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Selection of Dynamical Invariant Amplitudes and Universal Coupling Constants*

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Clebsch-Gordan coefficients, suitable for reducing in a completely symmetric manner the tensor product of an arbitrary number of "physical" representations of the Poincaré covering group $\overline{P}$, are described in a straightforward fashion. These Clebsch-Gordan coefficients are used to define invariant amplitudes for an arbitrary n-point process and to obtain the partial-wave expansion for the four-point amplitude. A natural definition of crossing leads to the conclusion that the invariant amplitudes introduced are crossing-symmetric. A new dynamical hypothesis is proposed based on the assumption that the above-mentioned crossing-symmetric invariant amplitudes are analytic functions on a certain $(3n-10)$-dimensional analytic subvariety of the space of complex four-momenta except for those singularities required by extended unitarity. Analyticity in the usual variables, namely, the scalar products of the complex four-momenta, seems to be ruled out. The advantage of the new hypothesis is that all theories constructed from basic three-point vertices would yield renormalized perturbation series by means of the successive pole approximation. Since spinor amplitudes are not employed, another method for introducing coupling constants and determining their dimensions is required. It is proposed that the appropriate universal coupling constants are the partial-wave amplitudes with given orbital angular momentum l and total spin s in a given standard channel for the three-point process. The kinematic factor which defines the dimension of the coupling constants is then $|Q|^l$ where Q^μ is the relative momentum four-vector in that channel. A qualitative examination of a number of common interactions suggests that this hypothesis is reasonable. The dependence of the kinematic factor on the masses of the particles of the three-point vertex is, in general, different from that given by spinor theories. This difference changes the meaning of universality as applied to coupling constants.

I. INTRODUCTION

Of the many unpleasant features of spinor field theories, perhaps the most annoying is the fact that most are nonrenormalizable. The theory of weak interactions is plagued by this problem. The source of the difficulty is the bad asymptotic behavior of the free particle propagators which is caused by the so-called spin projection operators. Moreover, this behavior becomes worse for higher spin.

S-matrix theory was established in an effort to circumvent the difficulties of field theory. But the central trouble-making feature was retained; namely, the representation of spin by spinor amplitudes. Indeed, S-matrix theories based on analyticity of spinor amplitudes and constructed from basic three-point amplitudes by means of the successive-pole approximation[1,2] lead to the same classification of renormalizable and nonrenormalizable theories as that found in local spinor field theories.

The principal reason for the bad asymptotic behavior is the fact that the spinor basis is not a unitary one; consequently, the spin sums which occur in the contribution of some intermediate state to the unitarity equations are polynomials in both the energy and the momentum variables and therefore grow without limit as the energy increases. On the other hand, if one could consistently employ a unitary basis such as those defined by pure boosts or helicity boosts for massive particles, the spin sum factors would be a product of unitary $D^{(J)}$ functions which would necessarily be bounded by unity throughout the physical region. However, because maximal analyticity consistent with extended unitarity in every channel and the crossing hypothesis constitute the essential dynamical ingredients of S-matrix theory, one must choose invariant amplitudes which have crossing matrices which are free of singularities.

In the following, a set of invariant amplitudes for an arbitrary n-point process is defined. These invariant amplitudes are just the usual T-matrix amplitudes (with the energy-momentum δ function removed) evaluated in a unitary spin basis and in the principal reference frame in which the four-momenta, q_i^μ $(i=1, \ldots, n)$, satisfy

$$\begin{aligned} &q_i \cdot q_i = m_i^2 , \\ &\sum_{i=1}^{n} q_i^\mu = 0 , \\ &M^{\mu\nu}(q) = \sum_{i=1}^{n} q_i^\mu q_i^\nu \\ &\qquad = \mathrm{diag}(\xi_{(0)}, -\xi_{(1)}, -\xi_{(2)}, -\xi_{(3)}) , \end{aligned} \tag{1.1}$$

where the "all-in" notation for the four-momenta has been used. Note that this principal reference frame is defined symmetrically with respect to the n particles. The fact that the Eqs. (1.1) also make sense for complex q_i^μ $(i=1,\ldots,n)$ suggests that the invariant amplitudes should be defined on the complex variety[3] $\mathcal{V}_n(q)$ consisting of the points $[q_i^\mu\ (i=1,\ldots,n)]$ in complex $4n$-dimensional space satisfying Eqs. (1.1). It is natural to define crossing symmetry in such a way that the analytic continuation from one physical region to another follows the principal frame of reference. With this definition of crossing, the above-mentioned invariant amplitudes are crossing-symmetric. This definition of crossing is not consistent with the usual one based on spinor amplitudes. Unfortunately, any attempt to decide this issue by direct appeal to experiment encounters the substantial difficulty of performing the analytic continuation from one physical region to another.

One must also decide which boost type to use. The most likely possibilities are pure boosts or helicity boosts for massive particles and helicity boosts for massless particles. It turns out that with the definition of crossing mentioned above, the principal helicity amplitudes are crossing-symmetric while the principal pure boost amplitudes are not, so that helicity boosts are preferred.

Following a discussion of revised axioms for analytic S-matrix theory, the introduction of coupling constants associated with basic three-point amplitudes is considered. The procedure proposed is inconsistent with that used in spinor field theory except possibly in certain special cases. The basic idea is that the kinematic factor which governs the dimension of the universal coupling constant should be $|Q|^l$ where $|Q|$ is the magnitude of the relative momentum four-vector of two of the particles and l is the orbital angular momentum between these two particles. A qualitative comparison of this hypothesis with a number of common spinor couplings is given.

II. THE CLEBSCH-GORDAN COEFFICIENTS

Physical particle states can be described in an elegant way by means of certain of the unitary irreducible representations (UIR) of the Poincaré covering group $\overline{P}$.[4] A brief summary of the relevant representations is given in Appendix A. The notational conventions used are those of Moussa and Stora.[5]

The problem of coupling together a number of particle states is frequently encountered; for example, in the construction of invariant amplitudes or in partial-wave analysis. In general, such problems can be reduced to the selection of the identity or "vacuum" representation from the tensor product of an arbitrary number of UIR of $\overline{P}$. Since all of the UIR of $\overline{P}$ can be obtained as induced representations, this problem can be solved quite generally with the aid of Mackey's[6] subgroup theorem. For the group $\overline{P}$, this analysis has been performed by Klink and Smith.[7] The decomposition of the tensor product leads to a direct integral over a space of double cosets. The selection of a double coset representative amounts to the choice of an over-all reference frame. The continuous degeneracy labels are just the four-momenta evaluated in this frame. An explicit expression for the Clebsch-Gordan coefficient which couples an arbitrary number of massive or massless particle states of either positive or negative energy to the vacuum states will be given in this section.

Although the expression is essentially that found in Ref. 7, it is presented here in the form that will be required in the following sections and with all particle states treated in a completely symmetric manner. The $\mathfrak{D}$ functions which appear in the Clebsch-Gordan coefficient are defined in Appendix A.

For a tensor product of

N_1 kets $|[m_iJ_i]p_i\lambda_i\rangle, \quad i=1,\ldots,N_1,$

N_2 kets $|p_j\lambda_j\rangle, \quad j=N_1+1,\ldots,N_1+N_2,$

N_3 bras $\langle[m_kJ_k]p_k\lambda_k|,$

$k=N_1+N_2+1,\ldots,N_1+N_2+N_3,$

N_4 bras $\langle p_l\lambda_l|,$

$l=N_1+N_2+N_3+1,\ldots,N_1+N_2+N_3+N_4,$

the appropriate Clebsch-Gordan coefficient is

$$\mathfrak{C}(\{\mu_k,\mu_i,q_l,q_k,q_j,q_i\}|\,p_l\lambda_l;[m_kJ_k]p_k\lambda_k;p_j\lambda_j;[m_iJ_i]p_i\lambda_i)$$
$$=N\int d^6\overline{A}\left[\prod_l \mathfrak{D}^{[\lambda_l]*}_{\hat{p}_l;p_l}(0,\overline{B}^{-1}(q_l)\overline{A})\prod_k \mathfrak{D}^{[m_kJ_k]*}_{\hat{p}_k\mu_k;p_k\lambda_k}(0,\overline{B}^{-1}(q_k)\overline{A})\right.$$
$$\left.\times\prod_j \mathfrak{D}^{[\lambda_j]}_{\hat{p}_j;p_j}(0,\overline{B}^{-1}(q_j)\overline{A})\prod_i \mathfrak{D}^{[m_iJ_i]}_{\hat{p}_i\mu_i;p_i\lambda_i}(0,\overline{B}^{-1}(q_i)\overline{A})\right], \quad (2.1)$$

where N is a normalization constant, * denotes complex conjugation, the $\overline{B}(q)$ are boosts appropriate to the particle type, and i,j,k,l range over the values indicated above. There are N_1+N_3 discrete degeneracy labels $\{\mu_k, \mu_i\}$. Each index μ ranges over the usual $2J+1$ values. Before specifying the measure on the space of continuous degeneracy labels $\{q_1, q_k, q_j, q_i\}$, we introduce the real symmetric 4×4 matrix[8]

$$M^{\mu\nu}(p)=\sum_{\text{all momenta}} p^{\mu}p^{\nu}\,. \tag{2.2}$$

The measure on the space of continuous degeneracy parameters is then

$$d\mu(q)=\prod_l d^4q_l\delta(q_l\cdot q_l)\Theta(q_l)\prod_k d^4q_k\delta(q_k\cdot q_k-m_k^2)\Theta(q_k)\prod_j d^4q_j\delta(q_j\cdot q_j)\Theta(q_j)\prod_i d^4q_i\delta(q_i\cdot q_i-m_i^2)\Theta(q_i)$$
$$\times\delta^4\Big(\sum_l q_l^{\mu}+\sum_k q_k^{\mu}-\sum_j q_j^{\mu}-\sum_i q_i^{\mu}\Big)\delta(M^{01}(q))\delta(M^{02}(q))\delta(M^{03}(q))\delta(M^{23}(q))\delta(M^{31}(q))\delta(M^{12}(q))\,. \tag{2.3}$$

The dimension of this space is $3(N_1+N_2+N_3+N_4)-10$. The above measure is suitable when $\text{rank}(M^{\mu\nu})=4$. If $\text{rank}(M^{\mu\nu})<4$, then the four-momenta span a space of lower dimension. In such a case, the momenta q may be chosen such that

$$q^2=0 \text{ for all } q \text{ when } \text{rank}(M^{\mu\nu})=3$$
$$\text{or} \tag{2.4}$$
$$q^1=0=q^2 \text{ for all } q \text{ when } \text{rank}(M^{\mu\nu})=2\,,$$

where q^1 and q^2 denote the first and second components of the four-vector q^{μ}. In case $\text{rank}(M^{\mu\nu})=2$, all particles are moving along the 3-axis and all spins are quantized along this axis. Orbital angular momentum projection along this axis must be zero and thus the spin projections must be conserved. This fact is contained in the Clebsch-Gordan coefficient.

The method may readily be extended to include the nonphysical UIR's of the group $\overline{P}_+^{\dagger}$. For the spacelike case $\hat{p}=(0,0,0,m)$ and the helicity boosts are well defined everywhere on the spacelike mass shell.

$$\overline{B}(p)=\begin{bmatrix}\left(\frac{p+p^0}{m}\right)^{1/2}\cos\frac{1}{2}\theta & -\left(\frac{m}{p+p^0}\right)^{1/2}\sin\frac{1}{2}\theta\, e^{-i\varphi}\\ \left(\frac{p+p^0}{m}\right)^{1/2}\sin\frac{1}{2}\theta\, e^{i\varphi} & \left(\frac{m}{p+p^0}\right)^{1/2}\cos\frac{1}{2}\theta\end{bmatrix}\,, \tag{2.5}$$

where $p=[m^2+(p^0)^2]^{1/2}$ and $p^1=p\sin\theta\cos\varphi$, $p^2=p\sin\theta\sin\varphi$, $p^3=p\cos\theta$. Moreover, the boost choice (2.5) is consistent with the conditions (2.4) in the case $\text{rank}(M^{\mu\nu})=2$, for then the operator J^{12} is diagonal and the angular momentum constraint retains its simple form. For the same reason, for the representations corresponding to null four-momentum $\hat{p}=(0,0,0,0)$, one should use a basis in which J^{12} and J^{03} are diagonal. Then when constructing the Clebsch-Gordan coefficient for the reduction of a tensor product including such states, one need only include the $\mathfrak{D}$ functions of the appropriate representations in the bases described above. Those UIR's which cannot be realized on spaces of square-integrable functions cannot be included in this way.

