*Work supported by the U. S. Atomic Energy Commission.

[1]N. N. Khuri, Phys. Rev. Lett. 30, 1094 (1973).
[2]A. Martin and F. Cheung, *Analyticity Properties and Bounds of the Scattering Amplitudes* (Gordon and Breach, New York, 1970), p. 32.
[3]See, for example, V. Singh and S. M. Roy, Phys. Rev. D 1, 2638 (1970); M. B. Einhorn and R. Blankenbecler, Ann. Phys. (N.Y.) 67, 480 (1971); S. M. Roy, Phys. Rep. 5C (1972); F. Yndurain, Rev. Mod. Phys. 44, 645 (1972); A. P. Balachandran, Maurice L. Blackmon, and Cassio Sigaud, Phys. Rev. D 7, 3420 (1973) and references contained therein.
[4]The "solution" (6) of Eq. (4) may not give an integer value for L, necessitating a slight modification of Eqs. (4), (5), and (7). For a discussion of this point, see the cited references. We ignore this inessential complication.

PHYSICAL REVIEW D VOLUME 8, NUMBER 11 1 DECEMBER 1973

Energy Sum Rule and the Approach to Scaling: An Inclusive and Exclusive View*

Dennis Sivers†
High Energy Physics Division, Argonne National Laboratory, Argonne, Illinois 60439
(Received 29 March 1973)

The approach to Feynman-Yang scaling of single-particle inclusive spectra is discussed in terms of the average fraction of c.m. energy per event carried off by each type of constituent. Experimental values for energy fractions in pp collisions are presented and compared with the results of a Mueller-Regge expansion for the invariant cross sections in the fragmentation region. By examining the behavior of energy fractions in an independent-emission model, it is shown that known dynamic properties of the exclusive components of an inclusive process provide a natural scale by which the energy fractions approach their asymptotic values. The existence of such a scale is shown to explain the rapid approach to asymptotic behavior of the inclusive $pp\to\pi$ spectra while the $pp\to\bar{p}$ spectrum rises to an order of magnitude between Brookhaven Alternating Gradient Synchrotron and CERN Intersecting Storage Rings energies.

I. INTRODUCTION

Based on the generalized optical theorem[1] there are two alternatives in constructing a model for an inclusive cross section. To describe a process $ab\to c+\text{anything}$, one can either construct a model for the various production processes which contribute a particle of type c in the final state or one can construct a model for the 6-point amplitude $A_{ab\bar{c}\to ab\bar{c}}$, the discontinuity of which is related through unitarity to the invariant cross section. In principle, either of these approaches provides an adequate and complete description of the underlying physics. In practice, there are a great many features of the data which cannot be described naturally from one point of view or the other. There is, however, a certain complementarity between the two approaches which can be exploited. For example, a Regge analysis of the 6-point amplitude can provide a great deal of information about the dependence of cross sections on the quantum numbers of a, b, and c.[2] However, one thing which we do not have in a Regge analysis is an *a priori* estimate of the relative importance of the leading poles and the correction terms including cuts and nonleading poles (daughters). Another thing we do not have immediately is the effect of certain kinematic constraints in the exclusive cross sections, such as the conservation of momentum and quantum-number conservation. These have to be imposed at a secondary level in the form of sum rules[3] which provide relations between Regge parameters.

Making a model for the production processes guarantees that the kinematics and kinematic reflections of the known dynamic constraints are incorporated properly. Because of the potential complexity of many-particle final states, however, the models whose consequences can be calculated are usually quite crude. Typically, they neglect clustering or resonance effects known to be important in individual exclusive cross sections. The simple multiperipheral models[4] and emission models[5,6] are not thought to provide an adequate description of each n-particle final state throughout the accessible region of phase space. Instead, the averaging process inherent in forming an inclusive cross section from the exclusive constituents is relied upon to make it possible to reconstruct a reasonable inclusive spectrum from simple approximations to the exclusive production processes.

In this paper we study a specific example of this type of complementarity between the Mueller-Regge approach and the exclusive component approach to inclusive cross sections. Using the energy sum rules to define "energy fractions," we examine the approach to scaling of pp spectra in the fragmentation region. The behavior of these energy fractions is found to be consistent with a Mueller-Regge analysis. The Mueller-Regge analysis does not, however, provide a complete understanding. In particular, the question of the sign of the leading nondiffractive Regge terms and their relative magnitude does not seem to be particularly well understood in terms of the proposed generalizations[7,8] of the Freund-Harari hypothesis.[9]

To examine this feature we therefore construct a microscopic model. The specific model for the exclusive cross sections we use is the independent-emission model (IEM).[5,6] It can be argued that this model presents the most unbiased kinematic reflections of the known dynamic principles. A reasonable list of such dynamics extracted empirically from data might include[5]:

(1) Each exclusive process conserves the additive quantum numbers Q, B, Y and obeys momentum conservation.

(2) The average transverse momentum of final-state particles is limited (0.3–0.5 GeV/c) and asymptotically independent of incident energy.

(3) There are two particles of the same type as the incident particles which carry off, on the average, half the available c.m. energy.

A more complete discussion on the way in which these dynamic features are implemented in an IEM can be found in Ref. 5. We here construct an IEM for a process which can be labeled $pp \to pp + n(\pi) + m(p\bar{p}$ pair). In studying the behavior of the energy fractions in this, essentially kinematic, model we see that the exclusive approach provides an explanation of the experimental fact that $pp \to p$ approaches its scaling limit from above while the inclusive spectra of nonleading particles approach scaling from below. The IEM also provides a natural energy scale by which the energy fractions for π and $\bar{p}$ approach their asymptotic limit. Defining $\kappa_x = (m_x{}^2 + p_T{}^2)^{1/2}$, we see that $s^{1/2}/2\lambda\langle\kappa_\pi\rangle$ and $s^{1/2}/2\lambda\langle\kappa_{\text{pair}}\rangle$ are variables which must be large in order for $pp \to \pi$ and $pp \to \bar{p}$, respectively, to be close to their scaling limit. The parameter λ is related to the shape of inclusive X spectra and is determined empirically to be near 5. The existence of this type of kinematic scale seems to be reflected in the data as it correctly explains the order-of-magnitude rise in the $\bar{p}$ spectrum between Brookhaven Alternating Gradient Synchrotron (AGS) and CERN Intersecting Storage Rings (ISR) energies. The IEM therefore provides a quantitative representation of the belief often expressed[10] that the slow approach to scaling of $pp \to \bar{p}$ is due to "threshold effects."

The plan of this paper is as follows: In Sec. II we discuss the energy-momentum sum rules and their role in constraining a Mueller-Regge analysis of inclusive cross sections. We then introduce data on the average fractions obtained, in some cases, by using the energy sum rule on published inclusive spectra. In Sec. III we discuss the IEM and show how the energy-momentum sum rules can be used to estimate errors produced in inclusive cross sections by the use of analytic approximation schemes.[11] We then examine the behavior of the energy fractions in the IEM and show how a kinematic energy scale is established. In Sec. IV we summarize our results and draw some conclusions.

