S-Wave Pion-Nucleon Scattering*†

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(Received 19 March 1973)

Integral equations for the off-mass-shell S-wave pion-nucleon scattering amplitudes are developed from the Low equation. The seagull terms, present in this formulation, are calculated using partially conserved axial-vector current and current algebra, with the assumption that the pion current is independent of second- and higher-order derivatives of the pion field. These terms provide the potential for the S-wave equations, as expected, but also modify the rescattering integral. With a cutoff assumption, we obtain numerical solutions valid for low energy. Agreement with experiment is surprisingly good.

I. INTRODUCTION

In the mid 1950's Chew and Low[1,2] developed a nonrelativistic formalism which successfully described the low-energy pion-nucleon P-wave interaction. Their integral equations, which neglected recoil, antinucleons, the π-π interaction, and inelastic effects, were based on the nonrelativistic reduction of the pseudoscalar coupling model, i.e.,

$$H_I \sim \frac{f_0}{\mu} \sum_{j=1}^{3} \int d^3x\, \rho(x) \left(\sigma_j \frac{\partial}{\partial x^j}\right) \vec{\tau} \cdot \vec{\phi}(x),$$

where $\rho(x)$ is the source density.

Drell, Friedman, and Zachariasen[3] attempted a description of S-wave scattering along similar lines by introducing an S-wave meson-nucleon interaction

$$H_I \sim \int d^3x\, d^3x'\, \rho(x)\rho(x') \times [\lambda_0^0 \vec{\phi}(x) \cdot \vec{\phi}(x') + \lambda^0 \vec{\tau} \cdot \vec{\phi}(x) \times \vec{\pi}(x')],$$

with $\vec{\pi}(x)$ the canonical momentum of the pion field. Although they were able to obtain a reasonable fit to the existing data by adjusting λ_0^0 and λ^0, it is desirable to obtain a more model-independent approach to the S-wave problem.

The more recent successes of current commutation relations and the hypothesis of partial conservation of the axial-vector current (PCAC) in the prediction of threshold pion-nucleon scattering results are well known.[4-10] The purpose of the present work is to demonstrate the role of the PCAC–current-algebra amplitude in the Chew-Low formalism and to predict the low-energy S-wave phase shifts.

In Sec. II, we develop a relationship between the seagull terms and the PCAC–current-algebra amplitude (we neglect Schwinger terms), based on the conjecture that the pion current is independent of second- and higher-order derivatives of the pion field. Completeness is used to develop integral equations for the scattering amplitude in the manner of Low.

In Sec. III, the low-energy S-wave scattering equations are examined in more detail. The seagull terms not only determine the S-wave potential, but also provide for an approximate subtraction. The resulting equations are solved numerically with a cutoff procedure and compared with experiment. Agreement with the experimental $I=\frac{3}{2}$ phase shift is quite good. The $I=\frac{1}{2}$ fit is not as impressive, but is not at significant variance with the data. A discussion of our results is contained in Sec. IV.

In Appendix A, we examine the validity of the assumption which leads to a subtracted form for our integral equations. Appendix B and Appendix C are used to evaluate various potential contributions.

II. THE SEAGULL TERMS AND THE SOFT-PION AMPLITUDE

We wish to consider low-energy, pion-nucleon scattering. Before beginning, it will be convenient to make the following definitions:

$$P=\tfrac{1}{2}(p_1+p_2), \quad Q=\tfrac{1}{2}(q_1+q_2),$$
$$\Delta = p_2 - p_1 = q_1 - q_2, \tag{1}$$
$$s=(P+Q)^2, \quad t=\Delta^2, \quad u=(P-Q)^2,$$

where p_1 and p_2 are the initial and final nucleon momenta, and q_1 and q_2 are the initial and final pion momenta. The Greek letters β and α shall be used to denote the initial and final pion isospin, respectively. For the most part, we will suppress the nucleon spin and isospin indices.

The S-matrix element[11] is

$$_{\text{out}}\langle p_2; q_2, \alpha | p_1; q_1, \beta\rangle_{\text{in}} = \langle q_2, \alpha | q_1, \beta\rangle \langle p_2 | p_1\rangle + (2\pi)^4 i\delta^{(4)}(p_2+q_2-p_1-q_1)\ {}_{\text{out}}\langle p_2; q_2, \alpha | j_\beta(0) | p_1\rangle ,$$

where $j_\beta(y) = (\Box + m_\pi^{\,2})\phi_\beta(y)$ is the pion current. Applying the reduction formula to the outgoing pion and retaining all equal-time commutators, which arise from time derivatives of the retarded commutator, we obtain

$$_{\text{out}}\langle p_2; q_2, \alpha | j_\beta(0) | p_1\rangle = \langle p_2 | \gamma(q_2) | p_1\rangle + i\int d^4x\, e^{iq_2\cdot x} \langle p_2 | \theta(x_0)[j_\alpha(x), j_\beta(0)] | p_1\rangle , \tag{2}$$

where

$$\langle p_2 | \gamma(q_2) | p_1\rangle = i\int d^4x\, e^{iq_2\cdot x}\left(\delta(x_0)\langle p_2 | [\dot\phi_\alpha(x), j_\beta(0)] | p_1\rangle + \frac{\partial}{\partial x^0}\{\delta(x^0)\langle p_2 | [\phi_\alpha(x), j_\beta(0)] | p_1\rangle\}\right) \tag{3}$$

are the so-called "seagull" terms. In the following, we delete the specification (out) from the bra $\langle p_2; q_2, \alpha|$.

The equal-time commutators in Eq. (3) describe contributions from diagrams in which the two pions interact at the same space-time point. Thus, in some sense, they contain information about interactions which are quadratic in the pion field and thereby assume an important role in the S-wave problem. Two distinct categories of such contributions can be given: (1) pion-pion interaction effects, and (2) direct couplings to the nucleon field which are quadratic in the pion field.

There are ample reasons for believing that the role of the pion-pion interaction in low-energy pion-nucleon scattering is the t-channel exchange of a scalar and vector meson.[12] The simplest couplings (to the pion field) consistent with invariance principles are

$$L_I(x) \sim G_S\,\phi_\alpha(x)\phi_\alpha(x)\sigma(x) + iG_V\epsilon_{\alpha\beta\gamma}\rho_\alpha(x)^\mu[\phi_\beta(x)\overleftrightarrow{\partial}_\mu\phi_\gamma(x)] .$$

With this expression, we find

$$\frac{\delta j_\beta}{\delta\phi_\alpha} \sim \delta_{\alpha\beta}G_S\sigma, \quad \frac{\delta j_\beta}{\delta(\partial_\mu\phi_\alpha)} \sim i\epsilon_{\alpha\beta\gamma}G_V\rho^\mu_\gamma$$

(where $\delta j/\delta a$ denotes the functional derivative of j with respect to a), and more significantly, that $\delta j_\beta/\delta(\partial_\mu\partial_\nu\phi_\alpha)$ and functional derivatives with respect to higher-order derivatives of the pion field vanish. Phenomenological analysis does not indicate a need for interactions in the second category.[13]

The forms of $\delta j/\delta\phi$ and of $\delta j/\delta(\partial_\mu\phi)$ depend upon the particular Lagrangian we choose, but the vanishing of higher-order terms is nearly universally true for model Lagrangians used for the description of the pion interaction.[14] We shall thus assume that when describing the analytic structure of the scattering amplitude in the low-energy region, it is sufficient to consider $j_\pi = j_\pi(\phi, \partial_\mu\phi)$. It follows that

$$\begin{aligned}\langle p_2 | \gamma_{\alpha\beta}(q_2) | p_1\rangle &= \int d^4x\, e^{iq_2\cdot x}\left[\left\langle p_2\left|\frac{\delta j_\beta(0)}{\delta\phi_\alpha}\delta^{(4)}(x)\right|p_1\right\rangle - \left\langle p_2\left|\frac{\delta j_\beta(0)}{\delta(\partial_\mu\phi_\alpha)}\frac{\partial}{\partial x^\mu}\delta^{(4)}(x)\right|p_1\right\rangle\right]\\ &= \left\langle P+\tfrac{1}{2}\Delta\left|\frac{\delta j_\beta}{\delta\phi_\alpha}(0)\right|P-\tfrac{1}{2}\Delta\right\rangle + i(Q-\tfrac{1}{2}\Delta)_\mu\left\langle P+\tfrac{1}{2}\Delta\left|\frac{\delta j_\beta}{\delta(\partial_\mu\phi_\alpha)}(0)\right|P-\tfrac{1}{2}\Delta\right\rangle\end{aligned} \tag{4}$$

and we conclude that the "seagull" terms are smoothly extrapolating functions of Q, and in fact are no more than linear in Q. It should be noted that Eq. (4) applies only at low energy and that we are making no assumptions as to the high-energy behavior of our amplitude and the need for subtractions. This expression provides the essential clue we shall use for our treatment of the soft-pion amplitude.