For the group $\overline{P}_+^{\dagger}$, the essential aspect of the Clebsch-Gordan coefficient is the choice of an over-all reference frame, and the diagonalization of the matrix $M^{\mu\nu}(p)$ accomplishes this objective in a completely symmetric manner. The same technique may clearly be used for the Euclidean and pseudo-Euclidean groups in n dimensions. In complex, n-dimensional space, one encounters two types of groups: the semidirect product of the group of translations with an orthogonal or pseudo-orthogonal group, or the semidirect product of the group of translations with a unitary or pseudounitary group. In either case, one usually uses a basis in which the translation operators are diagonal. The states are then labeled in part by complex n-vectors p^{μ}. For the case of the orthogonal or pseudo-orthogonal group, the over-all reference frame may be chosen by diagonalizing the matrix

$$M^{\mu\nu}(p)=\sum_{i=1}^{N} p_i^{\mu}p_i^{\nu}\,. \tag{2.6}$$

For the case of the unitary or pseudounitary group, the over-all reference frame may be chosen by diagonalizing the matrix

$$M^{\mu\nu}(p)=\sum_{i=1}^{N} p_i^{\mu *}p_i^{\nu} \tag{2.7}$$

and requiring that its eigenvectors be real. (* means complex conjugate.)

III. INVARIANT AMPLITUDES

In S-matrix theory, one adopts the view that all experiments are scattering experiments and that all the dynamical information about such experiments is contained in a unitary operator S which relates in-states to out-states. This operator transforms as a scalar under the Poincaré group. It is customary to introduce the transition operator T according to

$$S = 1 + 2\pi i T. \tag{3.1}$$

To predict the result of any given experiment one must compute the matrix element of T for the particular physical states involved. The problem is considerably simplified by the introduction of invariant amplitudes. In this section, the Clebsch-Gordan coefficients defined in Sec. II will be used to define invariant amplitudes. The details will be presented only for the case of two-particle quasielastic scattering of massive particles, but the extension to the general case is straightforward.

For this case, $\operatorname{rank}(M^{\mu\nu}) = 3$ except on the boundary of the physical region and the measure on the space of continuous degeneracy labels is

$$d\mu(q) = \prod_{i=1}^{4} d^4 q_i \delta(q_i \cdot q_i - m_i^2)\Theta(q_i)\, \delta^3(q_4 + q_3 - q_2 - q_1)\delta(q_4^2)\delta(q_3^2)\delta(q_2^2)\delta(q_1^2) \times \delta(q_1^1 q_1^3 + q_2^1 q_2^3 + q_3^1 q_3^3 + q_4^1 q_4^3)\delta(\omega_1 q_1^1 + \omega_2 q_2^1 + \omega_3 q_3^1 + \omega_4 q_4^1)\delta(\omega_1 q_1^3 + \omega_2 q_2^3 + \omega_3 q_3^3 + \omega_4 q_4^3). \tag{3.2}$$

The Clebsch-Gordan coefficient is

$$\mathcal{C}(\{\mu_i, q_i\} | [m_i J_i] p_i \lambda_i,\ i = 1, \ldots, 4) = N \int d^6\bar{A} [\delta^3(\hat{p}_4, B^{-1}(q_4) A p_4) D^{(J_4)*}_{\mu_4 \lambda_4}(\bar{B}^{-1}(q_4)\bar{A}\bar{B}(p_4))\delta^3(\hat{p}_3, B^{-1}(q_3) A p_3) D^{(J_3)*}_{\mu_3\lambda_3}(\bar{B}^{-1}(q_3)\bar{A}\bar{B}(p_3)) \times \delta^3(\hat{p}_2, B^{-1}(q_2) A p_2) D^{(J_2)}_{\mu_2\lambda_2}(\bar{B}^{-1}(q_2)\bar{A}\bar{B}(p_2))\delta^3(\hat{p}_1, B^{-1}(q_1) A p_1) D^{(J_1)}_{\mu_1\lambda_1}(\bar{B}^{-1}(q_1)\bar{A}\bar{B}(p_1))]. \tag{3.3}$$

Then the matrix element of T may be expanded in terms of invariant amplitudes $\mathcal{A}(\{\mu_i, q_i\})$ according to

$$\langle [m_4 J_4] p_4 \lambda_4; [m_3 J_3] p_3 \lambda_3 | T | [m_2 J_2] p_2 \lambda_2; [m_1 J_1] p_1 \lambda_1 \rangle = \sum_{\mu_i} \int d\mu(q) \mathcal{A}(\{\mu_i, q_i\}) \mathcal{C}(\{\mu_i, q_i\} | [m_i J_i] p_i \lambda_i,\ i = 1, \ldots, 4). \tag{3.4}$$

The δ functions in the Clebsch-Gordan coefficient imply

$$q_i^\mu = A^\mu{}_\nu p_i^\nu, \quad i = 1, \ldots, 4. \tag{3.5}$$

Six of the twelve δ functions are therefore used to fix A. Two more δ functions are required to determine the q_i^μ in terms of the p_i^μ. The remaining four δ functions express the energy-momentum conservation

$$\delta^4(p_4 + p_3 - p_2 - p_1).$$

Thus the expansion becomes

$$\langle [m_4 J_4] p_4 \lambda_4; [m_3 J_3] p_3 \lambda_3 | T | [m_2 J_2] p_2 \lambda_2; [m_1 J_1] p_1 \lambda_1 \rangle = \delta^4(p_4 + p_3 - p_2 - p_1) \sum_{\mu_4 \mu_3 \mu_2 \mu_1} \mathcal{A}_{\mu_4 \mu_3 \mu_2 \mu_1}(q_i) D^{(J_4)*}_{\mu_4\lambda_4}(\bar{B}^{-1}(q_4)\bar{A}\bar{B}(p_4)) D^{(J_3)*}_{\mu_3\lambda_3}(\bar{B}^{-1}(q_3)\bar{A}\bar{B}(p_3)) \times D^{(J_2)}_{\mu_2\lambda_2}(\bar{B}^{-1}(q_2)\bar{A}\bar{B}(p_2)) D^{(J_1)}_{\mu_1\lambda_1}(\bar{B}^{-1}(q_1)\bar{A}\bar{B}(p_1)), \tag{3.6}$$

where q_i^μ ($i = 1, \ldots, 4$) and $\bar{A}$ have the values determined by the δ-function constraints and the values of p_i^μ ($i = 1, \ldots, 4$).

The interpretation of this expansion is straightforward. Any set of amplitudes evaluated in a fixed frame of reference is an invariant set. The amplitudes $\mathcal{A}_{\mu_4\mu_3\mu_2\mu_1}(q)$ are just the T-matrix elements evaluated in the frame in which the scattering occurs in the 1-3 plane and in which $M^{\mu\nu}(q)$ is diagonal, and with the energy-momentum δ-function factor removed. The four-momenta q_i^μ in this frame are scalars and possess only two degrees of freedom as is evident from the measure $d\mu(q)$. The $D^{(J)}$ functions contain the Wigner rotations necessary to transform from a general frame to this fixed frame.

Evidently, one may define invariant amplitudes in infinitely many ways. The use of the matrix $M^{\mu\nu}(p)$ has the advantage of treating all particles similarly. It is interesting to note that this matrix is closely related to the matrix of scalars $S_{ij} = p_i \cdot p_j$. Both have essentially the same characteristic equation. (This is true for an arbitrary number of particles.) Although one may in principle employ any type of spin basis including a spinor basis, the use of a unitary spin basis for the specification of the dynamical amplitudes is preferred since the spin sum factors in the unitarity equation are then products of unitary $D^{(J)}$ functions which are necessarily bounded by unity throughout the physical region.

IV. CROSSING

We turn now to the derivation of the crossing matrix for the invariant amplitudes defined in Sec. III. The definition of crossing used here differs from that which is usually employed. A discussion of the usual definition of crossing is given in the text of Martin and Spearman,[9] who follow closely the work of Trueman and Wick.[10] The crossing properties depend on the type of boost used. The case of helicity boost proves to be the simplest and therefore will be discussed first.

In the following discussion, quantum numbers arising from internal symmetries are suppressed. Consider a process involving four massive particles. Equation (3.6) gives the invariant amplitude expansion in the α channel $(12\to 34)$

$$\begin{aligned}&\langle [m_4J_4]p_4\lambda_4;[m_3J_3]p_3\lambda_3|T|[m_2J_2]p_2\lambda_2;[m_1J_1]p_1\lambda_1\rangle\\&=\delta^4(p_4+p_3-p_2-p_1)\sum_{\mu_4\mu_3\mu_2\mu_1}\mathfrak{A}^{(\alpha)}_{\mu_4\mu_3\mu_2\mu_1}(q_i)D^{(J_4)*}_{\mu_4\lambda_4}(\overline{B}^{-1}(q_4)\overline{A}_{(\alpha)}\overline{B}(p_4))D^{(J_3)*}_{\mu_3\lambda_3}(\overline{B}^{-1}(q_3)\overline{A}_{(\alpha)}\overline{B}(p_3))\\&\qquad\times D^{(J_2)}_{\mu_2\lambda_2}(\overline{B}^{-1}(q_2)\overline{A}_{(\alpha)}\overline{B}(p_2))D^{(J_1)}_{\mu_1\lambda_1}(\overline{B}^{-1}(q_1)\overline{A}_{(\alpha)}\overline{B}(p_1)).\end{aligned}\tag{4.1}$$

The corresponding expansion in the β channel $(13\to 24)$ is

$$\begin{aligned}&\langle [m_4J_4]P_4\lambda_4;[m_2J_2]P_2\lambda_2|T|[m_3J_3]P_3\lambda_3;[m_1J_1]P_1\lambda_1\rangle\\&=\delta^4(P_4+P_2-P_3-P_1)\sum_{\mu_4\mu_2\mu_3\mu_1}\mathfrak{A}^{(\beta)}_{\mu_4\mu_2\mu_3\mu_1}(Q_i)D^{(J_4)*}_{\mu_4\lambda_4}(\overline{B}^{-1}(Q_4)\overline{A}_{(\beta)}\overline{B}(P_4))D^{(J_2)*}_{\mu_2\lambda_2}(\overline{B}^{-1}(Q_2)\overline{A}_{(\beta)}\overline{B}(P_2))\\&\qquad\times D^{(J_3)}_{\mu_3\lambda_3}(\overline{B}^{-1}(Q_3)\overline{A}_{(\beta)}\overline{B}(P_3))D^{(J_1)}_{\mu_1\lambda_1}(\overline{B}^{-1}(Q_1)\overline{A}_{(\beta)}\overline{B}(P_1)),\end{aligned}\tag{4.2}$$

where q_i and Q_i denote the four-momenta in the principal frames corresponding to the four-momenta p_i and P_i, respectively, and $A_{(\alpha)}$ and $A_{(\beta)}$ are defined by

$$q_i^\mu=A_{(\alpha)\nu}^{\ \mu}p_i^\nu\quad(i=1,\ldots,4),\qquad Q_i^\mu=A_{(\beta)\nu}^{\ \mu}P_i^\nu\quad(i=1,\ldots,4).\tag{4.3}$$

The procedure used to obtain the crossing matrix for the invariant amplitudes is the following:

(1) Analytically continue Eq. (4.1) from the α-channel physical region to the β-channel physical region.

(2) Relate the continued α-channel T-matrix element to the β-channel T-matrix element.

