) ,$$

and

$$\tau(h) = \alpha_P(h) - \alpha_M(h) .$$

We note that, for $h = 0$, $\tau(0) = \tau_0(0) = \alpha_P - \alpha_M$, and $\mu_{ij}(0) = \delta_{ij}\delta_{iP}$, $\nu_{ij}(0) = \delta_{ij}\delta_{iM}$, $i,j$ run over $P$ and $M$. From (2.4), (3.5), and (3.6), we then have

$$I^{(a,b)}(h, Y) = [\sigma_T^{(a,b)}(Y)]^{-1} \sum_{i,j} G_i^{(a)} B_{ij}(h, Y) G_j^{(b)}$$
$$\equiv [\sigma_T^{(a,b)}(Y)]^{-1} \sigma_T^{(a,b)}(h, Y), \quad (3.11)$$

where

$$\sigma_T^{(a,b)}(h, Y) = G_P^{(a)}(h) G_P^{(b)}(h) e^{Y[\alpha_P(h) - 1]} + G_M^{(a)}(h) G_M^{(b)}(h) e^{Y[\alpha_M(h) - 1]}, \quad (3.12)$$

$$G_P^{(a)}(h) G_P^{(b)}(h) = G_i^{(a)} \mu_{ij}(h) G_j^{(b)} , \quad (3.13)$$

$$G_M^{(a)}(h) G_M^{(b)}(h) = G_i^{(a)} \nu_{ij}(h) G_j^{(b)} . \quad (3.14)$$

The above factorizable form is possible because $\det\mu(h) = \det\nu(h) = 0$, and $\mu + \nu = I$. Using (3.8), (3.9), and (3.10), we find that it is consistent to choose

$$G_P^{(a)}(h) = \left(\frac{\tau + \tau_0}{2\tau}\right)^{1/2} G_P^{(a)} + \left(\frac{\tau - \tau_0}{2\tau}\right)^{1/2} G_M^{(a)} , \quad (3.15)$$

$$G_M^{(a)}(h) = -\left(\frac{\tau - \tau_0}{2\tau}\right)^{1/2} G_P^{(a)} + \left(\frac{\tau + \tau_0}{2\tau}\right)^{1/2} G_M^{(a)} , \quad (3.16)$$

for any $a$. Since $G_P^{(a)}(0) = G_P^{(a)}$, $G_M^{(a)}(0) = G_M^{(a)}$, we see that

$$\sigma_T^{(a,b)}(0, Y) = \sigma_T^{(a,b)}(Y) . \quad (3.17)$$

Near $z = 1$, or $h \simeq 0$, we have $\alpha_P(h) > \alpha_M(h)$. To obtain the asymptotic expression for the correlation parameters $C_k(Y)$, we only need the asymptotic behavior of $I(z, Y)$ near $z = 1$:

$$I^{(a,b)}(z, Y) \simeq \frac{G_P^{(a)}(h) G_P^{(b)}(h)}{G_P^{(a)} G_P^{(b)}} \exp[Y(\alpha_P(h) - \alpha_P)]$$
$$= \exp[Y A(h) + B(h)] . \quad (3.18)$$

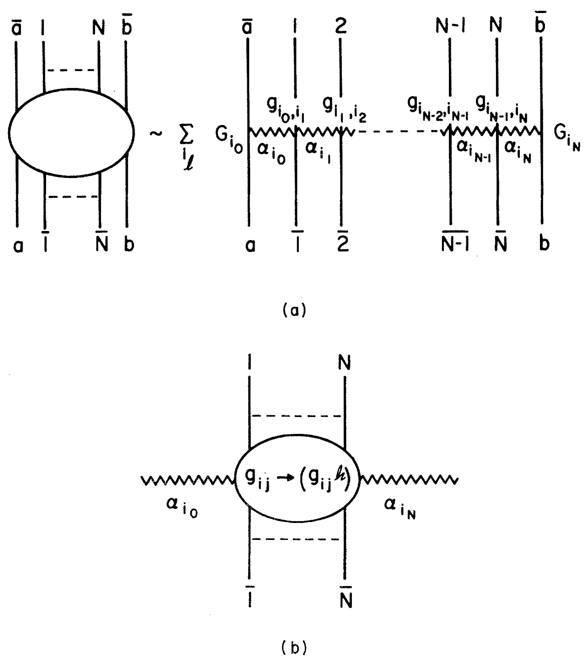


FIG. 1. (a) The multichannel Mueller-Regge model showing inclusive coupling constants and Regge exchanges. (b) The internal blob satisfying the integral equation.