We next define the off-mass-shell scattering amplitude

$$T_{\alpha\beta}(p_2, q_2; p_1, q_1) = \frac{2i}{f_\pi^{\,2}}(m_\pi^{\,2}-q_2^{\,2})(m_\pi^{\,2}-q_2^{\,2}) \times\int d^4x\, e^{iQ\cdot x}\langle p_2 | \theta(x_0)[\partial_\mu A^\mu_\alpha(\tfrac{1}{2}x), \partial_\nu A^\nu_\beta(-\tfrac{1}{2}x)] | p_1\rangle ,$$

in which we have used

$$\phi_\alpha(x) = \frac{\sqrt{2}}{f_\pi}\frac{\partial}{\partial x^\mu}A^\mu_\alpha(x)$$

as the definition of the interpolating pion field, and where f_π is the pion-decay constant. The use of the commutation relations

$$[A^0_\alpha(x), A^\nu_\beta(y)]\delta(x_0-y_0) = 2iG_V\epsilon_{\alpha\beta\gamma}V^\mu_\gamma(x) \times\delta^{(4)}(x-y),$$

$$[A^0_\alpha(x), \partial_\nu A^\nu_\beta(y)]\delta(x_0-y_0) = iG_S\sigma_{\alpha\beta}(x)\delta^{(4)}(x-y),$$

leads to the result

$$T_{\alpha\beta}(p_2, q_2; p_1, q_1) = \frac{2i}{f_\pi^2}(m_\pi^2 - q_1^2)(m_\pi^2 - q_2^2)\Big[-iG_S\langle p_2|\,\sigma_{\alpha\beta}(0)|p_1\rangle + 2G_V\epsilon_{\alpha\beta\gamma}q_{2\mu}\langle p_2|\,V_\gamma^\mu(0)|p_1\rangle + q_{1\nu}q_{2\mu}\int d^4x\, e^{iQ\cdot x}\langle p_2|\,\theta(x_0)[A_\alpha^\mu(\tfrac{1}{2}x), A_\beta^\nu(-\tfrac{1}{2}x)]|p_1\rangle\Big], \tag{5}$$

where $V_\gamma^\mu(x)$ is the isovector current density and $\sigma_{\alpha\beta}(x)$ a scalar operator.

Normally, one investigates the above expression in the limit of vanishing pion four-momenta to obtain a result for zero-mass pions and is then faced with the problem of an extrapolation in the mass variable. We propose, instead, to use the conjectured analyticity properties of the seagull terms to determine the role of the PCAC–current-algebra amplitude in physical scattering. Combining Eqs. (1), (2), (3), and (5), we obtain

$$\begin{aligned}\langle P+\tfrac{1}{2}\Delta|\,\gamma_{\alpha\beta}(Q-\tfrac{1}{2}\Delta)|P-\tfrac{1}{2}\Delta\rangle = \Big\{&[(m_\pi^2 - \tfrac{1}{4}\Delta^2 - Q^2)^2 - (Q\cdot\Delta)^2]\\ &\times\frac{2i}{f_\pi^2}\Big[-iG_S\langle P+\tfrac{1}{2}\Delta\,|\sigma_{\alpha\beta}(0)|\,P-\tfrac{1}{2}\Delta\rangle + 2G_V\epsilon_{\alpha\beta\gamma}Q_\mu\langle P+\tfrac{1}{2}\Delta\,|\,V_\gamma^\mu(0)|\,P-\tfrac{1}{2}\Delta\rangle\\ &+(Q+\tfrac{1}{2}\Delta)_\nu(Q-\tfrac{1}{2}\Delta)_\mu\int d^4x\,e^{iQ\cdot x}\langle P+\tfrac{1}{2}\Delta|\,\theta(x_0)[A_\alpha^\mu(\tfrac{1}{2}x), A_\nu^\beta(-\tfrac{1}{2}x)]\,|\,P-\tfrac{1}{2}\Delta\rangle\Big]\\ &-i\int d^4x\,e^{iQ\cdot x}\langle P+\tfrac{1}{2}\Delta\,|\,\theta(x_0)[j_\alpha(\tfrac{1}{2}x), j_\beta(-\tfrac{1}{2}x)]\,|\,P-\tfrac{1}{2}\Delta\rangle\Big\}\quad\text{(to order } Q).\end{aligned} \tag{6}$$

The curly brackets in this equation indicate that we are to expand the result in a power series in Q and retain only the first two terms in the expansion.

Inserting this expression into Eq. (2) and using completeness to do the x integrations, we obtain

$$\begin{aligned}\langle p_2; q_2, \alpha|\,j_\beta(0)|p_1\rangle = \Big\{&\frac{2}{f_\pi^2}[(m_\pi^2 - \tfrac{1}{4}\Delta^2 - Q^2)^2 - (Q\cdot\Delta)^2]\\ &\times\Big(+G_S\langle P+\tfrac{1}{2}\Delta\,|\sigma_{\alpha\beta}(0)|\,P-\tfrac{1}{2}\Delta\rangle + 2iG_V\epsilon_{\alpha\beta\gamma}Q_\mu\langle P+\tfrac{1}{2}\Delta\,|\,V_\gamma^\mu(0)|\,P-\tfrac{1}{2}\Delta\rangle\\ &-(2\pi)^3\sum_n\frac{\delta^{(3)}(\vec{P}+\vec{Q}-\vec{P}_n)}{P_0+Q_0-E_n+i\epsilon}\langle P+\tfrac{1}{2}\Delta\,|\,(Q+\tfrac{1}{2}\Delta)\cdot A_\alpha(0)|\,n\rangle\langle n|\,(Q-\tfrac{1}{2}\Delta)\cdot A_\beta(0)|\,P-\tfrac{1}{2}\Delta\rangle\Big)\\ &+(2\pi)^3\sum_n\frac{\delta^{(3)}(\vec{P}+\vec{Q}-\vec{P}_n)}{P_0+Q_0-E_n+i\epsilon}\langle P+\tfrac{1}{2}\Delta\,|\,j_\alpha(0)|n\rangle\langle n|\,j_\beta(0)|\,P-\tfrac{1}{2}\Delta\rangle\Big\}\quad\text{(to order } Q)\\ &-(2\pi)^3\sum_n\frac{\delta^{(3)}(\vec{P}+\vec{Q}-\vec{P}_n)}{P_0+Q_0-E_n+i\epsilon}\langle p_2|\,j_\alpha(0)|\,n\rangle\langle n|\,j_\beta(0)|\,p_1\rangle + \text{cross terms},\end{aligned} \tag{7}$$

where the sum is over a complete set of physical intermediate states.