(3) Express the continued α-channel $D^{(J)}$ functions (the spin basis) in terms of the β-channel $D^{(J)}$ functions.

(4) By comparing the resulting equation with Eq. (4.2), obtain the expression for the continued α-channel-invariant amplitude in terms of the β-channel-invariant amplitude; that is, the crossing matrix.

In order to relate the continued α-channel T-matrix element to the β-channel T-matrix element, we shall need the equations which give the transformation of these T-matrix elements under an arbitrary but fixed Lorentz transformation. The Jacob and Wick[11] phase convention for two-particle helicity states is not adopted here in order to treat all particles as symmetrically as possible. The ambiguity in the definition of the helicity boost for $\theta=\pi$ is resolved by choosing $\varphi=0$. The required transformation equations are

$$\begin{aligned}&\langle [m_4J_4]p_4\lambda_4;[m_3J_3]p_3\lambda_3|T|[m_2J_2]p_2\lambda_2;[m_1J_1]p_1\lambda_1\rangle\\&=\sum_{\lambda_4'\lambda_3'\lambda_2'\lambda_1'}\langle [m_4J_4]p_4'\lambda_4';[m_3J_3]p_3'\lambda_3'|T|[m_2J_2]p_2'\lambda_2';[m_1J_1]p_1'\lambda_1'\rangle\, D^{(J_4)*}_{\lambda_4'\lambda_4}(\overline{B}^{-1}(p_4')\overline{\Lambda}\,\overline{B}(p_4))D^{(J_3)*}_{\lambda_3'\lambda_3}(\overline{B}^{-1}(p_3')\overline{\Lambda}\overline{B}(p_3))\\&\qquad\times D^{(J_2)}_{\lambda_2'\lambda_2}(\overline{B}^{-1}(p_2')\overline{\Lambda}\overline{B}(p_2))D^{(J_1)}_{\lambda_1'\lambda_1}(\overline{B}^{-1}(p_1')\overline{\Lambda}\overline{B}(p_1))\end{aligned}\tag{4.4}$$

and

$$\begin{aligned}&\langle [m_4J_4]P_4\lambda_4;[m_2J_2]P_2\lambda_2|T|[m_3J_3]P_3\lambda_3;[m_1J_1]P_1\lambda_1\rangle\\&=\sum_{\lambda_4'\lambda_2'\lambda_3'\lambda_1'}\langle [m_4J_4]P_4'\lambda_4';[m_2J_2]P_2'\lambda_2'|T|[m_3J_3]P_3'\lambda_3';[m_1J_1]P_1'\lambda_1'\rangle D^{(J_4)*}_{\lambda_4'\lambda_4}(\overline{B}^{-1}(P_4')\overline{\Lambda}\overline{B}(P_4))D^{(J_2)*}_{\lambda_2'\lambda_2}(\overline{B}^{-1}(P_2')\overline{\Lambda}\overline{B}(P_2))\\&\qquad\times D^{(J_3)}_{\lambda_3'\lambda_3}(\overline{B}^{-1}(P_3')\overline{\Lambda}\overline{B}(P_3))D^{(J_1)}_{\lambda_1'\lambda_1}(\overline{B}^{-1}(P_1')\overline{\Lambda}\overline{B}(P_1)),\end{aligned}\tag{4.5}$$

where Λ is an arbitrary but fixed Lorentz transformation and $p_i'=\Lambda p_i$ and $P_i'=\Lambda P_i$ for $i=1,2,3,4$. Analytically continue Eq. (4.4) in the four-momenta from the α-channel physical region to the β-channel physical region. Choose Λ to be an infinitesimal transformation so that p_i and p_i' follow essentially the same path of continuation. Then under the continuation the four-momenta change according to

$$\begin{aligned} &p_4 \to P_4, \quad p_4' \to P_4', \quad p_3 \to -P_3, \quad p_3' \to -P_3', \\ &p_2 \to -P_2, \quad p_2' \to -P_2', \quad p_1 \to P_1, \quad p_1' \to P_1'. \end{aligned} \tag{4.6}$$

The behavior of the helicity boost $B_H(p)$ under the continuation $p \to -P$ depends on the path of continuation followed in the p^0 plane because of the dependence on $|\vec{p}| = [(p^0)^2 - m^2]^{1/2}$ [see Eqs. (A8), (A9)]. We choose the path shown in Fig. 1. In this case

$$\begin{aligned} &p^0 \to -P^0, \quad |\vec{p}| \to -|\vec{P}|, \\ &\theta \to \theta, \quad \varphi \to \varphi, \end{aligned} \tag{4.7}$$

and

$$B_H(p)^\mu{}_\nu \to -B_H(P)^\mu{}_\sigma R_3(\pi)^\sigma{}_\nu . \tag{4.8}$$

Then

$$\begin{aligned} D^{(J)}_{\lambda'\lambda}(B_H{}^{-1}(p')\Lambda B_H(p)) \to D^{(J)}_{\lambda'\lambda}(R_3{}^{-1}(\pi)B_H{}^{-1}(P')\Lambda B_H(P)R_3(\pi)) &= (-)^{\lambda'-\lambda} D^{(J)}_{\lambda'\lambda}(B_H{}^{-1}(P')\Lambda B_H(P)) \\ &= D^{(J)*}_{-\lambda',-\lambda}(B_H{}^{-1}(P')\Lambda B_H(P)). \end{aligned}$$

Thus the continued version of Eq. (4.4) is

$$\begin{aligned} &\langle [m_4J_4]P_4\lambda_4; [m_3J_3] - P_3 - \lambda_3 | T | [m_2J_2] - P_2 - \lambda_2; [m_1J_1]P_1\lambda_1 \rangle \\ &= \sum_{\lambda_4'\lambda_3'\lambda_2'\lambda_1'} \langle [m_4J_4]P_4'\lambda_4'; [m_3J_3] - P_3' - \lambda_3' | T | [m_2J_2] - P_2' - \lambda_2'; [m_1J_1]P_1'\lambda_1' \rangle \\ &\qquad \times D^{(J_4)*}_{\lambda_4'\lambda_4}(\overline{B}^{-1}(P_4')\overline{\Lambda}\,\overline{B}(P_4)) D^{(J_3)}_{\lambda_3'\lambda_3}(\overline{B}^{-1}(P_3')\overline{\Lambda}\,\overline{B}(P_3))\, D^{(J_2)*}_{\lambda_2'\lambda_2}(\overline{B}^{-1}(P_2')\overline{\Lambda}\,\overline{B}(P_2)) D^{(J_1)}_{\lambda_1'\lambda_1}(\overline{B}^{-1}(P_1')\overline{\Lambda}\,\overline{B}(P_1)). \end{aligned} \tag{4.9}$$

Comparison of Eq. (4.9) with Eq. (4.5) gives the crossing relation

$$\begin{aligned} &\langle [m_4J_4]P_4\lambda_4; [m_3J_3] - P_3 - \lambda_3 | T | [m_2J_2] - P_2 - \lambda_2; [m_1J_1]P_1\lambda_1 \rangle \\ &\qquad = \langle [m_4J_4]P_4\lambda_4; [m_2J_2]P_2\lambda_2 | T | [m_3J_3]P_3\lambda_3; [m_1J_1]P_1\lambda_1 \rangle \end{aligned} \tag{4.10}$$

up to a helicity-independent phase factor. This derivation of Eq. (4.10) is given in the text of Martin and Spearman[9] and is reproduced here for completeness.

The next step is the analytic continuation of Eq. (4.1) from the α-channel physical region to the β-channel physical region. We assume that the path of continuation may be chosen so that

$$\begin{aligned} &p_4 \to P_4, \quad q_4 \to Q_4, \quad p_3 \to -P_3, \quad q_3 \to -Q_3, \\ &p_2 \to -P_2, \quad q_2 \to -Q_2, \quad p_1 \to P_1, \quad q_1 \to Q_1, \end{aligned} \tag{4.11}$$

and so that the boosts $B_H(q_i)$ and $B_H(p_i)$ change in a similar fashion for each $i = 1, 2, 3, 4$. Then Eqs. (4.3) imply

$$A^\mu_{(\alpha)\nu} \to A^\mu_{(\beta)\nu} . \tag{4.12}$$

Thus Eq. (4.1) becomes

$$\begin{aligned} &\langle [m_4J_4]P_4\lambda_4; [m_3J_3] - P_3 - \lambda_3 | T | [m_2J_2] - P_2 - \lambda_2; [m_1J_1]P_1\lambda_1 \rangle \\ &\qquad = \delta^4(P_4 + P_2 - P_3 - P_1) \sum_{\mu_4\mu_3\mu_2\mu_1} \mathcal{A}^{(\alpha)}_{\mu_4,-\mu_3,-\mu_2,\mu_1}(Q_4, -Q_3, -Q_2, Q_1) \\ &\qquad\qquad \times D^{(J_4)*}_{\mu_4\lambda_4}(\overline{B}^{-1}(Q_4)\overline{A}_{(\beta)}\overline{B}(P_4)) D^{(J_3)}_{\mu_3\lambda_3}(\overline{B}^{-1}(Q_3)\overline{A}_{(\beta)}\overline{B}(P_3)) \\ &\qquad\qquad \times D^{(J_2)*}_{\mu_2\lambda_2}(\overline{B}^{-1}(Q_2)\overline{A}_{(\beta)}\overline{B}(P_2)) D^{(J_1)}_{\mu_1\lambda_1}(\overline{B}^{-1}(Q_1)\overline{A}_{(\beta)}\overline{B}(P_1)). \end{aligned} \tag{4.13}$$

Using Eq. (4.10), we can compare Eq. (4.13) with Eq. (4.2) and so obtain the crossing relation for the invariant amplitudes, namely,

$$\mathfrak{A}^{(\alpha)}_{\mu_4,-\mu_3,-\mu_2,\mu_1}(Q_4,\ -Q_3,\ -Q_2,\ Q_1)=\mathfrak{A}^{(\beta)}_{\mu_4,\mu_2,\mu_3,\mu_1}(Q_4,\ Q_2,\ Q_3,\ Q_1)\,. \tag{4.14}$$

Note that if an "all-in" formalism is adopted in which outgoing states are represented by states of negative energy, then the crossing matrix for the principal helicity amplitudes is the identity matrix up to a helicity-independent phase.

If pure boosts are used, the major change in the preceding argument is that the boosts behave differently under the analytic continuation $p\to -P$. Thus Eq. (4.8) and those aspects of the discussion which depend on it must be changed. The pure boost is given by Eqs. (A6), (A7):

$$B(p)^{\mu}{}_{\nu}=\begin{bmatrix} \frac{p^0}{m} & \frac{p^1}{m} & \frac{p^2}{m} & \frac{p^3}{m} \\ \frac{p^1}{m} & 1+\frac{p^1p^1}{m(m+p^0)} & \frac{p^1p^2}{m(m+p^0)} & \frac{p^1p^3}{m(m+p^0)} \\ \frac{p^2}{m} & \frac{p^2p^1}{m(m+p^0)} & 1+\frac{p^2p^2}{m(m+p^0)} & \frac{p^2p^3}{m(m+p^0)} \\ \frac{p^3}{m} & \frac{p^3p^1}{m(m+p^0)} & \frac{p^3p^2}{m(m+p^0)} & 1+\frac{p^3p^3}{m(m+p^0)} \end{bmatrix}. \tag{4.15}$$

The continued boost is given by

$$-B(-P)^{\mu}{}_{\nu}=\begin{bmatrix} \frac{P^0}{m} & \frac{P^1}{m} & \frac{P^2}{m} & \frac{P^3}{m} \\ \frac{P^1}{m} & -\left(1+\frac{P^1P^1}{m(m-P^0)}\right) & -\frac{P^1P^2}{m(m-P^0)} & -\frac{P^1P^3}{m(m-P^0)} \\ \frac{P^2}{m} & -\frac{P^2P^1}{m(m-P^0)} & -\left(1+\frac{P^2P^2}{m(m-P^0)}\right) & -\frac{P^2P^3}{m(m-P^0)} \\ \frac{P^3}{m} & -\frac{P^3P^1}{m(m-P^0)} & -\frac{P^3P^2}{m(m-P^0)} & -\left(1+\frac{P^3P^3}{m(m-P^0)}\right) \end{bmatrix}. \tag{4.16}$$

This boost is related to the pure boost $B(P)^{\mu}{}_{\nu}$ by

$$B(-P)^{\mu}{}_{\nu}=-B(P)^{\mu}{}_{\sigma}R^{\sigma}{}_{\nu}\,, \tag{4.17}$$

where

$$R^{\sigma}{}_{\nu}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2\frac{P^1P^1}{|\vec{P}|^2}-1 & 2\frac{P^1P^2}{|\vec{P}|^2} & 2\frac{P^1P^3}{|\vec{P}|^2} \\ 0 & 2\frac{P^2P^1}{|\vec{P}|^2} & 2\frac{P^2P^2}{|\vec{P}|^2}-1 & 2\frac{P^2P^3}{|\vec{P}|^2} \\ 0 & 2\frac{P^3P^1}{|\vec{P}|^2} & 2\frac{P^3P^2}{|\vec{P}|^2} & 2\frac{P^3P^3}{|\vec{P}|^2}-1 \end{bmatrix}. \tag{4.18}$$

Note that the limit of $R^{\sigma}{}_{\nu}$ as $|\vec{P}|\to 0$ depends on the direction from which $\vec{P}\to 0$.

