Relations Between Large-Angle Scattering, Form Factors, and Structure Functions*

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We study the large-momentum-transfer limit of electromagnetic form factors and the high-energy, large-momentum-transfer limit of exclusive and inclusive scattering under the physical approximation that hadronic states are composite systems which can be decomposed in this kinematic regime into two finite mass and spin constituents. One of these constituents (the parton) is pointlike in accordance with the results of deep-inelastic scattering, while the other (the core) is not. We study in this work only the constituent interchange contribution to scattering, as in a previous work by Gunion, Blankenbecler, and Brodsky. Our approach is to use a general, covariant decomposition of the composite-particle vertex, and then to study relations between the form factors and scattering amplitudes. Rather than taking a phenomenological approach, our aim is to make a careful study of the theoretical underpinnings of such relations. We find that the Drell-Yan-West relation between elastic form factors and deep-inelastic structure functions holds in general, and that the inclusive cross section is proportional to a deep-inelastic structure function. The power dependence of the exclusive and inclusive cross section on elastic form factors is not uniquely determined and we classify various theories according to this dependence. Basically the reason for this is that the elastic and inelastic form factor sample the off-shell behavior of the composite system only in the parton momentum, whereas scattering amplitudes depend intimately on the off-shell behavior of the hadronic vertex in both the core and parton. We also examine a noncovariant integral equation first given by Weinberg to study these problems.

I. INTRODUCTION

The area of strong-interaction physics which has perhaps been subject to the greatest neglect by theorists is the subject of high-energy scattering, both exclusive and inclusive, at large transverse momentum. At the same time such reactions may be the most significant ones as far as unveiling the underlying dynamical structure of hadrons. Large-transverse-momentum collisions involve small transverse distances and one begins to probe the inner core of the particles involved in the collision. Thus, even though the corresponding cross sections are extremely small, the rewards which would follow an experimental and theoretical effort are extremely great. In the last few years, an excellent example of this phenomenon has appeared in the form of deep-inelastic electroproduction.[1,2]

We may contrast this situation with the more intensely studied fixed-momentum-transfer high-energy collision. Cross sections for such reactions, which are governed by a peripheral or Regge description, are quite large and numerically account for almost the entire cross section. While we understand the qualitative systematics of such reactions comparatively well, they have not, in our opinion, given us a very deep insight into underlying hadronic structure.

The first question we may ask about large-transverse-momentum processes concerns the systematic behavior in the invariants. With regard to exclusive scattering, as shown in Fig. 1(a), it has been suggested[3-5] that the data[6] for pp elastic scattering may be characterized by scaling laws

$$\frac{d\sigma}{dt}=\frac{1}{s^N}f(\theta)\ , \tag{1.1a}$$

where

$$f(\theta)=(\sin\theta)^{-M}\ , \tag{1.1b}$$

and where $N\approx 12$, $M\geqslant 10$. With regard to inclusive scattering, as shown in Fig. 1(b), where data[7] are in a preliminary stage, there are theoretical reasons[8] for a scaling-type behavior of the form

$$E\frac{d\sigma}{d^3p}=\frac{1}{s^L}\bar{f}(x_1,x_2)\ , \tag{1.2a}$$

where

$$x_1=-\frac{t}{s},\quad x_2=-\frac{u}{s}\ . \tag{1.2b}$$

The power behavior of Eqs. (1.1) and (1.2) is in contrast with the exponential behavior familiar in finite-momentum-transfer experiments. Since exponential behavior of scattering is typically the property of a "soft" extended system, it is clear that there may be an important qualitative change in going to large momentum transfer. Indeed, it is a power behavior which is suggestive of point-like structure within the hadron, which strengthens our feeling that we may learn something of value about the underlying structure of hadrons.

The possibility of relating large-angle scattering to properties of the electromagnetic form factors of the scattered particles has been of interest for some time. It was first suggested by Wu and Yang[9] and then developed further[10] that the (purely absorptive) large-s,t exclusive scattering cross section behaves like

$$\frac{d\sigma}{dt} \sim [G_E(t)]^4 , \tag{1.3}$$

where $G_E(t)$ is the hadronic charge form factor. In this picture the scattering is both entirely diffractive and local within hadronic matter. When the matter distribution is equal to the charge form factor, Eq. (1.3) emerges. At large angles it is the overlap of the nuclear "soup" which determines the cross section.

More recently, it has been suggested[11] that the composite nature of the hadrons is crucial to the interpretation of form factors and deep-inelastic scattering experiments. Therefore it is tempting to apply these same ideas of compositeness to the large-s,t region. In particular one framework for the study of such reactions envisions hadrons as being primarily composed of two virtual components *of finite mass and spin* at large momentum transfer. One of these components is the point-like object which leads to large deep-inelastic cross sections, the parton. The other component is simply referred to as the core. Two courses of development now are open: The constituents can interact [as in Fig. 2(a)], or they can simply be exchanged [as in Fig. 2(b)]. There is no *a priori* way to decide which (or which mixture) of these approaches may describe the real world.

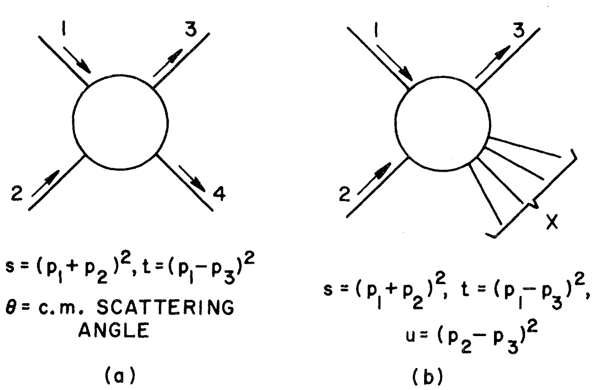

FIG. 1. Scattering diagram for (a) the exclusive process $1+2 \to 3+4$ and (b) the inclusive process $1+2 \to 3+X$.

The first approach is basically the one taken by Berman, Bjorken, and Kogut.[8] Although they were interested mainly in electromagnetic contributions to the scattering of the constituents, it is a very small step[12] to imagine the scattering is strong and to make guesses for the interactions of the constituents.

The possibility of constituent exchange was put forth in a primitive form by Wu[13] many years ago. In our context, this approach was initiated and followed vigorously by Blankenbecler, Brodsky, and Gunion[3,14] (referred to as BBG herein) in a series of papers. They used the Weinberg[15] rules, i.e., a form of old-fashioned perturbation theory, with a special ansatz for the composite hadron structure (the hadronic "wave function"). For large-angle elastic scattering they are led to a characterization of the amplitude by the relation

$$A(s,t) \underset{s,t,u\to\infty}{\sim} s\, G_E(s)\, G_E(t)\, G_E(u) + \text{perms.} \tag{1.4}$$

for the exclusive amplitude, and for the inclusive scattering cross section,

$$E\frac{d\sigma}{d^3p} \propto F_2\left(-\frac{t}{s+u}\right) G_E{}^2\left(\frac{s}{s+u}\, p_\perp{}^2\right) \times G_E{}^2\left(\frac{s}{s+u} p_\perp{}^2\right) + \text{perms.}, \tag{1.5}$$

where F_2 is the deep-inelastic form factor, exhibiting Bjorken scaling, and where G_E has a power-type falloff in its argument. Of course when the various scaling factors are included, the relations (1.4) and (1.5) feed into the phenomenological form of Eqs. (1.1) and (1.2) with different values of

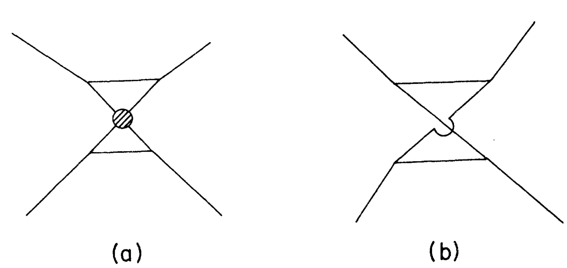

FIG. 2. (a) Four-point function in the case where constituents interact. (b) Four-point function in the interchange model without constituent interactions.

N and functions of θ. Moreover, they represent an extension of the Wu-Yang idea[9] that the large-s,t scattering amplitude can be related to the electromagnetic form factor. Note, however, that in the BBG approach, six powers of the form factor appear in the exclusive cross section rather than four.

The details of the BBG approach depend upon a bound-state wave function which follows from a Bethe-Salpeter-like wave equation in the infinite-momentum formalism. In Appendix A we examine this equation, which is unfortunately not covariant, in some detail. Another approach which remained within the two-component picture and constituent exchange was made by Landshoff and Polkinghorne (LP),[16] this time using a covariant picture of the bound-state vertex. They found that the details of their angular distributions did not agree completely with those of BBG.

With these results in hand, we decided to re-examine the whole question of the dependence of such relations on the details of the bound state vertex in a covariant approach. We retain the idea of a finite mass and spin parton and core as constituents, and of constituent exchange as the dominant process. *We find that in exclusive scattering not only the angular dependence but also the power dependence (i.e., the number of form factors appearing) as well are highly model-dependent; similar remarks apply to inclusive scattering.* More precisely, we find that the results depend critically upon the details of the off-mass-shell behavior of the hadronic vertex.

Our approach is to characterize the behavior of the hadron-core-parton vertex with core and parton off-shell by the Deser-Gilbert-Sudarshan[17] (DGS) representation. Such a representation, while not generally proven in axiomatic field theory, is known to be valid order-by-order in perturbation theory. Label the hadron line with momentum p, the parton line with momentum k, and the core line with momentum $p-k$, as in Fig. 3. Then for all spinless hadrons, the irreducible vertex is

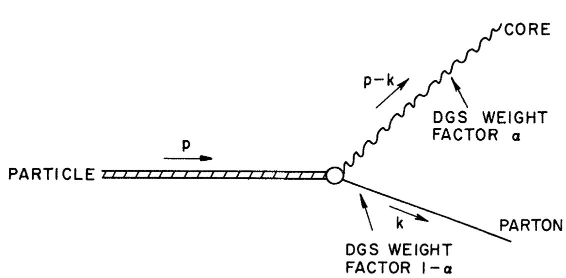

FIG. 3. Diagrammatic representation for the bound-state vertex function Γ with hadron of momentum p and core and parton of momenta $p-k$ and k.

given by

$$\Gamma(k,p-k;p) = \int_{\sigma^2_{\min}}^{\infty} d\sigma^2 \int_0^1 d\alpha\, H(\sigma^2,\alpha) \times[k^2(1-\alpha)+(p-k)^2\alpha-\sigma^2+i\epsilon]^{-1} \tag{1.6a}$$

$$\equiv \int dh(\sigma^2,\alpha) \times[k^2(1-\alpha)+(p-k)^2\alpha-\sigma^2+i\epsilon]^{-1}\,. \tag{1.6b}$$

$H(\sigma^2, \alpha)$ is the DGS spectral weight depending on a spectral mass variable σ^2 and a variable of finite support α. The support properties of H ensure the known singularities at normal and anomalous thresholds given by Feynman graphs. In Eq. (1.6b) we have defined a convenient abbreviation for the integration weight. We shall adopt the convention of associating the factor α with the core line and the factor $1-\alpha$ with the parton line.