II. THE ENERGY SUM RULES AND THE REGGE-MUELLER ANALYSIS OF INCLUSIVE CROSS SECTIONS

A. The Sum Rules

Define

$$f^c_{ab}(s, \vec{p}_c) = \sigma_{ab}{}^{-1} E_c \frac{d^3\sigma}{d^3p_c} \qquad (2.1)$$

as the normalized invariant cross section for the inclusive process $ab \to c(\vec{p}_c)$ + anything. Here, $s = (p_a + p_b)^2$ is the usual Mandelstam invariant. Based upon energy-momentum conservation, we have the relation

$$(P_a + P_b)^\mu \sigma_{ab} = \sum_c \int \frac{d^3p_c}{E_c} (P^\mu_c) E_c \frac{d^3\sigma}{d^3p_c} . \qquad (2.2)$$

This is one of a class of sum rules[3] which express the effects on inclusive spectra of the conservation laws operating on each exclusive component. Taking the $\mu = 0$ in (2.2) and evaluating the expression in the c.m. frame yields

$$\sqrt{s} = \sum_c \int \frac{d^3p_c}{E_c} (E_c) f^c_{ab}(s, \vec{p}_c) . \qquad (2.3)$$

The individual terms on the right-hand side of (2.3) are recognized as giving the average amount of c.m. energy per collision carried off by the constituents of type c. We therefore define the average fraction of c.m. energy per event as

$$\eta_c(s) = \int \frac{d^3p_c}{E_c} \frac{E_c}{\sqrt{s}} f^c_{ab}(s, \vec{p}_c) . \qquad (2.4)$$

The sum rule (2.3) now is recast to give the constraint that the sum of the fractional energies is one:

$$1=\sum_c \eta_c(s)\ . \tag{2.5}$$

The form of (2.4) is similar to that of the expressions

$$\langle n_c\rangle=\int\frac{d^3p_c}{E_c}f^c_{ab}(s,\vec{p}_c)\ , \tag{2.6}$$

$$\langle Q_c\rangle=\int\frac{d^3p_c}{E_c}Q_c f^c_{ab}(s,\vec{p}_c)\ , \tag{2.7}$$

giving the average number of particles of type c or the average charge carried off per event by constituent c, respectively.

We now introduce Feynman's[12] scaling parameter $x_c=2p_{cL}s^{-1/2}$, write $f^c_{ab}(s,\vec{p}_c)=f^c_{ab}(x,\vec{p}_T,s)$, and reexpress (2.4) as

$$\eta_c(s)=\tfrac{1}{2}\int dx\, d^2p_T f^c_{ab}(x,\vec{p}_T,s)\ . \tag{2.8}$$

Finally, we take into account the fact that the distributions in transverse momentum are sharply peaked for small $|\vec{p}_T|$ so that, although the limits of integration over d^2p_T nominally depend on s, for moderate or high values of s this integration becomes independent of energy. Since the limits on the x integration are independent of energy, the hypothesis of Feynman-Yang[12,13] scaling yields the result that the average fraction of c.m. energy per event carried off by a particular particle type approaches an asymptotic constant,

$$f^c_{ab}(x,\vec{p}_T;s)\sim f^c_{ab}(x,\vec{p}_T)\Rightarrow\eta_c(s)\sim\eta_c(\infty)\ . \tag{2.9}$$

Because of the different shapes of inclusive spectra, it is not always convenient to pick a "typical" value of x and $|\vec{p}_T|$ at which to study the approach to scaling of the $f^c_{ab}(x,\vec{p}_T)$ for different c. The behavior of the energy fractions, however, enables us to describe the approach to scaling of $ab\to c_1$, $ab\to c_2$, etc., on a more or less equivalent footing.

B. Mueller-Regge Analysis

The asymptotic values of the energy fractions as well as the energy behavior of the approach to these values can be studied through a Mueller-Regge analysis of the inclusive cross sections.[2] The starting point is the assumption that the invariant cross section (2.1) is related to an absorptive part of a forward 3, 3 amplitude[1]

$$E_c\frac{d^3\sigma}{d^3p_c}=\frac{1}{\mathfrak{M}^2}\,\mathrm{disc}_{\mathfrak{M}^2}[A_{ab\bar{c}\to ab\bar{c}}(s,p_c;\text{all }\Delta^2=0)]\ , \tag{2.10}$$

where $\mathfrak{M}^2=(p_a+p_b-p_c)^2$. This relation is then combined with the assumption of Regge asymptotic behavior of the amplitude in various channels. Taking particle c to be in the fragmentation region of particle b

$$s=(p_a+p_b)^2\to\infty\ ,\qquad \mathfrak{M}^2=(p_a+p_b-p_c)^2\sim(1-x_c)s\ , \tag{2.11}$$

we have the asymptotic expansion

$$2E_c\frac{d^3\sigma}{d^3p_c}\sim\sum_i\gamma^{ai}\beta_i^{b\bar{c}}(x_c,p_{cT}^2)\left(\frac{s}{s_c}\right)^{\alpha_i(0)-1}\ . \tag{2.12}$$

The expression is written down assuming the dominance of poles because it may be that, as in $2\to2$ scattering, the effect of poles and their absorptive cuts is approximated at a certain level by "effective poles." To the extent that the effective-pole approximation is adequate to describe both the asymptotic behavior of the total cross section and the asymptotic behavior of the invariant inclusive cross section in the fragmentation region, the sum rule (2.2) provides an important constraint on the effective-pole couplings.[14] To see this we insert a Regge asymptotic expansion for the total cross section into the left-hand side of (2.2) and the expansion (2.12) into the integral on the right-hand side:

$$\begin{aligned}\sum_i\gamma_i^a\gamma_i^b(s/s_0)^{\alpha_i(0)-1}&\\ =\tfrac{1}{2}\sum_c\int_{-1}^0 dx\int d^2p_T\sum_i\gamma^{bi}\beta^i_{a\bar{c}}(x,\vec{p}_T)(s/s_c)^{\alpha_i(0)-1}&\\ +\tfrac{1}{2}\sum_c\int_0^1 dx\int d^2p_T\sum_i\gamma^{ai}\beta^i_{b\bar{c}}(x,\vec{p}_T)(s/s_c)^{\alpha_i(0)-1}&\end{aligned} \tag{2.13}$$

The number of terms in the asymptotic expansion which are expected to be important depends on the kinematic region. To study the energy dependence of the terms on the right-hand side of the sum rule (2.13), we must be careful to treat properly the regions $x\cong0$ and $x\cong1$. Near $x=0$, a double-Regge expansion of the 6-point function (2.10) becomes valid and we have

$$E\frac{d^3\sigma}{d^3p_c}\sim\frac{1}{s}\sum_{i,j}\tilde{\gamma}_{ij}(\vec{p}_T)\,s_{a\bar{c}}^{\alpha_i(0)}\,s_{b\bar{c}}^{\alpha_j(0)}\ , \tag{2.14}$$

where $s_{a\bar{c}}=(p_a-p_c)^2$ and $s_{b\bar{c}}=(p_b-p_c)^2$. Consistency of this expansion with the single-Regge result (2.12) in the small-x region where

$$2p_a\cdot p_c\cong\frac{m_c^2+p_{cT}^2}{x}+xm_a^2+O(x^{-1}s^{-1/2})\ , \tag{2.15a}$$

$$2p_b\cdot p_c\cong sx+\frac{m_c^2+p_{cT}^2}{x}+O(x^{-1}s^{-1/2})\ , \tag{2.15b}$$

requires some structure in the $\beta^i_{b\bar{c}}(x,\vec{p}_T)$ (see Ref. 15):

$$\beta_{b\bar{c}}^{i}(x, \vec{p}_T) \underset{x\to 0}{\sim} \sum_j \gamma_{ij}(p_T^2)\, x^{\alpha_i(0)-\alpha_j(0)} . \qquad (2.16)$$

