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Implications of Classical Two-Tensor Gravity*

P. C. Aichelburg†
International Centre for Theoretical Physics, Trieste, Italy
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We discuss implications of f-g gravity on the level of classical field theory, with special emphasis on its metric structure. We show that although space-time is "bicausal," f-g theory reduces to general relativity in the limit of weak static fields. The discrepancy between the degrees of freedom of the linearized and full theory is discussed. Further, we review all known exact solutions and give a new exact nonvacuum solution.

I. INTRODUCTION

Recently there has been considerable interest in two-tensor theories of gravitation.[1-4] These theories assume that gravity is described by two tensor fields, thereby modifying Einstein's equations of general relativity (GR). The reason for introducing a second tensor field is not experimental, since GR is in good agreement with experiments, but theoretical.

(i) There exist in nature massive spin-2 fields such as the f-meson–hadron state. The inclusion of a spin-2 field into GR by coupling minimally its energy-momentum tensor to the gravitational field causes severe difficulties, as pointed out by Aragone and Deser.[5]

(ii) There have been attempts to assign a mass to the gravitational field,[6-8] which of course breaks general covariance (as in electrodynamics gauge invariance is broken by massive photons). If the theory is to be self-consistent to all orders in the field, the massless part of the Lagrangian must be the same as in the Einstein case. In Ref. 2 we have shown that the causal structure of this theory is then Riemannian and that such a theory is actually the limit of a generally covariant two-tensor theory.

This motiviates the study of two-tensor theories on the level of classical field theories. We shall concentrate here on the f-g two-tensor theory proposed by Isham, Salam, and Strathdee[1]; however, the main features of this theory are also found in theories with different interaction between the two tensor fields. Thus the aim of this paper is to consider the implications of a two-tensor theory for GR with special emphasis on its metric structure, and to give a review of what is so far known about the theory by summarizing previous papers and stating new results.

In Sec. II we start by giving the Lagrangian, introducing notations and drawing general conclusions. Section III is devoted to the vacuum field equations. We state a theorem about trivial solutions (i.e., $f_{\mu\nu}=g_{\mu\nu}$) which is important for linearizing the theory. We discuss in some detail the problem of internal degrees of freedom and show explicitly why the mass term introduces an additional degree of freedom in the full theory. Moreover, we consider the propagation of the f and g fields.

In Sec. IV we consider implications following from the coupling of the tensor field to matter. The question of which field might play the role of a space-time metric is discussed by looking at the motion of test particles. We show that in general there is no natural metric interpretation of the gravitational field in the f-g theory. We discuss what Hammel and Lubkin called the "local corruption of space-time."

By identifying the mass of the linearized massive field with that of the f meson and assigning definite values for the coupling constants, the Riemannian structure of space-time is recovered for weak *static* vacuum fields, at distances $> 10^{-12}$ cm from the source.

Finally, we turn our attention to exact solution of the field equations. We give a review of the known solutions, all of which belong to the Kerr-Schild type (or related), also encountered in GR. Further, we give a peculiar nonvacuum solution with no analog in GR.

These solutions have only a limited physical significance because they all possess a geodesic null congruence with vanishing optical scalars. So far no spherical symmetric solution is known.

II. CLASSICAL FORMULATION OF THE f-g THEORY

The f-g theory is a generally covariant theory with two dynamical, symmetric, and self-interacting tensor fields ($f_{\mu\nu}$ and $g_{\mu\nu}$). General covariance implies that no absolute objects[9] are contained, and dynamical means that hyperbolic propagation of both fields is possible. We therefore do not restrict the fields to have the same signature.[10] The main assumption is that matter can be divided into two classes (hadrons and leptons) on which gravity acts differently. While the energy-momentum tensor of hadronic matter is supposed to couple only to the f field, leptons couple only to the g field. The two kinds of matter interact gravitationally only via an f-g coupling. The theory is given by the following generally covariant Lagrangian density:

$$L=\frac{1}{\kappa_f^2}\sqrt{-f}\,R(f)+\frac{1}{\kappa_g^2}\sqrt{-g}\,R(g)+L_{fg}+L(f,\text{hadrons})+L(g,\text{leptons}). \quad (2.1)$$

Notation

$f_{\mu\nu}$ and $g_{\mu\nu}$ are the covariant components of the tensor fields, and we denote their inverses by $f^{\mu\nu}$ and $g^{\mu\nu}$:

$$f_{\mu\lambda}f^{\lambda\nu}=\delta^\nu_\mu,\quad g_{\mu\lambda}g^{\lambda\nu}=\delta^\nu_\mu .$$

