[1]A. Tomimatsu and H. Sato, Phys. Rev. Lett. 29, 1344 (1972).
[2]K. S. Thorne, in *High Energy Astrophysics*, edited by C. DeWitt, E. Schatzman, and P. Veron (Gordon and Breach, New York, 1967), Vol. III, p. 319, Ref. 9.
[3]R. H. Boyer and R. W. Lindquist, J. Math. Phys. 8, 265 (1967).

PHYSICAL REVIEW D VOLUME 8, NUMBER 2 15 JULY 1973

Restricted Proof that the Weak Equivalence Principle Implies the Einstein Equivalence Principle*

Alan P. Lightman[†] and David L. Lee[‡]
California Institute of Technology, Pasadena, California 91109
(Received 5 March 1973)

Schiff has conjectured that the weak equivalence principle (WEP: free-fall trajectories independent of test-body composition) implies the Einstein equivalence principle (EEP: all nongravitational laws of physics the same in every freely falling frame). This paper presents a proof of Schiff's conjecture, restricted to (i) test bodies, made of electromagnetically interacting point particles, that fall from rest in a static, spherically symmetric gravitational field; and (ii) theories of gravity within a certain broad class—a class that includes almost all complete relativistic theories that we have found in the literature, but with each theory truncated to contain only point particles plus electromagnetic and gravitational fields. The proof shows that every "nonmetric" theory in the class (every theory that violates EEP) must violate WEP. A formula is derived for the magnitude of the violation. Comparison with the results of Eötvös-Dicke-type experiments rules out various nonmetric theories, including those of Belinfante and Swihart and of Naida and Capella—theories that previously were believed to agree with all current experiments. It is shown that WEP is a powerful theoretical and experimental tool for constraining the manner in which gravity couples to electromagnetism in gravitation theories.

I. INTRODUCTION

In a previous paper[1] we have discussed the content and significance of Schiff's conjecture. In brief, the conjecture states that all theories of gravity which satisfy the weak equivalence principle[1] (WEP), i.e., predict a unique composition-independent trajectory for any test body at a given point of spacetime and with a given initial velocity through that point, must satisfy the Einstein equivalence principle (EEP), i.e., must show that the nongravitational[1] laws of physics are the same in every freely falling frame. When specialized to "relativistic theories of gravity"[1] (as will be done throughout this paper), Schiff's conjecture says that every theory satisfying WEP is necessarily a "metric theory."[1] Plausibility arguments (e.g., Refs. 1 and 2) have frequently been given for the conjecture, but there have been few detailed calculations that bear upon its validity or invalidity. Indeed, the conjecture is so sweeping that it will probably never be proved with complete generality. (Such a proof would require a moderately deep understanding of all gravitation theories that satisfy WEP—including theories not yet invented, and never destined to be invented. Such understanding is well beyond one's grasp in 1973.)

On the other hand, one can gain useful insight by proving restricted versions of the conjecture, and by searching for the most general versions that are provable. For example, one might first analyze test bodies with purely electromagnetic internal interactions and thereby attempt to show that particles and electromagnetism must interact with gravity in the manner of metric theories (EEP) in order that WEP be satisfied; next analyze purely nuclear systems and attempt to show that nuclear fields must couple to gravity metrically; etc. Unfortunately, for our purposes, nuclear interactions have not been given an adequate mathematical representation even in the absence of gravity; and the nonmetric theories known to us make no attempt to write down nuclear force laws. Hence our present program must end one way or another after the first stage. Even a general proof of the first stage (Schiff conjecture for bodies with internal electromagnetic interactions) is too much to expect. To make it manageable, one must assume some restricted (but hopefully quite general) form for the interactions. This we shall do in the present paper—with an interaction form general enough to include all metric theories plus almost all nonmetric theories we have found in the published literature. As a byproduct of our proof, we can rule out several nonmetric theories in the literature.

In order not to prejudice ourselves, the language and concepts used in the calculation will be those employed in standard classical field theory with gravity treated as just another ordinary field. In particular, we will not use such phrases as "curved spacetime" and will not make any coordinate transformations to real or pseudo- "freely falling frames." The concept of gravity as a metric phenomenon should be forced upon us by WEP.

As spelled out in Sec. II, we shall take a non-quantum-mechanical approach and shall use a particle rather than a fluid picture for the test body. Since the gravitation theories with which we attempt to tie in are largely classical theories, we feel that a classical approach is completely justified and perhaps essential. There are two reasons why a particle approach has been taken: first, more often than not, classical field theories formulate the interaction of gravity with matter in the form of point particles; second, a charged-particle approach allows one to deal with the exact "gravitationally modified Maxwell equations" of a given theory, rather than with their smeared-out averages.

Our calculation is not the first of its type. For several particular theories, and at lower orders of approximation, the acceleration of electromagnetic test bodies in a gravitational field has been previously calculated. Nordtvedt[3] and Belinfante and Swihart[4] have both done calculations, to first order in the gravitational field potential and squared particle velocities; Nordtvedt for general metric theories, and Belinfante and Swihart for their theory of gravity. In addition, Post[5] has done a calculation, at post-Newtonian order, of the acceleration of a confined quantity of electromagnetic energy in a gravitational field. Had his calculation been carried to higher order it is conceivable he could have obtained part of our result: that $\epsilon=\mu$ [cf. Eq. (21)].