We truncate the completeness sum, retaining only the single-nucleon and pion-nucleon intermediate-state contributions. As has been emphasized elsewhere,[1,3] there are no compelling reasons for believing that this is an especially good approximation. It is hoped that the larger energy denominator of the inelastic contributions make these terms insignificant at low energies.

The resulting expression constitutes the basis for our discussion of S-wave scattering.

III. THE S-WAVE EQUATIONS

Consider the matrix element $\langle p_2;q_2, \alpha|\,j_\beta(0)|p_1\rangle$ with p_1, p_2, and q_2 on the mass shell, but with all other restrictions removed. This object is a function of the three four-vectors, p_1, p_2, and q_2 only; and there is no reference to the "missing" pion momentum q_1. We are interested in low-energy phenomena and shall develop our theory from a nonrelativistic viewpoint. To that end, we make the change of variables

$$\vec{P}' = \vec{P} + \vec{Q},\qquad \vec{l}_1 = \frac{m\vec{q}_2 - m_\pi\vec{p}_1}{m+m_\pi},\qquad \vec{l}_2 = \frac{m\vec{q}_2 - m_\pi\vec{p}_2}{m+m_\pi} \tag{8}$$

where we have defined the "missing" pion three-momentum with $\vec{q}_1 \equiv \vec{p}_2 + \vec{q}_2 - \vec{p}_1$. With this definition, $\vec{l}_1$ and $\vec{l}_2$ can both be expressed as relative velocities and, hence, are Galilean invariants. Clearly then, $\langle p_2; q_2, \alpha|\,j_\beta(0)|p_1\rangle$ cannot depend upon $\vec{P}'$ if it is to be a Galilean-invariant object and we lose no generality by restricting ourselves to the $\vec{P}+\vec{Q}=0$ subspace of the $\vec{P},\vec{Q}$ plane. This justifies the change of notation

$$\langle p_2; q_2, \alpha | j_\beta(0) | p_1 \rangle = (\vec{l}_2; \tau_2 S_2, \alpha | j_\beta(0) | \vec{l}_1; \tau_1 S_1), \tag{9}$$

where S_2 and τ_2 (S_1 and τ_1) are the final (initial) spin and isospin quantum numbers of the nucleon.

We define the partial-wave decomposition of Eq. (9) by

$$(\vec{l}_2; \tau_2 S_2, \alpha | j_\beta(0) | \vec{l}_1; \tau_1 S_1) = \left(\hat{l}_2; \tau_2 S_2, \alpha \left| \sum_{l,\delta} h_l^\delta(l_2, l_1) Q_l^\delta \right| \hat{l}_1; \tau_1 S_1, \beta\right),$$

where the Q_l^δ are projection operators corresponding to a given combination of isospin, orbital, and total angular momentum. The states $|\hat{l}_2; \tau_1 S_1, \beta)$ are normalized according to

$$(\hat{l}_2; \tau_2 S_2, \alpha | \hat{l}_1; \tau_1 S_1, \beta) = \delta_{\tau_1, \tau_2} \delta_{S_1, S_2} \delta_{\beta, \alpha} \delta(\hat{\Omega}_{l_1} - \hat{\Omega}_{l_2}),$$

so that the phase shifts are given by

$$h_l(l_2, l_2) = (4\pi)^2 \frac{e^{2i\delta_l} - 1}{2i|\vec{l}_2|}.$$

Consider Eq. (7) in the one-meson approximation. We define the "potential"

$$\begin{aligned}
(\vec{l}_2; \tau_2 S_2 | W_{\alpha\beta} | \vec{l}_1; \tau_1 S_1)
= \Bigg\{ & \frac{2i}{f_\pi^2} (m_\pi^2 - \tfrac{1}{4}\Delta^2)^2 [-iG_S \langle P + \tfrac{1}{2}\Delta | \sigma_{\alpha\beta}(0) | P - \tfrac{1}{2}\Delta\rangle + 2G_V \epsilon_{\alpha\beta\gamma} Q_\mu \langle P + \tfrac{1}{2}\Delta | V_\gamma^\mu(0) | P - \tfrac{1}{2}\Delta\rangle] \\
& - (2\pi)^3 \sum_N \Bigg[\frac{\delta^{(3)}(\vec{P} + \vec{Q} - \vec{p}_N)}{P_0 + Q_0 - E_N + i\epsilon} \Bigg(-\frac{2}{f_\pi^2} (m_\pi^2 - \tfrac{1}{4}\Delta^2)^2 \langle P + \tfrac{1}{2}\Delta | (Q - \tfrac{1}{2}\Delta) \cdot A_\alpha(0) | N\rangle \langle N | (Q + \tfrac{1}{2}\Delta) \cdot A_\beta(0) | P - \tfrac{1}{2}\Delta\rangle \\
& \qquad + \langle P + \tfrac{1}{2}\Delta | j_\alpha(0) | N\rangle \langle N | j_\beta(0) | P - \tfrac{1}{2}\Delta\rangle \Bigg) + \text{cross term} \Bigg] \\
& - (2\pi)^3 \sum_{\pi, N} \Bigg[\frac{\delta^{(3)}(\vec{P} + \vec{Q} - \vec{p}_N - \vec{q}_\pi)}{P_0 + Q_0 - E_N - \omega_\pi + i\epsilon} \Bigg(-\frac{2}{f_\pi^2} (m_\pi^2 - \tfrac{1}{4}\Delta^2)^2 \langle P + \tfrac{1}{2}\Delta | (Q - \tfrac{1}{2}\Delta) \cdot A_\alpha(0) | \pi N\rangle \langle \pi N | (Q + \tfrac{1}{2}\Delta) \cdot A_\beta(0) | P - \tfrac{1}{2}\Delta\rangle \\
& \qquad + \langle P + \tfrac{1}{2}\Delta | j_\alpha(0) | \pi N\rangle_{P\,P} \langle \pi N | j_\beta(0) | P - \tfrac{1}{2}\Delta \Bigg) + \text{cross term} \Bigg] \Bigg\} \text{(to order } Q) \\
& - (2\pi)^3 \sum_N \Bigg[\frac{\delta^{(3)}(\vec{P} + \vec{Q} - \vec{p}_N)}{P_0 + Q_0 - E_N + i\epsilon} \langle p_2 | j_\alpha(0) | N\rangle \langle N | j_\beta(0) | p_1\rangle + \text{cross term} \Bigg],
\end{aligned} \tag{10}$$

where the subscript on $\langle P + \frac{1}{2}\Delta | j_\alpha(0) | \pi N\rangle_P$ indicates we have explicitly separated the P-wave contribution to the intermediate-state sum. The S-wave contribution will be treated in conjunction with the rescattering term. Higher partial waves are ignored. The partial-wave decomposition of this expression is given by

$$(\vec{l}_2; \tau_2 S_2 | W_{\alpha\beta} | \vec{l}_1; \tau_1 S_1) = (\hat{l}_2; \tau_2 S_2, \alpha | \sum_{l,\delta} W_l^\delta(l_2, l_1) Q_l^\delta | \hat{l}_1; \tau_1 S_1, \beta).$$