The terms with $\alpha_j(0) > \alpha_i(0)$ lead to singularities in the $\beta_{b\bar{c}}^{i}$. To interchange the order of the intergration with the limit in s in (2.13), we have to assume that such singularities are integrable or their contributions can be separated. Even granted this can be done, we see a suggestion in (2.16) that terms in (2.12) with low intercepts will be important at $x \cong 0$ so that the expansion (2.12) converges slowly in this kinematic region. Of course, the isolated point at $x = 0$ has anomalous energy behavior

$$E\frac{d^3\sigma}{d^3p_c}\bigg|_{x=0} = \sum_{i,j} \gamma_{ij}(p_T^2)\, s^{[\alpha_i(0)+\alpha_j(0)]/2-1} , \qquad (2.17)$$

but this does not contribute a finite amount to the integral.

Similarly, in the region $x \cong 1$, we have the triple-Regge expansion

$$E\frac{d^3\sigma}{d^3p_c} \sim \sum_{i,l} g(p_T^2)(1-x)^{\alpha_i(0)-2\alpha_l(p_T^2)}(s/s_c)^{\alpha_i(0)-1} \qquad (2.18)$$

so that the coefficient $\beta_i^{b\bar{c}}(x, \vec{p}_T)$ can be singular at $x = 1$ as well. Practically, this is of interest mainly for the case when b and c are the same particle so α_l can be a Pomeranchuk singularity. Since

$$x_{\max} = 1 - O(1/s) \qquad (2.19)$$

we can pick up a spurious s dependence from singularities in $\beta_i^{b\bar{c}}(x, \vec{p}_T)$ in this limit.[16] Assuming these spurious s dependences can be isolated, we can interchange the order of the integral and the limits on the asymptotic expansion and identify similar powers of s,

$$\gamma_i^a \gamma_i^b \left(\frac{s}{s_0}\right)^{\alpha_i} = \tfrac{1}{4}\sum_c \gamma_i^b \beta_{a\bar{c}}^{i}\left(\frac{s}{s_c}\right)^{\alpha_i} + \tfrac{1}{2}\sum_c \gamma_i^a \beta_{b\bar{c}}^{i}\left(\frac{s}{s_c}\right)^{\alpha_i} , \qquad (2.20)$$

where

$$\beta_{a\bar{c}}^{i} = \int d^2p_T \int_{-1}^{0} dx\, \beta_{a\bar{c}}^{i}(x, \vec{p}_T) . \qquad (2.21)$$

There are limitations to the validity of these expressions. Cuts are known to be important in the analysis 2-2 reactions where they invalidate factorization properties of amplitudes predicted on the basis of pole dominance.[17] The factorization implied by the manner in which (2.20) is written might therefore be wrong even if there is a matching of similar powers of s in (2.13). An interesting fact which follows immediately from (2.20) is that if the contribution of a single Regge pole or of an exchange-degenerate set of Regge poles to the total cross section vanishes, we have

$$0 = \sum_c (\gamma_i^b \beta_{a\bar{c}}^{i} + \gamma_i^a \beta_{b\bar{c}}^{i})\left(\frac{s}{s_c}\right)^{\alpha_i} . \qquad (2.22)$$

The sum over the contributions of that pole or set of poles to all inclusive single-particle spectra must then vanish. Some early analysis of discontinuities of 3-3 amplitudes[7,8,15] ignored these constraints and predicted that each term in the asymptotic expansion (2.13) would be positive so that scaling is approached from above in all reactions. From expressions like (2.22), we now know that there are certainly some Regge terms which are negative in the fragmentation region.[14] In some reactions such as NN and KN scattering, the highest non-Pomeron exchanges, the ρ-f-ω-A_2, approximately cancel in their contributions to the total cross section. For the sum of these poles, the nonleading corrections of $O(s^{-1/2})$ to the inclusive cross sections must satisfy the constraint (2.22). If these meson exchanges produce the dominant nonscaling contribution to (2.12), then the sign of the meson term in (2.13) for a given constituent c will determine whether the fraction of energy carried off by c is increasing with energy or decreasing with energy.

If there are some trajectories which decouple from all total cross sections but which are important in the approach to scaling of inclusive spectra such as the Q trajectory proposed by Chan Hong-Mo, then we again have the constraint (2.22) for the couplings of these trajectories. Although (2.20) and (2.22) are written down for integrated quantities, the proposal has been made[18] that (2.13) is satisfied "semilocally" in x and p_T so that constraints (2.20) and (2.22) are approximately valid with $\beta_{a\bar{c}}^{i}$ replaced by $\beta_{a\bar{c}}^{i}(\bar{x}, \vec{p}_T)$.

Finally, we point out that if cuts are not able to be simply accounted for by "effective poles" but the continuous spectrum of powers and (ln$s$) factors are important in the Regge expansions of $E\, d^3\sigma/d^3p$ in the fragmentation region (2.12), then we no longer gain any essential information by requiring that the coefficients of a given s behavior on the right- and left-hand side can be matched.

In writing the Regge expansion (2.12) we have explicitly included energy scales by writing the energy factors in the form (s/s_c), where s_c depends on the mass of c and other kinematic features of the process. In Sec. III we will give an argument based on the IEM fixing this s_c.

C. The Data on Energy Fractions

Published data[19–24] on invariant cross sections as a function of the Feynman scaling variable $x = 2p_L s^{1/2}$ provide us with an opportunity to test the Mueller-Regge ideas about the approach to the asymptotic values of the energy fractions. The integrals over d^2p_T and dx can be done in order to obtain the energy fraction (2.8). This provides a simple estimate of the approach to scaling of the various constituents. As is evident from the discussion above, it is not a substitute for a more detailed study of the approach to scaling on specific regions of x and $\vec{p}_T$, but it does provide information which is useful because of the constraints of the sum rule (2.3). We are now going to restrict attention to pp collisions where the widest energy range of data is available. Within the errors on the available data on inclusive spectra, it makes sense to approximate σ_{tot}^{pp} by

$$\sigma_{tot}^{pp}(s) = 40 \text{ mb}$$

over the entire energy range, $s \in (13,3000)$ GeV^2. This simplifies the Regge interpretation (2.13) and allows us to divide through a σ_{tot} as in (2.8) and still expect a Regge-like behavior of the energy fractions (2.8). The energy fractions for p, π^+, π^-, K^+, K^-, K_s, Λ, and $\bar{p}$ produced in pp collisions are given in Fig. 1.[19–24] The most striking feature of this graph is the fact that only the proton fraction is dropping to its asymptotic value. Since the final-state protons take the largest share of the energy, this fall is sufficient to account for the rise in the share of energy for all the other particles. In Sec. III we will discuss a simple independent-emission model which gives a "kinematic" explanation of this facet of the data in terms of the leading particle effect observed in the exclusive components. Nothing in the Regge-Mueller analysis tells us that the approach to scaling for all the produced or nonleading particles should be uniformly from below.