Section II of this paper gives an outline of the assumptions, procedure, and techniques of our calculation, including the results; Sec. III presents the details. Section IV compares the predictions for WEP violation with the results of Eötvös-Dicke-type experiments, and thereby rules out the nonmetric theories of Belinfante and Swihart,[4,6] Capella,[7] Naida,[8] and Whitehead.[9] Also discussed is the manner—both quantitative and qualitative—in which WEP is an experimental probe of the "gravitational-Maxwell equations," as contrasted to previously recognized experimental tests of those equations.

II. GENERAL FRAMEWORK AND RESULTS

In calculating the center-of-mass acceleration of an electromagnetic test body, we would like to set up a formalism which includes as many types of gravitation theories as possible, but which is not too complicated. In particular, our formalism should be able to deal with scalar, vector, tensor, scalar-tensor, etc. theories.

We have found that all of these different types of theories can be put into a somewhat universal form when describing a static, spherically symmetric (SSS) gravitational field—providing their dynamical law[1] for particle motion is derivable from a Lagrangian. (The restriction to SSS fields is certainly a limitation in principle, but it allows us to handle many different theories at once; and, as discussed in Sec. IV, is not a limitation in practice.) The quasiuniversal description of particles and electromagnetism in an SSS field is as follows:

The motion of charged particles under the joint action of gravity and the electromagnetic field A_μ can be derived from the Lagrangian[10]

$$\bar{L}=\sum_k \int [-m_{0k}(T-H\vec{v}_k{}^2)^{1/2}+e_k A_\mu v_k^\mu]dt\,, \qquad (1)$$

where we have used the bar above the L to indicate that $\bar{L}$ may be only a part of the total Lagrangian, and where the various symbols will be defined below. The "gravitationally modified Maxwell equations" (GMM: Maxwell's equations in the presence of a gravitational field) are of the form

$$\vec{\nabla}\cdot(\epsilon\vec{E})=4\pi\rho\,, \qquad (2)$$

$$\vec{\nabla}\times(\mu^{-1}\vec{B})=4\pi\vec{J}+\frac{\partial}{\partial t}(\epsilon\vec{E})\,. \qquad (3)$$

Definitions of the quantities in Eqs. (1) – (3) and of other quantities that will be used in the calculation are given below:

$x^l\equiv$ spatial coordinates; they are nearly Cartesian when gravity is weak,

$t\equiv$ a time coordinate associated with the static nature of the SSS field, nearly equal to proper time for slowly moving particles when gravity is weak,

$m_{0k}\equiv$ rest mass of particle k, a constant,
$e_k\equiv$ charge of particle k, a constant,
$x_k^\mu(t)\equiv$ world line of particle k,
$v_k^\mu\equiv dx_k^\mu/dt$,
$x^0\equiv t$,
$\vec{v}_k{}^2\equiv\delta_{ij}v_k^i v_k^j$ with δ_{ij} the 3-Kronecker δ,

$U(r)\equiv$ a gravitational potential equal to M_s/r, where M_s is a constant ("active gravitational mass") characterizing the source of the SSS field, and r is coordinate distance, $[(x-x_s)^2+(y-y_s)^2$

$+(z-z_s)]^{1/2}$, from source of field point,

$\vec{\nabla}, \vec{\nabla}\cdot\equiv$ the usual differential operators of gravity free Euclidean space,

$\vec{g}\equiv\vec{\nabla}U\equiv$ the gravitational acceleration to be expected if the theory in question were Newtonian theory,

$T, H, \epsilon, \mu\equiv$ functions of the gravitational potential U; functions that are arbitrary in this calculation but that have a specific form in each theory of gravity when the coordinate system has been suitably specified,

$A^\mu\equiv$ components of an electromagnetic vector potential, a four-vector,

$(\vec{A})^l\equiv A_l\equiv$ spatial part of vector potential,

$$\varphi\equiv -A_0,$$

$$\vec{J}\equiv\sum_k e_k\vec{v}_k\delta^3(\vec{x}-\vec{x}_k(t)), \tag{4a}$$

$$\rho\equiv\sum_k e_k\delta^3(\vec{x}-\vec{x}_k(t)), \tag{4b}$$

$$\vec{E}\equiv\vec{\nabla}A_0-\partial\vec{A}/\partial t, \tag{4c}$$

$$\vec{B}\equiv\vec{\nabla}\times\vec{A}. \tag{4d}$$

Although in most theories the form of $\overline{L}$ in Eq. (1) is typical only of SSS fields, it turns out that all of the results we shall obtain hold even if U is an arbitrary, but time-independent function of position.

For an SSS field in a given theory, T, H, ϵ, and μ will be particular functions of U (and hence of position). *Here we assume that T, H, ϵ, and μ have been given and we seek the relations among them, if any, that are required for compliance with* WEP. It is clear from Eq. (1) that we have sacrificed general covariance of the particle Lagrangian in order to encompass a wide range of theories.