With this definition, the one-meson approximation to Eq. (9) becomes

$$\begin{aligned}
(\vec{l}_2; \tau_2 S_2, \alpha | j_\beta(0) | \vec{l}_1; \tau_1 S_1) = {} & (\vec{l}_2; \tau_2 S_2 | W_{\alpha\beta} | \vec{l}_1; \tau_1 S_1) \\
& + \Bigg[\int \frac{d^3 l_n}{(2\pi)^3} \frac{m}{E_n \omega_n} \sum_{\tau_n, S_n, \gamma} \frac{{}_S(\vec{l}_n; \tau_n S_n, \gamma | j_\alpha(0) | \vec{l}_2; \tau_2 S_2)^* \, {}_S(\vec{l}_n; \tau_n S_n, \gamma | j_\beta(0) | \vec{l}_1; \tau_1 S_1)}{P_0 + Q_0 - E_n - \omega_n + i\epsilon} \\
& + \int \frac{d^3 l_n}{(2\pi)^3} \frac{m}{E_{n'} \omega_{n'}} \sum_{\tau_n, S_n, \gamma} \frac{{}_S(\vec{l}_n; \tau_n S_n, \gamma | j_\alpha(0) | \vec{\gamma}_2; \tau_2 S_2)^* (\vec{l}_n; \tau_n S_n, \gamma | j_\alpha(0) | \vec{\gamma}_1; \tau_1 S_1)}{P_0 - Q_0 - E_{n'} - \omega_{n'} - i\epsilon} \Bigg] \\
& \text{(to order } Q) \\
& - \int \frac{d^3 l_n}{(2\pi)^3} \frac{m}{E_n \omega_n} \sum_{\tau_n, S_n, \gamma} \frac{(\vec{l}_n; \tau_n S_n, \gamma | j_\alpha(0) | \vec{l}_2; \tau_2 S_2)^* (\vec{l}_n; \tau_n S_n, \gamma | j_\beta(0) | \vec{l}_1; \tau_1 S_1)}{P_0 + Q_0 - E_n - \omega_n + i\epsilon} \\
& - \int \frac{d^3 l_n}{(2\pi)^3} \frac{m}{E_{n'} \omega_{n'}} \sum_{\tau_n, S_n, \gamma} \frac{(\vec{l}_n; \tau_n S_n, \gamma | j_\beta(0) | \vec{\gamma}_2; \tau_2 S_2)^* (\vec{l}_n; \tau_n S_n, \gamma | j_\alpha(0) | \vec{\gamma}_1; \tau_1 S_1)}{P_0 - Q_0 - E_{n'} - \omega_{n'} - i\epsilon}
\end{aligned} \tag{11}$$

The presubscript S on the matrix elements within the bracket denotes that we are to consider only the S-wave intermediate-state contribution.

In the above expression, τ_n, S_n, and γ are the spin and isospin of the intermediate state,

$$\vec{\mathrm{l}}_n = \frac{m\vec{\mathrm{q}}_n - m_\pi \vec{\mathrm{p}}_n}{m+m_\pi}, \quad \vec{\mathrm{q}}_n = \vec{\mathrm{l}}_n + \frac{m_\pi}{m+m_\pi}(\vec{\mathrm{P}}+\vec{\mathrm{Q}}),$$

$$E_n = (m^2 + \vec{\mathrm{p}}_n^{\,2})^{1/2}, \quad E_{n'} = E_n(-\vec{\mathrm{Q}}),$$

$$\vec{\mathrm{p}}_n = \vec{\mathrm{l}}_n + \frac{m}{m+m_\pi}(\vec{\mathrm{P}}+\vec{\mathrm{Q}}),$$

$$\omega_n = (m_\pi^{\,2} + \vec{\mathrm{q}}_n^{\,2})^{1/2}, \quad \omega_{n'} = \omega_n(-\vec{\mathrm{Q}}),$$

and the relative momenta

$$\vec{\mathrm{l}}_1 = \frac{m\vec{\mathrm{Q}} - m_\pi \vec{\mathrm{P}}}{m+m_\pi} + \tfrac{1}{2}\vec{\Delta}, \quad \vec{\mathrm{l}}_2 = \frac{m\vec{\mathrm{Q}} - m_\pi \vec{\mathrm{P}}}{m+m_\pi} - \tfrac{1}{2}\vec{\Delta},$$

$$\vec{\gamma}_1 = -\frac{m\vec{\mathrm{Q}} + m_\pi \vec{\mathrm{P}}}{m+m_\pi} + \tfrac{1}{2}\vec{\Delta}, \quad \vec{\gamma}_2 = -\frac{m\vec{\mathrm{Q}} + m_\pi \vec{\mathrm{P}}}{m+m_\pi} - \tfrac{1}{2}\vec{\Delta}$$

are defined in accordance with Eq. (8).

Consider the integral

$$\frac{1}{(2\pi)^3}\left[\int d^3 l_n \frac{m}{E_n \omega_n} \sum_{\tau_n, S_n, \gamma} \frac{{}_S(\vec{\mathrm{l}}_n; \tau_n S_n, \gamma | j_\alpha(0) | \vec{\mathrm{l}}_2; \tau_2 S_2)^* \,{}_S(\vec{\mathrm{l}}_n; \tau_n S_n, \gamma | j_\beta(0) | \vec{\mathrm{l}}_1; \tau_1 S_1)}{P_0 + Q_0 - E_n - \omega_n + i\epsilon}\right] \quad \text{(to order } Q\text{)}.$$

As we shall impose the static limit, we neglect the $\vec{\mathrm{Q}}$ dependence of E_n and W_n. In addition, we shall assume that the S-wave projection of

$$\sum_{\tau_n, S_n, \gamma} {}_S(\vec{\mathrm{l}}_n; \tau_n S_n, \gamma | j_\alpha(0) | \vec{\mathrm{l}}_2; \tau_2 S_2)^* \,{}_S(\vec{\mathrm{l}}_n; \tau_n S_n, \gamma | j_\beta(0) | \vec{\mathrm{l}}_1; \tau_1 S_1)$$

is a slowly varying function of $\vec{\mathrm{Q}}$ (see Appendix A).

This allows us to express the fixed-source S-wave projections of Eq. (11) as

$$h_0^\delta(l_2, l_1) = W_0^\delta(l_2, l_1) - \frac{\omega_2^2}{(2\pi)^3}\int_0^\infty \frac{l_n^2 dl_n}{2\omega_n^3}\left[\frac{h_0^\delta(l_2, l_n)^* h_0^\delta(l_n, l_1)}{\omega_2 - \omega_n + i\epsilon} - \sum_\gamma A_{\delta\gamma}\frac{h_0^\gamma(l_1, l_n)^* h_0^\gamma(l_n, l_2)}{\omega_2 + \omega_n + i\epsilon}\right], \tag{12}$$

where

$$\omega_2 = (m_\pi^{\,2} + \vec{\mathrm{l}}_2^{\,2})^{1/2}, \quad \omega_n = (m_\pi^{\,2} + \vec{\mathrm{l}}_n^{\,2})^{1/2},$$

and where $A_{\delta\gamma}$ is the $l = 0$ crossing matrix.

If we define the isospin-symmetric and isospin-antisymmetric amplitudes in the customary manner,

$$h_0^{(+)}(l_2, l_1) = \tfrac{1}{3}h_0^{1/2}(l_2, l_1) + 2h_0^{3/2}(l_2, l_1),$$

$$h_0^{(-)}(l_2, l_1) = \tfrac{1}{3}h_0^{1/2}(l_2, l_1) - h_0^{3/2}(l_2, l_1),$$

etc., we obtain

$$h_0^{(+)}(l_2, l_1) = W_0^{(+)}(l_2, l_1) - \frac{\omega_2^2}{(2\pi)^3}\int_0^\infty \frac{l_n^2 dl_n}{2\omega_n^3}\,\frac{h_0^{(+)}(l_2, l_n)^* h_0^{(+)}(l_n, l_1) + 2h_0^{(-)}(l_2, l_n)^* h_0^{(-)}(l_n, l_1)}{\omega_2 - \omega_n + i\epsilon}$$
$$+ \frac{\omega_2^2}{(2\pi)^3}\int_0^\infty \frac{l_n^2 dl_n}{2\omega_n^3}\,\frac{h_0^{(+)}(l_1, l_n)^* h_0^{(+)}(l_n, l_2) + 2h_0^{(-)}(l_1, l_n)^* h_0^{(-)}(l_n, l_2)}{\omega_2 + \omega_n}, \tag{13a}$$

$$h_0^{(-)}(l_2, l_1) = W_0^{(-)}(l_2, l_1) - \frac{\omega_2^2}{(2\pi)^3}\int_0^\infty \frac{l_n^2 dl_n}{2\omega_n^3}\,\frac{h_0^{(-)}(l_2, l_n)^* h_0^{(+)}(l_n, l_1) + h_0^{(+)}(l_2, l_n)^* h_0^{(-)}(l_n, l_1) + h_0^{(-)}(l_2, l_n)^* h_0^{(-)}(l_n, l_1)}{\omega_2 - \omega_n + i\epsilon}$$
$$- \frac{\omega_2^2}{(2\pi)^3}\int_0^\infty \frac{l_n^2 dl_n}{2\omega_n^3}\,\frac{h_0^{(-)}(l_1, l_n)^* h_0^{(+)}(l_n, l_2) + h_0^{(+)}(l_1, l_n)^* h_0^{(-)}(l_n, l_2) + h_0^{(-)}(l_1, l_n)^* h_0^{(-)}(l_n, l_2)}{\omega_2 + \omega_n}. \tag{13b}$$