Let the unitary matrix which transforms a UIR of the group SU(2) into its conjugate[12] be $U^{(J)}$. Then

$$U^{(J)}D^{(J)}(U^{(J)})^{-1}=D^{(J)*}\,. \tag{4.19}$$

Instead of Eq. (4.10), one obtains

$$\begin{aligned}\langle [m_4 J_4] P_4 \lambda_4; [m_2 J_2] P_2 \lambda_2 | T | [m_3 J_3] P_3 \lambda_3; [m_1 J_1] P_1 \lambda_1 \rangle \\ = \sum_{\mu_2 \nu_2 \mu_3 \nu_3} \langle [m_4 J_4] P_4 \lambda_4; [m_3 J_3] - P_3 \mu_3 | T | [m_2 J_2] - P_2 \mu_2; [m_1 J_1] P_1 \lambda_1 \rangle D^{(J_3)*}_{\mu_3 \nu_3}(R_3^{-1}) U^{(J_3)}_{\nu_3 \lambda_3} D^{(J_2)}_{\mu_2 \nu_2}(R_2^{-1}) (U^{(J_2)}_{\nu_2 \lambda_2})^{-1},\end{aligned} \tag{4.20}$$

where R_i is the rotation defined by Eq. (4.18) for momentum P_i. The crossing relation for the invariant amplitudes is

$$\mathcal{A}^{(\alpha)}_{\lambda_4 \lambda_3 \lambda_2 \lambda_1}(Q_4, -Q_3, -Q_2, Q_1) = \sum_{\mu_2 \nu_2 \mu_3 \nu_3} \mathcal{A}^{(\beta)}_{\lambda_4 \mu_2 \mu_3 \lambda_1}(Q_4, Q_2, Q_3, Q_1) U^{(J_2)}_{\mu_2 \nu_2} D^{(J_2)}_{\nu_2 \lambda_2}(R_2) (U^{(J_3)}_{\mu_3 \nu_3})^{-1} D^{(J_3)*}_{\nu_3 \lambda_3}(R_3), \tag{4.21}$$

where R_i is now the rotation defined by Eq. (4.18) for momentum Q_i. Since Eq. (4.21) is rather more complicated than Eq. (4.14), helicity boosts are preferred.

For zero-mass particles the helicity boosts are most natural. The crossing relation is the same as that for massive particles when helicity boosts are used except that a helicity-dependent phase factor may be present because helicity is an invariant for a zero-mass particle. The comment concerning the use of the "all-in" notation applies here also.

The definition of crossing discussed above differs sharply from that usually employed. Group theory itself can only relate amplitudes which are evaluated at the same point in the space of scalars. Any attempt to relate amplitudes evaluated at different points in the space of scalars necessarily involves additional assumptions which are of a dynamical nature. The question is in which scheme is the functional dependence of this interpolation the simplest. In the scheme described above, the principal helicity amplitudes are selected as the fundamental dynamical amplitudes and the interpolation follows the principal reference frame. In the customary scheme, the covariant spinor amplitudes are chosen as the fundamental dynamical amplitudes by appeal to local spinor field theory. Invariant amplitudes which are free of kinematical singularities as defined with respect to the spinor amplitudes are introduced by expanding the spinor amplitude in a suitable spin basis of Lorentz covariants. The crossing properties of the invariant amplitudes then follow from those of the spinor amplitudes which are known from field theory.

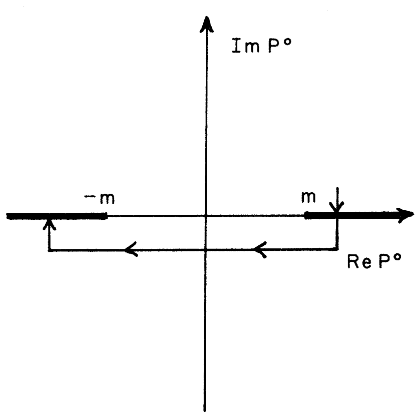

FIG. 1. The cut P^0 plane.

Although in principle the crossing relations given by the two schemes could be confronted with experiment, in practice the analytic continuation from one channel to another is extremely difficult to carry out. It would seem more promising to check the functional behavior of the interpolation within a given channel. For example, one could look at the energy dependence of the forward principal helicity amplitude or examine the dependence of the high-energy principal helicity amplitude on scattering angle. It may be that the functional dependence is simpler in the new scheme.

V. REVISED AXIOMS FOR ANALYTIC S-MATRIX THEORY

We are now in a position to state a set of revised axioms[13] for analytic S-matrix theory. The postulates concerning the linear superposition of states, the completeness of in and out states, the connectedness structure, and the unitarity of the S matrix remain intact. The points at issue are the following:

(1) Which of the many invariant amplitudes that can be associated with the connected part of an n-point process should be considered the dynamical one?

(2) What is the appropriate domain of definition of this amplitude; that is, which complex scalar variables should be used?

It is proposed to replace the usual axioms relating to these questions by the following:

(1) For the connected part of an n-point process, the appropriate dynamical amplitude is the principal helicity amplitude. The reason for this choice is that the crossing matrix in the "all-in" formalism is the identity matrix.

(2) The appropriate domain for the invariant amplitude is the complex analytic variety $\mathcal{V}_n(q)$

$$\mathcal{V}_n(q)=\left\{\{q_i\}\in C^{(4n)}\,\middle|\, q_i\cdot q_i=m_i^2,\quad i=1,\ldots,n;\ \sum_{i=1}^{n} q_i^\mu=0\,;\quad M^{\mu\nu}(q)=0 \text{ for } \mu\neq\nu\right\}\ . \tag{5.1}$$

The basic dynamical assumption is that the principal helicity amplitudes are analytic functions on $\mathcal{V}_n(q)$ except for those singularities required by extended unitarity in all channels. The consequences of this postulate could be explored by means of perturbation theory; that is, the successive pole approximation. Perturbation theory has not been thoroughly developed from the S-matrix viewpoint because the usual form of S-matrix theory would yield the same results as field theory and the field-theoretic methods are easier. The revised S-matrix axioms proposed above are not consistent with spinor field theory, so it becomes important to develop an adequate S-matrix perturbation theory.

It is expected that the new axioms will have the effect of eliminating ultraviolet divergences so that all theories constructed from basic three-point vertices by means of the successive pole approximation would have renormalized perturbation series so that the only undetermined constants are the coupling constants associated with the basic vertices. This statement can be made plausible by the following argument. Recall that the unitarity equation expresses the cut-discontinuity of an amplitude as a sum of terms each of which is a product of two amplitudes and a spin sum factor. In the new theory the spin sum factor is always bounded by unity throughout the physical region of the given channel because it is a product of unitary $D^{(J)}$ functions. Consequently, the asymptotic behavior of the discontinuity is governed by the high-energy behavior of the two amplitudes which in turn is governed by contributions to their crossed channels so that the asymptotic behavior is the result of some intermediate-state contribution in a crossed channel. However, when spinor amplitudes are used, the spin sum factor is not bounded by unity throughout the physical region but rather grows without limit polynomially in the energy. In theories corresponding to nonrenormalizable field theories this energy dependence of the spin sum factor forces the introduction of additional subtraction constants which cannot be determined. For a theory which corresponds to a renormalizable field theory, cancellations occur which remove this additional energy dependence. The important point is that in the new theory, the feature of bounded spin sums occurs regardless of spin.

The most promising application for the new theory is the theory of weak interactions which is presently plagued by ultraviolet divergences. In view of the sharp break with spinor field theory, it is important to reformulate quantum electrodynamics and gravitation as S-matrix theories. The problem of infrared divergences present in these theories could possibly be handled in the manner suggested by Barut[1] or in the way discussed by Storrow.[14] From the viewpoint of the hypothesis described above, the success of quantum electrodynamics is a result of fortuitous cancellations which improve the asymptotic behavior of the theory.

The space $\mathcal{V}_n(q)$ of scalar variables introduced above differs in some respects from the conventional one.[15] Usually, one considers the matrix of scalar products $S_{ij}=P_i\cdot P_j$ and defines an analytic variety $\mathcal{V}_n(S)$ consisting of those points in the $\frac{1}{2}n(n-1)$-dimensional complex space spanned by the variables S_{ij} for $i\neq j$ such that $\operatorname{rank}(S_{ij})\leq 4$ and energy and momentum are conserved. Then the invariant amplitudes are assumed to be defined on $\mathcal{V}_n(S)$. Both of the analytic varieties, $\mathcal{V}_n(q)$ and $\mathcal{V}_n(S)$, have complex dimension $(3n-10)$. The number of nonlinear constraints required to define $\mathcal{V}_n(S)$ is $\frac{1}{2}(n-5)(n-4)$ for $n\geq 6$ and no such constraints are required for $n=4,5$ while the number of nonlinear constraints required to define $\mathcal{V}_n(q)$ is $n+6$. Thus, for small n, $\mathcal{V}_n(S)$ is easier to work with than $\mathcal{V}_n(q)$. Another important difference follows from the fact that the variables S_{ij} are quadratic functions of the variables q_i^μ. Thus a function which is analytic on $\mathcal{V}_n(q)$ may be expected to have cut singularities if it is expressed as a function on $\mathcal{V}_n(S)$. This difference seems to be essential for the scheme we have proposed because the $D^{(J)}$ functions, which appear in the spin-sum factor in the unitarity equation, have arguments which can be readily expressed in terms of the variables q_i^μ but not in terms of the variables S_{ij}. A study of the Appendixes B and C will make this apparent.

VI. UNIVERSAL COUPLING CONSTANTS

The technique used to introduce coupling constants and to determine their dimensions in the context of spinor field theory and spinor S-matrix theory is well known. Since we have proposed to dispense with spinor amplitudes, a new rationale for introducing coupling constants and determining their dimensions must be found. It turns out that the question is of interest in its own right and leads to a hypothesis which is physically reasonable yet differs from the usual one except in special cases.

