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e^+e^- Annihilation in Gauge Theories of Strong Interactions*

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We discuss the behavior of the e^+e^- total annihilation cross section into hadrons in a class of non-Abelian vector gluon theories. Using the quasi-free-field behavior of these theories in the deep Euclidean region, we show that the asymptotic behavior of $\sigma_{\text{tot}}^{e^+e^-}$ is that of the parton model. Nonleading corrections fall off relative to this leading term like inverse powers of logarithms with calculable coefficients. The problem of making more detailed statements about the final state is discussed.

A very interesting calculation has recently been done by Politzer,[1] Gross and Wilczek,[2] and 't Hooft.[3] They have demonstrated that for a large class of non-Abelian gauge theories, the point $g=0$ is an ultraviolet-stable fixed point of the renormalization group.[4] This means that these theories can exhibit quasi-free-field behavior in the deep Euclidean region. Scaling up to logarithmic corrections occurs with the leading terms given by a one-loop perturbation theory calculation.

These theories are obviously attractive models for strong interactions providing some mechanism can be found to avoid the catastrophic infrared problems[5] of the gauge-symmetric theory while preserving perturbative renormalizability. Spontaneous breakdown via the introduction of scalar mesons and the Higgs mechanism is one obvious solution, but the necessary additional coupling constants can lead to a destabilization of the origin.[6] The possibility of a dynamical spontaneous symmetry breaking[7] is an attractive alternative but cannot be discovered perturbatively. In ultraviolet-stable and hence infrared-unstable theories, this is a strong-coupling problem probably as difficult as the calculation of short-distance behavior of theories with an ultraviolet-unstable origin.

In this note, we will discuss some physical consequences of ultraviolet stability which depend only on the short-distance behavior and not on the details of the spontaneous symmetry breaking and the mass-shell structure. In particular we discuss the behavior of the e^+e^- total annihilation cross section into hadrons: the leading scale-invariant term and the calculation of the next-to-leading corrections. We also discuss the problem of making more detailed statements about the final states in this process and the extension of this work to electroproduction.

Consider a non-Abelian gauge theory of the strong interactions involving multiplets of fermions as in the colored-quark model.[8] We assume that the generators of the strong gauge group commute with the electric charge, so that the strong gauge fields are neutral. If the weak interactions are described by a unified gauge model, we require that the full weak and electromagnetic gauge group commutes with the strong gauge group. For example in the colored-quark model, we can take the strong gauge group to be SU(3) on the color indices.

As discussed in Refs. 1 and 2, the renormalization-group equation for the gauge coupling constant g_λ defined by renormalizing at momentum λ is

$$\lambda^2 \frac{\partial}{\partial \lambda^2} g_\lambda = -(\tfrac{1}{2}b) g_\lambda{}^3 + O(g_\lambda{}^5) + O\left(\frac{m^2}{\lambda^2}\right), \tag{1}$$

where m is the mass scale of the fermions. The constant b is positive for a large class of Yang-Mills theories. In a theory without Higgs mesons, b is[1]

$$b = \frac{1}{16\pi^2}\left(\tfrac{11}{3}c_1 - \tfrac{4}{3}c_2\right), \tag{2}$$

where c_1 and c_2 are defined in terms of the structure constants of the strong gauge group f^{abc} and representation matrices $T^a{}_{ij}$ of the fermion multiplets:

$$f_{acd} f_{bcd} = c_1 \delta_{ab}, \quad \operatorname{Tr} T^a T^b = c_2 \delta^{ab}. \tag{3}$$

Integrating Eq. (1) gives

$$g_\lambda{}^2 \sim \frac{1}{b \ln(\lambda^2/m^2)} \quad \text{as } \lambda^2 \to \infty, \tag{4}$$

providing that the Gell-Mann–Low function[9] remains negative up to $g_\lambda = g =$ strong coupling constant.

To connect this behavior to $\sigma_{\text{tot}}^{e^+e^-}$, we consider the renormalized hadronic vacuum-polarization tensor

$$D_{\mu\nu}(k) = (k_\mu k_\nu - g_{\mu\nu} k^2) D(k^2/m^2, g^2),$$

with

$$D(k^2/m^2, g^2) = k^2 \int_{4m^2}^{\infty} \frac{dM^2}{M^2} \frac{\Pi(M^2)}{k^2 - M^2 + i\epsilon}.$$

The absorptive part $\Pi(M^2)$ is proportional to $\sigma_{\rm tot}^{e^+e^-}$:

$$\sigma_{\rm tot}^{e^+e^-}(k^2) = \frac{16\pi^3\alpha^2}{k^2}\Pi(k^2). \tag{6}$$

We begin by writing a renormalization-group equation[9] for $D(k^2/m^2, g^2)$. If $D(k^2/\lambda^2, m^2/\lambda^2, g_\lambda{}^2)$ is the analogous function computed by renormalizing at some Euclidean momentum point λ, it can easily be shown by making use of the electromagnetic Ward identity that

$$D\left(\frac{k^2}{m^2}, g^2\right) = D\left(\frac{k^2}{\lambda^2}, \frac{m^2}{\lambda^2}, g_\lambda{}^2\right) + K\left(\frac{m^2}{\lambda^2}\right), \tag{7}$$

where the momentum-independent term K is simply the difference between making the over-all subtraction at $k^2 = 0$ or $k^2 = -\lambda^2$. Differentiating Eq. (7) with respect to k^2 and setting $k^2 = -\lambda^2$,