While Eq. (1.6a) represents the original form of the DGS representation, specific properties of the spectral weight can change the form. For example, if H obeys some superconvergence relation in σ^2, i.e.,

$$\int d\sigma^2 H(\sigma^2,\alpha)(\sigma^2)^j = 0, \quad j<n\,, \tag{1.7}$$

then successive integration by parts of Eq. (1.6) allows us to write the vertex in the form

$$\Gamma(k,p-k;p) = \int dh(\sigma^2,\alpha) \times[k^2(1-\alpha)+(p-k)^2\alpha-\sigma^2+i\epsilon]^{-(n+1)}\,. \tag{1.8}$$

We shall use this form quite frequently. Other examples of the apparent transmutation of the representation will be seen in the remainder of the paper.

The results of LP follow from a special case of the representations we study. Since they devote a good deal of effort to phenomenological fits to the data, we feel that it would be repetitive to try to fit the data again. Instead, our emphasis will be on the credibility of these results in so far as they presume to represent a fundamental property of hadronic matter.

The results of BBG can be obtained only if the singularities in the relative energy variable, k_0 (or k_- in the infinite-momentum language), are neglected when loop integrations are performed with Eq. (1.6). (See the discussion in the last paper of BBG in Ref. 14.) This, as we discuss in Appendix A, corresponds to a static approximation to the bound-state vertex. In general, however, the k_0 dependence is important and leads to results which, even if in agreement with the results of BBG as far as the power dependence in s, t, and u is concerned, are in disagreement with the specific forms obtained by BBG.

In general we find that the power dependence of Eqs. (1.4) and (1.5) in the asymptotic variables is not obtained unless the DGS weights have very special behavior. As an independent exercise we examine under what conditions generalizations of Eqs. (1.4) and (1.5) are obtained when

$$G_E(t) \underset{t\to\infty}{\sim} \frac{1}{t^{n+1}}, \quad n \geq 0 .$$

We find among other results that in general the Drell-Yan-West relation[18] between the asymptotic behavior of the elastic form factor and the threshold dependence of deep-inelastic scattering holds since in the model as defined these two observables sample the same feature of the bound-state vertex functions (namely the cutoff in transverse momentum).

We hope that our results serve as a guide for cataloging possible relationships between electromagnetic form factors and structure functions and scattering cross sections, at least within the interchange picture. In particular the power dependence of BBG, in which the exclusive cross section is given by six powers of the elastic form factor, is satisfied under the assumption of a condition (see Sec. II) on the DGS weight which insures a strong symmetry of the bound-state vertex function in the off-shell behavior of core and parton. In our opinion the justification of this requirement seems to be questionable in view of the very different physics the core and parton are supposed to manifest. We also examine this question in light of a scaling behavior for the vertex function in Sec. II.

In Sec. II we study the elastic form factor, the exclusive cross section, and their various possible relations. In particular we study the case of general power behavior of the form factor,

$$G(t) \sim t^{-(n+1)}, \quad n \geq 0 .$$

Several examples are worked out explicitly. Although either a momentum-space technique or a Feynman parameter technique can be used here, we employ the latter.

In Sec. III we continue by studying deep-inelastic scattering (scaling properties, the Drell-Yan-West relation, etc.) and the inclusive process. Application of the interchange picture with the use of the vertex in Fig. 3 generally restricts one to the limiting case of inclusive scattering where the detected particle is well separated in momentum from the undetected particles. In this section we employ a momentum-space technique, although again the Feynman parameter method can be used as well.

Of course, all such calculations are sensible only if the constituents of hadrons actually exist as particles in appropriate asymptotic states. We feel in the absence of real constituents the calculations described here using such devices are highly academic. However, for a possible contrary point of view see the recent work of Johnson.[19]

II. ELECTROMAGNETIC FORM FACTORS AND EXCLUSIVE SCATTERING AMPLITUDES

We consider, in detail here, the calculations of the electromagnetic form factors and the scattering amplitudes in a model where the scattering is dominated by the parton-exchange process of Fig. 2(b). The calculations are performed in the approximation where a hadron is composed of a parton and core, both of low mass and spin. This section is written with all particles spinless; we briefly indicate in Appendix B how the formalism is changed by the inclusion of spin. The qualitative conclusions we draw are completely insensitive to spin. While we employ a Feynman-parameter technique in this section, the connection between

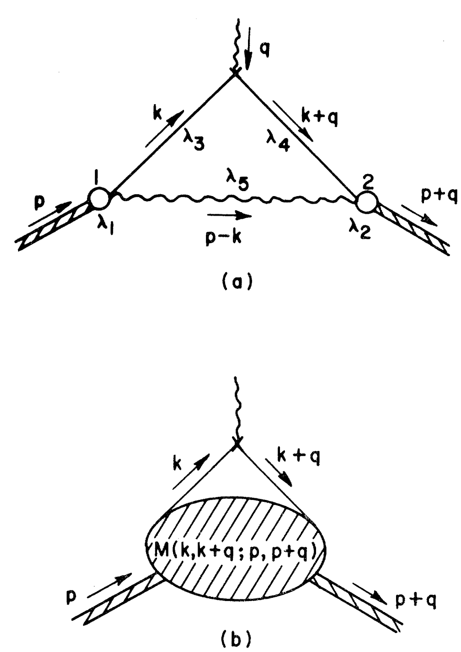

FIG. 4. (a) Electromagnetic form factor in a composite model with constituents parton and core. (b) More general representation for electromagnetic form factor in terms of parton-hadron scattering amplitude M.

this technique and momentum-space methods is by now well understood, and all the results we derive can be equally well derived with the latter procedure.

As we stated in the Introduction the bound-state vertex function which describes the hadron in terms of parton and core components as in Fig. 3 is given by a DGS form, Eq. (1.6a). This expression or the form in Eq. (1.8) can be inserted into expressions for the three- and four-point functions and the momentum integrations over the loop variables can then be performed analytically using the Feynman-parameter method. The Feynman-parameter technique is particularly well suited when DGS forms for the bound-state vertices are involved; in fact the DGS parameter α is itself one of the Feynman parameters in the integration over the parameter space if the DGS representation is derived in perturbation theory. The immediate advantage of Eqs. (1.6a) and (1.6b) is that an infinity of assumptions, all of which can be supported in perturbative models, can be made about the asymptotic behavior of the vertex function in the core and parton momenta by making a corresponding infinity of assumptions about the spectral weight H. In general the DGS representation, which respects analytic properties, is mute on the question of asymptotic properties until H is specified.

We now proceed with the electromagnetic form factor $G(t)$, where the photon attaches to all charge-carrying constituents and with a summation over such constituents. The leading behavior is the result of attachment to the parton line (or the sum of such parton lines), as in Fig. 4(a). We note in passing that in the more general framework shown in Fig. 4(b), where the leading behavior follows only from photon attachment to the parton lines, the gauge invariance of the hadronic current is assured by the simple requirement that the off-shell parton-hadron scattering amplitude M satisfy a simple exchange symmetry in the parton momentum,

$$M(k, k+q; p, p+q) = M(k+q, k; p, p+q) \ . \qquad (2.1)$$

The more specialized model we study of course obeys this requirement.

The expression corresponding to Fig. 4(a) is

$$G(q^2) = \text{const} \times \int d^4k \frac{\Gamma_1(k, p-k; p)\, \Gamma_2(k+q, p-k; p+q)}{(k^2-\mu^2+i\epsilon)\,[(p-k)^2-m^2+i\epsilon]\,[(k+q)^2-\mu^2+i\epsilon]} \ . \qquad (2.2)$$

In this expression (and in all the following) the parton and core masses are μ and m, respectively, and all external particles are on the mass shell, $p^2=(p+q)^2=M^2$.

Use of Eq. (1.8) for the vertices Γ_i in Eq. (2.2) yields the result up to an over-all constant

$$\begin{aligned} G(q^2) = \int \Big[\prod_{i=1}^{2} dh(\alpha_i, \sigma_i^2) \Big] \int d^4k\, \{(k^2-\mu^2+i\epsilon)\,[(p-k)^2-m^2+i\epsilon] \\ \times [(k+q)^2-\mu^2+i\epsilon]\,(k^2-2p\cdot k\,\alpha_1-\sigma_1^2+M^2\alpha_1+i\epsilon)^{n_1+1} \\ \times [(k+q)^2-2(k+q)\cdot(p+q)\,\alpha_2-\sigma_2^2+M^2\alpha_2+i\epsilon]^{n_2+1}\}^{-1} \ , \end{aligned} \qquad (2.3)$$

and use of the Feynman identity allows the k-integration to be done analytically. We find

$$G(q^2) = \int \Big[\prod_{i=1}^{2} dh(\alpha_i, \sigma_i^2) \Big] d\Lambda\, \lambda_1^{n_1}\lambda_2^{n_2} \{ q^2[\lambda_2(1-\alpha_2)+\lambda_4][\lambda_1(1-\alpha_1)+\lambda_3] - \sigma_1^2\lambda_1 - \sigma_2^2\lambda_2 - \tilde{M}^2\}^{-3-n_1-n_2} \ . \qquad (2.4)$$

In this expression, $d\Lambda$ is the usual Feynman-parameter measure, $d\Lambda = \prod_i d\lambda_i\, \delta(\sum \lambda_i - 1)$. There are five parameters involved; these are labelled in Fig. 4(a). In particular λ_1 and λ_2 combine the DGS denominators associated with the vertices 1 and 2. The numerator factor $\lambda_1^{n_1}\lambda_2^{n_2}$ appears because of the higher powers of the DGS denominators. Finally $\tilde{M}^2$ is a finite nonvanishing term involving the external, parton, and core masses whose precise form is not required here.

The corresponding expression for the scattering amplitude in the interchange model is given for the $A(u, t)$ graph (we follow the nomenclature of BBG in this) indicated in Fig. 5 by the expression

$$\begin{aligned} A(u,t) = (2\pi)^{-4} \int d^4k\, \Gamma_1(k, p_A-k; p_A)\, \Gamma_2(k+q, p_B-k-q; p_B)\, \Gamma_3(k+q, p_A-k; p_A+q)\, \Gamma_4(k, p_B-k-q; p_B-q) \\ \times \{(k^2-\mu^2+i\epsilon)\,[(p_A-k)^2-m^2+i\epsilon]\,[(k+q)^2-\mu^2+i\epsilon]\,[(p_B-k-q)^2-m^2+i\epsilon]\}^{-1} \ , \end{aligned} \qquad (2.5)$$

where as usual $s=(p_A+p_B)^2$, $t=q^2$, $u=(p_B-p_A-q)^2$. The hadronic vertices Γ_i are again to be given by Eq.

(1.8). Once more we use the Feynman identity to perform the loop integration over k, and we obtain again up to an over-all constant

$$A(u,t)=\int d\Lambda\left[\prod_{i=1}^{4}dh(\alpha_i,\sigma_i^2)\lambda_i^{n_i}\right]\Big\{t[\lambda_5+\lambda_1(1-\alpha_1)+\lambda_4(1-\alpha_4)][\lambda_7+\lambda_2(1-\alpha_2)+\lambda_3(1-\alpha_3)]+u(\lambda_6+\lambda_4\alpha_4+\lambda_2\alpha_2)(\lambda_8+\lambda_1\alpha_1+\lambda_3\alpha_3)-\tilde{M}^2-\sum\sigma_i^2\lambda_i\Big\}^{-6-\Sigma_{i=1}^{4}n_i}. \quad (2.6)$$

As indicated in Fig. 5, λ_i, $i=1,\ldots,4$, combine the DGS denominators, while $\lambda_5,\ldots,\lambda_8$ combine the appropriate propagators. $\tilde{M}^2$ is (another) finite nonvanishing function of M^2, μ^2, and m^2 whose explicit form we shall not require.