Also,

$$f=\det f_{\mu\nu}\quad\text{and}\quad g=\det g_{\mu\nu}.$$

By defining two symmetric connections $\Gamma^\sigma_{\mu\nu}(f)$ and $\Gamma^\sigma_{\mu\nu}(g)$ from f and g as in Riemannian geometry, one can build up all geometrical objects used there. For example, $R_{\mu\nu}(f)$ denotes the Ricci tensor formed with respect to the f field. Two types of covariant differentiation can be defined, which we denote by | and ; when taken with respect to f or g, respectively. Note that raising and lowering of indices is obtained with the help of f (g) for quantities formed from f (g) only. Greek indices take the values 0, 1, 2, 3 and Latin indices 1, 2, 3, and a comma denotes partial differentiation.

The first two terms are the free-field parts of the f and g field, respectively, and are chosen to be the same as in GR in order to obtain second-order hyperbolic equations for both fields. L_{fg} describes the interaction between the f and g fields and represents a "generally covariant mass term" containing no derivative couplings:

$$L_{fg}=\frac{M^2}{4\kappa_f^2}(-f)^{1/2}(f^{\alpha\beta}-g^{\alpha\beta})(f^{\lambda\sigma}-g^{\lambda\sigma})\times(g_{\alpha\lambda}g_{\beta\sigma}-g_{\alpha\beta}g_{\lambda\sigma}). \quad (2.2)$$

The last two terms in the Lagrangian (2.1) give the coupling of f to hadronic and g to leptonic matter. Variation of these terms with respect to $f_{\mu\nu}$ and $g_{\mu\nu}$ defines the covariantly conserved energy-momentum tensors $T^H_{\mu\nu}$ and $T^L_{\mu\nu}$ for hadrons and leptons. Assuming that the equations of motion for matter are satisfied, one has

$$\delta\int L(f,\text{hadrons})\,d^4x=\tfrac{1}{2}\int T^H_{\mu\nu}\sqrt{-f}\,\delta f^{\mu\nu}d^4x$$

and (2.3)

$$\delta\int L(g,\text{leptons})\,d^4x=\tfrac{1}{2}\int T^L_{\mu\nu}\sqrt{-g}\,\delta g^{\mu\nu}d^4x .$$

Therefore, hadronic (leptonic) matter couples minimally to f (g). We note that the theory does not include nongravitational interaction between hadrons and leptons, which would spoil the covariant conservation laws

$$T^{H}_{\mu\nu}{}^{|\nu}=T^{L}_{\mu\nu}{}^{;\nu}=0 \quad (2.4)$$

following from Eq. (2.3). Two sets of field equations are derived from (2.1) by variation with respect to $g^{\mu\nu}$ and $f^{\mu\nu}$,

$$\frac{1}{\kappa_f^2}G_{\mu\nu}(f)+(-f)^{-1/2}\left(\frac{\partial L_{fg}}{\partial f^{\mu\nu}}\right)=-\tfrac{1}{2}T^H_{\mu\nu},$$
$$\frac{1}{\kappa_g^2}G_{\mu\nu}(g)+(-g)^{-1/2}\left(\frac{\partial L_{fg}}{\partial g^{\mu\nu}}\right)=-\tfrac{1}{2}T^L_{\mu\nu}. \quad (2.5)$$

$G_{\mu\nu}$ is the Einstein tensor and

$$\frac{\partial L_{fg}}{\partial f^{\mu\nu}}=\left(-\tfrac{1}{2}\delta^\alpha_\mu L_{fg}+\frac{M^2}{2\kappa_f}\mathfrak{F}^\alpha_\mu\right)f_{\alpha\nu},$$
$$\frac{\partial L_{fg}}{\partial g^{\mu\nu}}=-\frac{M^2}{2\kappa_f}\mathfrak{F}^\alpha_\mu g_{\alpha\nu}, \quad (2.6)$$

where

$$\mathfrak{F}^\alpha_\mu\equiv(-f)^{1/2}\frac{1}{\kappa_f}\left[g_{\mu\rho}f^{\rho\sigma}g_{\sigma\tau}f^{\tau\alpha}-g_{\mu\rho}f^{\rho\alpha}(g_{\kappa\lambda}f^{\lambda\kappa}-3)\right].$$