Comparing (3.18) with (2.9), we find that the linear term of the correlation can be computed easily by expanding $[\alpha_P(h) - \alpha_P]$ in powers of $h$. After some algebra, we obtain

$$a_1 = g_{PP},$$
$$a_k = (k-1)!\,[P_{k-2}(X) - XP_{k-1}(X)](\sqrt{\zeta})^k (2L)^{k-1}, \quad k \geq 2 \tag{3.19}$$

where

$$\zeta = g_{MP}{}^2 + \tfrac{1}{4}(g_{PP} - g_{MM})^2,$$
$$X = (g_{MM} - g_{PP})/2\sqrt{\zeta},$$
$$L = (\alpha_P - \alpha_M)^{-1} \equiv \tau(0)^{-1},$$

and $P_k(X)$ is the Legendre polynomial. In the weak coupling limit[2] where couplings involving $\alpha_P$ are small, i.e., $|g_{MM}| \gg |g_{PP}| + |g_{MP}|$, (3.19) reduces to the Eq. (20) of Ref. 2 :

$$a_n = n!\,L^{n-1}(g_{MP})^2(g_{MM})^{n-2}. \tag{3.20}$$

The constants $b_k$ of the correlation parameters get contributions not only from the central region, but also from the fragmentation regions. Therefore, the function $B(h)$ given by (3.18), which follows from the Mueller-Regge expansion (3.2), will in general be modified by assumptions made concerning the behavior of the inclusive spectra in the fragmentation regions. For instance, the most general criterion for the Mueller-Regge expansion of the single-particle spectrum at sufficiently high energies is

$$e^Y \frac{d\sigma}{dy_1} \to \begin{cases} f^{(a)}(y_1)e^{\alpha_0 Y}, & 0 \leq y_1 \leq \Delta \\ F_1^{(a,b)}(y_1, Y), & \Delta \leq y_1 \leq Y - \Delta \\ f^{(b)}(Y - y_1)e^{\alpha_0 Y}, & Y - \Delta < y_1 \leq Y \end{cases} \tag{3.21}$$

where $\Delta$ is a finite constant of the order $L$—the correlation length of outgoing particles—and $F_1^{(a,b)}(y, Y)$ is given by (3.2). The functions $f^{(a)}$ and $f^{(b)}$ are arbitrary except for the conditions that they should approach the central region smoothly and should exhibit leading-particle effect, e.g., the triple-Regge behavior [Fig. 2(a)]

$$f^{(a)}(y) \sim \sum_{ij} g_{ij,0}(1 - e^{-y})^{-[\alpha_i(t) + \alpha_j(t)] + \alpha_0},$$

where $g_{ij,0}$ are triple-Regge couplings to the leading pole $\alpha_0$. Since $C_1(Y)$ is related to the integral over (3.21), it is easily seen that, although the term linear in $Y$ is unmodified, the constant $b_1$ depends on the choice of $f^{(a)}$ and $f^{(b)}$.

Since we are ignoring transverse momenta, it is difficult to construct realistic representations of $f^{(a)}$ and $f^{(b)}$. As a "minimal" modification, we assume that the Mueller-Regge expansion works for the entire phase space and represents the leading-particle effect by adding appropriate $\delta$-function terms at the edge of the phase space. For instance, we replace $f^{(a)}(y_1)e^{\alpha_0 Y}$ by $\xi_a \delta(y_1) \times F_0^{(a,b)}(Y - y_1)$, and $f^{(b)}(Y - y_1)e^{\alpha_0 Y}$ by $\xi_b \delta(Y - y_1) \times F_0^{(a,b)}(y_1)$, where $\xi_a$ and $\xi_b$ are numerical constants. For the two-particle cross section, in addition to two single-$\delta$-function terms, we also add terms which are products of two $\delta$ functions :

$$\xi_a \xi_b [\delta(y_1)\delta(Y - y_2)F_0{}^{(a,b)}(y_2) + \delta(y_2)\delta(Y - y_1)F_0{}^{(a,b)}(y_1)].$$

These terms will then simulate the contributions from the di-triple-Regge regions [Fig. 2(b)].