Figure 2 shows the energy fractions for p, π^+, and π^- plotted against $s^{-1/2}$. Under the assumption that the dominant corrections to the asymptotic values are given by meson exchanges with $\alpha_M(0) = \frac{1}{2}$, these fractions should approximately fall on straight lines from (2.12) and (2.13). The figure shows that it is possible that the dominant corrections are of the form $s^{-1/2}$, but the errors are too large to conclude that other terms are negligible. In the analysis in Sec. III we will see that we expect terms of $O(s^{-1})$ to be important.

When a complete set of final particles has been detected, we can "test" the sum rule (2.5). Since in current data some particles cannot be detected,

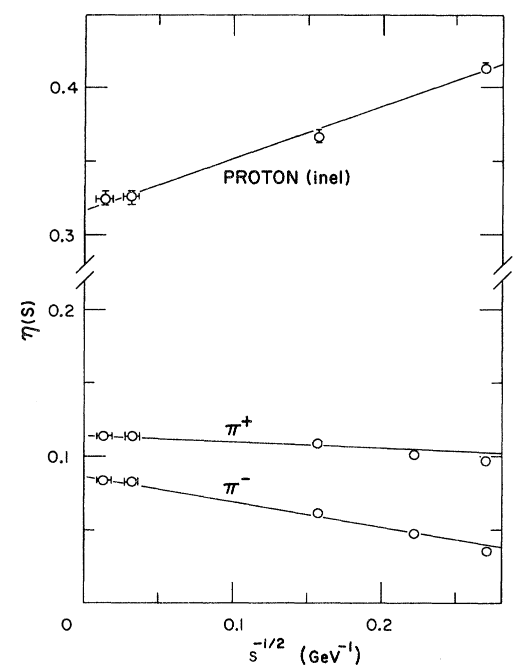

FIG. 1. Experimental values of energy fractions in pp collisions. The data on inclusive X distributions can be found in Refs. 19–24. Errors include estimates of extrapolation errors in instances where the inclusive distribution was not measured over the entire kinematic region. Hand-drawn curves are to guide the eye.

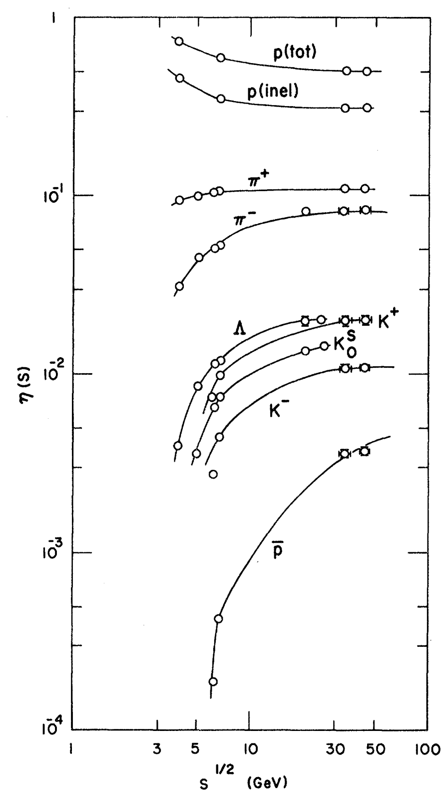

FIG. 2. Values of energy fractions for $pp \to p$, $pp \to \pi^+$, and $pp \to \pi^-$ plotted against $s^{-1/2}$. Under the assumption that the approach to scaling in the fragmentation region is predominantly controlled by meson exchanges with $\alpha_M(0) \cong \frac{1}{2}$, this should be approximately linear.

TABLE I. Energy fraction carried off by neutrons in pp collisions, assuming that π^0's and n's take off the bulk of the unobserved energy and $\eta_{\pi^0} = \frac{1}{2}(\eta_{\pi^+} + \eta_{\pi^-})$.

24 GeV/c	ISR
0.10 ± 0.02	0.12 ± 0.04

a more meaningful utilization of (2.5) is to determine the energy dependence of the energy fraction carried off by the unobserved particles. Assuming that the only important omissions from Fig. 1 are π^0's and n's and that

$$\eta_{\pi^0}(s) = \tfrac{1}{2}\left[\eta_{\pi^+}(s) + \eta_{\pi^-}(s)\right] , \tag{2.23}$$

as occurs in some models, we can determine $\eta_n(s)$ from (2.5). This gives $\eta_n(s)$ in Table I. The errors are large but it appears likely that $pp \to n$ approaches scaling from below or that there are other important neutral constituents.

One point has to do with the variable in which scaling is achieved. The inclusive spectrum $pp \to \pi^+$ attains its asymptotic value very rapidly[19] in terms of the variable

$$\begin{aligned} x' &= \frac{p_L}{p_{L\,\max}} \\ &\cong \frac{2p_L}{(s - 4m_p^2)^{1/2}} \\ &= \frac{x}{(1 - 4m_p^2/s)^{1/2}} \end{aligned} \tag{2.24}$$

However, it is clear from the definition of the energy fraction that we want the integration over x rather than x' in order to define η_c in (2.8), so we do have an s dependence for the energy fraction carried off by π^+'s.

D. The Harari-Freund Hypothesis

Many features of 2-2 scattering data can be neatly and simply expressed in terms of the relation conjectured by Harari and Freund[9] between s-channel resonances and crossed-channel Regge poles:

$$\begin{aligned} &\mathrm{Im}f(s\text{-channel resonances}) \\ &\qquad \cong \mathrm{Im}f(t\text{-channel Regge poles}), \\ &\mathrm{Im}f(s\text{-channel background}) \\ &\qquad \cong \mathrm{Im}f(t\text{-channel Pomeranchuk exchange}). \end{aligned} \tag{2.25}$$

The equality is in the sense that at fixed t, semilocal averaging over regions in s will produce cancellations in finite-energy sum rules.[25] The absence of resonances in exotic channels such as pp and Kp then leads to patterns of exchange degeneracy among the non-Pomeranchuk t-channel Regge exchanges. This hypothesis leads to a natural interpretation of the fact that for exotic channels, the total cross section is approximately constant over a wide range of energies. Since the resonance contributions to total cross sections should be positive, this also helps to explain why total cross sections for nonexotic processes approach their asymptotic forms from above. In exotic channels the Regge-pole contributions are predicted to be largely real and incoherent with the Pomeron so that little structure in $d\sigma/dt$ is expected, while in nonexotic channels dip structure is expected. The systematics of this prediction are also verified by experiment.[17] Finally, the form of elastic polarizations constitutes another success of this scheme.

All of the predictions of the duality scheme are not successful, however. Polarization in $\pi^- p \to \pi^0 n$ and $\bar{K}N \to \pi\Lambda$, $\pi\Sigma$ presents problems. The necessity of bringing in exchange-degeneracy breaking effects, usually Pomeron-Regge cuts, at some level is recognized. The empirically observed systematic pattern

$$\frac{d\sigma}{dt}(\text{real channel}) > \frac{d\sigma}{dt}(\text{rotating phase channel}) \tag{2.26}$$

rules out simple models of absorptive cuts but these relations can be understood in certain "unitarized" dual models.[26]

The success of the predictions in 2-2 scattering while not unqualified[27] has led to attempts to generalize or extend the scheme into the analysis of inclusive cross sections.