If the so-called "s-t graph" is labelled as in Fig. 6, then we find that $A(s,t)$ is given by Eq. (2.6) with the replacement of u by s and the interchange of λ_6 and λ_8. Similar results hold for the remaining permutations of parton and core lines in which $m^2 \to \mu^2$.

With Eqs. (2.4) and (2.6) in hand we can turn to a discussion of asymptotic behavior and possible relations between the electromagnetic form factor and the four-point function. Asymptotic behavior follows from the application of the usual end-point considerations for the Feynman parameter integral. Such techniques are well-known in the literature; the text[20] by Eden *et al.* is an excellent survey. Before proceeding with the general case we illustrate the technique by considering the case $n_i=0$, $i=1,2$, in the form factor. From Eq. (2.4) the asymptotic behavior in q^2 is $G(q^2)\sim(q^2)^{-3}$ in all regions of Feynman parameter space where the coefficient of q^2 is not zero. However, when

$$[\lambda_2(1-\alpha_2)+\lambda_4][\lambda_1(1-\alpha_1)+\lambda_3]=O(1/q^2) \quad (2.7)$$

the integral can be enhanced. It is maximally enhanced if any of the following sets of parameters are $O(1/q^2)$ (vanishing):

$$(\lambda_2,\lambda_4),\quad (1-\alpha_2,\lambda_4),$$
$$(\lambda_1,\lambda_3),\quad (1-\alpha_1,\lambda_3). \quad (2.8)$$

From the vanishing of any single set from Eq. (2.8) we obtain the result

$$G(q^2)=O((1/q^2)^2). \quad (2.9)$$

The simultaneous vanishing of four scaling sets from Eq. (2.8) adds logarithms to the above result (see Ref. 20):

$$G(q^2)\sim(1/q^2)^2\ln^3(-q^2).$$

In giving this result we tacitly assumed the DGS spectral weights did not vanish near $\alpha=0$ and $\alpha=1$. If on the contrary

$$\int d\sigma^2 H(\alpha,\sigma^2)\underset{\alpha\to 0}{\sim}\alpha^q \underset{\alpha\to 1}{\sim}(1-\alpha)^p, \quad (2.10)$$

where p and q are both >0, then the relative importance of the second and fourth and the first and third sets of parameters in Eq. (2.8) changes. We shall see this in more detail below, where not only logarithms but also powers can be affected.

To complete our discussion of the form factor, we consider $n_i>0$. The numerator factors $\lambda_1^{n_1}\lambda_2^{n_2}$ in Eq. (2.7) are now important. The enhancement of Eq. (2.7) that is most significant is that associated with the smallest exponent of the full numer-

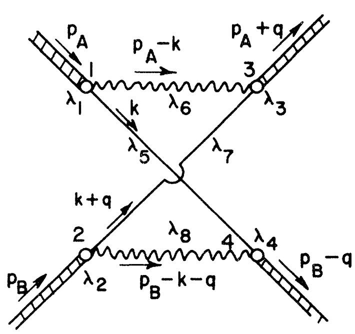

FIG. 5. The (u,t) four-point function for elastic scattering in the interchange model.

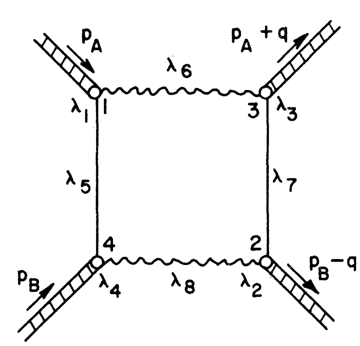

FIG. 6. The (s,t) four-point function for elastic scattering.

ator

$$(1-\alpha_1)^{p_1}(1-\alpha_2)^{p_2}\lambda_1{}^{n_1}\lambda_2{}^{n_2} .$$

Note that the behavior of H near $\alpha=0$, which is associated with the off-shell behavior of the core line, is not of interest, since α_i is by itself not a coefficient of q^2. In other words, α_i is not contained in any of the scaling sets in Eq. (2.8). We obtain up to logarithms

$$G(q^2) \underset{q^2\to\infty}{\sim} (1/q^2)^{a+2} , \qquad (2.11)$$

where

$$a=\min(n_1, n_2, p_1, p_2) . \qquad (2.12)$$

The power in Eq. (2.11) is $a+2$ because there are always at least two parameters which must vanish to make the coefficient of q^2 vanish, as in Eq. (2.7). If of the set $\{n_1, n_2, p_1, p_2\}$ from which a is chosen, l values can all give a, then Eq. (2.11) is augmented by $l-1$ powers of logarithms, as in Eq. (3.4.9) of Ref. 20. (See Appendix C for more details.) Primarily we shall be concerned with the power behavior in the asymptotic region.

Anticipating the computations of Sec. III, we remark here that the Drell-Yan-West relation[18] [if the asymptotic behavior of the elastic form factor is $(1/q^2)^N$, then the threshold dependence of the structure function F_2 in the Bjorken scaling variable is $(1-\omega)^{2N-1}$] will hold automatically within the physical context of the covariant core-parton picture.

There is one further qualification that is necessary before leaving the form factor. Namely, unless there are some special circumstances all regions of the Feynman-parameter space contribute additively. We take the region that contributes the dominant asymptotic behavior. It is possible that regions of comparable importance could cancel; however, this would require quite restrictive conditions on the DGS weight function.

We next proceed with the four-point function, Eq. (2.6). We are interested in the kinematic region of large t, u, and s. We work at fixed angle, so that $t=cu$, where c is a nonzero constant. Then both factors

$$[\lambda_5+\lambda_1(1-\alpha_1)+\lambda_4(1-\alpha_4)] \times[\lambda_7+\lambda_2(1-\alpha_2)+\lambda_3(1-\alpha_3)] \qquad (2.13)$$

and

$$(\lambda_6+\lambda_4\alpha_4+\lambda_2\alpha_2)(\lambda_8+\lambda_1\alpha_1+\lambda_3\alpha_3)$$

are coefficients of t and each must be of order t^{-1} to enhance the asymptotic behavior from $(1/t)^{6+\sum_{i=1}^4 n_i}$. In this case there are many different possible scalings of the Feynman parameters. When the numerator factors $\prod_{i=1}^4 \lambda_i{}^{n_i}$ together with the numerator factors in H, Eq. (2.10), are taken into account, then there will only be some particular subsets of scalings which minimize the falloff of $A(u, t)$.

The fewest possible parameters which go to zero to make the coefficients of t and u in Eq. (2.13) vanish are five. In any case at least one of the pairs

$$\begin{aligned}&(\lambda_5, \lambda_6), (\lambda_5, \lambda_8) ,\\ &(\lambda_7, \lambda_6), (\lambda_7, \lambda_8)\end{aligned} \qquad (2.14)$$

must be zero; these parameters are not associated with numerator factors. To minimize the effect of the numerator we choose the quantity b as follows:

$$\begin{aligned}b=\min\{&n_1+n_2+n_4, n_1+n_3+n_4,\\ &n_1+n_2+n_3, n_2+n_3+n_4,\\ &n_4+n_1+q_2, n_4+n_2+p_1, n_4+p_1+q_2 ,\\ &n_1+n_4+q_3, n_1+n_3+p_4, n_1+p_4+q_3 ,\\ &n_2+n_3+q_4, n_2+n_4+p_3, n_2+p_3+q_4 ,\\ &n_3+n_2+q_1, n_3+n_1+p_2, n_3+p_2+q_1\} .\end{aligned} \qquad (2.15)$$

When b is chosen to be particular set, then the scaling set referred to which vanishes is that set associated with the parameters in b plus the corresponding set in Eq. (2.14). For example, if $b=n_1+n_2+n_4$, then the scaling set is ($\lambda_5, \lambda_6, \lambda_1, \lambda_2, \lambda_4$). [This procedure defines a "d line" in the sense of Ref. 20 although in our work some of the parameters refer to vertices.] Once we have chosen b, then up to logarithms the asymptotic behavior of $A(u, t)$ is

$$A(u, t) \underset{u,t\to\infty}{\sim} (1/t)^{5+b} . \qquad (2.16)$$

If several of the 16 sets in Eq. (2.15) are of the same size, then simultaneous scaling sets occur and logarithms are possible. The maximum number of simultaneous scaling sets is five with a corresponding maximum number of logarithms of four. We refer the reader to Appendix C for details.

The remaining four-point graphs are treated in a similar fashion. We remark that for the (s, t) graph in Fig. 6, which is essentially the same as for $A(u, t)$ with the replacement $u\to s$, it appears that because s and t are of opposite sign for physical scattering angle, the two terms of Eq. (2.13) can cancel for nonzero values of the various Feynman parameters. However, this is a simple pole, not two pinching singularities, so it can be avoided by suitable distortion of the contour in (α, λ) space. Once again, only the end-point singularities contribute.

We now turn to the physical implications of these results for phenomenological relations of the type Eqs. (1.1) and (1.4). A striking fact is that, in general, the asymptotic behavior of the four-point function involves not only n and p but q as well. The quantities q_i govern the behavior of the various vertices Γ_i in the asymptotic off-shell momentum of the core line, since α_i is the coefficient of the off-shell core mass in Γ_i. This is in striking contrast with the form factor, where only the quantities p_i, which govern the off-shell behavior of the vertices in the parton momentum, come into play. In other words, the large-q^2 behavior of the form factor only samples the behavior of the bound-state wave function in the parton line. Moreover, the quantity b which appears in the four-point function, Eqs. (2.15) and (2.16), always involves *at least* one n_i, the power of the DGS denominator, while the quantity a which appears in the three-point function, Eqs. (2.11) and (2.12), can involve either p_i or n_i.

Therefore we obtain a result like Eq. (1.4) if and only if

$$p_i, q_i \geq n_i \ , \tag{2.17}$$

modulo logarithms. In other words, the asymptotic behavior of the vertex function must be the same in both the parton and core momenta, when the off-shell variables are taken large one at a time. In view of the different roles the core and parton play in this theory, this is a rather strong restriction. In particular, if Eq. (2.17) holds, then up to logarithms we obtain

$$G_i(t) \underset{t\to\infty}{\sim} (1/t)^{n_i+2} \tag{2.18}$$

and

$$A(s,t,u) \underset{s,t,u\to\infty}{\sim} t(1/t)^{n_i+n_j+n_k+6} f(\theta) = tG_i(t)\,G_j(t)\,G_k(t) f(\theta) \ , \tag{2.19}$$

where $i \neq j \neq k$ and i,j,k are chosen to label the three hadrons with the slowest falloff in their form factors.

If Eq. (2.17) does not hold, then the relation expressed by Eq. (2.19) is broken in a variety of ways. The four-point function can either fall off faster than or slower than t times three powers of the form factor. For example, if $n_i = 10$, $p_i = q_i = 0$ for all i, then $a = p_i$ and $G_i(t) \sim (1/t)^2$, while $b = 10$ and $A \sim t(1/t)^{16}$. On the other hand, if $n_i = 5$, $p_i = 10$, $q_i = 0$ for all i, then $a = n_i$ and $G_i(t) \sim (1/t)^7$, while $b = 10$ and $A \sim t(1/t)^{16}$. While these examples are purely fanciful in the sense that we have not written a field theory which leads to them they are basically not more arbitrary without further physical assumptions than Eq. (2.17), which insures the BBG results, Eqs. (2.18) and (2.19).