An advantage of the twice-subtracted form of Eqs. (13) over the normal static-limit S-wave equations of the Chew-Low type is that it minimizes the importance of the high-energy behavior of the potential. At threshold these equations become exact, within the limits of the static approximation and the neglect of inelastic effects, and are similar in form to the mass variable dispersion relations of Fubini and Furlan:[15]

$$T^{(+)}(q^2 = m_\pi^{\,2}) = T^{(+)}(q^2 = 0) + \frac{m_\pi^{\,2}}{(2\pi)^3}\int \frac{(\omega^2 - m_\pi^{\,2})^{1/2}}{\omega(\omega^2 - m_\pi^{\,2})}\,\mathrm{Im}\,T^{(+)}\,d\omega,$$

$$T^{(-)}(q^2 = m_\pi^{\,2}) = T^{(-)}(q^2 = 0) + \frac{m_\pi^{\,3}}{(2\pi)^3}\int \frac{(\omega^2 - m_\pi^{\,2})^{1/2}}{\omega^2(\omega^2 - m_\pi^{\,2})}\,\mathrm{Im}\,T^{(-)}\,d\omega,$$

where $T^{(\pm)}$ are the scattering amplitudes.

The potential forms $W_0^{(\pm)}(l_2, l_1)$ are evaluated in Appendix B and Appendix C. The results are indicated below:

$$\langle\sigma\rangle = \frac{2G_S}{f_\pi^2} G_\sigma(-l_1^2 - l_2^2)[m_\pi^2 + \tfrac{1}{4}(l_1^2 + l_2^2)]^2 \tau^{(+)} , \tag{14a}$$

$$\langle V\rangle = \frac{2F_1(-l_1^2 - l_2^2)}{f_\pi^2}[m_\pi^2 + \tfrac{1}{4}(l_1^2 + l_2^2)]^2 \left[Q_0 + \left(1 - \frac{2g}{m} Q_0\right)\frac{l_1^2 + l_2^2}{4m}\right]\tau^{(-)} , \tag{14b}$$

$$\begin{aligned}\langle AA - jj\rangle_N = &-2\{g_{NN\pi}[-\tfrac{1}{4}(l_1^2 + l_2^2)]\}^2 \frac{Q_0}{m_\pi^2 + \frac{1}{4}(l_1^2 + l_2^2)}\left(\frac{P_0}{[m^2 + \frac{1}{4}(l_1^2 + l_2^2)]^{1/2}} - 1\right)\tau^{(-)} \\ &+ \frac{\{g_A[-\frac{1}{4}(l_1^2 + l_2^2)]\}^2}{f_\pi^2}[m_\pi^2 + \tfrac{1}{4}(l_1^2 + l_2^2)]^2\left(-\frac{Q_0}{2m}\{P_0 + [m^2 + \tfrac{1}{4}(l_1^2 + l_2^2)]^{1/2} - 2m\} - \frac{l_1^2 + l_2^2}{2m}\right)\tau^{(-)} \\ &+ \frac{\{g_A[-\frac{1}{4}(l_1^2 + l_2^2)]\}^2}{f_\pi^2}[m_\pi^2 + \tfrac{1}{4}(l_1^2 + l_2^2)]^2\, 2m\left(1 - \frac{m}{[m^2 + \frac{1}{4}(l_1^2 + l_2^2)]^{1/2}}\right)\tau^{(+)} .\end{aligned} \tag{14c}$$

In this expression, $\langle\sigma\rangle$ refers to the σ contribution, $\langle V\rangle$ to the isovector-current contribution, and $\langle AA - jj\rangle$ to the nucleon intermediate-state contribution. $\tau^{(+)}$ and $\tau^{(-)}$ represent the isospin-symmetric and isospin-antisymmetric projection operator, respectively.

Corrections to the static limit have been included in $W_0^{(\pm)}$ as they play an important role, both at threshold and at higher energies. The σ and isovector contributions [Eqs. (14a) and (14b), respectively] and the first term in Eq. (14c) dominate at threshold. The remaining two terms in Eq. (14c) are small at low energy but serve to "smooth" the potential in the intermediate energy range.

Little is known of the functional dependence of $G_\sigma(x)$ and $g_{NN\pi}(x)$. We shall assume they are slowly varying functions over the region of interest and take

$$g_{NN\pi}(x) \simeq \sqrt{2}\,\frac{m m_\pi^2}{f_\pi}\, g_A(0)$$

and[16]

$$\frac{G_S G_\sigma(x)}{m} \simeq -0.469 .$$

For the electromagnetic form factors, we choose the dipole fit of Hofstadter *et al.*,[17]

$$F_1(x) = \tfrac{1}{2}(1 - x/37.5 m_\pi^2)^{-2} ,$$

$$F_2(x) = g F_1(x) ,$$

with $g = 1.85$. For $g_A(x)$, Gleeson *et al.*[18] give

$$g_A(x) = g_A(0)\,(1 - x/M_A^2)^{-2} ,$$

with $M_A \sim 7.86 m_\pi^2$.[19]

In view of the approximations used in obtaining $W_0^{(\pm)}$, we do not expect a great deal of accuracy at higher energies. We therefore cut off the potentials at the point where the σ and isovector contributions are exactly canceled by Eq. (14c), i.e.,

$$W^{(\pm)}(l_2, l_1) = \begin{cases} W_0^{(\pm)}(l_2, l_1) & \text{for } W_0^{(\pm)}(l_2, l_1) \lessgtr 0 , \\ 0 & \text{for } W_0^{(\pm)}(l_2, l_1) \gtrless 0 . \end{cases} \tag{15}$$

This occurs at a pion kinetic energy of about 300 MeV for the physical potential. It is hoped that our ignorance of the detailed behavior of $W_0^{(\pm)}$ at higher energies will have little effect on low-energy results due to the twice-subtracted nature of Eqs. (13).

Equations (13), with the potential as given by Eqs. (14) and (15), have been solved numerically with a simple iterative procedure. As a first estimate for $h_0^{(\pm)}(l_2, l_1)$, we use the potential $W_0^{(\pm)}(l_2, l_1)$, with good convergence after four or five interactions. The phase shifts we obtain are compared with the experimental data in Figs. 1 and 2. The predicted scattering lengths are $a^{(-)}$

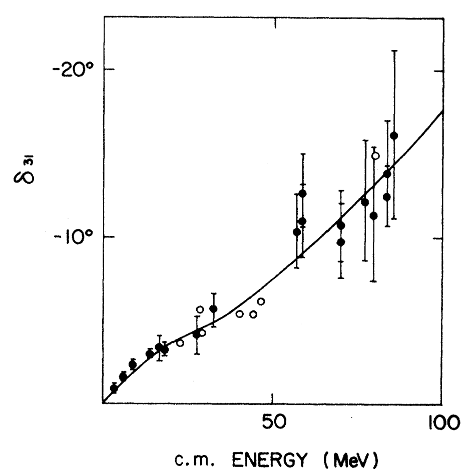

FIG. 1. Comparison of the $I = \frac{3}{2}$, S-wave phase shift with experiment.