The first suggestion presented was based on the idea that the $\mathfrak{M}^2$ discontinuity in (2.10) could be separated into resonance contributions and background in a manner similar to (2.25). This suggests that a criterion for "early scaling" might be that $ab\bar{c}$ have exotic quantum numbers.[7] However, this suggestion was questioned by others who pointed out that resonances in other channels could produce some energy dependence.[8] The generalization of the Freund-Harari hypothesis to inclusive distributions has proved therefore to be not completely straightforward. The only complete formulation of duality incorporating some form of diffraction is the generalization of the Veneziano model to dual loops. This led to attempts to use dual-loop diagrams to keep track of the quantum numbers and formulate appropriate criteria for early scaling. There are two slightly different interpretations of the loop-graph model for vacuum exchange which lead to slightly different conclusions. It should be noted that Tye and Veneziano[18] have combined this type of analysis of dual graphs with a study of the constraint of the sum rules and shown that the apparent negative sign of the leading correction terms for nonleading particles in pp scattering can be accommodated within the framework of dual models.

In terms of an analysis of the quantum numbers of $ab\bar{c}$, we note that both $pp\bar{p}$ and $pp\bar{n}$ channels are nonexotic. If the original consideration of Chan Hong-Mo *et al.*[7] about the contributions to the discontinuity in $\mathfrak{M}^2$ remains valid for nonexotic channels, these distributions should both drop to their asymptotic values in the fragmentation region. The behavior of the fraction $\eta_p(s)$ supports this for the proton, but the fraction $\eta_n(s)$ obtained by assuming the unobserved particles were predominately π^0's and n's, and $\eta_{\pi^0}=\frac{1}{2}(\eta_{\pi^+}+\eta_{\pi^-})$ in Table I is consistent with being constant or rising. The errors are, of course, large. A more accurate analysis and a test of the assumptions seem appropriate as well as experiments detecting neutrons directly.

III. THE ENERGY SUM RULE IN INDEPENDENT-EMISSION MODELS

To supplement the analysis of the approach to scaling in terms of the Regge behavior of a 3-3 amplitude, we can try to understand the energy behavior of the exclusive components from which an inclusive cross section is constructed. The model for the exclusive processes which will be discussed here is the independent-emission model.[5,6] This model consists of phase space weighted to reflect well-established dynamical features extracted from data. The treatment of energy-momentum conservation in an IEM constitutes one concrete advantage of the approach over other simple models for exclusive processes. For example, a simple multiperipheral model (such as that discussed, for example, by Arnold)[28] can be manipulated analytically to give inclusive distributions only in the strong-ordered limit. In this limit the leading particles take almost all the momentum and the produced, or secondary, particles are in the central region. If one wants to obtain inclusive distributions outside the central region in order to discuss the energy-momentum sum rules, then Monte Carlo calculations[29] are necessary in the multiperipheral model.

In contrast, in the independent-emission approach, the use of analytic approximation schemes[11] enables one to incorporate energy-momentum constraints without the use of Monte Carlo calculations. In the limit of high incident momentum one can obtain analytically inclusive distributions valid in the bulk of the kinematically allowed region. We are, therefore, in a position to discuss the behavior of the energy fractions (2.4), (2.8) and relate this to the approach to scaling of the inclusive distributions. From the IEM, we then have some idea of how scaling is approached as a consequence of kinematic reflections of known features in the data on exclusive cross sections.

A. The Energy Sum Rule and Errors in Analytic Approximation Schemes

One problem in the straightforward use of analytic approximation schemes[11] to obtain inclusive distributions in an IEM is how to handle the errors that these schemes contain. The approximation formulas used involve statistical estimates valid when the number of produced particles is large and the effect of the correction terms cannot strictly be neglected even at the highest available energies. For example, Chen and Peierls[30] have done a detailed comparison of Monte Carlo calculations of transverse phase space with the available approximation schemes. They find such schemes introduce important errors in the transition region between low energies, where the phase space available is essentially three-dimensional, and high energies where the phase space is essentially one-dimensional. The energy range over which this transition occurs depends on the multiplicity so there are some consequences for extracting inclusive distributions. What we would now like to show is that, in spite of the errors in the approximation schemes, a simple IEM with only one type of particle produces a single-particle inclusive distribution which, at high energy, satisfies the energy sum rule within terms of $O(1/\bar{n}(s)^2)$. This gives us an estimate of the accuracy to which we can believe the results on energy fractions in a system of many constituents.

To illustrate the use of the analytic-approximation scheme and the energy sum rule, we first form the generating function

$$Q(z,P)=\sum_{n=2}^{\infty}\frac{z^n}{n!}\,\Omega_n(P)\quad, \tag{3.1}$$

where

$$\Omega_n(P)=\int\left[\prod_{j=1}^{n}\frac{d^3q_j}{2\omega_j}f(q_{jT})\right]\delta^{(4)}\left(P-\sum_{j=1}^{n}q_j\right). \tag{3.2}$$

We take the Laplace transform of (3.1) to get

$$Q(z,\alpha)=\exp[z\Phi(\alpha_T,\alpha_L)]\quad, \tag{3.3}$$

where we have exponentiated the sum in (3.1) with the understanding that terms in z^0 and z^1 are not actually present. In (3.3), $\alpha_L=(\alpha_0{}^2-\alpha_1{}^2)^{1/2}$ and $\alpha_T=(\alpha_2{}^2+\alpha_3{}^2)^{1/2}$ and

$$\Phi(\alpha_T, \alpha_L) = \pi \int d(q_T{}^2) f(q_T) I_0(\alpha_T q_T) \times K_0(\alpha_L(m^2 + q_T{}^2)^{1/2}) . \quad (3.4)$$

The basic approximation in the analytic estimate of (3.1) is a steepest-descents approximation to the inverse Laplace transform. In a frame where P_T is small and P_1 is zero we find α_L and α_T, which give a stationary phase by solving the equations

$$\frac{-\partial}{\partial \alpha_L} \ln Q(1, \alpha_T, \alpha_L) = P_0 , \quad (3.5a)$$

$$\frac{\partial}{\partial \alpha_T} \ln Q(1, \alpha_T, \alpha_L) = |\vec{P}_T| . \quad (3.5b)$$

The inverse transform then is approximately

$$Q(z, P) = \frac{\exp(\alpha_L P^0 - \alpha_T P^T)}{(2\pi)^2 (\det \beta)^{1/2}} \exp[z\Phi(\alpha_L, \alpha_T] \times [1 + R(z, \alpha_T, \alpha_L)] , \quad ((3.6)$$

where α_L and α_T are implicit functions of P through (3.5) and $\det B = 4B_{00} B_{11} B_{TT}{}^2$, where

$$B_{00} = \frac{\partial^2}{\partial \alpha_L{}^2} \Phi(\alpha_L, \alpha_T) , \quad (3.7a)$$

$$B_{11} = \frac{s^{1/2}}{\alpha_L} , \quad (3.7b)$$

$$B_{TT} = \frac{|\vec{P}_T|}{\alpha_T} . \quad (3.7c)$$

We will evaluate the sum rule (2.3) in the case where $f(q_T{}^2)$ is peaked sharply enough near $q_T{}^2 = 0$ and P_0 is large enough so that we can use the small argument expansion of the I_0 and K_0 inside the integral to write

$$\Phi(\alpha_T, \alpha_L) \cong -(g_0 \ln c\alpha_L + \tfrac{1}{4} \alpha_T{}^2 g_0 \langle q_T{}^2 \rangle \ln c\alpha_L) , \quad (3.8)$$

where

$$g_0 = \pi \int d(q_T{}^2) f(q_T{}^2) , \quad (3.8a)$$

$$g_0 \langle q_T{}^2 \rangle = \pi \int d(q_T{}^2) f(q_T{}^2) q_T{}^2 , \quad (3.8b)$$

$$g_0 \ln c = \pi \int d(q_T{}^2) f(q_T{}^2) [\gamma + \ln \tfrac{1}{2} (m^2 + q_T{}^2)^{1/2}] . \quad (3.8c)$$