The T-matrix element for a given channel of a three-point process evaluated in the principal frame of reference has the invariant amplitude expansion

$$\langle [m_3 J_3] p_3 \lambda_3 | T | [m_2 J_2] p_2 \lambda_2 ; [m_1 J_1] p_1 \lambda_1 \rangle = \delta^4(p_3 - p_2 - p_1) \sum_{\mu_3 \mu_2 \mu_1} \mathfrak{A}_{\mu_3 \mu_2 \mu_1} D^{(J_3)*}_{\mu_3 \lambda_3}(\bar{B}^{-1}(q_3)\bar{A}\bar{B}(p_3)) D^{(J_2)}_{\mu_2 \lambda_2}(\bar{B}^{-1}(q_2)\bar{A}\bar{B}(p_2)) D^{(J_1)}_{\mu_1 \lambda_1}(\bar{B}^{-1}(q_1)\bar{A}\bar{B}(p_1)), \quad (6.1)$$

where $\mathfrak{A}_{\mu_3 \mu_2 \mu_1}$ depend only on the masses and are therefore constants. From the normalization condition (A3), it follows that each state has the dimension of a length L. The left-hand side of (6.1) therefore has the dimension L^3. The δ^4 function on the right-hand side of (6.1) has the dimension L^4 so that the constants $\mathfrak{A}_{\mu_3 \mu_2 \mu_1}$ have the dimension L^{-1}. However, the electromagnetic coupling constant is dimensionless and the gravitational coupling constant has the dimensions of a length L^1. It follows that some dimensional kinematic factor depending both on the masses of the particles present and on the nature of the physical process must be present.

We note that a partial-wave amplitude corresponding to a definite orbital angular momentum l has a natural dimension. The allowed values of orbital angular momentum depend on the channel examined. An examination of known interactions suggests that the defining channel is the one in which a "current" absorbs or emits an "exchanged" particle as shown in Fig. 2. If all of the particles are massive, it is natural to analyze the amplitude $\mathfrak{A}_{\mu_3 \mu_2 \mu_1}$ into partial waves with total orbital angular momentum l and total spin s. Then the partial-wave amplitude with given (l, s) may be factored into an amplitude $g(l, s)$ and a kinematic factor $|Q|^l$ where Q^μ is the relative momentum four-vector defined by Eq. (B8). The factor $|Q|^l$ has the dimension L^{-l} so that the amplitude $g(l, s)$ has the dimension L^{l-1}. For electrodynamics $l=1$ and for gravitation $l=2$ so that the corresponding coupling constants have their usual dimension for these two cases.

We now consider a number of familiar three-point vertices. The particles are numbered as in Fig. 2. First consider the case in which particles 1 and 3 have $J^P = 0^\pm$ and particle 2 has $J^P = 1^-$. The usual spinor coupling for this case is

$$\mathfrak{L}_{\text{int}} = g i \varphi^\dagger \overleftrightarrow{\partial}^\mu \varphi V_\mu , \quad (6.2)$$

where g is a dimensionless coupling constant. Conservation of angular momentum requires that the particles 1 and 2 are in a state of relative orbital angular momentum $l=1$ so that the coupling constant is also dimensionless according to the new scheme.

If particles 1 and 3 have $J^P = \frac{1}{2}^+$ and particle 2 has $J^P = 1^-$, then again parity requires $l=1$ for both the spin-nonflip and the spin-flip amplitudes which are therefore characterized by dimensionless coupling constants. For simplicity take $m_1 = m_3$. Then the usual spinor interaction is

$$\mathfrak{L}_{\text{int}} = g \bar{\psi} \gamma^\mu \psi V_\mu , \quad (6.3)$$

where g is a dimensionless coupling constant. For the Dirac spinors, we shall employ the conventions used in the text of Gasiorowicz.[16] Using the Gordon decomposition of the current

$$\bar{\psi} \gamma^\mu \psi = \frac{i}{2M} \bar{\psi} \overleftrightarrow{\partial}^\mu \psi + \frac{\partial_\nu}{2M} (\bar{\psi} \sigma^{\mu\nu} \psi) , \quad (6.4)$$

the interaction (6.3) may be written

$$\mathfrak{L}_{\text{int}} = g \frac{i}{2M} \bar{\psi} \overleftrightarrow{\partial}^\mu \psi V_\mu + g \frac{\partial_\nu}{2M} (\bar{\psi} \sigma^{\mu\nu} \psi) V_\mu . \quad (6.5)$$

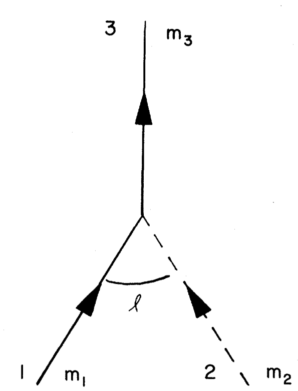

FIG. 2. Three-point vertex.

The first term is analogous to (6.2) and corresponds to the spin-nonflip amplitude. The second term corresponds to the spin-flip amplitude. The factor $2M$ is due to the fact that spin-$\frac{1}{2}$ states are usually normalized according to

$$\langle [M\tfrac{1}{2}]p'\lambda' | [M\tfrac{1}{2}]p\lambda\rangle = \delta_{\lambda'\lambda}\frac{\omega}{M}\,\delta^3(p'-p) \qquad (6.6)$$

instead of (A3). A factor $1/(2M)^{1/2}$ is associated with each spinor field. It is important to note that both terms of (6.5) correspond to coupling with particles 1 and 2 in a state of relative orbital angular momentum $l=1$ and therefore should have dimensionless coupling constants. The usual practice is to associate a coupling constant with the dimension of a length with an interaction having the structure of the second term of (6.5). This procedure is misleading. For example, the coupling constant for the electromagnetic spin-flip amplitude is customarily stated in terms of the magnetic moment $e/2M$. However, this constant varies widely between the leptons and the baryons. It is evident that e and not $e/2M$ is the appropriate universal coupling constant. Moreover, the minimal electromagnetic coupling becomes much more comprehensible if it is expressed by saying that the spin-nonflip and the spin-flip amplitudes both contribute and that the corresponding dimensionless coupling constants are both equal to e.

If particles 1 and 3 have $J^P=0^\pm$ or $J^P=\frac{1}{2}^+$ and particle 2 has $J^P=2^+$, then particles 1 and 2 must be in a state of relative orbital angular momentum $l=2$. Thus the coupling constant must have the dimension of a length. This result agrees with our expectation in the case of gravitation. We note, however, that in the cases of electromagnetism and gravitation the zero-mass limits are nontrivial.

Next, consider the case in which particles 1 and 3 have $J^P=\frac{1}{2}^+$ and particle 2 has $J^P=0^-$. The well-known example is pion-nucleon coupling. The usual spinor interaction is

$$\mathcal{L}_{\text{int}} = gi\bar{\psi}\gamma_5\psi\varphi\,, \qquad (6.7)$$

where g is dimensionless. Conservation of parity requires that the relative orbital angular momentum of particles 1 and 2 should be $l=1$ so that the new scheme also requires a dimensionless coupling constant.

Finally, we examine the case in which particles 1 and 3 have $J^P=\frac{1}{2}^+$ while particle 2 has $J^P=0^+$. Then parity requires that $l=0$. Thus the coupling constant should have the dimension of an inverse length. However, the usual spinor interaction is

$$\mathcal{L}_{\text{int}} = g\bar{\psi}\psi\varphi\,, \qquad (6.8)$$

where g is a dimensionless constant. Here again, it would seem that the abnormal normalization of spin-$\frac{1}{2}$ states (6.6) is misleading. The fact that $l=0$ and there is no spin-flip contribution suggests a comparison with φ^3 theory which has a coupling constant with the dimension of an inverse length.

For a given three-point vertex, the kinematic factor $|Q|^l$ is not in general the same as that given by the corresponding spinor coupling even in those cases in which the two approaches give the same dimension for the coupling constant. It follows that the meaning of universality for the coupling constants is different in the two approaches. There are only two theories for which firm experimental evidence concerning universality exists, namely, electrodynamics and gravitation. Unfortunately, for these theories the three-point process is not physical and one must employ a limiting procedure. In the limit $|Q|\to 0$ and in the limiting principal frame the process corresponds to the absorption (or emission) of a zero-momentum photon or graviton from the massive particle which is at rest. This process is reminiscent of the static Coulomb or Newtonian potential. Since the boost which takes the zero-momentum photon or graviton into a real photon or graviton must be singular, a reasonable limit probably exists. Moreover, since one of the masses is zero and the other two are equal, it is not unreasonable to expect that the kinematic factors will come out correctly as well.

The phenomenology of the less well-known theories will be affected by the proposed change in the kinematic factor. Conclusions concerning the structure of a theory based on polarization and angular correlation experiments would not be affected much. However, conclusions based on observed decay rates and cross sections depend on both the universality assumption and the form of the kinematic factors in an essential way and consequently may be expected to change. For example, in the theory of weak interactions the conclusion that the pseudoscalar coupling is excluded[17] is based on both the observed decay ratio

$$\frac{\Gamma(\pi\to e+\nu)}{\Gamma(\pi\to\mu+\nu)} \qquad (6.9)$$

and the presumed universality of the four-fermion interaction structure. In the theory of strong interactions, the definition of coupling constants would be different, so that the discussion of such topics as SU(3) symmetry breaking would be affected.

Finally, we note that the relationship of this hypothesis for the introduction of coupling constants with crossing symmetry needs to be clarified. In connection with this problem we offer the following observations. For the coupling of a massive neutral particle with $J^P=1^-$ to a charged particle with $J^P=\frac{1}{2}^+$, one finds that in the "exchange" channel there are two partial-wave amplitudes which contribute, a spin-nonflip and a spin-flip amplitude. Both of these amplitudes are associated with orbital angular momentum $l=1$ and are dimensionless. Symbolically,

$$\mathcal{A}^{(1)}=|Q|\,(g^{(1)}_{\mathrm{SNF}}+g^{(1)}_{\mathrm{SF}})\,. \tag{6.10}$$

The "minimal" coupling hypothesis corresponds to the assumption that

$$g^{(1)}_{\mathrm{SNF}}=g^{(1)}_{\mathrm{SF}}=e\,. \tag{6.11}$$

If the neutral particle has $J^P=2^+$, then both the partial-wave amplitudes are associated with orbital angular momentum $l=2$ and have the dimension of a length. In this case

$$\mathcal{A}^{(2)}=|Q|^2(g^{(2)}_{\mathrm{SNF}}+g^{(2)}_{\mathrm{SF}}) \tag{6.12}$$

and the "minimal" coupling hypothesis corresponds to the assumption

$$g^{(2)}_{\mathrm{SNF}}=g^{(2)}_{\mathrm{SF}}=k\,. \tag{6.13}$$

If the partial-wave analysis is performed in the annihilation channel, one finds that the coupling occurs in the spin-triplet state and that for $\mathcal{A}^{(1)}$ either $l=0$ (s wave) or $l=2$ (d wave) is possible while for $\mathcal{A}^{(2)}$ either $l=1$ (p wave) or $l=3$ (f wave) is possible. Symbolically,

$$\mathcal{A}^{(1)}=g^{(1)}_s+|Q'|^2g^{(1)}_d \tag{6.14}$$

and

$$\mathcal{A}^{(2)}=|Q'|g^{(2)}_p+|Q'|^3g^{(2)}_f\,. \tag{6.15}$$

In order to introduce coupling constants analogous to e and k one may impose the conditions

$$g^{(1)}_s g^{(1)}_d=e^2\,,\qquad g^{(2)}_p g^{(2)}_f=k^2 \tag{6.16}$$

and

$$|Q'|=[g^{(1)}_s/g^{(1)}_d]^{1/2}\,,\qquad |Q'|=[g^{(2)}_p/g^{(2)}_f]^{1/2}\,. \tag{6.17}$$

Then

$$\mathcal{A}^{(1)}=|Q'|\,(e+e)\,,\qquad \mathcal{A}^{(2)}=|Q'|^2(k+k)\,. \tag{6.18}$$

In the above discussion, many details have been omitted. Still, one may hope that further analysis would lead to a deeper understanding of the meaning of universal coupling constants and their relationship to the crossing hypothesis.

VII. CONCLUSIONS AND COMMENTS

It is customary to assign a fundamental role to the use of the finite-dimensional, nonunitary, irreducible representations of the homogeneous Lorentz group in the description of particle interactions involving particles with spin. The main reason for this practice is the success of quantum electrodynamics. However, this theory is a rather special case, and in the general case the nonunitary nature of the representations seems to seriously distort the role of spin in particle interactions.