Because L_{fg} is a covariant density it follows from Noether's theorem[11] that

$$\left(\frac{\partial L_{fg}}{\partial f^{\mu\nu}}\right)^{|\nu}+\left(\frac{\partial L_{fg}}{\partial g^{\mu\nu}}\right)^{;\nu}=0, \quad (2.7)$$

and therefore the field equation does not require separate covariant conservation of $T^H_{\mu\nu}$ and $T^L_{\mu\nu}$. This is in contrast to the Einstein equations. Conversely, if Eq. (2.4) is valid, the mass term of the field equation imposes four conditions on f and g of the form

$$\left(\frac{\partial L_{fg}}{\partial f^{\mu\nu}}\right)^{|\nu}=\left(\frac{\partial L_{fg}}{\partial g^{\mu\nu}}\right)^{;\nu}=0. \quad (2.8)$$

If $f_{\mu\nu}(x)=g_{\mu\nu}(x)$, then the mass term in Eqs. (2.5) vanishes, as can be seen from Eqs. (2.6), and the field equations decouple into two Einstein equations. Therefore, any solution of the Einstein equation is also a solution to the system (2.5) with $f_{\mu\nu}=g_{\mu\nu}$ and identical source term

$$\kappa_f^{\,2} T^H_{\mu\nu} = \kappa_g^{\,2} T^L_{\mu\nu} \ . \tag{2.9}$$

We call these solutions *trivial*. In particular, the Schwarzschild solution of GR is a spherical symmetric solution for f-g theory if hadronic and leptonic energy is related by Eq. (2.9). However, as we shall see from the linearized theory, for this case the source does not couple to the "massive," i.e., short-range part of the field.

III. THE VACUUM EQUATIONS

In this section we discuss consequences of the vacuum equations. By vacuum we understand the absence of all matter and fields except the f and g fields, i.e., $T^H_{\mu\nu} = T^L_{\mu\nu} = 0$.

We first state a theorem concerning trivial solutions.

Theorem I. If for a vacuum solution either $R_{\mu\nu}(f) = 0$ *or* $R_{\mu\nu}(g) = 0$, then $f_{\mu\nu} = g_{\mu\nu}$.

We shall only sketch the proof[12]:

Let $R_{\mu\nu}(g) = 0$; then from Eqs. (2.5) it follows that $\mathfrak{F}^\alpha_\mu = 0$, multiplying by $g^{\mu\rho} f_{\rho\alpha}$ gives $g_{\lambda\sigma} f^{\lambda\sigma} = 4$ and then $g_{\lambda\beta} f^{\beta\alpha} = \delta^\alpha_\lambda$.

This theorem has important consequences. Suppose that, in vacuum, $R_{\mu\nu\lambda\rho}(g) = 0$; it then follows that also $R_{\mu\nu\lambda\rho}(f) = 0$. The coupling between the fields is therefore so rigid that if one field is "flat" the other is also forced to be flat.

A still stronger consequence is that if one chooses for $R_{\mu\nu\lambda\rho}(g) = 0$ a coordinate system where $g_{\mu\nu} = \eta_{\mu\nu}$ (where $\eta_{\mu\nu}$ stands for the signature of the field) then in the *same* coordinate system $f_{\mu\nu} = \eta_{\mu\nu}$. This implies that all solutions obtainable by expansion around flatness can be written in the form

$$\begin{aligned} f_{\mu\nu} &= \eta_{\mu\nu} + \kappa_f F_{\mu\nu}(x) , \\ g_{\mu\nu} &= \eta_{\mu\nu} + \kappa_g h_{\mu\nu}(x) , \end{aligned} \tag{3.1}$$

where $\kappa_f F_{\mu\nu}$ and $\kappa_g h_{\mu\nu}$ are small deviations from $\eta_{\mu\nu}$. This is a nontrivial result since one could imagine the η term of one field to be nondiagonal or to have different signature from the other.

A. Linear Theory

We now proceed to linearize the field equations by substituting (3.1) into the Lagrangian (1.1) and retaining at most quadratic terms in the field variables. The result is

$$\begin{aligned} L &= L(F) + L(h) + L(F,h) \\ &= \tfrac{1}{4}(F_{\mu\nu,\lambda}F^{\mu\nu,\lambda} - 2F_{\mu\nu,\lambda}F^{\nu\lambda,\mu} - F_{,\lambda}F^{,\lambda} + 2F_{,\lambda}F^{\nu\lambda}{}_{,\nu}) + \tfrac{1}{4}(h_{\mu\nu,\lambda}h^{\mu\nu,\lambda} - 2h_{\mu\nu,\lambda}h^{\nu\lambda,\mu} - h_{,\lambda}h^{,\lambda} + 2h_{,\lambda}h^{\nu\lambda}{}_{,\nu}) \\ &\quad + \frac{M^2}{4\kappa_f}(\kappa_f F^{\alpha\beta} - \kappa_g h^{\alpha\beta})(\kappa_f F^{\lambda\sigma} - \kappa_g h^{\lambda\sigma})(\eta_{\alpha\lambda}\eta_{\beta\sigma} - \eta_{\alpha\beta}\eta_{\lambda\sigma}) , \end{aligned}$$