The above construction is certainly heuristic, but as we shall see later, it is supported by the MPM. In general, after performing the phase-space integration, the $N$th moment of the distribution is changed to

$$D_N \to D_N + (\xi_a + \xi_b)ND_{N-1} + \xi_a \xi_b N(N-1)D_{N-2}, \quad N \geq 2. \tag{3.22}$$

The new generating function, which we denote by

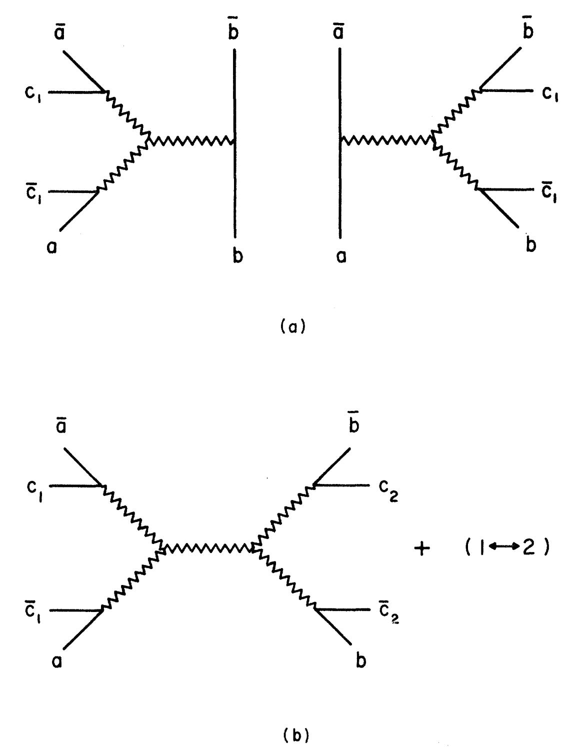


FIG. 2. (a) Triple-Regge corrections. (b) Di-triple-Regge corrections.

$$I_R(z,Y)=[(1-\xi_a)+\xi_a z][(1-\xi_b)+\xi_b z]\,I(z,Y)\,. \tag{3.23}$$

As we have noted, the leading-particle effects have not changed the $Y$-dependent part of our multiplicity generating function. Only those constants, $b_k$, have been altered. We are still left with two constants $\xi_a$, $\xi_b$ which can be fixed by energy-momentum sum rules. We shall later show that the Chew-Pignotti model requires $\xi_a=\xi_b=1$, which can satisfy the energy-momentum sum rules only in the weak-coupling limit.

## IV. MULTIPERIPHERAL MODEL AND MULTIPLICITY DISTRIBUTION

The multiplicity distribution can now be obtained from the generating function (3.24) by a direct differentiation. However, the resulting expressions are so complicated that it is difficult to extract "physical" information from them. On the other hand, if one is able to derive the *same* multiplicity distribution from an "exclusive" multiparticle model, the physical basis of the original "inclusive" model then becomes clear. We shall do so next by considering a two-channel multiperipheral model under the $CP$ approximation.[3-5]

The partial cross section for the process

$$a+b\to 0+1+\cdots+(N+1)\,,\quad N=0,1,\ldots \tag{4.1}$$

is given by

$$\sigma_{N+2}\simeq\frac{e^{-Y}}{(N+2)!}\int d\Phi_{N+2}|T(a+b\to N+2)|^2\,. \tag{4.2}$$

We assume the usual factorization approximation for production amplitudes (Fig. 3),

$$T_{N+2}\sim\sum_{i_l=0,1}v_{i_0}^{(a)}\lambda_{i_0,i_1}\cdots\lambda_{i_{N-1},i_N}v_{i_N}^{(b)}\theta(y_1-y_0)\cdots\theta(y_{N+1}-y_N)\delta(y_0)\delta(y_{N+1}-Y)$$
$$\times\exp[\alpha_{i_0}(y_1-y_0)+\cdots+\alpha_{i_N}(y_{N+1}-y_N)]+\text{permutations of }(0,1,\ldots,N+1)\,, \tag{4.3}$$