The momentum-space description of an elementary system is well understood. Unfortunately, no satisfactory space-time description of an elementary system is known yet. For this reason, it was decided to work within the general structure of the S-matrix formalism. For a given process one must choose a set of invariant amplitudes which are considered dynamical. One must use a representation in which the particle momenta are diagonal so that energy-momentum conservation may be easily imposed. However, the usual reliance on a spinor spin basis was rejected. The decision to work with a unitary spin basis, such as one defined by pure boosts or helicity boosts, was motivated by a desire to obtain bounded spin sums. To define an invariant amplitude one has only to choose a fixed frame of reference. The principal reference frame was chosen because it is defined symmetrically with respect to all particles involved in the process. Since the principal reference frame can also be defined for complex values of the particle momenta, it was natural to define crossing so that the analytic continuation followed this frame. With this definition, it turns out that the principal helicity amplitudes were the dynamical invariant amplitudes. The natural domain for these functions is the analytic variety $\mathcal{V}_n(q)$ defined by the conditions (1.1).

Although this hypothesis is simple in its basic concept, it is in certain respects more difficult to work with than the usual one. The definition of the analytic variety $\mathcal{V}_n(q)$ involves nonlinear constraints even for $n=4, 5$. In the usual theory such nonlinear constraints appear only for $n\geq 6$. An analysis of the structure of $\mathcal{V}_n(q)$ similar to that given by Jacobson[18] for the usual manifold of scalars would be relatively straightforward. Obtaining an explicit analytic atlas might be somewhat less trivial. In order to obtain an integral representation for complex functions on $\mathcal{V}_n(q)$, one must employ the theory of complex exterior differential forms.[19] Even then there remains the task of finding explicit expressions for the spectral differential forms from the unitarity equations. In addition, there is the problem of proving that the analyticity assumptions are consistent with the unitarity equations. Since the Wigner rotations which appear are rather complicated, it would probably be necessary to employ more sophisticated mathematical techniques to understand their structure. The main reason for facing these hard problems is the expectation that the new formalism would yield renormalizable theories regardless of the spin of the particles involved. It is also interesting to note that the principal helicity amplitudes have simple properties with respect to both crossing and particle statistics. It does not seem possible to construct such amplitudes in the usual formalism.[20]

Since a fundamental role for spinor amplitudes was rejected and since the usual method for introducing universal coupling constants depends on the use of the spinor spin basis, it was necessary to develop a new approach to this question. In a basic three-point amplitude, two particles are coupled to form a third particle. If the process occurs with the two particles in a state of orbital angular momentum l, then it is natural to introduce a kinematic factor $|Q|^l$ where $|Q|$ is the magnitude of the relative momentum of the two particles. With this idea in mind, the procedure proposed for identifying universal coupling constants may be described as follows. First, select an appropriate channel of the three-point process. Then analyze the amplitude into partial-wave amplitudes with orbital angular momentum l and total spin s. Write the partial-wave amplitude for given l and s as a product of $|Q|^l$ and a constant g. Then g is taken to be the universal coupling constant. The dimension of g is L^{l-1}.

An examination of a number of common interactions indicates that this hypothesis is a plausible one. The fact that the kinematic factor is in general different from the usual one means that the meaning of universality of coupling constants is changed. If the hypothesis were accepted, the phenomenology of strong and weak interactions would have to be reexamined. Conclusive experimental evidence for universality is available only for the theories of electrodynamics and gravitation. Unfortunately, in these cases it is necessary to use a limiting procedure. Even so, it is not unreasonable to expect that the hypothesis will work in these cases since the limiting process has a Coulomb or Newtonian character. Finally, the relationship of this hypothesis to crossing symmetry needs to be clarified.

In this paper we have proposed new axioms for analytic S-matrix theory and new criteria for identifying universal coupling constants. Although we have been explicit enough to define the general formalism, we have not presented any detailed formulas for specific processes such as πN scattering and Compton scattering. We expect to do this in subsequent work. We anticipate no essential difficulty in obtaining the expressions for the single-pole contributions to four-point amplitudes which involve only massive particles. As we have pointed out above, processes involving zero-mass particles require special handling; nevertheless, the pole terms could also be worked out for these cases. Evaluating the contributions from two-particle intermediate states would be more difficult. The cut discontinuities can be evaluated readily enough. However, since $\mathcal{V}_4(q)$ is more complicated than $\mathcal{V}_4(S)$, the integral representation for functions defined on $\mathcal{V}_4(q)$ will be much more difficult to write down than the Mandelstam[21] representation.

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APPENDIX A: THE PHYSICAL STATES

A brief summary of those UIR of $\overline{P}$ which are used to describe particle states is given in this appendix. For additional information concerning the notational conventions used in this paper, the reader should consult Ref. 5.

The elements of the proper, orthochronous Poincaré covering group $\overline{P}_+^\dagger$ are faithfully represented by the matrices

$$(a,\overline{A}) \equiv \begin{pmatrix} \overline{A} & H(a)\overline{A}^{\dagger -1} \\ 0 & \overline{A}^{\dagger -1} \end{pmatrix} , \qquad \text{(A1)}$$

where $\overline{A} \in \mathrm{SL}(2, C)$, a^μ is a four-vector, and $H(a)$ is the associated 2×2 Hermitian matrix

$$H(a) = \begin{pmatrix} a^0 + a^3 & a^1 - ia^2 \\ a^1 + ia^2 & a^0 - a^3 \end{pmatrix} . \qquad \text{(A2)}$$

A denotes the Lorentz transformation corresponding to $\overline{A}$.

The eigenket corresponding to a particle of mass m, spin J, momentum p, and spin-projection λ is normalized according to

$$\langle [mJ]p'\lambda' | [mJ]p\lambda \rangle = \delta_{\lambda'\lambda} 2\omega_p \delta^3(\vec{p}\,' - \vec{p}) . \qquad \text{(A3)}$$

The UIR of $\overline{P}$ characterized by $[mJ]$ is given by

$$U(a,\bar{A})|[mJ]p,\lambda\rangle = e^{i(Ap)\cdot a}\sum_{\lambda'}|[mJ]Ap,\lambda'\rangle D^{(J)}_{\lambda'\lambda}(\bar{B}^{-1}(Ap)\bar{A}\,\bar{B}(p)), \tag{A4}$$

where $D^{(J)}$ denotes the well-known UIR of the group SU(2) and $\bar{B}(p)$ denotes one of the boosts defined below.

The matrix elements of the unitary operator $U(a,\bar{A})$ are the distributions

$$\begin{aligned}\mathfrak{D}^{(m,J)}_{p\lambda;q\mu}(a,\bar{A}) &= \langle[mJ]p,\lambda|U(a,\bar{A})|[mJ]q,\mu\rangle\\ &= e^{ip\cdot a}2\omega_p\,\delta^3(p,A\,q)D^{(J)}_{\lambda\mu}(\bar{B}^{-1}(p)\bar{A}\bar{B}(q)),\end{aligned} \tag{A5}$$

where $\omega_p = (m^2+\vec{p}^{\,2})^{1/2}$.

The meaning of the spin projection eigenvalue λ is determined by the type of boost employed. For massive particles, the two most common choices are the pure boost and the helicity boost. The pure boost is given by

$$\bar{B}(p) = [2m(\omega_p+m)]^{-1/2}\begin{pmatrix}\omega_p+m+p^3 & p^1-ip^2\\ p^1+ip^2 & \omega_p+m-p^3\end{pmatrix}. \tag{A6}$$

The corresponding tensor form is

$$B(p)^{\mu}{}_{\nu} = \begin{bmatrix}\dfrac{\omega_p}{m} & \dfrac{p^1}{m} & \dfrac{p^2}{m} & \dfrac{p^3}{m}\\[2ex] \dfrac{p^1}{m} & 1+\dfrac{p^1p^1}{m(m+\omega_p)} & \dfrac{p^1p^2}{m(m+\omega_p)} & \dfrac{p^1p^3}{m(m+\omega_p)}\\[2ex] \dfrac{p^2}{m} & \dfrac{p^2p^1}{m(m+\omega_p)} & 1+\dfrac{p^2p^2}{m(m+\omega_p)} & \dfrac{p^2p^3}{m(m+\omega_p)}\\[2ex] \dfrac{p^3}{m} & \dfrac{p^3p^1}{m(m+\omega_p)} & \dfrac{p^3p^2}{m(m+\omega_p)} & 1+\dfrac{p^3p^3}{m(m+\omega_p)}\end{bmatrix}. \tag{A7}$$

The helicity boost is given by

$$\bar{B}_H(p) = (2m(m+\omega_p))^{-1/2}\begin{pmatrix}(m+\omega_p+|\vec{p}|)\cos(\tfrac{1}{2}\theta) & -(m+\omega_p-|\vec{p}|)\sin(\tfrac{1}{2}\theta)e^{-i\varphi}\\ (m+\omega_p+|\vec{p}|)\sin(\tfrac{1}{2}\theta)e^{i\varphi} & (m+\omega_p-|\vec{p}|)\cos(\tfrac{1}{2}\theta)\end{pmatrix} \tag{A8}$$

with the corresponding tensor form

$$B_H(p)^{\mu}{}_{\nu} = R_3(\varphi)^{\mu}{}_{\alpha}R_2(\theta)^{\alpha}{}_{\beta}R_3(-\varphi)^{\beta}{}_{\gamma}B_3(|\vec{p}|)^{\gamma}{}_{\nu}, \tag{A9}$$

where $B_3(|\vec{p}|)$ denotes a pure boost along the z axis and $R_2(\theta)$ and $R_3(\varphi)$ denote rotations about the y and z axes, respectively. The angles θ and φ are defined by

$$p^1 = |\vec{p}|\sin\theta\cos\varphi,\quad p^2 = |\vec{p}|\sin\theta\sin\varphi,\quad p^3 = |\vec{p}|\cos\theta. \tag{A10}$$

For a massless particle with helicity λ, the eigenkets are normalized according to

$$\langle p',\lambda|p,\lambda\rangle = 2\omega_p\,\delta^3(\vec{p}\,'-\vec{p}), \tag{A11}$$

where $\omega_p = |\vec{p}|$. The standard momentum vector is chosen to be $\hat{p} = (m,0,0,m)$ where m is an arbitrary positive number. The corresponding UIR of $\bar{P}$ is given by

$$U(a,\bar{A})|p,\lambda\rangle = \exp[i(Ap)\cdot a]|Ap,\lambda\rangle\exp[i\lambda\chi(\bar{B}^{-1}(Ap)\bar{A}\bar{B}(p))] \tag{A12}$$

and the matrix elements of the unitary operator $U(a,\bar{A})$ are the distributions

$$\mathfrak{D}^{(\lambda)}_{p;q}(a,\bar{A}) = \langle p,\lambda|U(a,\bar{A})|q,\lambda\rangle = e^{ip\cdot a}2\omega_p\delta^3(p,Aq)\exp[i\lambda\chi(\bar{B}^{-1}(p)\bar{A}\bar{B}(q))], \tag{A13}$$

where the helicity boost is given by

$$\bar{B}(p) = \begin{bmatrix}\left(\dfrac{\omega_p}{m}\right)^{1/2}\cos(\tfrac{1}{2}\theta) & -\left(\dfrac{m}{\omega_p}\right)^{1/2}\sin(\tfrac{1}{2}\theta)e^{-i\varphi}\\[2ex] \left(\dfrac{\omega_p}{m}\right)^{1/2}\sin(\tfrac{1}{2}\theta)e^{i\varphi} & \left(\dfrac{m}{\omega_p}\right)^{1/2}\cos(\tfrac{1}{2}\theta)\end{bmatrix}, \tag{A14}$$

with θ and φ defined as in (A10) with $\omega_p = |\vec{p}|$. The angle $\chi(\overline{B}^{-1}(p)\overline{A}\overline{B}(q))$ of the little-group element is defined by

$$\overline{B}^{-1}(p)\overline{A}\overline{B}(q) = \begin{pmatrix} e^{i\chi/2} & (\alpha + i\beta)e^{-i\chi/2} \\ 0 & e^{-i\chi/2} \end{pmatrix}. \tag{A15}$$