where

$$F = F^\nu_\nu \quad \text{and} \quad h = h^\nu_\nu . \tag{3.2}$$

One can further introduce "diagonalizing" fields $\tilde{F}^{\mu\nu}$ and $\tilde{h}^{\mu\nu}$ by writing

$$\begin{aligned} (\kappa_f^{\,2} + \kappa_g^{\,2})^{1/2}\tilde{F}^{\mu\nu} &= \kappa_f F^{\mu\nu} - \kappa_g h^{\mu\nu} , \\ (\kappa_f^{\,2} + \kappa_g^{\,2})^{1/2}\tilde{h}^{\mu\nu} &= \kappa_g F^{\mu\nu} + \kappa_f h^{\mu\nu} , \end{aligned} \tag{3.3}$$

and finally arrive at the decoupled linear field equations of the form

$$(\Box + \tilde{M}^2)\tilde{F}^{\mu\nu} = 0, \quad \tilde{F}^{\mu\nu}{}_{,\nu} = \tilde{F} = 0 , \tag{3.4}$$

$$\Box(\tilde{h}^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}\tilde{h}) = 0, \quad \tilde{h}^{\mu\nu}{}_{,\nu} = \tfrac{1}{2}\tilde{h}^{,\mu} , \tag{3.5}$$

where $\tilde{M}^2 = M^2(1 + \kappa_g^{\,2}/\kappa_f^{\,2})$ is the renormalized mass of the $\tilde{F}$ field; we have used the conditions for pure spin-2, following from the mass term, to simplify the massive equation. In Eqs. (3.5) we have taken advantage of gauge invariance under transformations

$$\tilde{h}^{\mu\nu} \to \tilde{h}^{\mu\nu} + \Lambda^{\mu,\nu} + \Lambda^{\nu,\mu}$$

to impose the Hilbert gauge.

B. Internal Degrees of Freedom of the Free Fields

The L_{fg} term in the linearized Lagrangian becomes a Pauli-Fierz mass term leading to a massive pure spin-2 field. The linear theory thus contains 5 (from $\tilde{F}^{\mu\nu}$) + 2 (from $\tilde{h}^{\mu\nu}$) = 7 independent polarization modes or degrees of freedom.

While a decomposition into massive and massless parts of the fields is possible in the linear theory, this is not so in the full theory. It is impossible to find a covariant combination of f and g such that the field equations decouple into massive and massless parts. The reason for this is simply that one cannot build from *one* tensor field a covariant mass term.[13] (Note that the cosmological term in GR is not a mass term.) Nevertheless, it is possible to obtain the total number of degrees of freedom also in the full theory.

Like the Einstein equations, the f-g theory contains redundant field components. In GR there exists the Arnowitt-Deser-Misner (ADM) formalism[14] to eliminate systematically constraint and gauge variables by making a [3+1 (space + time)]-dimensional decomposition of the field, and thereby extracting the independent dynamical modes of the theory. For the Einstein Lagrangian the quantities $G=(-g^{00})^{1/2}$ and $G_i=g_{0i}$, $i=1,2,3$, play the role of Lagrangian multipliers leading to four constraint equations on the remaining six quantities g_{ij}, together with their canonical momenta π^{ij}. The 4 constraints +4 gauge conditions, due to coordinate invariance, reduce the dynamical degrees of freedom from 6 to 2.

Lawrence and Toton[15] have generalized the ADM formalism for the f-g theory. They give the Lagrangian (1.1) in a (3+1)-dimensional decomposition and show explicitly that the mass term imposes only four constraints. The reason for this is that the corresponding eight quantities, G, G_i and $F=(-f^{00})^{1/2}$, $F_i=f_{0i}$, appear quadratically in the mass term. Variation with respect to, say, G and G_i leads to equations which determine them. Reintroducing for G and G_i into the Lagrangian leaves the remaining four quantities F, F_i linear, i.e., as Lagrangian multipliers, thus leading to four constraint equations. Again, general covariance allows one to fix a coordinate system by imposing four conditions. Therefore, the total of 24 dynamical variables, namely, g_{ij}, f_{ij}, and their canonical momenta $\pi^{ij}(g)$ and $\pi^{ij}(f)$, are related by eight equations, leading to eight unconstrained pairs of canonical variables. In contrast to the linear theory, the full theory has 8 degrees of freedom.