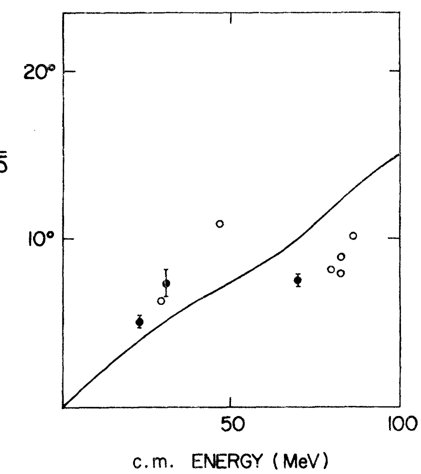

FIG. 2. Comparison of the $I=\frac{1}{2}$, S-wave phase shift with experiment.

$=0.086 m_\pi^{-1}$ and $a^{(+)}=-0.023 m_\pi^{-1}$, or, equivalently, $a^{1/2}=0.149 m_\pi^{-1}$ and $a^{3/2}=-0.109 m_\pi^{-1}$. For comparison, Roper *et al.*[20] obtain $a^{1/2}=0.157 m_\pi^{-1}$ and $a^{3/2}=-0.097 m_\pi^{-1}$ from their 0–100-MeV fit.

IV. CONCLUSION

We have developed a prescription for extracting the informational content of the off-mass-shell scattering amplitude, based on the conjecture that the pion current is independent of second- and higher-order derivatives of the pion field. This led to the development of integral equations for the "half" off-mass-shell amplitude in the manner of the Chew-Low approach. The seagull terms in this formulation provide the potential for the S-wave equations as expected, but also provide for subtractions.

We examined the σ, isovector, and nucleon-pole contributions in considerable detail, isolating what we believe to be the dominant terms. Our treatment of the pion-nucleon intermediate-state contribution to these terms is more simplistic, but the excellence of the $I=\frac{3}{2}$ phase-shift fit to experiment is a strong argument in favor of our neglect of the P-wave structure of these terms.

Our techniques, of course, can be applied to other interactions involving pions. As an example, we note that for photomeson production, the assumption of minimal electromagnetic interaction should establish the conjectured analytic properties of the seagull terms [with $j_\pi = j_\pi(\phi, \partial_\mu \phi)$].

ACKNOWLEDGMENTS

The author would like to thank Professor C. A. Levinson for suggesting this problem and Professor M. K. Banerjee for important discussions and encouragement. The author would also like to thank Dr. M. Feigenbaum for reading the final manuscript.

APPENDIX A

Let

$$\begin{aligned} g(\vec{Q}) &= \sum_{\tau_n, S_n, \gamma} {}_S(\vec{1}_n;\, \tau_n S_n, \gamma\,|\, j_\alpha(0)\,|\,\vec{1}_2;\, \tau_2 S_2)^* \\ &\quad \times {}_S(\vec{1}_n;\, \tau_n S_n, \gamma\,|\, j_\beta(0)\vec{1}_1;\, \tau_1 S_1) \\ &= a + \vec{b}\cdot\vec{Q} + \cdots . \end{aligned} \tag{A1}$$

Then, Eq. (12) amounts to the neglect of the static-limit S-wave projections of

$$\int d^3 l_n \frac{m}{E_n \omega_n}\left[\frac{g(\vec{Q}) - a - \vec{b}\cdot\vec{Q}}{P_0 - \omega_n - E_n} + Q_0 \frac{g(\vec{Q}) - a}{(P_0 - \omega_n - E_n)^2}\right] + \text{cross term}. \tag{A2}$$

Consider the form of Eqs. (13). Experimentally, it is found that

$$h_0^{(-)}(0,0) \sim 10 h_0^{(+)}(0,0),$$

so that the "dispersive" corrections are dominated by $h^{(-)}$. If as a first approximation for the antisymmetric isospin amplitude

$$(\vec{1}_n;\, \tau_n S_n, \gamma\,|\, j_\beta(0)\,|\,\vec{1}_1;\, \tau_n S_1) = (\vec{1}_n;\, \tau_n S_n\,|\, W_{\alpha\beta}\,|\,\vec{1}_1;\, \tau_1 S_1),$$

it is found that the error introduced by the neglect of Eq. (A2) is easily within the error inherent in the static approximation at low energies.

APPENDIX B

We present in this appendix the details of the calculation of S-wave contributions of the σ and isovector matrix elements. The relevant form factors are defined by

$$\langle p_2 |\sigma_{\alpha\beta}(0)| p_1\rangle = \bar{U}(p_2)[G_\sigma(\Delta^2)\delta_{\alpha\beta}]U(p_1),$$

$$\langle p_2 | V^\mu_\gamma(0) | p_1\rangle = \bar{U}(p_2)\left(\gamma^\mu F_1(\Delta^2) + i\frac{\Delta_\nu}{m}\sigma^{\mu\nu}F_2(\Delta^2)\right)\tfrac{1}{2}\tau_\gamma U(p_1).$$

$U(p_2)$ and $U(p_1)$ are the relativistic four-component nucleon spinors, and $F_1(\Delta^2)$ and $F_2(\Delta^2)$ are the electromagnetic form factors. We shall assume

$$F_2(\Delta^2) = (\mu_p - \mu_n)F_1(\Delta^2),$$

where μ_p and μ_n are the proton and neutron magnetic moments.

$$\left\{\frac{2i}{f_\pi^2}[(m_\pi^2-\tfrac{1}{4}\Delta^2-Q^2)^2-(Q\cdot\Delta)^2][-iG_S\langle P+\tfrac{1}{2}\Delta|\sigma_{\alpha\beta}(0)|P-\tfrac{1}{2}\Delta\rangle+2G_V\epsilon_{\alpha\beta\gamma}Q_\mu\langle P+\tfrac{1}{2}\Delta|V_\gamma^\mu(0)|P-\tfrac{1}{2}\Delta\rangle]\right\}\text{ (to order } Q)$$
$$=\overline{U}(P+\tfrac{1}{2}\Delta)\left\{\frac{(m_\pi^2-\frac{1}{4}\Delta^2)^2}{f_\pi^2}\left[2G_S G_\sigma(\Delta^2)\tau^{(\pm)}-8G_V\frac{P\cdot Q}{m}F_2(\Delta^2)\tau^{(-)}+4g_V[F_1(\Delta^2)+2F_2(\Delta^2)]\not{Q}\tau^{(-)}\right]\right\}U(P-\tfrac{1}{2}\Delta), \tag{B1}$$

with

$$\tau^{(+)}=\delta_{\alpha\beta}, \qquad \tau^{(-)}=i\epsilon_{\alpha\beta\gamma}\tau_\gamma=[\tau_\alpha,\tau_\beta].$$

In the nonrelativistic limit, Eq. (B1) becomes

$$\frac{(m_\pi^2-\frac{1}{4}\Delta^2)^2}{f_\pi^2}\frac{[(E_1+m)(E_2+m)]^{1/2}}{2m}\chi_2^+\Bigg\{2g_S G_\sigma(\Delta^2)\tau^{(+)}+4gF_1(\Delta^2)\left(-2g\frac{P\cdot Q}{m}+(1+2g)(P_0+Q_0-m)\right)\tau^{(-)}$$
$$+\frac{\vec{\sigma}\cdot\vec{l}_2\vec{\sigma}\cdot\vec{l}_1}{(E_1+m)(E_2+m)}\Bigg[-2g_S G_\sigma(\Delta^2)\tau^{(+)}$$
$$+4g_V F_1(\Delta^2)\left(2\frac{P\cdot Q}{m}g+(1+2g)(P_0+Q_0+m)\right)\Bigg]\Bigg\}\chi_1, \tag{B2}$$

where $g=\mu_p-\mu_n$ and where χ_2^+ and χ_1 are nonrelativistic two-dimensional Pauli spinors.