$$\lambda^2 \frac{\partial}{\partial\lambda^2} D\left(-\frac{\lambda^2}{m^2}, g^2\right) = \psi(g_\lambda{}^2) + O\left(\frac{m^2}{\lambda^2}\right), \tag{8}$$

where

$$\psi(g_\lambda{}^2) = \frac{\partial}{\partial x} D(-x, 0, g_\lambda{}^2)\bigg|_{x=1} \tag{9}$$

and use has been made of the existence of the zero-mass limit when renormalization is performed at a Euclidean point.[10] In perturbation theory

$$\psi(g_\lambda{}^2) = A(1 + Bg_\lambda{}^2 + \cdots), \tag{10}$$

with the constant A being given by a simple one-loop calculation. The calculation of the constant B involves the two loop graphs of Fig. 1 and is nontrivial. This, however, is essentially the old Jost-Luttinger calculation[11] in quantum electrodynamics. The result is

$$B = \frac{3}{16\pi^2} c_3, \tag{11}$$

where

$$T^a{}_{ij} T^a{}_{jk} = c_3 \delta_{ik}. \tag{12}$$

By using the asymptotic behavior of $g_\lambda{}^2$ given by Eq. (4), the large-λ^2 behavior of $D(-\lambda^2/m^2, g^2)$ can be found by integrating Eq. (8). This is in turn related to the asymptotic behavior of $\sigma_{\rm tot}^{e^+e^-}$ through the spectral integral representation. Alternatively, one can simply take the discontinuity of Eq. (7) and evaluate it at $k^2 = +\lambda^2$. The result is

$$\Pi\left(\frac{\lambda^2}{m^2}, g^2\right) = \Pi(1, 0, g_\lambda{}^2) + O\left(\frac{m^2}{\lambda^2}\right)$$
$$= \text{const}\times(1 + Bg_\lambda{}^2 + \cdots) + O\left(\frac{m^2}{\lambda^2}\right), \tag{13}$$

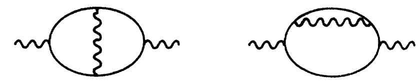

FIG. 1. Graphs contributing to the constant B.

where B is the same constant that appears in Eq. (10). The leading and nonleading terms in $\sigma_{\rm tot}^{e^+e^-}$ can now be written down using Eqs. (4) and (6):

$$\sigma_{\rm tot}^{e^+e^-}(k^2) = \frac{4\pi\alpha^2}{3k^2}\left(\sum_i Q_i{}^2\right)\left[1 + \frac{C}{\ln(k^2/m^2)} + \cdots\right], \tag{14}$$

where $\sum_i Q_i{}^2$ is the sum of the squares of the fermion charges in units of e and

$$C = \frac{9C_2}{11C_1 - 4C_2}. \tag{15}$$

Thus one finds a slow (logarithmic) approach to an asymptotic scale-invariant form given by the naive parton model.[12] Note that the positive constant C is coupling-constant-independent and (thanks to the Jost-Luttinger calculation) given purely by group theory. For the example of three triplets of fractionally charged quarks

$$\sum Q_i{}^2 = 2,$$
$$c_1 = 3, \quad c_2 = \tfrac{3}{2}, \quad c_3 = \tfrac{4}{3}, \tag{16}$$
$$C = \tfrac{4}{9}.$$

Additional corrections to $\sigma_{\rm tot}^{e^+e^-}$ can be calculated with enough work. The next correction is of the form $\ln(\ln k^2)/\ln^2 k^2$ with a coefficient given by the $g_\lambda{}^5$ (two-loop) correction to Eq. (1). Sitting under the logarithmic corrections are polynomially suppressed terms starting out like $[(k^2/m^2)\ln(k^2/m^2)]^{-1}$. Thus the picture that emerges here is one of three energy regions: first the low-energy region where $k^2 \lesssim m^2$ (with m taken to be on the order of hadronic masses), then a region of logarithmic energy variation, and finally the scale-invariant result emerging at high energies.

Can one say anything about e^+e^- hadrons beyond the behavior of the total cross section? In general the answer is no, at least not without learning (or postulating) a great deal more about these theories, since statements about n-particle cross sections, multiplicities, momentum distributions, etc., depend upon their long-range or mass-shell structure.

One feature of the final states which is part of the traditional parton-model folklore is the existence of jets. In our results, the leading, scale-invariant term in $\sigma_{\rm tot}^{e^+e^-}$ corresponds to the production of two "bare" partons which subsequently develop into physical hadrons. It seems reasonable to conjecture that each bare parton is the nucleus of a jet, but details of transverse-momentum distribution and the quantum numbers of the jets are beyond the scope of short-distance physics.

Even if an asymptotic two-jet structure does emerge, it will probably only do so at very high energies. The logarithmically suppressed terms in $\sigma_{\text{tot}}^{e^+e^-}$ correspond to the production of more than two bare partons. For instance, the graphs in Fig. 1 may contribute to a three-jet structure. It seems very unlikely that in this "logarithmic" energy range any sort of jet structure would actually be observed.

An analysis similar to the one given here can also be applied to inelastic lepton scattering. This is a more complicated problem and not as directly accessible by these techniques since it is not a short-distance effect. It involves one on-shell particle (the target hadron) and large longitudinal distances. Progress can be made by using Wilson's operator-product expansion for the two lightlike separated currents. Renormalization-group equations can be written down for each term in the expansion which in turn is related to a structure-function moment. This formalism has been developed for example by Christ, Hasslacher, and Mueller[13] and can be used to analyze these theories with ultraviolet stability at $q=0$.

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