Thus, in this case, the nonlinearities in the mass term increase the number of degrees of freedom by one.[7,14] Following Deser, we show now why this happens for the f-g theory. If we decompose the linearized mass term, we obtain

$$L_{fg}=\frac{1}{\kappa_f^2}\frac{M^2}{4}\left[(\kappa_f F^{ik}-\kappa_g h^{ik})^2-(\kappa_f F^{ii}-\kappa_g h^{ii})^2+2(\kappa_f F^{00}-\kappa_g h^{00})(\kappa_f F^{ii}-\kappa_g h^{ii})-(\kappa_f F^{0i}-\kappa_g h^{0i})^2\right]. \tag{3.6}$$

In the linearized version the quantities F^{00} and h^{00} appear linearly in the free part of $L(F)$ and $L(h)$, respectively, as can be seen from (3.2) if one introduces the canonical momenta $\pi^{ij}(F)$ and $\pi^{ij}(h)$ of F_{ij} and h_{ij}. But they also appear linearly in (3.6) and, therefore, lead immediately to the following constraints:

$$\begin{aligned}\nabla^2 F_{ii}-F_{ij,ij}&=-M^2\left(F_{ii}-\frac{\kappa_g}{\kappa_f}h_{ii}\right),\\ \nabla^2 h_{ii}-h_{ij,ij}&=M^2\frac{\kappa_g}{\kappa_f}\left(F_{ii}-\frac{\kappa_g}{\kappa_f}h_{ii}\right),\end{aligned} \tag{3.7}$$

where the left-hand side follows from the free Lagrangian.

From (3.6) we see that F^{0i} and h^{0i} occur quadratically, as in the full theory. Variation with respect to them gives six equations containing F^{0i} and h^{0i}. These equations can be separated into three equations defining three of the quantities F^{0i} and h^{0i},

$$\frac{M^2}{4}\frac{\kappa_g}{\kappa_f}(\kappa_f F^{0i}-\kappa_g h^{0i})=-\kappa_g\pi^{ij}(F)_{,j}+\kappa_f\pi^{ij}(h)_{,j}, \tag{3.8}$$

and three constraint equations of the form

$$\kappa_g\pi^{ij}{}_{,j}(F)+\kappa_f\pi^{ij}{}_{,j}(h)=0. \tag{3.9}$$

In the linearized theory the number of constraints is five instead of four. Choosing the corresponding Lagrangian multipliers imposes five more conditions. The total number of unconstrained variables is therefore 24 − 10 = 14, i.e., seven canonical pairs of independent modes. We should mention that in Refs. 7 and 13 it is emphasized that this extra degree of freedom, probably present in any nonlinear Einstein-like massive field theory, causes severe difficulties with the positive definiteness of the energy.

C. Propagation of Fields

It is well known that the causal structure of GR is governed by the null cone of the metric $g_{\mu\nu}(x)$. In particular, disturbances of the g field itself propagate along the characteristic surfaces $\omega(x)$ = const of the Einstein equations, defined by (see Refs. 16 and 17)

$$g^{\mu\nu}\omega_{,\mu}\omega_{,\nu}=0. \tag{3.10}$$

In the f-g theory the characteristic surfaces for the g field are again given by (3.10) whereas for the f field one has

$$f^{\mu\nu}\omega_{,\mu}\omega_{,\nu}=0. \tag{3.11}$$

The mathematical reason for this is that the mass term contains no derivative and therefore does not change the coefficients of highest derivative in the field equations. Hence, there are at each point in space-time two null cones along which disturbances of the gravitational field propagate. In the next section, when considering the coupling to matter and other fields, we shall see that the two cones are also essential for the propagation of other fields and the motion of particles.

The phenomenon of two (or more) propagation cones can occur already in Lorentz-covariant theories, with derivative couplings.[18] Whether this always breaks causality is an open question.

IV. INTERACTION WITH MATTER

A. Motion of Test Particles

In order to obtain an understanding of the metrical structure of the theory consider now the interaction of g and f with matter. In GR all matter (and fields) couples minimally to the gravitational field. This ensures that test particles follow geodesics in a Riemannian space-time and that fields propagate along the g null cone. In the two-tensor theory, hadrons (or hadronic fields) and leptons also couple minimally, but to different fields. Assuming the existence of test particles with only hadronic or leptonic mass, we have, e.g., for a hadronic point particle m_H, the energy-momentum tensor

$$T^{\mu\nu}_H = m_H \int d\lambda \, \frac{\delta^4(x-\xi(\lambda))\dot{\xi}^\mu \dot{\xi}^\nu}{[f_{\rho\sigma}(x)\dot{\xi}^\mu\dot{\xi}^\nu]^{1/2}} \,, \tag{4.1}$$

where $\xi^\mu(\lambda)$ is the world line of the particle with arbitrary parameter λ and $\dot{\xi}^\mu = d\xi^\mu/d\lambda$. From the covariant conservation law $T^H_{\mu\nu}{}^{|\nu} = 0$ then follows the equation of "geodesic" motion,

$$\ddot{\xi}^\mu + \Gamma^\mu_{\nu\rho}(f)\dot{\xi}^\nu\dot{\xi}^\rho = 0 \,, \tag{4.2}$$

with the affine parametrization $\dot{\xi}^\nu f_{\nu\rho}\dot{\xi}^\rho = 1$.