The partial-wave projections of

$$(\vec{l}_2;\tau_2 S_2|W_{\alpha\beta}|\vec{l}_1;\tau_1 S_1)=\chi_2^+(W_1+\vec{\sigma}\cdot\hat{l}_2\;\vec{\sigma}\cdot\hat{l}_1W_2)\chi_1 \tag{B3}$$

are given by

$$W_{l\pm}=\tfrac{1}{2}\int_{-1}^{1}[W_1P_l(Z)+W_2P_{l\pm1}(Z)]\,dZ, \tag{B4}$$

where

$$l\pm=J\pm\tfrac{1}{2}.$$

As a general rule, the contribution of W_2 to $W_0^\delta(l_2,l_1)$ is of the order

$$\frac{l_1l_2}{(E_1+m)(E_2+m)}O(l_1l_2).$$

Since

$$\frac{l_1l_2}{(E_1+m)(E_2+m)}\lesssim 0.04$$

for $l_{1,2}\lesssim 3m_\pi$, we neglect this contribution.

Expanding W_1 in powers of $\cos\theta$ and projecting out the S-wave contributions, we find that

$$W_0(l_2,l_1)=[m_\pi^2+\tfrac{1}{4}(l_1^2+l_2^2)]^2\left\{\frac{2g_S}{f_\pi^2}G_\sigma(-l_1^2-l_2^2)\tau^{(+)}+\frac{4g_V}{f_\pi^2}F_1(-l_1^2-l_2^2)\left[\omega_2+\frac{l_1^2+l_2^2}{4m}\left(1-\frac{2g}{m}\omega_2\right)\right]\tau^{(-)}\right\}. \tag{B5}$$

In arriving at this expression, we have neglected terms involving $(\cos\theta)^n$ for $n\geq 2$, for simplicity. The most important of these terms is

$$\tfrac{1}{12}(l_1l_2)^2\{\;\;\},$$

where the curly brackets are the same as in Eq. (B5). This is completely negligible at small energies and its neglect introduces no more than a 10% error at $l_{1,2}\sim 3m_\pi$. Further, we have not imposed the strict static limit, but have kept those contributions which can become important at higher energies.

APPENDIX C

In this appendix, we calculate the contribution of

$$\Bigg\{-(2\pi)^3\sum_N\Bigg[\frac{\delta^{(3)}(\vec{P}+\vec{Q}-\vec{p}_N)}{P_0+Q_0-E_N+i\epsilon}\Big(\langle p_2|j_\alpha(0)|N\rangle\langle N|j_\beta(0)|p_1\rangle$$
$$-\frac{2}{f_\pi^2}(m_\pi^2-q_1^2)(m_\pi^2-q_2^2)\langle p_2|q_2\cdot A_\alpha(0)|N\rangle\langle N|q_1\cdot A_\beta(0)|p_1\rangle\Big)+\text{cross term}\Bigg]$$
$$-(2\pi)^3\sum_{\pi,N}\Bigg[\frac{\delta^{(3)}(\vec{P}+\vec{Q}-\vec{p}_N-\vec{q}_\pi)}{P_0+Q_0-E_N-\omega_\pi+i\epsilon}\Big(\langle p_2|j_\alpha(0)|\pi N\rangle_p\;{}_p\langle\pi N|j_\beta(0)|p_1\rangle$$
$$-\frac{2}{f_\pi^2}(m_\pi^2-q_1^2)(m_\pi^2-q_2^2)\langle p_2|q_2\cdot A_\alpha(0)|\pi N\rangle\langle\pi N|q_1\cdot A_\beta(0)|p_1\rangle\Big)+\text{cross term}\Bigg]\Bigg\}\text{(to order } Q), \tag{C1}$$

to $W_0^\delta(l_2, l_1)$.

We first determine the nucleon intermediate-state contribution

$$W_0(l_2, l_1)_{\text{N.P.}} = (2\pi)^3 \sum_N \left[\frac{\delta^{(3)}(P+Q-p_N)}{P_0+Q_0-E_N+i\epsilon}\left(\langle p_2|j_\alpha(0)|N\rangle\langle N|j_\beta(0)|p_1\rangle - \frac{2}{f_\pi^2}(m_\pi^2-q_1^2)(m_\pi^2-q_2^2)\langle p_2|q_2\cdot A_\alpha(0)|N\rangle \times\langle N|q_1\cdot A_\beta(0)|p_1\rangle\right) + \text{cross term}\right], \tag{C2}$$

keeping only those terms which are independent of or linear in Q. The intermediate-state sum is given by

$$\sum_N |N\rangle\langle N| = \sum_{\text{spin,isospin}} \int \frac{d^3p_N}{(2\pi)^3}\frac{m}{E_N}|p_N\rangle\langle p_N| \ . \tag{C3}$$

The form factors

$$_N\langle p'|A_\alpha^\mu(0)|p\rangle_N = \tfrac{1}{2}\,\overline{U}(p')[g_A(q^2)\gamma^\mu + h_A(q^2)q^\mu]\gamma^5\tau_\alpha U(p)\ , \tag{C4a}$$

$$_N\langle p'|j_\alpha(0)|p\rangle_N = \tfrac{1}{2}\,i\,\overline{U}(p')[g_{NN\pi}(q^2)\gamma^5\tau_\alpha]U(p)\ , \tag{C4b}$$

where $q=p'-p$, are related by

$$2mg_A(q^2)+q^2h_A(q^2)=\sqrt{2}\,f_\pi\,\frac{g_{NN\pi}(q^2)}{(m_\pi^2-q^2)}\ . \tag{C5}$$

Inserting Eqs. (C3) and (C4) into Eq. (C2), and utilizing Eq. (C5), we find that

$$\begin{aligned}
W_0(l_2, l_1)_{\text{N.P.}} = \frac{1}{4}\,&\frac{1}{E_N(P_0+Q_0-E_N+i\epsilon)}\\
&\times \overline{U}(p)\left[2g_{NN\pi}(k_1^2)g_{NN\pi}(k_2^2)\left(\frac{(m_\pi^2-q_2^2)(m_\pi^2-q_1^2)}{(m_\pi^2-k_2^2)(m_\pi^2-k_1^2)}-1\right)(-\not p_N+m)\right.\\
&-\frac{1}{f_\pi^2}(m_\pi^2-q_2^2)(m_\pi^2-q_1^2)(q_1-k_1)\cdot k_1(q_2-k_2)\cdot k_2 h_A(k_1^2)h_A(k_2^2)(-\not p_N+m)\\
&-\frac{2}{f_\pi^2}(m_\pi^2-q_1^2)(m_\pi^2-q_2^2)\left(\frac{(q_2+k_2)\cdot k_2}{(m_\pi^2-k_1^2)}h_A(k)g_{NN}(k)-\frac{(q_1-k_1)\cdot k_1}{(m_\pi^2-k_2^2)}h_A(k_1^2)g_{NN}(k_2^2)\right)(-\not p_N+m)\\
&-\frac{2m}{f_\pi^2}(m_\pi^2-q_1^2)(m_\pi^2-q_2^2)g_A(k_1^2)g_A(k_2^2)\\
&\times[(-\not p_N+m)(\not q_1+2m)+(\not q_2+2m)(-\not p_N+m)-(\not q_2+2m)(-\not p_N+m)(\not q_1+2m)]\\
&+\frac{1}{f_\pi^2}(q_2\cdot k_2)(m_\pi^2-q_1^2)(m_\pi^2-q_2^2)h_A(k_2^2)g_A(k_1^2)(-\not p_N+m)(\not q_1+2m)\\
&\left.+\frac{1}{f_\pi^2}(q_1\cdot k_1)(m_\pi^2-q_1^2)(m_\pi^2-q_2^2)h_A(k_1^2)g_A(k_2^2)(\not q_2+2m)(-\not p_N+m)\right]\tau_\alpha\tau_\beta U(p)\\
&+\text{cross term}\ ,
\end{aligned} \tag{C6}$$

where

$$k_1=p_N-p_1,\quad k_2=p_2-p_N,\quad p_N=([m+(\vec{P}+\vec{Q})^2]^{1/2},\ \vec{P}+\vec{Q})\ . \tag{C7}$$

An explicit calculation indicates that Eq. (C6) is a well-defined function of Q. We shall assume that, for low energies,

$$h_A(x) \simeq \frac{f_\pi}{\sqrt{2}\, m_\pi^2} \frac{g_{NN\pi}(m_\pi^2)}{(m_\pi^2 - x)} , \tag{C8}$$

i.e., that h_A is dominated by its nearest singularity.