We would like to emphasize the role played by the off-shell behavior of the core line. This behavior is not involved with or determined by the asymptotic behavior of the elastic form factor (nor, as we shall see in Sec. III, of the inelastic form factor). However, it plays a role equivalent to that of the parton line in the behavior of the four-point function, and hence must be considered carefully when relations between form factors and cross sections are under consideration.

We can examine the relative importance of the asymptotic behavior of the bound-state vertex function in parton and core line in a more quantitative manner. If we introduce a scaling variable $\rho = k^2/k'^2$, $k'^2 = (k-p)^2$ in Eq. (1.8), we obtain the result for large k^2 at fixed ρ of the form

$$\Gamma(k^2, k'^2) = \int dh(\sigma^2, \alpha)\,[k^2(1-\alpha) + k'^2\alpha - \sigma^2 + i\epsilon]^{-(n+1)} \underset{\substack{k^2\to\infty \\ \rho \text{ fixed}}}{\longrightarrow} \left(\frac{1}{k'^2}\right)^{n+1} \int_0^1 \frac{d\alpha\, g(\alpha)}{[\rho(1-\alpha) + \alpha - \sigma^2/k'^2]^{n+1}} \ , \tag{2.20}$$

which is well defined for $\rho > 0$ and where $g(\alpha) = \int d\sigma^2 H(\sigma^2, \alpha)$, which is assumed to be convergent.

Now it is obvious that Eq. (2.20) defines a scaling-type behavior for the function $(k^2)^{n+1}\Gamma(k^2, k'^2)$. What is important for the question of the validity of phenomenological relations like Eq. (2.19) is the behavior of the asymptotic form in Eq. (2.20) as ρ approaches 0, $k'^2 \gg k^2$, or ρ approaches ∞, $k^2 \gg k'^2$. In particular we have the following results neglecting mass terms:

$$\begin{aligned} \Gamma(k^2, k'^2) &\underset{\substack{k^2\to\infty \\ \rho\to 0}}{\longrightarrow} \left(\frac{1}{k'^2}\right)^{n+1} \int_0^1 \frac{g(\alpha)}{\alpha^{n+1}} \ , \\ \Gamma(k^2, k'^2) &\underset{\substack{k'^2\to\infty \\ \rho\to\infty}}{\longrightarrow} \left(\frac{1}{k^2}\right)^{n+1} \int_0^1 \frac{g(\alpha)}{(1-\alpha)^{n+1}} \ . \end{aligned} \tag{2.21}$$

If conditions like those of Eq. (2.17) are valid, i.e.,

$$g(\alpha) \underset{\alpha\to 0}{\sim} \alpha^q,\ g(\alpha) \underset{\alpha\to 1}{\sim} (1-\alpha)^p, \ \ p,q > n$$

then the integrals in Eq. (2.21) are well defined up to perhaps logarithmic corrections of the form $\ln(\sigma^2/k^2)$ if $p, q = n$. This constitutes the special requirement that insures Eq. (2.19) and gives the symmetric off-shell behavior (one variable large at a time).

For the particular case of $n = 0$, the results of Eq. (2.21) imply the existence of a free-field light-cone product for the sources of the parton and core $J_p(x)$ and $J_c(0)$, respectively, at $x^2 \to 0$, if they exist, by rather standard arguments:

$$J_p(x)J_c(0) \underset{x^2\to 0}{\sim} \frac{1}{x^2+i\epsilon x_0} O(x)$$
$$+ \text{ less singular terms}, \qquad (2.22)$$

and the expressions in Eq. (2.20) and in the various ρ limits of Eq. (2.21) are the matrix elements of the local operator $O(x)$ on the light cone. For $n>0$ the $1/x^2$ term in Eq. (2.22) is absent and we are relying on the less singular behavior of the expansion in Eq. (2.22) to give the first nonvanishing term.

The upshot of this discussion is that those field theories that have the possibility when treated in a bound-state approximation of yielding a meaningful light-cone expansion with well defined $\langle 0|O(x)|p\rangle$ will give relations of the type given by Eq. (2.19) with $G(t)\to(1/t)^2$ for $n=0$. The resulting scaling in electroproduction and Drell-Yan-West relations etc. will all follow. We are well aware that all of this assumes that the core is described by a local operator and probably is much more complicated itself.

We would also like to add that when the composite hadron system is a boson composed only of a quark-antiquark pair, it becomes more reasonable that the vertex function is symmetric in the core and parton off-shell behavior, insuring Eq. (2.19). From our point of view, and given that nucleon form factors seem to fall off more quickly than those of bosons, the phenomenology of BBG then equivalently becomes more reasonable when (at least one) pion or kaon is involved in the large-angle scattering.

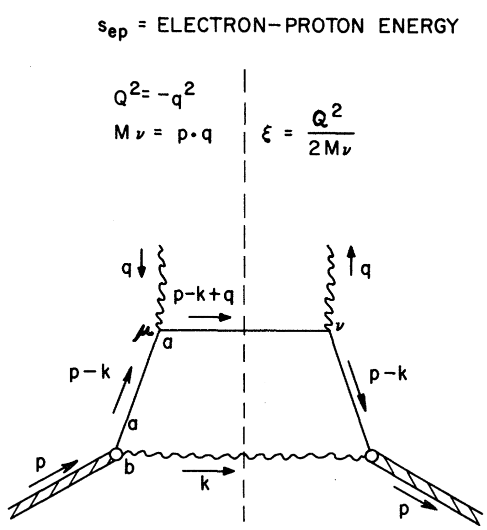

FIG. 7. The diagrammatic representation for the off-shell Compton amplitude $W_{\mu\nu}(q,p)$ in the parton-plus-core model.

III. DEEP-INELASTIC AND INCLUSIVE SCATTERING

Within the assumptions of our approach, the deep-inelastic (DI) scattering cross section, $e+p\to e+X$, is described by the imaginary part of the forward Compton amplitude with photons of mass $-Q^2 = q^2$, $q^2<0$. We find it convenient to discuss this matter in momentum space[21] rather than Feynman parameter language. Notation and variables are defined in Fig. 7. In general, the Bjorken scaling variable $\xi=\omega^{-1}$ runs between 0 and 1 in the physical region, and in the deep-inelastic region $\xi \gg O(M^2/Q^2)$.

The graph of Fig. 7 commits one to a picture in which the parton and core ("a" and "b" respectively in the figure) appear in the asymptotic state. To avoid this one would have to go to a picture as in Fig. 8. This approach would take us out of the set of assumptions we have set for ourselves in that a new vertex is necessary to describe a final state which is not a simple bound state of parton and core.

The invariant decomposition of the amplitude $M_{\mu\nu}$ into invariant form factors $W_{1,2}(q^2,\nu)$ is too well known to repeat here. A convenient frame[22] for evaluation of the form factors is as follows:

$$p=(W,0,M^2/W)\ ,$$
$$q=(O(1/W),Q_\perp,2M\nu/W), \qquad (3.1)$$
$$k=(xW,k_\perp,y/W)\ ,$$

where $W^2=s_{ep}\gg Q^2$, $2M\nu\gg M^2$, and we have written the four-vectors in a $(+,\perp,-)$ notation. The form factor W_2 is uniquely projected in this frame by computing the $\mu=\nu=+$ component of $M_{\mu\nu}$. If λ is the coupling strength of the parton to the electromagnetic current,

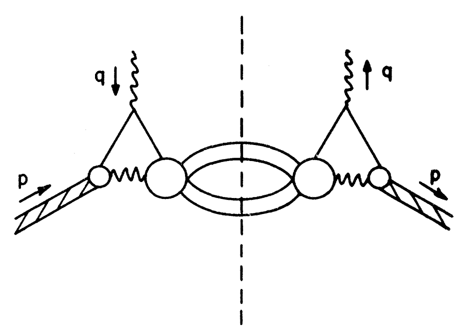

FIG. 8. Off-shell Compton amplitude with final-state interactions.

$$W_2(q^2,\nu)=\lambda^2 \text{ constant}\times\int dx\,d^2k_\perp dy\,4(1-x)^2\,\delta(k^2-m^2)\,\delta((p-k+q)^2-\mu^2)$$
$$\times[(p-k)^2-\mu^2+i\epsilon]^{-2}\prod_{i=1}^{2}dh(\alpha_i,\sigma_i^2)[\alpha_i k^2+(1-\alpha_i)(k-p)^2-\sigma_i^2+i\epsilon]^{-(n+1)}. \tag{3.2}$$

In this expression a factor $4(1-x)^2$ is due to the plus-component coupling of the (spinless) parton line to the electromagnetic current, and we have used the vertex function as given by Eq. (1.8). In addition $d^4k=\frac{1}{2}dxdyd^2k_\perp$, and a vector v has mass $v^2=v_+v_--v_\perp^2$.

We now proceed to perform the y integration by using $\delta(k^2-m^2)$ to set $y=(k_\perp^2+m^2)/x$. We then have

$$W_2(q^2,\nu)=\text{const}\times\int_0^1 dx(1-x)^2x^{2n+3}\int d^2k_\perp\frac{1}{[k_\perp^2+m^2(1-x)+\mu^2x-M^2x(1-x)]^2}$$
$$\times\left[\prod_{i=1}^{2}dh(\alpha_i,\sigma_i^2)[k_\perp^2(1-\alpha_i)+(1-\alpha_i)(1-x)(m^2-M^2x)-\alpha_i m^2x-\sigma_i^2x]^{-(n+1)}\right]$$
$$\times\,\delta\!\left((1-x)\left(2M\nu-\frac{k_\perp^2+m^2}{x}+M^2\right)-(k_\perp-Q_\perp)^2-\mu^2\right). \tag{3.3}$$

The $k_\perp$ integral now converges regardless of the behavior of the $H^{(i)}$ in $1-\alpha_i$ near $\alpha_i=1$. We may then ignore the $k_\perp$ (and mass) dependence in the δ function. Define

$$I^{\text{DI}}(x,\tilde{M}^2)\equiv\int d^2k_\perp\frac{1}{[k_\perp^2+m^2(1-x)+\mu^2x-M^2x(1-x)]^2}$$
$$\times\left[\prod_{i=1}^{2}dh(\alpha_i,\sigma_i^2)[k_\perp^2(1-\alpha_i)+(1-\alpha_i)(1-x)(m^2-M^2x)-\alpha_i m^2x-\sigma_i^2x]^{-(n+1)}\right], \tag{3.4}$$

which is a regular and nonvanishing function of x and masses (generically denoted by $\tilde{M}^2$). Then

$$W_2(q^2,\nu)=\text{const}\times\int_0^1 dx(1-x)x^{2n+3}\,\delta((1-x)2M\nu-Q^2)\,I^{\text{DI}}(x,\tilde{M}^2). \tag{3.5}$$

This result exhibits Bjorken scaling, with the δ function setting $1-x=Q^2/2m\nu\equiv\xi$ and no independent Q^2 or ν dependence in νW_2:

$$\nu W_2(\xi)=\text{const}\times(1-\xi)^{2n+3}\,\xi^2 I^{\text{DI}}((1-\xi),\tilde{M}^2). \tag{3.6}$$

The crucial factor in obtaining the scaling result is, of course, the convergence of the transverse-momentum integral.