Similarly, one obtains for leptonic test particles with world line $\zeta^\mu(\sigma)$ the equation

$$\zeta''^\mu + \Gamma^\mu_{\nu\rho}(g)\zeta'^\nu\zeta'^\rho = 0, \tag{4.3}$$

where $\zeta' = d\zeta/d\sigma$ and the parameter σ is chosen such that $\zeta'^\mu g_{\mu\nu}\zeta'^\nu = 1$. Therefore, in general the two types of matter will move along different "geodesics." Matter containing a mixture of hadronic and leptonic substances will not follow along geodesics of either field.

B. The Linearized Nonvacuum Equations

The linearized interaction term of the Lagrangian (2.1) is

$$L(f,\ \text{hadrons}) + L(g,\ \text{leptons}) = \kappa_f f_{\mu\nu} T^{\mu\nu}_H + \kappa_g h_{\mu\nu} T^{\mu\nu}_L \tag{4.4}$$

and the diagonalized field equations become

$$\Box(\tilde{h}^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}\tilde{h}) = \tfrac{1}{2}\kappa_f\kappa_g(\kappa_f^2+\kappa_g^2)^{-1/2}(T^{\mu\nu}_L + T^{\mu\nu}_H) \tag{4.5}$$

and

$$\Box\tilde{F}^{\mu\nu} - \tilde{F}^{,\mu\nu} + \tilde{M}^2(\tilde{F}^{\mu\nu} - \eta^{\mu\nu}\tilde{F}) = \tfrac{1}{2}\kappa_f^2(\kappa_f^2+\kappa_g^2)^{-1/2}\left(T^{\mu\nu}_H - \frac{\kappa_g^2}{\kappa_f^2}\,T^{\mu\nu}_L\right) \tag{4.6}$$

with only $\tilde{F}^{\mu\nu}{}_{,\nu} = 0$, while $\tilde{F} \neq 0$ since it couples to the trace of the energy-momentum tensor. We notice that both hadrons and leptons couple to $\tilde{h}^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ (to the $\tilde{h}^{\mu\nu}$ field with equal strength, while to $\tilde{F}^{\mu\nu}$ with the ratio κ_g^2/κ_f^2).

We have not yet specified the coupling constants κ_f, κ_g and the mass M. All that has been said up to now is independent of their values. This, however, becomes important when the connection between theory and experiments is established. If the f-g theory is to be realistic it should reproduce the classical experiments of GR. If we now follow Salam *et al.*[1] and identify the $\tilde{F}$ field with the spin-2 f meson of mass $M \approx 1500$ MeV, Eq. (4.6) tells us that the f meson couples to the energy-momentum tensor of the other hadrons.[19] The coupling constant

$$\tilde{\kappa}_f = \kappa_f^2(\kappa_f^2 + \kappa_g^2)^{-1/2}$$

is estimated to be a few BeV. Because Eq. (4.5) is identical to the linearized Einstein equation, it is natural to identify

$$\tilde{\kappa}_g = \kappa_g\kappa_f(\kappa_f^2 + \kappa_g^2)^{-1/2}$$

with the usual gravitational constant. Thus the ratio of the coupling constants is $\tilde{\kappa}_g/\tilde{\kappa}_f = \kappa_g/\kappa_f \approx 10^{-19}$ which completely suppresses the coupling of $\tilde{F}$ to leptons (f-meson dominance).

Solutions of Eq. (4.6) for a bounded static source show the typical $e^{-\tilde{M}r}/r$ falloff outside the source. Therefore, a weak static F field goes to zero within a distance $\sim 10^{-12}$ cm from its source. The motion of macroscopic bodies in weak static gravitational fields is governed by the long-range component $\tilde{h}$.