APPENDIX B: REDUCTION OF THE TWO-PARTICLE STATE AND THE PARTIAL-WAVE EXPANSION

In this appendix, the Clebsch-Gordan coefficients (for the special case $n=3$) defined in Sec. II are used to reduce the two-particle state and thereby obtain the partial-wave expansion for the four-point amplitude. This example illustrates the details of applying the Clebsch-Gordan coefficients in a rather interesting case. Moreover, the expansion obtained differs somewhat from that given by the usual l-s, multipole and helicity couplings. A discussion of these other approaches to the reduction of two-particle states may be found in the work of Moussa and Stora.[5] It is interesting to note that crossed-channel expansion in terms of the principal or discrete series of representations of the group O(1, 2) can be carried out in much the same way. The Clebsch-Gordan coefficient needed for the reduction of the two-particle state is

$$\begin{aligned}\mathfrak{C}([mJ]p\lambda:\{qq_1q_2\mu\mu_1\mu_2\}|[m_1J_1]p_1\lambda_1;[m_2J_2]p_2\lambda_2) \\ = N\int d^6\overline{A}\,[\delta^3(\hat{p}, B^{-1}(q)Ap)D^{(J)*}_{\mu\lambda}(\overline{B}^{-1}(q)\overline{A}\overline{B}(p))\delta^3(\hat{p}_1, B^{-1}(q_1)Ap_1)D^{(J_1)}_{\mu_1\lambda_1}(\overline{B}^{-1}(q_1)\overline{A}\overline{B}(p_1)) \\ \times\delta^3(\hat{p}_2, B^{-1}(q_2)Ap_2)D^{(J_2)}_{\mu_2\lambda_2}(\overline{B}^{-1}(q_2)\overline{A}\overline{B}(p_2))]\,.\end{aligned} \tag{B1}$$

The measure on the space of continuous degeneracy labels is

$$\begin{aligned}d\mu(q,q_1,q_2) = d^4q\,\delta(q\cdot q - m^2)\Theta(q)\delta(q^0 - q_1{}^0 - q_2{}^0)d^4q_1\delta(q_1\cdot q_1 - m_1{}^2)\Theta(q_1)\delta(q^3 - q_1{}^3 - q_2{}^3) \\ \times d^4q_2\delta(q_2\cdot q_2 - m_2{}^2)\Theta(q_2)\delta(q^0q^3 + q_1^0q_1^3 + q_2^0q_2^3)\delta(q^1)\delta(q_1^1)\delta(q_2^1)\delta(q^2)\delta(q_1^2)\delta(q_2^2)\,.\end{aligned} \tag{B2}$$

In this case, the space of continuous degeneracy labels reduces to a single point and the $\{q, q_1, q_2\}$ are determined by $\{m, m_1, m_2\}$. One may write

$$\overline{A} = \overline{R}_z\overline{R}_y\overline{R}_x\overline{B}_z\overline{B}_y\overline{B}_x\,. \tag{B3}$$

Since all the momenta q, q_1, q_2 are along the z axis, $\overline{R}_z$ is undetermined by the δ functions in the Clebsch-Gordan coefficient. Integration over this parameter yields the Kronecker $\delta_{\mu,\mu_1+\mu_2}$. The other five parameters of $\overline{A}$ are determined by the δ functions leaving four δ functions which express energy-momentum conservation. The two-particle state decomposes according to

$$\begin{aligned}|[m_1J_1]p_1\lambda_1;[m_2J_2]p_2\lambda_2\rangle \\ = \sum_{\mu\mu_1\mu_2}\int d\mu\,(qq_1q_2)\sum_J\int dm^2\sum_\lambda\int\frac{d^3p}{2\omega_p}|[mJ]p\lambda:\{\mu\mu_1\mu_2qq_1q_2\}\rangle \\ \times\mathfrak{C}([mJ]p\lambda:\{\mu\mu_1\mu_2qq_1q_2\}|[m_1J_1]p_1\lambda_1;[m_2J_2]p_2\lambda_2)\,.\end{aligned} \tag{B4}$$

Only one point contributes to $\int d\mu(qq_1q_2)$. The $\sum_\mu$ can be carried out using $\delta_{\mu,\mu_1+\mu_2}$ and $\int dm^2\,d^3p/2\omega_p$ can be carried out using the energy-momentum δ functions. Then one obtains

$$\begin{aligned}|[m_1J_1]p_1\lambda_1;[m_2J_2]p_2\lambda_2\rangle \\ = \sum_{J,\lambda,\mu_1\mu_2}|[mJ]p\lambda:\mu_1\mu_2\rangle D^{(J)*}_{\mu_1+\mu_2,\lambda}(\overline{B}^{-1}(q)\overline{A}\overline{B}(p))D^{(J_1)}_{\mu_1\lambda_1}(\overline{B}^{-1}(q_1)\overline{A}\overline{B}(p_1))D^{(J_2)}_{\mu_2\lambda_2}(\overline{B}^{-1}(q_2)\overline{A}\overline{B}(p_2)),\end{aligned} \tag{B5}$$

where q, q_1, q_2, and $\overline{A}$ have the values required by the δ functions. Using the unitarity of the $D^{(J)}$ function one has

$$\begin{aligned}|[m_1J_1]p_1\lambda_1;[m_2J_2]p_2\lambda_2\rangle \\ = \sum_{J\lambda\mu_1\mu_2}|[mJ]p\lambda:\mu_1\mu_2\rangle D^{(J)-1}_{\lambda,\mu_1+\mu_2}(\overline{B}^{-1}(q)\overline{A}\overline{B}(p))D^{(J_1)}_{\mu_1\lambda_1}(\overline{B}^{-1}(q_1)\overline{A}\overline{B}(p_1))D^{(J_2)}_{\mu_2\lambda_2}(\overline{B}^{-1}(q_2)\overline{A}\overline{B}(p_2))\,.\end{aligned} \tag{B6}$$

The expression for the inverse relation is

$$|[mJ]p\lambda:\{qq_1q_2\mu\mu_1\mu_2\}\rangle$$
$$=\sum_{\lambda_1}\int\frac{d^3p_1}{2\omega_{p_1}}\sum_{\lambda_2}\int\frac{d^3p_2}{2\omega_{p_2}}|[m_1J_1]p_1\lambda_1;[m_2J_2]p_2\lambda_2\rangle\mathfrak{C}^*([mJ]p\lambda:\{qq_1q_2\mu\mu_1\mu_2\}|[m_1J_1]p_1\lambda_1;[m_2J_2]p_2\lambda_2)\,. \tag{B7}$$

Introduce the new variables

$$P=p_1+p_2\,,\quad Q=\frac{1}{2}\left[(p_1-p_2)-\frac{(p_1-p_2)\cdot P}{P\cdot P}P\right]\,,\quad Q\cdot Q=-\frac{\lambda(m_1^2,m_2^2,P\cdot P)}{4P\cdot P}\,, \tag{B8}$$

where

$$\lambda(z_1,z_2,z_3)=z_1^2+z_2^2+z_3^2-2(z_1z_2+z_2z_3+z_3z_1)\,. \tag{B9}$$

For fixed P, Q depends on only two angles which may be chosen to be the polar angles of Q in the rest frame of P. Corresponding to this change of variables, introduce the notation

$$M^2=P\cdot P \tag{B10}$$

and

$$\frac{d^3p_1}{2\omega_{p_1}}\frac{d^3p_2}{2\omega_{p_2}}=dM^2\frac{d^3P}{2\omega_P}d^2Q \tag{B11}$$

and define

$$|M,P,Q,\lambda_1,\lambda_2,[m_1J_1],[m_2J_2]\rangle=|[m_1J_1]p_1\lambda_1;[m_2J_2]p_2\lambda_2\rangle\,. \tag{B12}$$

Then the energy-momentum δ function gives $P=p=p_1+p_2$, $m=M$ and

$$|[mJ]p\lambda:\mu_1\mu_2\rangle=\sum_{\lambda_1\lambda_2}\int d^2Q|m,p,Q,\lambda_1,\lambda_2,[m_1J_1],[m_2J_2]\rangle$$
$$\times D^{(J)}_{\mu_1+\mu_2,\lambda}(\overline{B}^{-1}(q)\overline{A}\overline{B}(p))D^{(J_1)-1}_{\lambda_1,\mu_1}(\overline{B}^{-1}(q_1)\overline{A}\overline{B}(p_1))D^{(J_2)-1}_{\lambda_2,\mu_2}(\overline{B}^{-1}(q_2)\overline{A}\overline{B}(p_2))\,, \tag{B13}$$

where $\overline{A}$, p_1, and p_2 are to be expressed in terms of p and Q. The reader is warned that no attempt has been made to obtain the correct relative normalization which may be absorbed into the measure d^2Q.

The partial-wave expansion for the four-point function is obtained by combining the result derived above with the Wigner-Eckart theorem. The result is

$$\langle[m_4J_4]p_4\lambda_4;[m_3J_3]p_3\lambda_3|T|[m_2J_2]p_2\lambda_2;[m_1J_1]p_1\lambda_1\rangle$$
$$=\delta^4(p_4+p_3-p_2-p_1)\sum_{J,\mu_4\mu_3\mu_2\mu_1}T^J_{\mu_4\mu_3\mu_2\mu_1}(\omega)D^{(J)}_{\mu_4+\mu_3,\mu_2+\mu_1}(\overline{B}^{-1}(q_{43})\overline{A}_{43}\overline{A}_{21}{}^{-1}\overline{B}(q_{21}))$$
$$\times D^{(J_4)*}_{\mu_4\lambda_4}(\overline{B}^{-1}(q_4)\overline{A}_{43}\overline{B}(p_4))D^{(J_3)*}_{\mu_3\lambda_3}(\overline{B}^{-1}(q_3)\overline{A}_{43}\overline{B}(p_3))$$
$$\times D^{(J_2)}_{\mu_2\lambda_2}(\overline{B}^{-1}(q_2)\overline{A}_{21}\overline{B}(p_2))D^{(J_1)}_{\mu_1\lambda_1}(\overline{B}^{-1}(q_1)\overline{A}_{21}\overline{B}(p_1))\,. \tag{B14}$$

As discussed in Sec. III, the invariant amplitudes are just the T-matrix elements in the frame in which $M^{\mu\nu}(p)$ is diagonal; that is, *in this frame*

$$\langle[m_4J_4]p_4\lambda_4;[m_3J_3]p_3\lambda_3|T|[m_2J_2]p_2\lambda_2;[m_1J_1]p_1\lambda_1\rangle=\delta^4(p_4+p_3-p_2-p_1)\mathfrak{A}_{\lambda_4\lambda_3\lambda_2\lambda_1}(p_i)\,, \tag{B15}$$

whence

$$\mathfrak{A}_{\lambda_4\lambda_3\lambda_2\lambda_1}(p_i)=\sum_{J,\mu_4\mu_3\mu_2\mu_1}T^{(J)}_{\mu_4\mu_3\mu_2\mu_1}(\omega)D^{(J)}_{\mu_4+\mu_3,\mu_2+\mu_1}(\overline{B}^{-1}(q_{43})\overline{A}_{43}\overline{A}_{21}{}^{-1}\overline{B}(q_{21}))$$
$$\times D^{(J_4)*}_{\mu_4\lambda_4}(\overline{B}^{-1}(q_4)\overline{A}_{43}\overline{B}(p_4))D^{(J_3)*}_{\mu_3\lambda_3}(\overline{B}^{-1}(q_3)\overline{A}_{43}\overline{B}(p_3))$$
$$\times D^{(J_2)}_{\mu_2\lambda_2}(\overline{B}^{-1}(q_2)\overline{A}_{21}\overline{B}(p_2))D^{(J_1)}_{\mu_1\lambda_1}(\overline{B}^{-1}(q_1)\overline{A}_{21}\overline{B}(p_1))\,. \tag{B16}$$

In these formulas, the invariant amplitude is defined on the variety $\mathcal{V}_4(p)$ as defined in the Introduction and the argument of the partial-wave amplitude is taken to be the total energy $\omega = \omega_1 + \omega_2 = \omega_3 + \omega_4$ in the special reference frame. The analyticity properties of the partial-wave amplitudes are to be derived from the analyticity properties of the invariant amplitudes which in turn are to be determined by the requirements of extended unitarity.