We note two features of Eq. (3.6):

(i) The threshold behavior, for $\xi\approx1$, is given by $(1-\xi)^{2a+3}$, where $a=\min(p,n)$, and where p is defined by Eq. (2.10). To see this, note that in the definition of I^{DI}, Eq. (3.4), the integrand is undefined at $x=0$ (corresponding to $\xi=1$) when $1-\alpha_i=0$. Instead, only the quantity $x^{2n+3}I^{\text{DI}}$ is defined. If $p>n$, then the region $1-\alpha_i=0$ causes no difficulty at $x=0$, and the threshold dependence x^{2n+3} is retained. If on the other hand $p<n$, then computation of the integrals in Eq. (3.4) near $\alpha=1$ changes the threshold dependence to x^{2p+3}. Recall from Sec. II that the elastic form factor is given by $G(t)\sim(1/t)^{a+2}$ [see Eqs. (2.11) and (2.12)]. The combination of these two results is precisely the Drell-Yan-West relation, which we now see follows *automatically* within the context of any covariant core-parton picture.

(ii) As $\xi\to0$, which corresponds to Regge behavior in the Compton amplitude, νW_2 vanishes as $\xi^2=(Q^2/2M\nu)^2$. This corresponds to the familiar leading pole at $J=-1$ when spinless particles are involved.

We note that the other inelastic form factor $W_1(q^2,\nu)$ is given by

$$W_1(q^2,\nu)=0, \tag{3.7}$$

in accordance with the Callan-Gross relation[23] for spinless particles. Of course inclusion of spin-$\frac{1}{2}$ partons will change the result in Eqs. (3.6) and (3.7) because of changed numerator factors. We shall not go through this exercise in detail, as it is well known in the literature (see, e.g., Ref. 22).

In this last respect and in others, we do not attempt to make this approach into a complete theory of deep-inelastic scattering. For example, for a description including Regge behavior (moving poles), one would have to relax our assumptions about the core and treat the core in a more general way.[16] Instead we are content with the treatment as it stands, for it is sufficient to lead us to comparison with the inclusive scattering process.

The inclusive process $1+2\to 3+X$ at large momentum transfer and high energy can be computed only at a kinematic boundary within the framework of the core-parton assumptions we have described. Figure 9(a) shows the nature of the pseudo-two-body process (we refer to this as the "unassociated" inclusive process) which can be computed, in one of core-parton permutations which is allowed. Figure 9(b) shows by contrast what the calculation of the full inclusive cross section (we refer to this as the "associated" inclusive process) entails within the core-parton framework; we see that a new vertex is required to describe core + parton → particle + X. Fortunately the kinematics of the detected particle allows us to identify the unassociated case. We simply require that in the center-of-mass system the detected particle, which we recall has large transverse momentum, carry the same energy as the incident particles, namely $\frac{1}{2}s^{1/2}$. If we denote the undetected mass by M_m, then this is the limit where $M_m^2/s \ll 1$, which for small transverse momentum is known as the triple Regge region.

There is no *a priori* argument to see that the

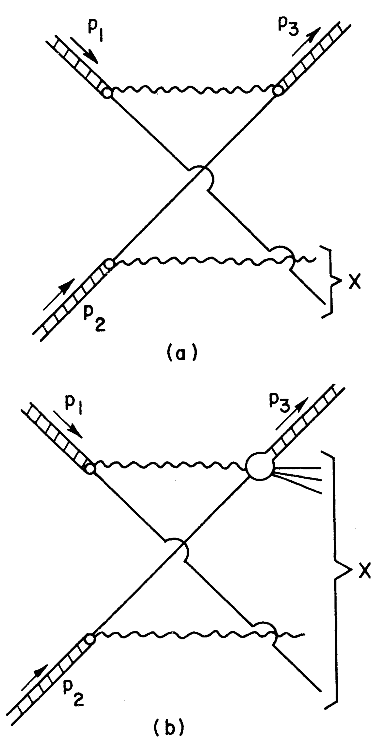

FIG. 9. (a) Unassociated inclusive scattering. (b) Associated inclusive scattering where the detected particle 3 and system X overlap kinematically.

amplitude represented by Fig. 9(a) is larger (or smaller) than that given by Fig. 9(b). At least for spinless particles one may argue that in the construction of the new vertex for Fig. 9(b) more denominator factors are required and that in the limit of all kinematic variables large these factors may damp this contribution. But even if one believed this argument it loses force when spin is introduced.

In the usual fashion, the unassociated inclusive cross section of Fig. 9(a) equals, up to kinematic factors, the discontinuity of a forward 3-to-3 amplitude, as shown and labelled in Fig. 10, and the vertices also labelled by number in that diagram. The vertices 1, 3, 4, and 6 are distinguished from the vertices used in Sec. II in that two lines are on-shell. This only indicates that some of the α_i will become ineffective as far as the asymptotic behavior of the expression $|\mathfrak{M}|^2$ corresponding to Fig. 10 goes. In our discussion of the deep-inelastic cross sections such a vertex appears, but it is only the core which goes on-shell (corresponding to an α_i factor in the DGS representation), whereas here the core is on-shell on one side of the graph while the parton line (corresponding to a $1-\alpha_i$ factor) is on-shell on the other side. We shall see the consequences of this below.

A convenient frame for computation is

$$p_1 \approx \left(W, 0, \frac{M^2}{M}\right), \quad p_2 \approx \left(\frac{M^2}{W}, 0, W\right),$$

$$p_3 \approx \left(-\frac{u}{W}+\frac{M^2 a}{W}, \frac{(tu)^{1/2}}{W}\left(1+\frac{M^2}{s}\delta\right), \frac{-t+M^2 b}{W}\right), \tag{3.8}$$

$$p_4 \approx \left(\frac{s+u}{W}+\frac{M^2 c}{W}, -\frac{(tu)^{1/2}}{W}\left(1+\frac{M^2}{s}\delta\right), \frac{s+t+M^2 d}{W}\right),$$

$$k=(xW, k_\perp, y/W)\ ,$$

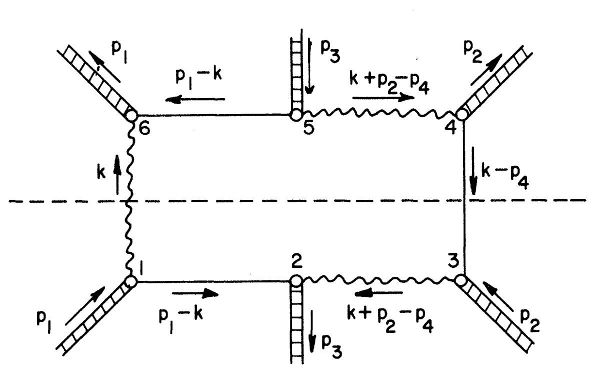

FIG. 10. The six-point function for the inclusive process in Fig. 9(a) in the parton-plus-core approximation.

in the $(+,\perp,-)$ notation. a, b, c, d, and δ are finite functions of s, t, and u. We assume for convenience that the scattering plane is fixed, so the $\perp$ components are fixed in, say, the x direction. We have $W^2=s=(p_1+p_2)^2$, $t=(p_1-p_3)^2$, $u=(p_1-p_4)^2$. It is convenient also to define the variables $x_1=-t/s$, $x_2=-u/s$ lying between 0 and 1. Then a, b, c, d, and δ are finite functions of x_1 and x_2. Unlike the exclusive scattering case these are now independent, and $x_1+x_2\lesssim 1$; $x_1+x_2=1$ is the exclusive scattering limit. The angle at which particle 3 is detected is

$$\cos\theta=\frac{t-u}{t+u}, \tag{3.9}$$

which is, of course, the same form as in the exclusive case.

As for deep-inelastic scattering, the fact that we are dealing with a discontinuity makes it convenient to work with momentum-space techniques rather than Feynman-parameter methods.

With these preliminaries over, we have

$$\begin{aligned}|\mathfrak{M}|^2=\text{const}\int d^4k\left[\prod_{i=1}^{6}dh(\alpha_i,\sigma_i^2)\right]\\
&\times\{[(1-\alpha_1)(p_1-k)^2+\alpha_1 m^2-\sigma_1^2]^{n_1+1}[\alpha_2(k+p_2-p_4)^2+(1-\alpha_2)(p_1-k)^2-\sigma_2^2]^{n_2+1}\\
&\times[\alpha_3(k+p_2-p_4)^2+(1-\alpha_2)\mu^2-\sigma_3^2]^{n_3+1}[\alpha_4(k+p_2-p_4)^2+(1-\alpha_4)\mu^2-\sigma_4^2]^{n_4+1}\\
&\times[\alpha_5(k+p_2-p_4)^2+(1-\alpha_5)(p_1-k)^2-\sigma_5^2]^{n_5+1}[(1-\alpha_6)(p_1-k)^2+\alpha_6 m^2-\sigma_6^2]^{n_6+1}\\
&\times[(p_1-k)^2-\mu^2]^2[(k+p_2-p_4)^2-m^2]^2\}^{-1}\delta(k^2-m^2)\,\delta((k-p_4)^2-\mu^2)\ ,\end{aligned} \tag{3.10}$$

where as usual all denominator factors have a small imaginary part. Recall $d^4k=\frac{1}{2}dx\,dy\,d^2k_\perp$. Then perform the y integration by using the first δ function to set $y=(k_\perp^2+m^2)/x$. We also have

$$(p_1-k)^2\rightarrow-\frac{1}{x}[k_\perp^2+m^2(1-x)-M^2x(1-x)],$$

$$(k+p_2-p_4)^2\rightarrow(x+x_2-1)\left(-t+\frac{k_\perp^2}{x}\right)-\left[k_\perp+\left(\frac{tu}{s}\right)^{1/2}\right]^2+\text{mass terms},$$

$$(k-p_4)^2\rightarrow(x+x_2-1)\left(\frac{k_\perp^2}{x}-s-t\right)-\left[k_\perp+\left(\frac{tu}{s}\right)^{1/2}\right]^2+\text{mass terms}.$$

The integral over $k_\perp$ converges without reference to the second δ function. This convergence already is sufficient from the $(p_1-k)^2-\mu^2$ propagator factor alone. As in the deep-inelastic structure function, the α_1 and α_6 vertex factors also enter into this convergence, depending on the behavior of $H^{(1)}$ or $H^{(6)}$ near $\alpha_1=0$ or $\alpha_6=0$, respectively.

We can then ignore $k_\perp^2$, σ_i^2, mass terms, etc. compared to the large variables s, t, and u. For example,

$$\alpha_5(k+p_2-p_4)^2+(1-\alpha_5)(p_1-k)^2-\sigma_5^2\approx\alpha_5[(s+t)(1-x)+u-(1-x_2)s]+\tilde{M}'^2,$$

where $\tilde{M}'^2$ is a generic mass term, a finite nonvanishing function of α_i, σ_i^2, and rest masses, which we shall sometimes include as a reminder that α_i integrations (in the above expression we refer to α_5) all converge. The remaining $k_\perp$ dependence can then be written as an integral which, including the $\int dh(\alpha_1,\sigma_1^2)$ and $\int dh(\alpha_6,\sigma_6^2)$ factors, we recognize as precisely $I^{\rm DI}(x,\tilde{M}^2)$, Eq. (3.4).