C. Geometrical Interpretation

We have seen that at each point in space-time one has two propagation cones. One is given by $f_{\mu\nu}$ along which disturbances of the f field and of fields minimally coupled to f propagate. The other is defined by $g_{\mu\nu}$, for propagation of g and fields coupled to it. Particles will in general not follow geodesics of either fields. This implies that one *cannot* naturally assign to either field (or some combination of them) a metrical interpretation.[20] One could think of taking, for example, the g field to be the metric of space-time. Then leptons would move along geodesics while hadrons would not. The hadronic mass would play the role of a charge, in analogy to electrically charged particles in GR. The important difference in the f-g theory lies in the existence (for this case) of purely hadronic matter. (The same argument applies for any combination of the fields.) Moreover, the theory is "bicausal" at each space-time point, in the sense of having two fundamental propagation velocities.

For weak static fields $\tilde{F}^{\mu\nu} \approx 0$ at distances $> 10^{-12}$ cm from the source. From the definition of $\tilde{F}^{\mu\nu}$ [Eqs. (3.3) and (3.1)] it follows, for regions where $\tilde{F}^{\mu\nu}=0$, that $f_{\mu\nu}=g_{\mu\nu}$. This means that hadrons and leptons move along the *same* geodesics given by the Riemannian metric

$$g_{\mu\nu}=\eta_{\mu\nu}+\kappa_g h_{\mu\nu} \simeq \eta_{\mu\nu}+\tilde{\kappa}_g \tilde{h}_{\mu\nu}, \tag{4.7}$$

where $\tilde{h}_{\mu\nu}$ is given by the linearized Einstein equation (4.5). We note that for strong fields (i.e., where linearization in terms of F and h is not possible) in regions where $\tilde{F}^{\mu\nu}=0$ the f-g equations reduce to the (full) Einstein equations. The Schwarzschild metric is then the only spherically symmetric vacuum solution.

Since all classical tests of GR are in a weak static gravitational field, the f-g theory agrees with the classical experiments.[21]

However, the f-g theory gives results different from those of GR when (even weak) radiation fields are considered. While for static sources we have asymptotically $f_{\mu\nu}=g_{\mu\nu}$, for radiation this is not the case. In the next section we shall give an exact radiation solution for which $f_{\mu\nu}\neq g_{\mu\nu}$ everywhere. In principle, radiation experiments could test between f-g theory and GR, e.g., a hadronic harmonic oscillator will only be excited by an f wave and remain unaffected by g waves.

V. EXACT SOLUTIONS

The f-g equations show all the difficulties of the Einstein equations plus the coupling due to the mass term. In GR, symmetry assumptions on the metric have been very successful in finding exact solutions. This is not yet the case for the f-g theory. However, imposing algebraic relations between f and g fields in order to simplify the mass term has turned out to be profitable.

The simplest relation is $f_{\mu\nu}=g_{\mu\nu}$; for it the mass term vanishes and one obtains Einstein's solutions. The generalization of this relation to $f_{\mu\nu}=\alpha(x)g_{\mu\nu}$ leads to a theorem formulated by Pirani,[22] which we state without proof.

Theorem II. The only vacuum solutions where f and g are conformally related are trivial, i.e., $\alpha(x)=1$.

All exact solutions known so far can be written in the form

$$f_{\mu\nu}=g_{\mu\nu}\pm l_\mu l_\nu, \tag{5.1}$$

where l_μ is a null vector field with respect to g and, therefore, also to f, i.e., $g^{\mu\nu}l_\mu l_\nu=f^{\mu\nu}l_\mu l_\nu=0$. Geometrically (5.1) means that the two null cones hyperosculate each other along l_μ.

A. Vacuum Solutions

The simplest solution (and first found[23]) of the type (5.1) are the pp waves known from GR.[24] They can be written in the form

$$g_{\mu\nu}dx^\mu dx^\nu=2G(dx^0)^2-2dx^0dx^3-(dx^1)^2-(dx^2)^2, \tag{5.2}$$

$$f_{\mu\nu}dx^\mu dx^\nu=2F(dx^0)^2-2dx^0dx^3-(dx^1)^2-(dx^2)^2,$$

where G and F are functions of x^0, x^1, and x^3 only. The null vector of (5.1) is $l_\mu \sim \delta^0_\mu$, and from the field equations (2.5) only the $\mu=\nu=0$ component remains, leading to simple linear equations:

$$\Delta_2 F=M^2(F-G),$$

where

$$\Delta_2=\frac{\partial^2}{(\partial x^1)^2}+\frac{\partial^2}{(\partial x^2)^2}, \qquad \Delta_2 G=-\frac{\kappa_g^{\,2}}{\kappa_f^{\,2}}M^2(F-G). \tag{5.3}$$

Physically, solutions of Eqs. (5.3) represent waves of the f and g fields propagating in the (same) l_μ direction.

One knows from GR that the null congruence to which l_μ is tangent is goedesic with vanishing optical scalars because l_μ is covariantly constant with respect to *both* fields. For both fields the Weyl tensor is of Petrov type N, thus they are pure radiation fields.