or, alternatively, we assume directly that

$$|T_{N+2}|^2\sim\sum_{i_l=0,1}V_{i_0}^{(a)}\Lambda_{i_0,i_1}\cdots\Lambda_{i_{N-1},i_N}V_{i_N}^{(b)}\theta(y_1-y_0)\cdots\theta(y_{N+1}-y_N)\exp[2\alpha_{i_0}(y_1-y_0)+\cdots+2\alpha_{i_N}(y_{N+1}-y_N)]$$
$$+\text{permutations of }(0,1,\ldots,N+1)\,, \tag{4.4}$$

where $v_i$'s and $\lambda_{ij}$'s are *exclusive* coupling constants, $V_i=v_i^{\,2}$, $\Lambda_{ij}=\lambda_{ij}^{\,2}$, $\alpha_0$, $\alpha_1$ are *input* poles, and we have also neglected cross terms in (4.4).[8] With (4.4), (4.2) can be evaluated under the $CP$ approximation, whereas the phase space is given by

$$\int d\Phi_{N+2}\sim e^{-Y}\int_0^Y dy_N\int_0^{y_N}dy_{N-1}\cdots\int_0^{y_2}dy_1\,. \tag{4.5}$$

After some very tedious calculation, we find

$$\sigma_2=V_0^{(a)}V_0^{(b)}e^{(\beta_0-1)Y}+V_1^{(a)}V_1^{(b)}e^{(\beta_1-1)Y}\,,$$

$$\sigma_3=[V_0^{(a)}V_0^{(b)}\Lambda_{00}Y+(V_0^{(a)}V_1^{(b)}+V_1^{(a)}V_0^{(b)})(\Lambda_{01})(\beta_0-\beta_1)^{-1}]e^{(\beta_0-1)Y}+[0\leftrightarrow 1]\,,$$

$$\sigma_{N+2}=\Bigg\{\Bigg[V_0^{(a)}V_0^{(b)}\sum_{m=1}\sum_{p=0}\sum_{n=0}W_n^m+(V_0^{(a)}V_1^{(b)}+V_1^{(a)}V_0^{(b)})\left(\frac{\Lambda_{00}}{\Lambda_{01}}\right)\sum_{m=1}\sum_{p=0}\sum_{n=-1}W_{n+1}^{m-1}$$
$$+V_1^{(a)}V_1^{(b)}\left(\frac{\Lambda_{00}}{\Lambda_{01}}\right)^2\sum_{m=2}\sum_{p=0}\sum_{n=-2}W_{n+2}^{m-2}\Bigg]\Omega_{n,m,p}^{N}(Y)+V_0^{(a)}V_0^{(b)}(\Lambda_{00})^NY^N\Bigg\}e^{(\beta_0-1)Y}+\{0\leftrightarrow 1\}\,, \tag{4.6}$$

where

$$\Omega_{n,m,p}^{N}(Y)=\delta_{2m+p+n,N}(\Lambda_{00})^n(\Lambda_{01})^{2m}(\Lambda_{11})^p(-1)^{n+m}W_p^{m-1}\sum_{l=0}^{N-m-p}\frac{(-1)^l}{l!(\beta_0-\beta_1)^{N-l}}\,W_{m+p-1}^{N-m-p-l}\,Y^l\,,$$

$$W_n^m=\frac{(m+n)!}{m!\,n!}\,,$$

and

$$\beta_i \equiv 2\alpha_i - 1\ , \quad i = 0, 1\ . \tag{4.7}$$

It is easy to see that, when $V_1 = \Lambda_{01} = \Lambda_{11} = 0$, $\sigma_{N+2}$ reduces to a simple Poisson distribution.

The multiplicity generating function for this MPM is

$$I_M(z, Y) = [\sigma_T^{(a,b)}(Y)]^{-1}\left(\sum_{N=0}^{\infty} \sigma_{N+2}^{(a,b)} z^{N+2}\right)$$
$$\equiv [\sigma_T^{(a,b)}(Y)]^{-1}\tilde{\sigma}_T^{(a,b)}(z, Y)\ , \tag{4.8}$$

where

$$\sigma_T^{(a,b)}(Y) = \tilde{\sigma}_T^{(a,b)}(z=1, Y) \tag{4.9}$$

and $\tilde{\sigma}_T^{(a,b)}(z, Y)$ can be calculated from (4.6). It turns out that this is really unnecessary since $\sigma_T^{(a,b)}(z, Y)$ can be directly written down from the known result of a two-channel *CP* MPM.[5] The only difference is that we should multiply the exclusive couplings $V_i$, $\Lambda_{ij}$, etc. by a factor $z$, and $\tilde{\sigma}_T^{(a,b)}(z, Y)$ is basically the total cross section [Eq. (2.20) of Ref. 5]:

$$\tilde{\sigma}_T^{(a,b)}(z, Y) = (z)^2\{\tilde{G}_+^{(a)}(z)\tilde{G}_+^{(b)}(z)e^{Y[\tilde{\alpha}_+(z)-1]} + \tilde{G}_-^{(a)}(z)\tilde{G}_-^{(b)}(z)e^{Y[\tilde{\alpha}_-(z)-1]}\}\ , \tag{4.10}$$

where

$$\tilde{\alpha}_\pm(z) = \tfrac{1}{2}(\tilde{\beta}_0 + \tilde{\beta}_1) \pm [\tfrac{1}{4}(\tilde{\beta}_0 - \tilde{\beta}_1)^2 + \Lambda_{01}{}^2 z^2]^{1/2}\ , \tag{4.11}$$

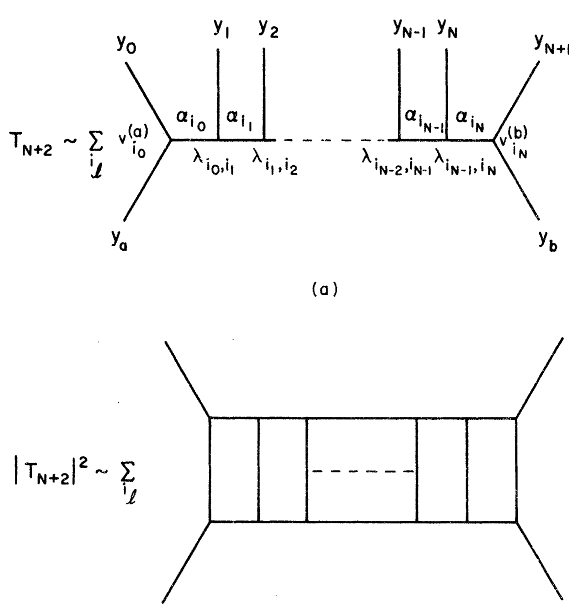


FIG. 3. (a) The diagram for the multiperipheral production of $(N+2)$ particles. (b) The square of $|T_{N+2}|$ without cross terms.

$$\tilde{G}_\pm^{(a)}(z) = \pm\left(\frac{\tilde{\delta} \pm \tilde{\delta}_0}{2\tilde{\delta}}\right)^{1/2} V_0^{(a)} + \left(\frac{\tilde{\delta} \mp \tilde{\delta}_0}{2\tilde{\delta}}\right)^{1/2} V_1^{(a)}\ , \tag{4.12}$$

and $\tilde{\beta}_i$, $\tilde{\delta}_0$, $\tilde{\delta}$ are functions of $z$ given by

$$\tilde{\beta}_i(z) = \beta_i + \Lambda_{ii} z\ , \tag{4.13}$$

$$\tilde{\delta}_0(z) = \tilde{\beta}_0(z) - \tilde{\beta}_1(z)\ , \tag{4.14}$$

$$\tilde{\delta}(z) = \tilde{\alpha}_+(z) - \tilde{\alpha}_-(z)\ . \tag{4.15}$$

Owing to the identical structure of the MPM integral equation and the multiplicity generating function, (3.5), it is perhaps not surprising that (4.10)–(4.15) can also be obtained from (3.6)–(3.14) by the following substitution:

$$h \to z, \quad G_i \to V_i\ , \quad g_{ij} \to \Lambda_{ij}\ , \quad (P, M) \to (0, 1)\ ,$$
$$\alpha_P \to \beta_0, \quad \alpha_M \to \beta_1, \quad G_P^{(a)}(h) \to \tilde{G}_+^{(a)}(z),$$
$$G_M^{(a)}(h) \to \tilde{G}_-^{(a)}(z)\ .$$