Furthermore, we then have

$$\delta((k-p_4)^2-\mu^2)\rightarrow\delta\left((1-x-x_2)(s+t)-\frac{tu}{s}\right)=\frac{1}{s+t}\,\delta\left(x-1+\frac{x_2}{1-x_1}\right). \tag{3.11}$$

Combining our results, we have

$$\begin{aligned}|\mathfrak{M}|^2=\frac{\text{const}}{s+t}\int dx\,x^{2n+3}\,\delta\left(x-1+\frac{x_2}{1-x_1}\right)I^{\rm DI}(x,\tilde{M}^2)\left[\prod_{i=2}^{5}dh(\alpha_i,\sigma_i^2)\right]\\
&\times\{[\alpha_2(1-x)\,t+\tilde{M}'^2]^{n_2+1}[\alpha_3(1-x)\,t+\tilde{M}'^2]^{n_3+1}\\
&\times[\alpha_4(1-x)\,t+\tilde{M}'^2]^{n_4+1}[\alpha_5(1-x)\,t+M'^2]^{n_5+1}[(1-x)\,t+M'^2]^2\}^{-1}\ .\end{aligned} \tag{3.12}$$

We recognize in this expression the structure function for particle 1, $\nu W_2(\xi)$, as in Eq. (3.6), evaluated at

$$\xi_0 \equiv \frac{x_2}{1-x_1} \tag{3.13}$$

and divided by a factor ξ_0^2. This factor in $\nu W_2(\xi_0)$ is directly traced to the electromagnetic coupling of the current. Clearly such a factor is absent in the inclusive cross section. This factor, which gives the behavior of νW_2 as $\xi_0 \to 0$ and therefore reflects just the Regge behavior, is spin-dependent. A treatment of νW_2 which included diffractive behavior would give $\nu W_2 \to$ const as $\xi_0 \to 0$, and there would then be no functional difference between νW_2 and the factor appearing in the inclusive distribution. Thus

$$\begin{aligned}|\mathfrak{M}|^2 &= \text{const} \times \frac{1}{s+t}\frac{1}{(\xi_0 t)^2}\frac{1}{\xi_0^2}\,\nu W_2(\xi_0)\prod_{i=2}^{5}\left[\int dh(\alpha_i,\sigma_i^2)(\alpha_i\,\xi_0 t+\tilde{M}^2)^{-(n_i+1)}\right]\\ &= \text{const}\times\frac{\nu W_2(\xi_0)}{(s+t)t^2}\,\xi_0^{-(\Sigma n_i+8)}\prod_{i=1}^{2}\int dh(\alpha_i,\sigma_i^2)(\alpha_i t+\tilde{M}^2)^{-(n_i+1)}\ .\end{aligned} \tag{3.14}$$

In this remarkably simple expression we have thus far been required to make no assumptions at all about the behavior of the $H^{(i)}$. Note that the leading behavior (the t dependence) of the denominators comes from the $(k+p_2-p_4)^2$ factor in the vertex functions 2 through 5. Since this corresponds to a core line, only α_i is involved, not $1-\alpha_i$.

To characterize the actual asymptotic behavior of Eq. (3.14), we note that again the dependence of $H^{(i)}$ on α_i is crucial, as enhancements from $\alpha_i = O(1/t)$ are possible. Since the region $1-\alpha_i \to 0$ contains no new features, write as in Eq. (2.10)

$$H^{(i)}(\alpha_i,\sigma_i^2) = \alpha_i^{q_i}\bar{H}^{(i)}(\alpha_i,\sigma_i^2)\ , \tag{3.15}$$

where $\bar{H}^{(i)}$ is nonvanishing and finite as $\alpha_i \to 0$. (Recall also that vertex 2 is identical to 5, and 3 to 4.) Then let

$$r_i \equiv \min(q_i, n_i)\ . \tag{3.16}$$

We have

$$\int dh(\alpha_i,\sigma_i^2)(\alpha_i\,t+\tilde{M}^2)^{-(n_i+1)} = \text{const}\times\ t^{-(r_i+1)}\ , \tag{3.17}$$

in the manner discussed in detail in Sec. II. To get the inclusive cross section $E\,d^3\sigma/d^2p$ we divide the quantity $|\mathfrak{M}|^2$ by a single power of s. Thus

$$E\frac{d^3\sigma}{d^3p} = \text{const}\times\frac{1}{s(s+t)}\left(\frac{1}{t^{\Sigma r_i+6}}\right)\times\xi_0^{-(\Sigma n_i+8)}\,\nu W_2(\xi_0)\ , \tag{3.18}$$

where ξ_0 is given by Eq. (3.13). (The full answer comes from summing the above term with a corresponding term with $t \leftrightarrow s$, as well as another pair which exchanges the role of p_1 and p_2, so that deep-inelastic scattering of particle 2 is also relevant.)

To compare the power dependence of Eq. (3.18) to the elastic form factors, we note that *a priori* there is no connection, because r_i, Eq. (3.16), requires knowledge of the off-shell behavior in the core line, whereas according to Eqs. (2.11) and (2.12) the off-shell behavior in the parton line is of importance for the form factor. If, comparing Eqs. (3.16) and (2.12), we have

$$p_i = q_i \quad or \quad p_i, q_i \geq n_i\ , \tag{3.19}$$

then Eq. (3.18) becomes for fixed angle

$$E\frac{d^3\sigma}{d^3p} = \text{const}\times G_3^2(t)\,G_2^2(t)\,\nu W_2^{(1)}(\xi_0)\ , \tag{3.20}$$

where the subscripts on G and the superscript on νW_2 labels the particle referred to. Equation (3.19) again expresses a strong symmetry in parton and core lines. If it did not hold, then although the power dependence of the cross section is well defined, the relation expressed by Eq. (3.20) fails. Equation (3.20) is in agreement with the work of BBG[3] and LP.[4]

If one were to decide that the rather special type of inclusive scattering we are studying, as in Fig. 9(a), were an important component of the total single-particle distribution at large s and t, then one could ask about two-particle spectra, etc. This would entail "opening up" the on-shell core or parton line. For example, referring to vertex 1 in Fig. 10, one might require knowledge of the full amplitude parton + particle 1 → particle $+X$. Alternatively, partial knowledge of the discontinuity in the parton + particle elastic scattering amplitude would suffice. Since the imaginary part of the parton-particle elastic scattering amplitude occurs in deep-inelastic scattering, and in particular partially inclusive deep-inelastic scattering, $e+p \to e+\text{hadron}+X$, picks out the required discontinuity, one may be able to relate the two-particle distribution to quantities appearing in the partial deep-inelastic process. Such a relation would presumably be subject to the caution we prescribe for the entire subject of this paper.

Furthermore, referring to vertex 3 of Fig. 10, one also requires knowledge of the more unfamiliar core-particle amplitude. Landshoff and Polkinghorne[16] have noticed that if particle 3 is a meson, then in the usual quark model it is not unreasonable that the core itself be a parton (quark). They have then gone on to make a more detailed picture of several-particle inclusive reactions at large s and t based upon their ideas of parton-particle scattering. We shall not pursue this subject further in the present work.

ACKNOWLEDGMENTS

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APPENDIX A

In this appendix we review and discuss the wave equation and corresponding solution—the "wave function"—used by BBG to describe the vertex of Fig. 3. Consider the T matrix for a "parton" of momentum k and mass μ and "core" of momentum $p-k$ and mass m (both spinless for simplicity) to scatter into a state p (not necessarily a bound state),

$$T(k,p-k;p)=\int d^4x_1\, d^4x_2\, e^{-ik\cdot x_1}\, e^{-i(p-k)\cdot x_2} \times\langle 0|(J_p(x_1)\,J_c(x_2))_+\,|p\rangle\ . \qquad \text{(A1)}$$

If we define

$$T(k,p-k;p)\equiv(2\pi)^4\,\delta(p-k-(p-k))\ t(k,p-k;p)\ , \qquad \text{(A2)}$$

then we shall assume that t satisfies a (covariant) Bethe-Salpeter equation of the form

$$t(k,p-k;p)=V(k,p-k;p) - i(2\pi)^4\int d^4q\,\Delta_p(q)\,\Delta_c(p-q) \times V(q,p-q;p)\,t(q,p-q;p)\ , \qquad \text{(A3)}$$

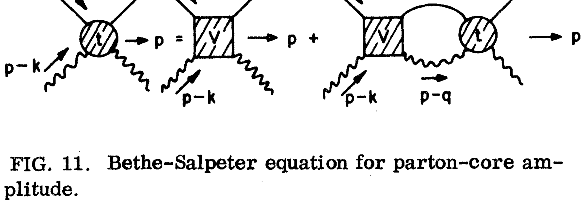

FIG. 11. Bethe-Salpeter equation for parton-core amplitude.

as shown in Fig. 11. Δ_p and Δ_c are the propagator functions for parton and core, respectively.

The vertex function is defined as the residue of the dynamical bound-state pole at $p^2=M^2$,

$$\Gamma(k,p-k;p)\equiv\lim_{p^2\to M^2}(p^2-M^2)\,t(k,p-k;p)\ , \qquad \text{(A4)}$$

and this function satisfies a homogeneous equation,

$$\Gamma(k,p-k;p)=-i(2\pi)^{-4}\int d^4q\,\Delta_p(q)\,\Delta_c(p-q) \times V(q,p-q;p)\,\Gamma(q,p-q;p)\ . \qquad \text{(A5)}$$

This equation can be the starting point of a full theory of the vertex function. For illustration we shall take the potential to be a single (spinless) particle exchange,

$$V=[(k-q)^2-m_0{}^2+i\epsilon]^{-1}\ ; \qquad \text{(A6)}$$

a spectral decomposition of the full V into the form of Eq. (A6) is always possible.

Although Eq. (A5) is written in a Feynman-diagram form, we can easily rewrite it in the language of old-fashioned perturbation theory, or alternatively in the language of the infinite-momentum-frame perturbation theory, better known[15] as Weinberg's rules. The work of BBG is carried out in the infinite-momentum-frame formalism, so we shall concentrate on this approach.

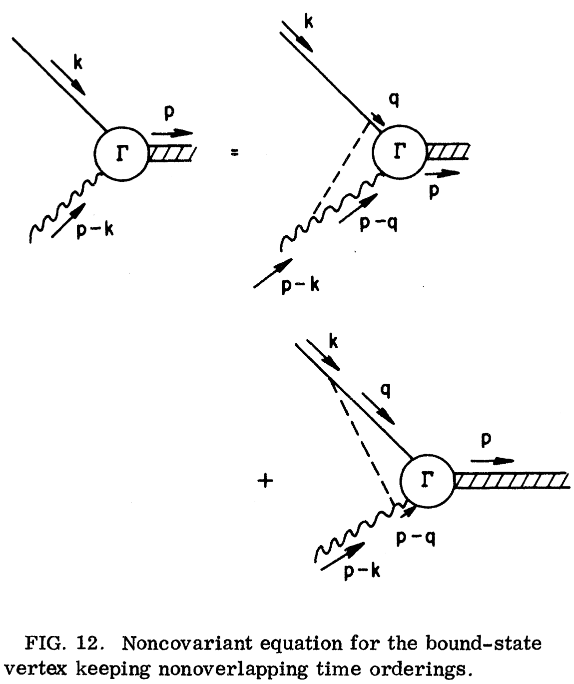

FIG. 12. Noncovariant equation for the bound-state vertex keeping nonoverlapping time orderings.