Equations (B14) and (B16) have been written for the case of pure boosts. If helicity boosts are used a few sign changes must be made because of the different meaning of the spin projection labels.

APPENDIX C: KINEMATICS

A special dynamical role has been assigned to the helicity amplitudes evaluated in the principal reference frame in which $M^{\mu\nu}(q)$ is diagonal. The purpose of this appendix is to discuss the transformation from a general reference frame to this principal frame. We first make some observations concerning the case of the general n-point amplitude and then describe certain special cases which can be worked out explicitly.

Let p_i^μ and q_i^μ $(i=1, \ldots, n)$ denote the four-momenta of the particles in an n-point amplitude relative to the general and principal reference frames, respectively. The matrix $M^{\mu\nu}(p)$ is defined by

$$M^{\mu\nu}(p) = \sum_{i=1}^{n} p_i^\mu p_i^\nu \tag{C1}$$

and the q_i^μ are defined by

$$q_i^\mu = A^\mu{}_\nu p_i^\nu \quad (i=1, \ldots, n), \tag{C2}$$

where $A^\mu{}_\nu$ is the Lorentz transformation which diagonalizes $M^{\mu\nu}(p)$. The additional conditions required if rank $(M) < 4$ are given in Eq. (2.4). Define the scalar quantities

$$\begin{aligned} M_1 &= \mathrm{Tr}(M), \\ M_2 &= \mathrm{Tr}(MM), \\ M_3 &= \mathrm{Tr}(MMM), \\ M_4 &= \mathrm{Tr}(MMMM). \end{aligned} \tag{C3}$$

Then the characteristic equation for $M^{\mu\nu}(p)$ is

$$\xi^4 - C_1\xi^3 + C_2\xi^2 - C_3\xi + C_4 = 0, \tag{C4}$$

where

$$\begin{aligned} C_1 &= M_1, \\ C_2 &= \tfrac{1}{2}(M_1^2 - M_2), \\ C_3 &= \tfrac{1}{6}(M_1^3 - 3M_1M_2 + 2M_3), \\ C_4 &= \tfrac{1}{24}(M_1^4 - 6M_1^2M_2 + 8M_1M_3 + 3M_2^2 - 6M_4). \end{aligned} \tag{C5}$$

Note that there is an intimate connection between the matrix $M^{\mu\nu}(p)$ and the matrix S_{ij} of scalar products

$$S_{ij} = p_i \cdot p_j. \tag{C6}$$

One can readily show that

$$\begin{aligned} M_1 &= \mathrm{Tr}(S), \\ M_2 &= \mathrm{Tr}(SS), \\ M_3 &= \mathrm{Tr}(SSS), \\ M_4 &= \mathrm{Tr}(SSSS). \end{aligned} \tag{C7}$$

Consequently, the nontrivial part of the characteristic equation for S_{ij} is given by Eq. (C4).

Denote by $\xi_{(\alpha)}$ $(\alpha = 0, 1, 2, 3)$ the solutions to Eq. (C4) with $\xi_{(0)} > 0$ and $\xi_{(k)} < 0$ $(k = 1, 2, 3)$ in the physical regions. Then

$$M^{\mu\nu}(q) = \mathrm{Diag}(\xi_{(0)}, -\xi_{(1)}, -\xi_{(2)}, -\xi_{(3)}) \tag{C8}$$

and

$$\begin{aligned} \xi_{(0)} &= \sum_{i=1}^{n} q_i^0 q_i^0, \\ \xi_{(k)} &= -\sum_{i=1}^{n} q_i^k q_i^k \quad (k = 1, 2, 3). \end{aligned} \tag{C9}$$

Once the solutions $\xi_{(\alpha)}$ $(\alpha = 0, 1, 2, 3)$ of the quartic equation (C4) are known, the Lorentz transformation $A^\mu{}_\nu$ which diagonalizes $M^{\mu\nu}(p)$ and the momenta q_i^μ $(i=1, \ldots, n)$ can be computed by well-known standard techniques.

In certain special cases, the characteristic equation (C4) reduces to a quadratic. For an arbitrary 3-point function, rank $(M) = 2$. If the particle masses are denoted by m_i $(i = 1, 2, 3)$ then the characteristic equation is

$$\xi^2 - (m_1^2 + m_2^2 + m_3^2)\xi - \tfrac{3}{4}\lambda(m_1^2, m_2^2, m_3^2) = 0, \tag{C10}$$

where

$$\lambda(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3^2 - 2(z_1z_2 + z_2z_3 + z_3z_1). \tag{C11}$$

The eigenvalues are

$$\begin{aligned} \xi_{(0)} &= \tfrac{1}{2}\{(m_1^2 + m_2^2 + m_3^2) \\ &\quad + [(m_1^2 + m_2^2 + m_3^2)^2 + 3\lambda(m_1^2, m_2^2, m_3^2)]^{1/2}\}, \\ \xi_{(3)} &= \tfrac{1}{2}\{(m_1^2 + m_2^2 + m_3^2) \\ &\quad - [(m_1^2 + m_2^2 + m_3^2)^2 + 3\lambda(m_1^2, m_2^2, m_3^2)]^{1/2}\}, \\ \xi_{(1)} &= \xi_{(2)} = 0. \end{aligned} \tag{C12}$$

The momenta q_i^μ $(i = 1, 2, 3)$ are given by

$$q_i^0=\frac{m_i}{\sqrt{2}}\left[\frac{(m_1^2+m_2^2+m_3^2)+\lambda(m_1^2,m_2^2,m_3^2)/m_i^2}{[(m_1^2+m_2^2+m_3^2)^2+3\lambda(m_1^2,m_2^2,m_3^2)]^{1/2}}+1\right]^{1/2},$$

$$|q_i^3|=\frac{m_i}{\sqrt{2}}\left[\frac{(m_1^2+m_2^2+m_3^2)+\lambda(m_1^2,m_2^2,m_3^2)/m_i^2}{[(m_1^2+m_2^2+m_3^2)^2+3\lambda(m_1^2,m_2^2,m_3^2)]^{1/2}}-1\right]^{1/2}, \qquad (C13)$$

$$q_i^1=q_i^2,$$

where the signs of the space components for the case in which $m_3>m_1+m_2$, $m_2>m_1$ are $q_1^3>0$, $q_2^3<0$, and $q_3^3>0$. For the special case $m_3=M$ and $m_1=m_2=m$,

$$q_1^0=q_2^0=\tfrac{1}{2}M, \quad q_3^0=M,$$
$$q_1^3=-q_2^3=[(\tfrac{1}{2}M)^2-m^2]^{1/2}, \quad q_3^3=0. \qquad (C14)$$

For the general four-point function the eigenvalue equation is

$$\xi^3-C_1\xi^2+C_2\xi-C_3=0, \qquad (C15)$$

where

$$C_1=s+t+u=m_1^2+m_2^2+m_3^2+m_4^2,$$
$$C_2=(st+tu+us)-\tfrac{3}{4}\lambda(m_1^2,m_2^2,m_3^2,m_4^2), \qquad (C16)$$
$$C_3=stu-(as+bt+cu)$$

in which

$$\lambda(z_1,z_2,z_3,z_4)=z_1^2+z_2^2+z_3^2+z_4^2-2(z_1z_2+z_1z_3+z_1z_4+z_2z_3+z_2z_4+z_3z_4) \qquad (C17)$$

and

$$C_1a=(m_1^2m_2^2-m_3^2m_4^2)(m_1^2+m_2^2-m_3^2-m_4^2),$$
$$C_1b=(m_1^2m_3^2-m_2^2m_4^2)(m_1^2+m_3^2-m_2^2-m_4^2), \qquad (C18)$$
$$C_1c=(m_1^2m_4^2-m_2^2m_3^2)(m_1^2+m_4^2-m_2^2-m_3^2)$$

and

$$s=(p_1+p_2)^2=(p_3+p_4)^2,$$
$$t=(p_1+p_3)^2=(p_2+p_4)^2, \qquad (C19)$$
$$u=(p_1+p_4)^2=(p_2+p_3)^2,$$

where the "all-in" notation has been used for the momenta p_i^μ $(i=1,2,3,4)$. The equation for the boundary of the physical region is given by the condition rank $(M)=2$; that is, $C_3=0$.

For certain special mass ratios, the cubic equation (C15) factors. The simplest case is equal-mass scattering. The case in which the masses are equal in pairs can also be worked out explicitly. These cases include a number of interesting processes such as nucleon-nucleon scattering and pion-nucleon scattering.

For equal-mass scattering which we denote by $(aa\to aa)$, the principal reference frame is just the symmetrically oriented center-of-mass frame shown in Fig. 3. Thus in this case, the center-of-mass helicity amplitudes are crossing-symmetric. This result is in sharp contrast with that obtained from spinor field theory.

If the masses are equal in pairs, there are two

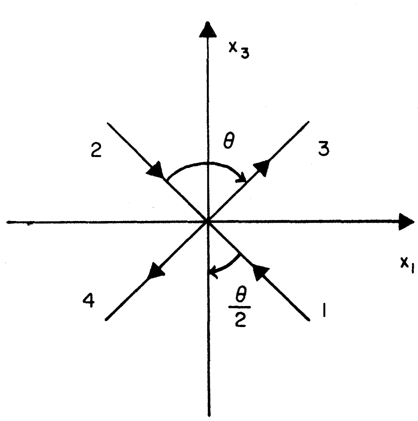

FIG. 3. The principal frame $aa\to aa$.

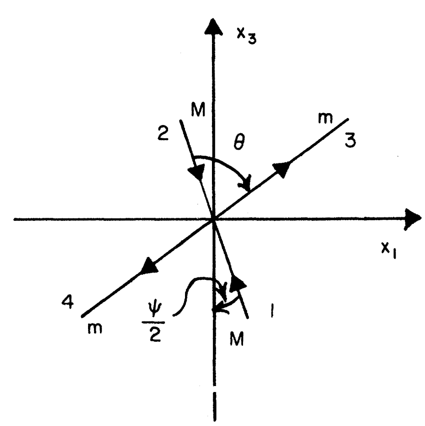

FIG. 4. The principal frame $aa\to bb$.

cases to consider which we denote by $(aa \to bb)$ and $(ab \to ab)$. In the first case, the principal reference frame is again a center-of-mass frame but is not so symmetrically oriented as in the case of equal-mass scattering. This frame is illustrated in Fig. 4. The angle is determined by the condition $M^{12}(q)=0$. In the second case $(ab \to ab)$, the principal frame differs from the symmetrically oriented center-of-mass frame by a boost in the negative z direction. This frame is shown in Fig. 5.

We are now in a position to evaluate the Wigner rotations which occur in the partial-wave expansion (B16) for the kinematic cases discussed above. For the cases $(aa \to aa)$ and $(aa \to bb)$, only the argument of the intermediate $D^{(J)}$ function differs from the identity rotation if helicity boosts are used. This nontrivial Wigner rotation is just a rotation about the x_2 axis through an angle θ, the center-of-mass scattering angle. For the case $(ab \to ab)$, the situation is somewhat more complicated and the computation rapidly becomes laborious.

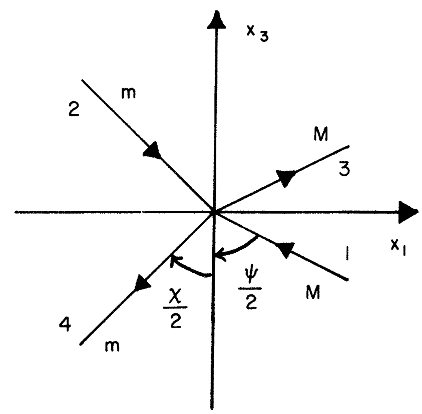

FIG. 5. The principal frame $ab \to ab$.

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