Mansouri and Urbantke[25] have generalized the above solutions to the Kerr-Schild[26] form

$$g_{\mu\nu}dx^\mu dx^\nu=\eta_{\mu\nu}+2G l_\mu l_\nu, \qquad f_{\mu\nu}dx^\mu dx^\nu=\eta_{\mu\nu}+2F l_\mu l_\nu \tag{5.4}$$

with $l_\mu \eta^{\mu\nu} l_\nu=0$ and l_μ not necessarily covariantly constant.

In GR the Kerr-Schild class contains important solutions such as the Schwarzschild, Kerr, and Vaidya metric. In f-g theory the result is rather poor. All solutions of the form (5.4) have algebraic type N and are plane-fronted waves, i.e., fields with a geodesic null congruence with vanishing optical scalars. The difference to pp waves is that the rotation does not vanish in general. Finally, Urbantke[27] studied general vacuum fields under the condition (5.1), where the previous solutions are contained as sub-cases.

Summarizing the general properties of all the vacuum solutions so far known, one has that:

(a) l_μ is a tangent vector to a congruence of null geodesics with vanishing twist, shear, and expansion, for *both* fields.

(b) Both fields have algebraic special Weyl tensors of the *same* type, with l_μ a multiple principal null direction.

B. A Nonvacuum Solution

We give a prescription for obtaining a nonvacuum solution from the special vacuum solution (5.2).

Assume that the f and g fields are of the form (5.2), but with $G\equiv 0$, i.e.,

$$g_{\mu\nu}dx^\mu dx^\nu = 2dx^0 dx^3 - (dx^1)^2 - (dx^2)^2 . \qquad (5.5)$$

This implies $R_{\mu\nu\rho\sigma}(g)=0$, and the g field is flat, written in radiation coordinates. In vacuum it follows from Theorem I or Eq. (5.3) that $F\equiv 0$ as well. However, taking an energy-momentum tensor of the radiation form

$$T^L_{\mu\nu} = \tau(x) l_\mu l_\nu, \quad l_\mu = \delta^0_\mu , \qquad (5.6)$$

which couples only to the g field, while $T^H_{\mu\nu}=0$, we get nontrivial solutions. The equations can simply be read off from Eqs. (5.3) by adding (5.6) and setting $G\equiv 0$,

$$\Delta_2 F = M^2 F, \quad M^2 F = -\tfrac{1}{2}\kappa_f^2 \tau , \qquad (5.7)$$

so that $\tau(x)$ must satisfy the equations

$$\Delta_2 \tau = M^2 \tau \quad \text{and} \quad \frac{\partial}{\partial x^3}\tau(x) = 0 \qquad (5.8)$$

for covariant conservation of (5.6). The solution to (5.8) will be similar to the solutions of the decoupled vacuum equations. We show elsewhere[28] that explicit solutions taking for $T^L_{\mu\nu}$ and electromagnetic null field can be obtained. Thus we have a solution where the source of the g equations leaves the g field flat, i.e.,

$$R_{\mu\nu\rho\sigma}(g) = 0 \quad \text{but} \quad T^L_{\mu\nu}(g) \neq 0 .$$

Finally, we should mention that Aragone and Chela-Flores[29] have tried to find a spherical symmetric vacuum solution for the f field by setting $g_{\mu\nu}=\eta_{\mu\nu}$. Since our Theorem I also requires $f_{\mu\nu}=\eta_{\mu\nu}$, they also had to put $\kappa_g=0$, so that only the f equation remained. They succeeded in giving a solution in the form of a power series showing the expected Yukawa behavior.

VI. CONCLUSION

We have seen that f-g gravity, although a generally covariant theory, cannot be geometrized. The reason for this is the nonuniversal coupling of all matter and fields. Moreover, the concept of causal metrical structure, the main element of classical field theories, breaks down due to the existence of two propagation cones at each space-time point.

In spite of these drastic consequences the theory is capable of reducing to GR for present experimental situations. The large value of the f-meson mass and weak static field approximation makes this possible. Differences from GR come into play for strong or radiation fields.

The linearized theory is a "normal" massless and massive pure spin-2 field theory which has 1 degree of freedom less than the nonlinear theory. We have shown how this degree is constrained in the linear theory. Further, we have summarized the exact solution known at present, giving also a nonvacuum field. None of these fields is spherical symmetric. The knowledge of an exact spherical symmetric solution would show the implications of f-g gravity for gravitational collapse, giving an answer to Salam's question, whether two-tensor gravity becomes repulsive for short distances.

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