The evaluation of this expansion is relatively straightforward, but tedious. The method is identical to that of Appendix B. The result is

$$\begin{aligned} W_0(l_2, l_1)_{\text{N.P.}} = & -2[g_{NN\pi}(-\tfrac{1}{4}(l_1^2+l_2^2))]^2 \frac{Q_0}{m_\pi^2 + \frac14(l_1^2+l_2^2)} \left(\frac{P_0}{[m^2+\frac14(l_1^2+l_2^2)]^{1/2}} - 1\right)\tau^{(-)} \\ & + \frac{[g_A(-\frac14(l_1^2+l_2^2))]^2}{f_\pi^2}[m_\pi^2+\tfrac14(l_1^2+l_2^2)]^2 \\ & \times\left[\left(-\frac{l_1^2+l_2^2}{2m} - \frac{Q_0}{2m}\{P_0 + [m+\tfrac14(l_1^2+l_2^2)]^{1/2} - 2m\}\right)\tau^{(-)} + 2m\left(1 - \frac{m}{[m^2+\frac14(l_1^2+l_2^2)]^{1/2}}\right)\tau^{(+)}\right], \end{aligned} \tag{C9}$$

where the N.P. subscript on W_0 indicates that this is the nucleon-pole contribution to the potential. To simplify the calculation of Eq. (C9), we have expanded in powers of $\cos\theta$ and neglected terms of order $\cos^2\theta$ and higher. These terms are even less important than comparable terms discussed in Appendix B.

The leading term in Eq. (C9) is unimportant at intermediate energies but makes a large contribution to the isospin-antisymmetric effective range. The remaining two terms are unimportant at threshold but moderate the behavior of $W_0^{(\pm)}$ at higher energies. All other contributions from Eq. (C6) are dominated by these terms.

Consider now the pion-nucleon intermediate-state contribution to Eq. (C1),

$$\begin{aligned} -(2\pi)^3 \sum_{\pi, N}\Bigg[\frac{\delta^{(3)}(P+Q-p_N-q)}{P_0+Q_0-E_N-\omega_\pi+i\epsilon}\Big(\langle p_2|j_\alpha(0)|\pi N\rangle_p\, {}_p\langle\pi N|j_\beta(0)|p_1\rangle \\ -\frac{2}{f_\pi^2}(m_\pi^2-q_1^2)(m_\pi^2-q_2^2)\langle p_2|q_2\cdot A_\alpha(0)|\pi N\rangle\langle\pi N|q_1\cdot A_\beta(0)|p_1\rangle\Big) + \text{cross term}\Bigg]. \end{aligned} \tag{C10}$$

We shall assume these terms are well approximated by the $I=\frac32$, $J^P=\frac32^+$ N_1^* resonance in the generalized Born approximation. The relevant form factors are defined by[20]

$$\begin{aligned} \langle N^*(p')|A^\mu(0)|N(p)\rangle = \Psi_\rho(p')\Big[&\mathcal{G}_A(q^2)g^{\mu\rho} + \mathcal{F}_A(q^2)r^\mu q^\rho \\ &+\mathcal{H}_A(q^2)P'^\mu q^\rho \\ &+\mathcal{L}_A(q^2)\left(\frac{q^\mu q^\rho}{q^2} - g^{\mu\rho}\right)\Big]U(p), \end{aligned}$$

$$\langle N^*(p')|j(0)|N(p)\rangle = \Psi_\rho(p')[g_{N^*N\pi}(q^2)q^\rho]U(p),$$

with

$$q = p' - p ,$$

$$p' = (p'+p) - (p'+p)\cdot(p'-p)\frac{q}{q^2} ,$$

$$r^\mu = \epsilon^{\mu\nu\lambda\sigma}\gamma_\nu(p'+p)_\lambda q_\sigma\gamma^5 .$$

In the generalized Born approximation, Eq. (C10) becomes

$$\begin{aligned} &\frac{2i}{f_\pi^2}(m_\pi^2-q_1^2)(m_\pi^2-q_2^2) \\ &\quad\times \bar U(p_2)\{[\mathcal{G}_A(q_2^2)q_2^\mu]\Lambda_{\mu\nu}[\mathcal{G}_A(q_1^2)q_1^\nu]\}U(p_1) \\ &\quad -i\bar U(p_2)\{[-ig_{N^*N}(q_2^2)q_2^\mu]\Lambda_{\mu\nu}[ig_{N^*N}(q_1^2)q_1^\nu]\}U(p), \end{aligned}$$

where $\Lambda_{\mu\nu}$ is the N_1^* propagator. This vanishes as a consequence of the PCAC relation

$$\mathcal{G}_A(q^2) = \frac{f_\pi}{\sqrt2}\frac{g_{N^*N\pi}(q^2)}{(m_\pi^2-q^2)} .$$

*Research sponsored by the U. S. Atomic Energy Commission under Contract No. AT-(40-1) 4317.

†Based in part on a thesis by R. H. Hackman to the University of Maryland in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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PHYSICAL REVIEW D VOLUME 8, NUMBER 11 1 DECEMBER 1973

Statistical Approach to Parton Models: Application to 90° Cross Sections*

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(Received 28 June 1973)

We present a parton statistical model in which the pairing interaction has been explicitly taken into account. Using a generalized equidistant model for the density of states we calculate the 90° differential elastic two-body cross section in terms of an incoherent sum of direct-channel resonances. Encouraging agreement with the data for $\pi^{\pm}p$ and $p\bar{p}$ elastic reaction is obtained. Two main results are deduced from our analysis: (a) an approximate exponential decrease of $d\sigma/d\Omega|_{90^\circ}$; (b) a break structure in $d\sigma/d\Omega|_{90^\circ}$. The position of the break is fixed at the energy where the temperature of the system reaches its critical value, i.e., a phase transition occurs.

I. INTRODUCTION

The description of interacting hadrons as a many-body system has been advanced recently in two different directions: (1) the statistical bootstrap model (SBM) of Hagedorn and Frautschi[1,2]; (2) the parton picture suggested by Feynman.[3]

The main assumptions of the statistical bootstrap model are as follows.

(a) Resonances rise indefinitely with mass.

(b) The bootstrap hypothesis: Resonances are built from each other, i.e., the constituents are the hadrons themselves.

(c) The mutual interaction of hadrons can be completely represented by resonance formation.

(d) The only other effect of interactions is to confine the constituents within a characteristic volume.

In the framework of this model the density of hadron states with mass m is predicted to be[4,5]

$$\rho(m)=\frac{a}{m^3}e^{m/T_0}, \tag{1}$$

where a is a constant and T_0 is the so-called critical temperature and is given approximately by the mass of the pion. Since $\rho(m)$ is connected to