Substituting (4.10) into (4.8), we find that $I_M(z, Y)$ has the same mathematical structure as $I_R(z, Y)$, (3.24), with $\xi_a = \xi_b = 1$. Therefore, by a proper identification between parameters of the MPM and that of the Mueller-Regge model, $I_M(z, Y)$ can then be made equal to $I_R(z, Y, \xi_a = \xi_b = 1)$. First, by comparing the total cross sections in these two approaches, we see that

$$\alpha_P = \tilde{\alpha}_+(1)\ , \quad \alpha_M = \tilde{\alpha}_-(1) \tag{4.16}$$

and

$$\begin{aligned} G_P^{(a)} &= G_P^{(a)}(h=0) \\ &= \tilde{G}_+^{(a)}(z=1)\ , \\ \tilde{G}_M^{(a)} &= G_M^{(a)}(h=0) \\ &= \tilde{G}_-^{(a)}(z=1)\ . \end{aligned} \tag{4.17}$$

After some algebra, one can show that the identification becomes complete if

$$\begin{pmatrix} G_P^{(a)} \\ G_M^{(a)} \end{pmatrix} = U(1) \begin{pmatrix} V_0^{(a)} \\ V_1^{(a)} \end{pmatrix}\ , \tag{4.18}$$

$$\begin{pmatrix} g_{PP} & g_{PM} \\ g_{MP} & g_{MM} \end{pmatrix} = U(1) \begin{pmatrix} \Lambda_{00} & \Lambda_{01} \\ \Lambda_{10} & \Lambda_{11} \end{pmatrix} U(1)^{-1}\ , \tag{4.19}$$

where

$$U(z) = \begin{pmatrix} \left(\frac{\tilde{\delta} + \tilde{\delta}_0}{2\tilde{\delta}}\right)^{1/2} & \left(\frac{\tilde{\delta} - \tilde{\delta}_0}{2\tilde{\delta}}\right)^{1/2} \\ -\left(\frac{\tilde{\delta} - \tilde{\delta}_0}{2\tilde{\delta}}\right)^{1/2} & \left(\frac{\tilde{\delta} + \tilde{\delta}_0}{2\tilde{\delta}}\right)^{1/2} \end{pmatrix} \tag{4.20}$$

is chosen to be a proper orthogonal matrix.

## V. DISCUSSION

We have constructed and solved the integral equation of the multiplicity generating function within the Mueller-Regge model. The assumption that there exists an isolated factorizable pole in the "complex $J$ plane" is shown to lead to the short-range correlation picture. These results are then identified with that of an *equivalent* Chew-Pignotti multiperipheral model. These two approaches are shown to be related by a matrix transformation, which can be chosen to be a proper orthogonal matrix as demonstrated in Ref. 5. Although this is done for the two-channel case, generalization to the $N$ channel is straightforward.

Our starting point is the Mueller-Regge assumption on the inclusive spectra. From the exclusive approach, the Mueller-Regge assumptions should then follow from the input of Chew-Pignotti model. For instance, the single-particle cross section can be obtained directly within the $CP$ MPM by removing the integration over $y$ from the diagram in Fig. 4. This has been done in Ref. 5, and, not too surprisingly, the result agrees with that given by (3.2) and (4.18)–(4.20). We also note that we automatically recover the δ-function terms from the end diagrams with weighting factors $\xi_a = \xi_b = 1$, i.e., the leading particles near $y_1 = 0$ and $y_1 = Y$ contribute to $d\sigma/dy_1$:

$$\delta(y_1)\sigma_T(Y) + \delta(y_1 - Y)\sigma_T(Y),$$

so that the contribution to $F_1^{(a,b)}$ is

$$\delta(y_1)F_0^{(a,b)}(Y - y_1) + \delta(Y - y_1)F_0^{(a,b)}(y_1).$$

In the $CP$ MPM, δ functions result from the approximation that end particles carry off all the incoming energy, while in the Mueller-Regge model, δ functions emerge from the inadequacy of the double-Regge expansion near the phase-space boundaries. Their connection can best be seen via the inclusive energy-momentum sum rules. Without these δ-function terms, these sum rules will impose an undesired constraint on inclusive couplings $g_{ij}$; e.g., in the one-channel case, we would obtain

$$1 \simeq g_{PP}(1 - e^{-Y}).$$

After adding the δ-function terms, we then have

$$1 \simeq g_{PP}(1 - e^{-Y}) + \xi_a e^{-Y} + \xi_b,$$

$$1 \simeq g_{PP}(1 - e^{-Y}) + \xi_b e^{-Y} + \xi_a.$$

In the weak-coupling limit where $g_{PP} \to 0$, these sum rules are satisfied with $\xi_a = \xi_b = 1$. Similar results also hold for $n$-particle spectra. Consequently, in both the inclusive and exclusive models, the δ functions, with $\xi_a = \xi_b = 1$, provide, in effect, a zeroth-order (in $g_{PP}^2$) approximation to energy-momentum conservation. One way to go beyond the weak-coupling limit is, therefore, to set $\xi_a < 1$ and $\xi_b < 1$. (A convenient starting point is the diagonalization of inclusive sum rules previously discussed by us.)[9]