It is now well known that the rules of old-fashioned perturbation theory in the infinite-momentum frame follow from picking up poles in the q_- variable, where $d^4q=\frac{1}{2}d^2q_\perp\,dq_+\,dq_-$. Define the four-vectors in an appropriate frame by

$$\begin{aligned} p&=(W,0,M^2/W)\ ,\\ k&=(xW,\vec{k},(\vec{k}^2+\mu^2)/(xW))\ ,\\ p-k&=((1-x)W,-\vec{k},(\vec{k}^2+m^2)/[(1-x)W])\ ,\\ q&=(yW,\vec{q},\overline{q}/W)\ ,\end{aligned}\tag{A7}$$

where we use a (+, ⊥, −) notation. To perform the q_- integration we would now need full knowledge of the singularities of $\Gamma(q,p-q;p)$. However, if we arbitrarily ignore the singularities of the vertex function, then the q_- singularities are explicitly displayed. We find, according to well-known application of Cauchy's theorem,

$$\Gamma(k_\perp,x)=\frac{1}{2(2\pi)^3}\int_0^1 dy\int d^2q_\perp\,\frac{1}{y(1-y)(y-x)}\,\frac{1}{M^2-(q_\perp{}^2+\mu^2)/y-(q_\perp{}^2+m^2)/(1-y)}\,\mathcal{V}(k_\perp,q_\perp;x,y)\Gamma(q_\perp,y)\ ,\tag{A8}$$

where

$$\mathcal{V}(k_\perp,q_\perp;x,y)=\theta(x-y)\left[\frac{q_\perp{}^2+\mu^2}{y}+\frac{(k_\perp-q_\perp)^2+m_0{}^2}{x-y}-\frac{k_\perp{}^2+m^2}{x}-M^2\right]^{-1}+\theta(y-x)\left[\frac{(k_\perp-q_\perp)^2+m_0{}^2}{y-x}-\frac{q_\perp{}^2+m^2}{1-y}+\frac{k_\perp{}^2+m^2}{1-x}-M^2\right]^{-1}\ .\tag{A9}$$

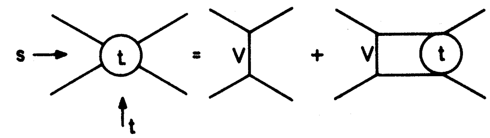

FIG. 13. Bethe-Salpeter equation in the one-meson-exchange approximation for the kernel.

In Eq. (A8) we have recognized that in infinite-momentum language only $k_\perp$ and x are independent variables. Equation (A8) is precisely the form of equation used by BBG, when we define the particle "wave-function" $\psi(k_\perp,x)$ by

$$\psi(k_\perp,x)\equiv\left(M^2-\frac{q_\perp{}^2+\mu^2}{y}-\frac{q_\perp{}^2+m^2}{1-y}\right)\Gamma(k_\perp,x)\ .\tag{A10}$$

Equation (A8) has a precise graphical meaning. Figure 12 shows this equation, where the two terms of $\mathcal{V}(k_\perp,q_\perp;x,y)$ in Eq. (A9) correspond to the two + − component orderings shown in the figure. This equations was first put forth by Weinberg[15] in his paper on the infinite-momentum rules. He fully recognized its noncovariant nature.

Of course a noncovariant equation can still be very useful if it is approximately true in some kinematic regime. Therefore we investigated various limits of this equation. To begin with, let us for completeness consider the full Bethe-Salpeter equation, as shown for spinless particles in Fig. 13. The fourth-order time-ordered graphs which we included in the modified equation [analogous to Eq. (A8)] are shown in Figs. 14(a)–14(d). At the same time the remaining graphs are shown in Figs. 14(e)–14(f). If we label the lines of all the graphs as in Fig. 14(a), and if the energy of the line labeled by i is E_i, then according to the rules of old-fashioned perturbation theory, the denominator structure of each graph is

$$\begin{aligned}&(\mathrm{a})\ \ (E_a+E_b-E_1)(E_b+E_d-E_1-E_2)(E_c+E_d-E_4)\ ,\\ &(\mathrm{b})\ \ (E_a+E_d-E_2)(E_b+E_d-E_1-E_2)(E_b+E_c-E_3)\ ,\\ &(\mathrm{c})\ \ (E_a+E_d-E_2)(E_b+E_d-E_1-E_2)(E_c+E_d-E_4)\ ,\\ &(\mathrm{d})\ \ (E_a+E_b-E_1)(E_b+E_d-E_1-E_2)(E_b+E_c-E_3)\ ,\\ &(\mathrm{e})\ \ (E_a+E_d-E_2)(E_a+E_c+E_1-E_3)(E_b+E_c-E_3)\ ,\\ &(\mathrm{f})\ \ (E_a+E_b-E_1)(E_a+E_c+E_2-E_4)(E_c+E_d-E_4)\ .\end{aligned}\tag{A11}$$

A convenient infinite-momentum frame for computation is

$$\begin{aligned} p_1&=\left(xP+\frac{m^2}{2xP}\,,0,\,xP\right),\\ p_2&=\left((1-x)P+\frac{m^2}{2(1-x)P}\,,0,\,(1-x)P\right),\\ p_3&=\left(\overline{x}P+\frac{m^2+q_\perp{}^2}{2\overline{x}P}\,,\,q_\perp,\,\overline{x}P\right),\\ p_4&=\left((1-\overline{x})P+\frac{m^2+q_\perp{}^2}{2(1-\overline{x})P}\,,\,-q_\perp,\,(1-\overline{x})P\right),\end{aligned}\tag{A12}$$

where we have labelled a four-vector by (energy, perpendicular component, z component), and where we envision $P\to\infty$. We have

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ &= m^2[x(1-x)]^{-1} , \\ t &= (p_1 - p_3)^2 \\ &= -q_\perp{}^2 + \left(\frac{m^2}{x} - \frac{q_\perp{}^2 + m^2}{\bar{x}}\right)(x - \bar{x}) , \end{aligned} \tag{A13}$$

and we label the integrated loop 3-momentum by

$$(p_{b\perp}, p_{b3}) \equiv (k_\perp, k_3) = (k_\perp, yP) . \tag{A14}$$

The other 3-vectors are determined by 3-momentum conservation, and the energy of the internal lines is determined by putting them on the mass shell.

We can now directly compute the energy denominators (A11) in terms of external and internal momentum variables. We find that only graphs (b), (c), (d), and (e) have denominator factors of order P^0 for a finite range of y; therefore the corresponding terms are the only ones which survive when $P \to \infty$. Equivalently we can start with the Feynman graph for the box and perform a k_- integration. The analytic expression we find is exactly the same as the form given by the old-fashioned perturbation theory.

We can now distinguish various limits of the external variables.

(1) s, t finite. All graphs contribute here. The analog to Eq. (A8) for the scattering amplitude fails in this order.

(2) $s \to \infty$; t, $q_\perp{}^2$ finite. This is the high-energy small-angle limit. In this limit the finite integration range of y in Fig. 14(e) vanishes. Thus the Eq. (A8) analog is approximately true in this order, since graphs (b), (c), and (d) are consistent with the equation.

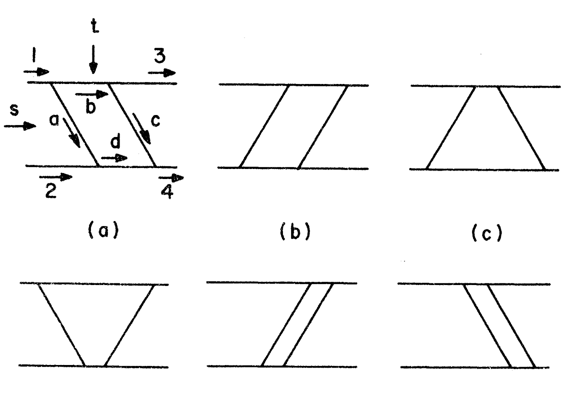

FIG. 14. Time-ordered graphs in fourth-order corresponding to the energy denominators in Eq. (A11); (a) through (d) are contained in Eq. (A8); (e) and (f) are not contained in Eq. (A8).

(3) $s \to \infty$; t/s, $q_\perp{}^2/s$ finite. This is the large-angle scattering limit. In this case the y integration range for graph (e) remains nonzero. However, the integration over $d^2k_\perp$ gives a result of order $(\ln s)/q_\perp{}^4$ compared to $O(1/q_\perp{}^2)$ for the remaining three graphs. Thus the analog of (A8) is approximately true in fourth order.

(4) s finite, $|q_\perp{}^2| \to \infty$. This is a limit which is appropriate for the applications of the body of this paper. We envision $q_\perp \to \infty$ within a graph (such as the form factor or four-point function) while s is fixed at the binding energy of the composite system. The same remarks hold here as for the limit (3). Thus in fourth order the equation used by BBG would appear to be justified.

In any application to strong interactions we must go beyond fourth order. We find when we do this that the equation fails in any limit. For example, in sixth order we compared the contributions of the old-fashioned perturbation graphs represented by Figs. 15(a) and 15(b). Figure 15(a) is not contained in the equation, whereas Fig. 15(b) is. We found that in the various limits the contributions of each graph were of comparable size, including logarithms.

As we mentioned earlier we can uncover exactly the form of BBG with regard to the expression for the four-point function [e.g., Fig. 6] or three-point function [e.g., Fig. 5(a)] if we neglect the k_- (or k_0) singularities of the vertex function $\Gamma(k, p-k; p)$ in performing the loop integration. More precisely if we see a form like

$$\Delta_p(k)\Delta_c(p-k)\,\Gamma(k, p-k; p) = \frac{1}{k^2 - \mu^2 + i\epsilon}\,\frac{1}{(p-k)^2 - m^2 + i\epsilon}\,\Gamma(k, p-k; p) \tag{A15}$$

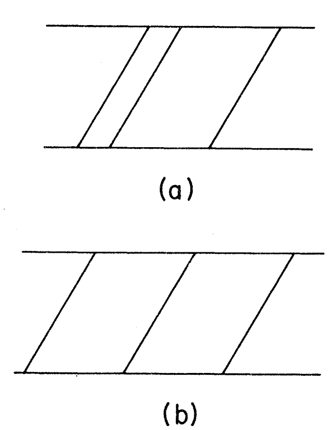

FIG. 15. (a) Time-ordered graph in sixth order not contained in the integral equation. (b) Time-ordered graph in sixth order contained in the integral equation.

as in Eqs. (2.4), (2.6), or (A3), we keep the poles of the Green's functions $\Delta_p(k)$ and $\Delta_c(p-k)$ but neglect the poles of $\Gamma(k, p-k; p)$ in the k_0 or k_-.