The solution to Eqs. (3.8) is then

$$\alpha_T \cong \frac{2P_T}{g_0 \langle q_T{}^2 \rangle \ln(P_0/cg_0)} , \quad (3.9a)$$

$$\alpha_L \cong \left[\frac{P_T{}^2}{g_0 \langle q_T{}^2 \rangle \ln^2(P_0/cg_0)} \right] \frac{1}{P_0} , \quad (3.9b)$$

so that

$$\Phi(\alpha_L, \alpha_T) \cong g_0 \ln\left(\frac{P_0}{cg_0}\right) + \frac{P_T{}^2}{g_0 \langle q_T{}^2 \rangle \ln^2(P_0/cg_0)} \quad (3.10)$$

and

$$Q(z, P_0, P_T) \cong \frac{(cg_0)^2 \exp\{g_0\} (P_0/cg_0)^{g_0 - 2}}{(2\pi)^2 \langle q_T{}^2 \rangle \ln(P_0/cg_0)} \times \left[1 - \frac{P_T{}^2}{g_0 \langle q_T{}^2 \rangle \ln(P_0/cg_0)} + \cdots \right] . \quad (3.11)$$

The sum rule (2.3) reduces in the case of a single component to

$$\sqrt{s} = \int \frac{d^3 q}{2\omega} \omega f(q_T) \frac{Q(1, m_L, P_T)}{Q(1, \sqrt{s}, 0)} , \quad (3.12)$$

$$\sqrt{s} = \int \frac{dq_L}{2} \frac{\ln(\sqrt{s}/cg_0)}{\ln(m_L/cg_0)} \left(\frac{m_L{}^2}{s}\right)^{g_0/2 - 1} \times \left[\pi \int_0^1 d(q_T{}^2) f(q_T{}^2) \times \left(1 - \frac{q_T{}^2}{g_0 \langle q_T{}^2 \rangle \ln(m_L/c g_0)} + \cdots \right) \right] . \quad (3.13)$$

In view of (3.8a) and (3.8b) the integral over $d(q_T{}^2)$ can be evaluated. Using $(m_L{}^2/s) \cong (1 - x)$ and

$$\ln\left(\frac{m_L}{cg_0}\right) = \ln \frac{\sqrt{s}}{cg_0} + \tfrac{1}{2} \ln(1 - x) , \quad (3.14)$$

we have

$$1 = \tfrac{1}{2} g_0 \int_0^1 dx (1 - x)^{g_0/2 - 1} \left[1 - \frac{1}{g_0 \ln(\sqrt{s}/cg_0)} \right] \times \left[1 - \frac{1}{2} \frac{\ln(1 - x)}{\ln(\sqrt{s}/cg_0)} + O\left(\frac{1}{\ln^2(\sqrt{s}/cg_0)} \right) \right] \quad (3.15)$$

and the terms of $O(1/\ln(\sqrt{s}/cg_0))$ cancel so the sum rule is satisfied. To match up terms in higher powers of $(1/\ln s)$ we would have to make more explicit assumptions about the $q_T{}^2$ behavior of $f(q_T{}^2)$ and solve more complicated expressions in the Eqs. (3.8). If we are interested in the properties of strict one-dimensional phase space where the Laplace transform can be inverted analytically, Campbell[31] has shown that the energy sum rule is satisfied with corrections of $O(1/s)$.

These considerations lend substance to our belief that the normalized inclusive single-particle distributions in an IEM from analytic approximation schemes correctly integrate to reproduce the energy fractions. This has consequences for the numerical calculation of distributions in more complicated IEM's,[5,6,32] but it also means that we can indeed learn something about the approach to scaling in this model by examining the energy behavior of the energy fractions and we now turn to this problem.

B. A Multicomponent IEM and Kinematic Features in the Approach to Scaling

Having examined the energy sum rule in a simple IEM, we now want to discuss the behavior of the energy fractions in a more realistic model with more than one type of particle. As an example, we will construct a model for the production of π's and $\bar{p}$'s and examine the scale for the s behavior of $\eta_\pi(s)$ and $\eta_{\bar{p}}(s)$. To simplify, we will neglect isospin and the production of strange particles. We will enforce baryon conservation by constructing a matrix element for the process $pp \to pp + n\pi + m(p\bar{p})$. Since the results will be simple kinematic reflections of the dynamic features we input into the matrix element, we have to be explicit what these features will be. The conditions we impose here will be:

(1) Each exclusive process conserves the additive quantum numbers Q, B, Y and obeys energy-momentum conservation.

(2) The average transverse momentum of final-state particles is limited (0.3–0.5 GeV/c) and asymptotically independent of incident energy.

(3) There are two particles of the same type as the incident particles which carry off, on the average, half the available c.m. energy.

An additional approximation we make for computational simplicity is to neglect the integrations over the relative momenta of the $p\bar{p}$ pair and characterize this process as the production of a massive particle with $M_{\rm PR} \cong 2$ GeV/c. The modulus squared of the matrix element will then be of the form

$$|M_{n,m}|^2 = \prod_{i=1}^{2} f_N(p_{iT}) e^{2\lambda s^{-1/2} E_i} \prod_{j=1}^{n} f_n(q_{jT}) \times \prod_{k=1}^{m} f_{\rm PR}(p_{kT}), \qquad (3.16)$$

where the f's are again sharply peaked near $p_T = 0$. In place of the generating function (3.3) we then have

$$Q(z_\pi, z_{\rm PR}, \alpha) = [\Psi_N(\xi)]^2 \exp[z_\pi \phi_\pi(\alpha) + z_{\rm PR}\phi_{\rm PR}(\alpha)], \qquad (3.17)$$

where $\xi = (\alpha - 2\lambda s^{-1/2})$. We use the sharply peaked $f_i(q_T)$ and (3.4) to write

$$\Psi_N(\xi) \cong G K_0(\xi\langle\kappa_N\rangle), \qquad (3.18a)$$

$$\phi_\pi(\alpha) \cong g_\pi K_0(\alpha\langle\kappa_\pi\rangle), \qquad (3.18b)$$

$$\phi_{\rm PR}(\alpha) \cong g_{\rm PR} K_0(\alpha\langle\kappa_{\rm PR}\rangle). \qquad (3.18c)$$

Since we are studying the approach to scaling we do not yet make the small-argument approximation of $K_0(x)$. In (3.18) $\kappa_i = (m_i{}^2 + p_{iT}{}^2)^{1/2}$ is the transverse mass and $\langle\kappa_i\rangle$ is some average value of this mass. The important kinematic distinction between the production of π's and the production of heavy pairs is just given by

$$\langle\kappa_\pi\rangle \approx 0.3 \text{ GeV}, \qquad (3.19a)$$

$$\langle\kappa_{\rm PR}\rangle > M_{\rm PR} \gtrsim 2 \text{ GeV}. \qquad (3.19b)$$