Because of the difference in the vertex structure, the incorporation of the isospin could be done more easily in the Mueller-Regge model than in the MPM. In fact a set of Chan-Paton factors,[10] which is compatible with the charge sum rules to all orders, has been found for the asymptotic terms of the correlations. These functions, being less sensitive to leading-particle effects, provide us with suitable candidates for a charge-sum-rule bootstrap.[11]

The Mueller-Regge hypothesis is probably a

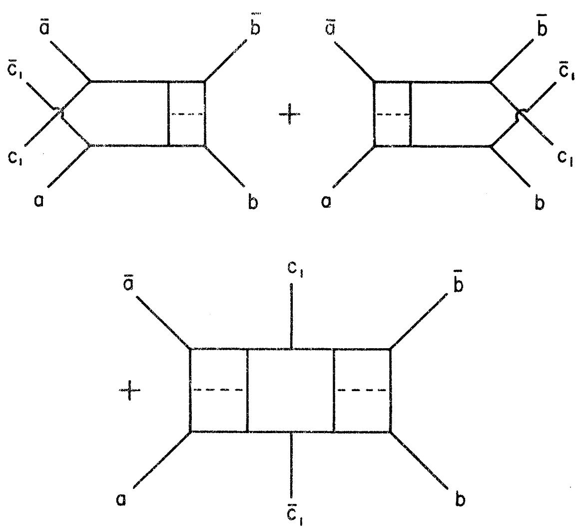


FIG. 4. The single-particle distribution from the MPM.

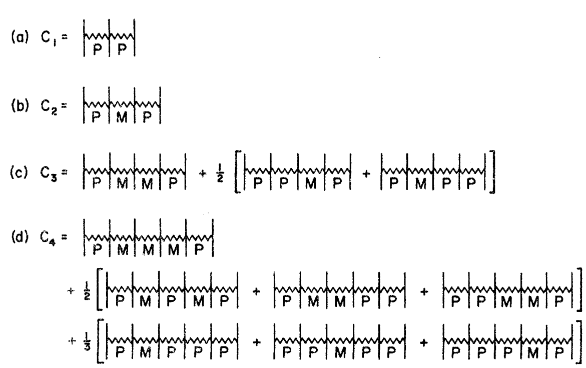


FIG. 5. The coefficients $a_k$ of $C_k(Y) = a_k Y + b_k$ can be obtained by calculating the coefficient of the term linear in $Y$ from each diagram shown. (Higher order contributions from some of these diagrams are canceled by other diagrams not shown here.)

meaningful notion if one is willing to accept the approximate concept of a two-component picture, where one separates the production mechanism into the short-range-correlated (SRC) and the diffractive components (Ref. 2). Since our present analysis applies only to the SRC component, various tests can be found to obtain limits on the diffractive contribution. One thing of immediate interest is the $\pi^-$-$\pi^-$ correlation in the $p$-$p$ collision. If the diffractive component is negligible, the asymptotic behavior of $C_2$ is given by the diagram Fig. 5(b), where the Pomeranchuk is excluded from the internal blob. If the concept of exchange degeneracy is applied to this particular diagram, we have

$$C_2^{\pi^-\pi^-} \to \text{constant}$$

as $Y \to \infty$. Violation of this prediction can either be used to estimate the diffractive contribution or be used as an indication for the breakdown of exchange degeneracy.

*Note added in proof.* After the submission of this paper we received a report by W. A. Bardeen and R. D. Peccei [Phys. Lett. 45B, 353 (1973)] who have independently arrived at results very similar to ours.

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[11]The charge sum rules for the correlations are particularly suitable for the inclusive bootstrap program because they relate directly average quantities which are less sensitive to approximation violating energy-momentum conservation. In general, daughters should be included if we want a consistent bootstrap on higher-order sum rules.