We now examine the physical nature of that approximation and its limitations. It is useful to use the Low equation which follows from completeness to evaluate the time-ordered product for $\Gamma(k, p-k; p)$ of Eqs. (A1) and (A4) ($\vec{p}, \vec{k}$ are three-vectors here)

$$\Gamma(k, p-k; p) = (2\pi)^3 \sum_n \frac{\delta^3(\vec{P}_n - \vec{k}) \langle 0|J_p(0)|n\rangle\langle n|J_c(0)|p\rangle}{k_0 - E_n(\vec{k}) + i\epsilon} + \sum_m \frac{\langle 0|J_c(0)|m\rangle\langle m|J_p(0)|p\rangle}{k_0 + E_n(\vec{k}) - E(\vec{p}) + i\epsilon}, \quad \text{(A16)}$$

where

$$E(\vec{p}) = (\vec{p}^{\,2} + M^2)^{1/2}, \quad E_n(\vec{k}) = (\vec{k}^2 + M_n^2)^{1/2}, \quad \text{(A17)}$$

M_n being the mass of the intermediate state in Eq. (A16). The nature of the approximation necessary to recover the wave function of BBG is the same as to obtain the bound-state integral Eq. (A8). Namely in the loop integration the most important contributions come from the region where

$$E_n \gg k_0 \quad \text{and} \quad E(\vec{p}) - E_n(\vec{k}) \gg k_0 . \quad \text{(A18)}$$

The energy denominators associated with the vertex function itself are neglected; this amounts to a static approximation for the bound-state wave function in which the relative time variable in Eq. (A1) is set to zero. Within the spirit of this approximation the most important singularities in the k_0 plane are those associated with the product of the two propagator functions in Eqs. (A15) and (A5) and are given by the form

$$\Delta_c(p-k)\Delta_p(k) \simeq \frac{1}{2\omega_p(\vec{k})}\frac{1}{2\omega_c(\vec{p}-\vec{k})} \times \frac{1}{k_0 - \omega_p(\vec{k}) + i\epsilon} \times \frac{1}{k_0 - E(\vec{p}) - \omega_c(\vec{k}-\vec{p}) + i\epsilon}, \quad \text{(A19)}$$

where

$$\omega_p(\vec{k}) = (\vec{k}^2 + \mu^2)^{1/2}, \quad \omega_c(\vec{p}-\vec{k}) = [(\vec{p}-\vec{k})^2 + m^2]^{1/2} .$$

It follows then that the most important energy denominator is the form

$$E(\vec{p}) - \omega_p(\vec{k}) - \omega_c(\vec{p}-\vec{k}) .$$

This type of approximation of keeping the energy denominators from the Green's function is certainly good in the nonrelativistic limit and indeed this is the manner in which the nonrelativistic Schrödinger equation is derived from the Bethe-Salpeter Eq. (A5) in the static limit.

In the work of BBG a similar set of approximations is made in the infinite-momentum limit. These approximations do not lead to results of the same form as in Sec. II; however, these arguments do not preclude numerical accuracy in some kinematic regions. In particular they have neglected terms which contain the hadron bound-state binding energy compared to rest masses of hadrons as a measure of the numerical accuracy.

APPENDIX B

We want to very briefly indicate how the procedures are amplified by the inclusion of spin. For example, consider the vertex of Fig. 3 when the hadron is spinless but the core and parton each has spin $\frac{1}{2}$. Such a situation may be appropriate when the hadron is a pion and the core and parton each have quark quantum numbers. We decompose the vertex of Fig. 3 into the irreducible form

$$\Gamma(k, p-k; p) = F_0(k, p-k; p) + \not{k} F_1(k, p-k; p) + \not{p} F_2(k, p-k; p) + [\not{k}, \not{p}] F_3(k, p-k; p) . \quad \text{(B1)}$$

The procedure is now to write a separate DGS decomposition for each one of the F_i.

The form factor G, for example, is then extracted from the irreducible pion electromagnetic vertex function $\Gamma_\mu(q^2)$ by

$$\Gamma_\mu(q^2) = (2p+q)_\mu G(q^2) ,$$

as in Fig. 4(a). The loop integral now involves a trace over the spin-$\frac{1}{2}$ constituent momenta. Because of the large number of cross terms in the F_i, it is clear the algebra quickly escalates. Nevertheless, it is also clear that the physical principles we have discussed in this paper concerning the off-shell behavior of core and parton continue to be true, so that our general conclusions are unchanged. Corresponding assumptions about the F_i of Eq. (B1) can be made which retain the essential model dependence of our results in the text.

It may also be appropriate to remark that if the hadronic states themselves have spin, then there may be added requirements, e.g., scaling between the electric and magnetic form factors of a spin-$\frac{1}{2}$ particle. As BBG have correctly pointed out, such requirements only affect the choice of spin for the constituents. Our general conclusions are again unchanged.

APPENDIX C

In this brief appendix we list a complete set of scalings of the Feynman parameters associated with the asymptotic behavior of the form factor and box graphs of Figs. 4 and 5.

1. Form Factor

Recall that the form factor in parametric form is given by Eq. (2.4) which we write here for completeness:

$$G(q^2) = \int \prod_{i=1}^{2} dh(\alpha_i, \sigma_i^2)\, d\Lambda \times \{q^2[\lambda_2(1-\alpha_2)+\lambda_4][\lambda_1(1-\alpha_1)+\lambda_3] - \sigma_1{}^2\lambda_1 - \sigma_2{}^2\lambda_2 - \tilde{M}^2\}^{-3} \qquad \text{(C1)}$$

Here we suppress numerator effects and set $n_i = 0$ since we merely desire to discuss d lines of comparable importance; as far as the power dependence of Eq. (2.11), see the discussion after Eq. (2.11).

The techniques we follow are exactly those of Ref. 20. The technique of letting groups of Feynman parameters vanish simultaneously is called scaling. Our first scaling will be defined by the change of variables

$$\lambda_1 = \rho_1 \lambda_1^1, \quad \lambda_3 = \rho_1 \lambda_3^1, \qquad \text{(C2)}$$

where

$$\lambda_3^1 + \lambda_1^1 = 1 .$$

This variable transformation has the Jacobian

$$d\lambda_1 d\lambda_3 = \rho_1\, d\rho_1\, \delta(\lambda_1^1 + \lambda_3^1 - 1)\, d\lambda_1^1\, d\lambda_3^1 \qquad \text{(C3)}$$

and the scaling operation is indicated by the notation

$$\rho_1 : \lambda_1, \lambda_3 . \qquad \text{(C4)}$$

The single parameter ρ_1 going to zero sets λ_1 and λ_3 to zero.

Next we perform the scalings

$$\rho_2:\ \lambda_2, \lambda_4, \quad \tau_1:\ 1-\alpha_1, \lambda_3^1, \quad \tau_2:\ 1-\alpha_2, \lambda_4^1 . \qquad \text{(C5)}$$

We finally obtain therefore for the form factor the expression

$$G(q^2) = \int \prod_{i=1}^{2} dh(\alpha_i, \sigma_i^2)\, d\tilde{\Lambda}\, \rho_1 d\rho_1 \rho_2 d\rho_2 \tau_1 d\tau_1 \tau_2 d\tau_2 \times (\rho_1\rho_2\tau_1\tau_2 g q^2 + h)^{-3}, \qquad \text{(C6)}$$

where $d\tilde{\Lambda}$ is the appropriate Feynman measure for the irrelevant Feynman parameters, defined by expressions like Eq. (C3), h is a function whose details need not concern us, and g is a nonvanishing function of the parameters after the scale transformation has been made.

The expression in Eq. (C6) immediately enables us to read off the relevant asymptotic behavior for the case $\rho_1 = \rho_2 = \tau_1 = \tau_2 = 0$. From Eq. (3.49) of Ref. 20 we immediately have from the region of small ρ_1, ρ_2, τ_1, and τ_2 the asymptotic behavior:

$$G(q^2) \sim \int \prod_{i=1}^{2} dh(\alpha_i, \sigma_i^2) d\Lambda \frac{1}{gh} \left(\frac{1}{q^2}\right)^2 (\ln q^2)^3 . \qquad \text{(C7)}$$

Of course the reader should remember that such fussiness about complete sets of scalings and the number of logarithms is irrelevant if a single p or n of Eqs. (2.4) or (2.10) is smaller than any of the others. Then one scaling gives a result which dominates asymptotically by powers.

2. Scattering Amplitude

The box graph of Fig. 5 is only slightly more complicated than the form factor of Fig. 4. Recall, from Eq. (2.6), that at fixed angle the expression for $A(u,t)$ is given by (suppressing numerators and setting $n_i = 0$)

$$A(u,t) = \int \prod_{i=1}^{4} dh(\alpha_i, \sigma_i^2) d\Lambda \Big\{ t\{[\lambda_5 + \lambda_1(1-\alpha_1) + \lambda_4(1-\alpha_4)][\lambda_7 + \lambda_2(1-\alpha_2) + \lambda_3(1-\alpha_3)] + c[\lambda_6 + \lambda_4\alpha_4 + \lambda_2\alpha_2][\lambda_8 + \lambda_1\alpha_1 + \lambda_3\alpha_3]\} - \sum_{i=1}^{4} \sigma_i{}^2 \lambda_i - \tilde{M}^2 \Big\}^{-6}, \qquad \text{(C8)}$$

where $c = u/t$, and we denote the terms in square brackets as factors 1, 2, 3, and 4, respectively.

The cardinal rule of the scaling trick is not to violate the δ function of the Feynman-parameter integration, Eq. (C3), yet still simultaneously set as many parameters to zero as possible until the remaining coefficient of t is nonzero.

Firstly, the sets of Feynman parameters of Eq. (C8) that can simultaneously vanish must exclude one of the factors 1, 2, 3, or 4 of Eq. (C8). This follows from the fact that $1 = \sum_i \lambda_i$ is a constraint on the parameter integration. As an example, we consider in detail the set of scalings that gives vanishing factors (1, 3, and 4). The number of parameters that can simultaneously vanish is five. A complete set of scalings are defined by the notation of Eq. (C5).

$$\rho_1: \lambda_1, \lambda_2, \lambda_4, \lambda_5, \lambda_6,$$
$$\rho_2: \lambda_1^1, \lambda_4^1, \lambda_5^1, \lambda_6^1, \alpha_2,$$
$$\rho_3: \lambda_1^2, \lambda_3, \lambda_6^2, 1-\alpha_1, \alpha_2^1,$$
$$\text{or}\begin{cases}\rho_4: \lambda_1^3, \lambda_4^2, \lambda_5^2, \lambda_8^1, \alpha_3,\\ \rho_5: \lambda_1^3, \lambda_3^2, \lambda_5^2, \lambda_1^8, (1-\alpha_1)^1,\end{cases} \tag{C9}$$
$$\rho_6: \lambda_1^4, \lambda_5^3, \lambda_8^2, \alpha_3^1, (1-\alpha_4).$$

The scalings ρ_4 and ρ_5 are connected by *or* since the scalings cannot be implemented simultaneously. The superscripts in Eq. (C9) merely indicate the fact that the variable has been part of more than one simultaneous scaling set and indicates the number of times that variable has been part of variable transformation of the type given explicitly for the form factor in Eq. (C2).

The resulting structure then takes the form:

$$A(u,t)=\int\prod_{i=1}^4 dh(\alpha_i,\sigma_i^2)\int d\tilde{\Lambda}\prod_i(\rho_i)^4 d\rho_i \times\frac{1}{(t\rho_1\rho_2\rho_3\rho_{4,5}\rho_6 g+h)^6}, \tag{C10}$$

where g and h are the scaled functions of the parameters. This expression has the asymptotic behavior

$$A(u,t)\sim\int\prod_{i=1}^4 dh(\alpha_i,\sigma_i^2)\int d\tilde{\Lambda}\times\frac{1}{gh}\frac{1}{t^5}(\ln t)^4. \tag{C11}$$

There is completely similar asymptotic behavior for the other sets leaving 1, 3, or 4 finite and their treatment is exactly the same.

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