The behavior of the energy fractions now comes directly from the equation

$$\frac{2GK_1(\xi\langle\kappa_N\rangle)\langle\kappa_N\rangle}{K_0(\xi\langle\kappa_N\rangle)} + g_\pi\langle\kappa_\pi\rangle K_1(\alpha\langle\kappa_\pi\rangle) + g_{\rm PR}\langle\kappa_{\rm PR}\rangle K_1(\alpha\langle\kappa_{\rm PR}\rangle) = \sqrt{s}, \qquad (3.20)$$

analogous to (3.8) which is used to determine the stationary phase point for the estimate of the inverse Laplace transform. The terms in (3.20) are readily identified as the c.m. energies carried off by the leading protons, the π's, and the massive pairs, respectively. We see that in order for the leading particles to carry off a nondropping fraction of the energy, we obviously must have $\xi \gg \alpha$. As discussed in detail in Ref. 5 the solution to (3.20) is given by

$$\alpha = 2\lambda s^{-1/2}[1 + c/\ln s + O(1/(\ln s)^2)]. \qquad (3.21)$$

The scale of the energy fractions

$$\eta_\pi(s) \cong \frac{g_\pi\langle\kappa_\pi\rangle}{\sqrt{s}} K_1(\alpha\langle\kappa_\pi\rangle), \qquad (3.22a)$$

$$\eta_{\rm PR}(s) \cong \frac{g_{\rm PR}\langle\kappa_{\rm PR}\rangle}{\sqrt{s}} K_1(\alpha\langle\kappa_{\rm PR}\rangle) \qquad (3.22b)$$

is therefore roughly determined by $2\lambda\langle\kappa_\pi\rangle s^{-1/2}$ and $2\lambda\langle\kappa_{\rm PR}\rangle s^{-1/2}$, respectively. Asymptotically

$$K_1(x) \underset{x\to 0}{\sim} \frac{1}{x},$$

so

$$\eta_\pi(\infty) = \frac{g_\pi}{2\lambda}, \qquad (3.23a)$$

$$\eta_{\rm PR}(\infty) = \frac{g_{\rm PR}}{2\lambda}. \qquad (3.23b)$$

The parameter λ which sets the scale of energy not surprisingly appears in the limiting inclusive distributions for the π's and pairs. In this model (2.1) is

$$f^\pi(x, s) \sim \tfrac{1}{2} g_\pi e^{-\lambda|x|}, \qquad (3.24a)$$

$$f^{PR}(x,s) \sim \tfrac{1}{2} g_{PR} e^{-\lambda |x|} . \qquad (3.24b)$$

Using (2.8), we see that Eq. (3.24) can be integrated to give (3.23) provided $e^{-\lambda} \ll 1$. The empirical observation of steep slopes for inclusive π^+, π^-, K^-, K_0^s, and $\bar{p}$ distributions then indicates that λ is very large. The best value is around $\lambda = 5$.[5] This means that $2\lambda\langle\kappa_\pi\rangle \cong 3$ GeV and $2\lambda\langle\kappa_{PR}\rangle \cong 20$ GeV. We have to reach a c.m. energy much above 20 GeV before we can expect an approach to the asymptotic value of η_{PR}.

Figure 3 shows the behavior of η_π and η_{PR} when (3.20) is solved numerically. The fractions are normalized to $\eta_\pi(\infty) = 0.1$ and $\eta_{PR}(\infty) = 0.013$ and are compared with the data on $\eta_\pi = \frac{1}{2}(\eta_{\pi^+} + \eta_{\pi^-})$ and $\eta_{PR} = 2\eta_{\bar{p}}$ taken from Sec. II.

We see from Fig. 3 that these simple kinematic considerations are all that are needed to explain the order-of-magnitude change of $\eta_{\bar{p}}$ from $P_{lab} = 24$ GeV/c to ISR momenta, while being consistent with the rough features of the energy dependence of $\eta_\pi(s)$. Certainly, at this level of sophistication we do not expect to understand more complicated features such as the difference between the approach to scaling of π^+ and π^-. The production of $K\bar{K}$ pairs ($\langle\kappa_{K\bar{K}}\rangle \geq 1$ GeV) can also be treated in the same way as $p\bar{p}$ pairs in this approach and we get the right magnitude of the scale for the approach to asymptotic behavior of $pp \to \bar{K}$.

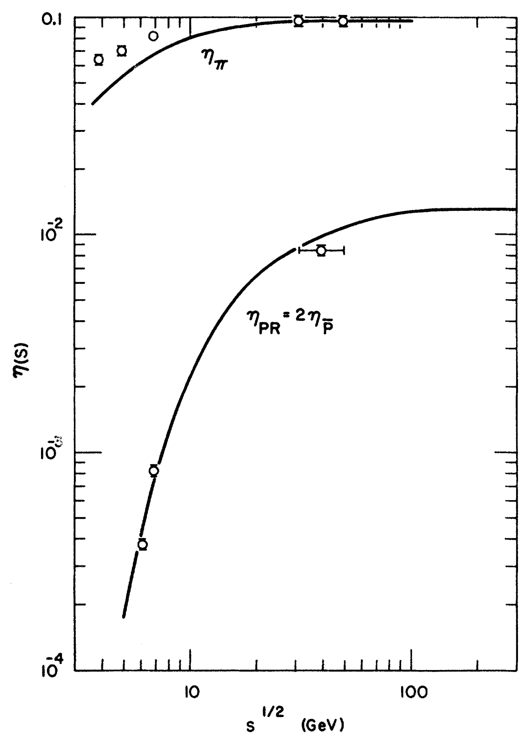

FIG. 3. The curves are the energy fractions for π's and $p\bar{p}$ pairs calculated by solving the IEM equation (3.20). The coupling constant g_{PR} is determined by matching with $\eta_{\bar{p}}$ at $P_{lab} = 24$ GeV/c. The data shown are $\eta_\pi = \frac{1}{2}(\eta_{\pi^+} + \eta_{\pi^-})$ and $\eta_{PR} = 2\eta_{\bar{p}}$ from Sec. II. The order-of-magnitude rise of $\eta_{\bar{p}}$ between $P_{lab} = 24$ and ISR momenta is understood in terms of the "kinematic" features of the IEM.

An interesting point to note from Fig. 3 is that $\eta_{\bar{p}}(s)$ is roughly 60% of its asymptotic value at ISR momenta. This is consistent with estimates of the $s \to \infty$ value of $f_{pp}^{\bar{p}}(x, \vec{p}_T, s)$ obtained by requiring $f_{pp}^{p}(0, \vec{p}_T, s) - f_{pp}^{\bar{p}}(0, \vec{p}_T, s) \sim 0$ and noting that at ISR $f_{pp}^{p}(0, \vec{p}_T, s) \approx 2 f_{pp}^{\bar{p}}(0, \vec{p}_T, s)$. A tentative conclusion is that the observed approach to asymptotic behavior of $f_{pp}^{\bar{p}}(x, \vec{p}_T, s)$ is completely consistent with what is expected as a kinematic reflection of known dynamics, namely limited transverse momenta and the leading-particle effect.

If we write the nonleading corrections to η_c (neglecting logarithmic factors) in the form $(s/s_c)^{-1}$, we then have

$$\begin{aligned} s_c &= 4\lambda^2\langle\kappa_c\rangle^2 \\ &\cong 100\langle\kappa_c\rangle^2 . \end{aligned} \qquad (3.25)$$

The leading-particle effect represented by the parameter λ crowds produced particles into the central region and the phase space available to them grows slowly. The fact that this simple parameterization gives a reasonable description of the leading-particle effect is therefore crucial to our argument of the existence of a scale. We note here that this method of incorporating the leading particle effect has been found to give a good description of the average proton energy as a function of charged prongs.[5] The form of the inclusive distributions (3.24) is in approximate agreement with data and therefore it is perhaps not surprising that we get such a dramatic energy dependence for $pp \to \bar{p}$ purely from kinematic reflections.

IV. SUMMARY AND CONCLUSIONS

The main result of this paper was the presentation of the c.m. energy fractions for pp collisions extracted from data on inclusive cross sections. These energy fractions are useful in discussing the approach to Feynman-Yang scaling. In our discussion of the behavior of these energy fractions we found an example of the complementarity of the Mueller-Regge and the exclusive component approach to inclusive reactions. The features of the data are consistent with an analysis assuming forms of Regge behavior in the discontinuity of a forward 3-3 amplitude but we do not have any feeling for the comparative size or the sign of the various terms in the asymptotic expansion. Turning to a model for the exclusive cross sections we find an explanation of the fact that $pp \to p$ is falling to its asymptotic value while other spectra are rising. The leading particle effect observed in exclusive final states is shown to set an energy scale for the approach to asymptotic behavior of the energy fractions. This scale explains quantitatively the approach to scaling of both $pp \to \pi$ and $pp \to \bar{p}$.

Note added in proof. A similar discussion of the threshold rise of the $\bar{p}$ yield has been presented by R. Jengo, A. Krzywicki, and B. Petersson, Phys. Lett. 43B, 397 (1973).

---

*Work performed under the auspices of the U. S. Atomic Energy Commission.

†Present address: Stanford Linear Accelerator Center, Stanford University, Stanford, Calif. 94305.

[1]A. H. Mueller, Phys. Rev. D 2, 2963 (1970).

[2]See, for example, the reviews by M. B. Einhorn [NAL Report No. 73/14/THY, 1973 (unpublished)], and R. N. Cahn [LBL Report No. 1007, 1972 (unpublished)].

[3]T. T. Chou and C. N. Yang, Phys. Rev. Lett. 25, 1072 (1972); C. E. DeTar, D. Z. Freedman, and G. Veneziano, Phys. Rev. D 4, 906 (1971).

[4]C. E. DeTar, Phys. Rev. D 3, 128 (1971); N. Bali, A. Pignotti, and D. Steele, *ibid.* 3, 1167 (1971); R. F. Amann, Phys. Rev. Lett. 26, 1349 (1971).

[5]D. Sivers and G. H. Thomas, Phys. Rev. D 6, 1961 (1972).

[6]E. H. de Groot and Th. W. Ruijgrok, Nucl. Phys. B27, 45 (1971); G. H. Thomas, Phys. Rev. D 7, 2058 (1973).

[7]Chan Hong-Mo, C. S. Hue, C. Quigg, and Jiunn-Ming Wang, Phys. Rev. Lett. 26, 672 (1972).

[8]J. Ellis, J. Finkelstein, P. H. Frampton, and M. Jacob, Phys. Lett. 35B, 227 (1971); M. B. Einhorn, M. B. Green, and M. A. Virasoro, *ibid.* 37B, 292 (1971); S.-H. H. Tye and G. Veneziano, *ibid.* 38B, 30 (1972).

[9]H. Harari, Phys. Rev. Lett. 20, 1395 (1968); P. G. O. Freund, *ibid.* 23, 235 (1968).

[10]See, for example, Chan Hong-Mo, in *Proceedings of the Fourth International Conference on High Energy Collisions, Oxford, 1972,* edited by J. R. Smith (Rutherford High Energy Laboratory, Chilton, Didcot, Berkshire, England, 1972).

[11]F. Lurçat and P. Mazur, Nuovo Cimento 31, 149 (1964); A. Krzywicki, J. Math. Phys. 6, 485 (1965).

[12]R. P. Feynman, Phys. Rev. Lett. 23, 1415 (1969).

[13]J. Benecke, T. T. Chou, C. N. Yang, and E. Yen, Phys. Rev. 188, 2159 (1969).

[14]L. Caneschi, Phys. Lett. 37B, 288 (1972).

[15]R. C. Brower and J. Ellis, Phys. Rev. D 5, 2253 (1972).

[16]C. E. Jones, F. E. Low, S.-H. H. Tye, G. Veneziano, and J. E. Young, Phys. Rev. D 4, 1033 (1972).

[17]See, for example, R. J. N. Phillips, Rutherford report, 1972 (unpublished).

[18]S.-H. H. Tye and G. Veneziano, Nuovo Cimento 14A, 711 (1973).

[19]The $\pi^+\pi^-$ and p distributions at $P_{\rm lab} = 6.6$ GeV/c are from Eugene R. Gilbert, LBL Report No. LBL-749, 1972 (unpublished).

[20]The distributions at 12 GeV/c and π^+, π^-, K^+, K^- at 24 GeV/c are from H. J. Mück *et al.* [Bonn-DESY-Hamburg-Max Planck Collaboration, Report No. DESY-Fi-72/1, 1972 (unpublished); Phys. Lett. 39B, 303 (1972)].

[21]Charged particle distributions at 19 GeV/c and 24 GeV/c are from J. V. Allaby *et al.*, CERN-Rome collaboration, in *Proceedings of the Fourth International Conference on High Energy Collisions, Oxford, 1972,* edited by J. R. Smith (Rutherford High Energy Laboratory, Chilton, Didcot, Berkshire, England, 1972).

[22]The Λ, K_s distributions include data in Berger *et al.*, Phys. Rev. Lett. 28, 322 (1972).

[23]Values at 205 GeV/c: private communication from J. Whitmore; G. Charlton *et al.*, Phys. Rev. Lett. 30, 574 (1973).

[24]The ISR distributions are from L. G. Ralnev *et al.*, Phys. Rev. Lett. 27, 68 (1971); M. G. Albrow *et al.*, Phys. Lett. 40B, 136 (1972); G. Neuhofer *et al.*, *ibid.* 38B, 51 (1972).

[25]R. Dolen, D. Horn, and C. Schmid, Phys. Rev. 166, 1768 (1968).

[26]E. L. Berger and G. F. Fox, Phys. Lett. 36B, 389 (1971).

[27]For an explicitly nondual approach to 2-body phenomenology, see B. J. Hartley and G. L. Kane, Phys. Lett. 39B, 531 (1972).

[28]R. C. Arnold, Argonne Report Nos. ANL-HEP 7242 and ANL/HEP 7250 (unpublished).

[29]J. Friedman, C. Risk, and D. B. Smith, Phys. Rev. Lett. 28, 191 (1972).

[30]M.-S. Chen and R. Peierls, Phys. Rev. D 7, 2183 (1973).

[31]D. Campbell, Phys. Rev. D 7, 2793 (1973).

[32]E. H. deGroot, Nucl. Phys. B48, 295 (1972).