Note that Eqs. (2)–(3) can be reinterpreted (different physics; same mathematical representation) as the usual Maxwell equations for a permeable medium in which the free sources originate from charged particles labeled by k. Thus ϵ and μ play the role of "gravitationally induced dielectric and permeability parameters," respectively. We require that T, H, ϵ, μ all approach unity as U vanishes so that the special relativistic limit is maintained.

Given the SSS restriction, one may ask how general are Eqs. (1)–(3). Except in the most general (nonmetric) case of Jordan's theory,[11] which is incomplete[1] in the sense that it involves unspecified processes of particle creation, all theories we know of which are complete enough to formulate the interaction of the electromagnetic field with gravity have GMM equations of the form of Eqs. (2)–(3).[12] In fact, the "ϵ-μ formulation" of the sourceless Maxwell equations in metric theories has sometimes been used in calculations.[13] The particle Lagrangian $\overline{L}$ [cf. Eq. (1)] also appears to be fairly general, except for a class of theories discussed by Naida[8] which includes the theory of Capella.[7] We treat the Capella-Naida theory on an individual basis in Sec. IV, using the methods developed in this section. We point out that it is sometimes necessary to perform a reformulation (same theory; new "mathematical representation") of a theory in order to put it into the form of Eqs. (1)–(3) (see, for example, the Belinfante-Swihart theory as analyzed in Ref. 14). Finally, we should emphasize that, even more important than the generality of Eqs. (1)–(3), are the techniques and methods developed in this section, since they can also be applied on an individual basis to that handful of theories which is not included in Eqs. (1)–(3). We now proceed with an outline of our calculations.

Variation of Eq. (1) yields an expression for the acceleration of the kth particle, which, together with Eqs. (2) and (3) constitutes three coupled equations. We seek a perturbation solution. There are two obvious, small dimensionless quantities in which one could expand: the gravitational potential U and the squared particle velocities $\vec{v}_k^{\,2}$. Since we prefer a result correct to all orders in the gravitational potential, we expand only in $\vec{v}_k^{\,2}$ and leave T, H, ϵ, and μ as arbitrary functions of U. We do, however, expand these latter functions in a Taylor series about the instantaneous center of mass of the test body (defined below), i.e.,

$$T=T_0+(\vec{g}\cdot\vec{x})T_0'+\cdots, \tag{5}$$

where

$$T'\equiv dT/dU \quad\text{and}\quad T_0'\equiv(dT/dU)_{\vec{x}=0}. \tag{6}$$

We shall assume that the body is small enough so that second derivatives of U make negligible contributions. Indeed, this is part of the definition of "test body" (Ref. 1) and is a necessary and integral qualification in Schiff's conjecture.

We define the center of mass for the test body by the following sequence of equations:

$$\begin{aligned} m_k = {} & m_{0k}\{1+F[U(\vec{x}_k)]\}+\tfrac{1}{2}m_{0k}\vec{v}_k^{\,2}\{1+G[U(\vec{x}_k)]\} \\ & +\tfrac{1}{2}e_k\sum_i e_i|\vec{x}_{ik}|^{-1}\{1+K[U(\vec{x}_i)]+S[U(\vec{x}_k)]\} \\ & +O(m_0v^4), \end{aligned} \tag{7}$$

$$\vec{x}_{ik}\equiv\vec{x}_i-\vec{x}_k, \qquad M\equiv\sum_k m_k, \tag{8}$$

$$\vec{X}_{\text{c.m.}}\equiv M^{-1}\sum_k m_k\,\vec{x}_k. \tag{9}$$

Here F, G, K, S are again arbitrary functions of the potential U. (Whenever two indices, e.g., i and k, occur in terms, in double or single sums, it is always assumed that $i \neq k$ in the sum.) Any credible result should be independent of the particular definition of the center of mass as long as it remains inside of the body, that is, the result should not depend on the specific forms of the functions F, G, K, and S.

We now assume that at $t=0$, the center of mass of the test body is momentarily at rest, at the origin of the coordinate system,

$$(\vec{X}_{\text{c.m.}})_{t=0} = (d\vec{X}_{\text{c.m.}}/dt)_{t=0} = 0 . \tag{10}$$

By differentiating Eq. (9) twice and combining with Eqs. (10), we obtain for the instantaneous center-of-mass acceleration

$$\vec{A}_{\text{c.m.}} = M^{-1}\Big(\sum_k \ddot{m}_k \vec{x}_k + 2\sum_k \dot{m}_k \vec{v}_k + \sum_k m_k \vec{a}_k\Big), \tag{11}$$

where

$$\vec{A}_{\text{c.m.}} \equiv d^2\vec{X}_{\text{c.m.}}/dt^2 ,$$

$$\vec{a}_k \equiv d\vec{v}_k/dt,$$

$$\dot{m}_k \equiv dm_k/dt , \quad \text{etc.}$$

Return for a moment to the details of the expansion scheme. Our expansion is in the quantity

$$v^2 \equiv (\text{typical squared particle velocity}) \gtrsim v_k^2 . \tag{12a}$$

The virial theorem guarantees that

$$v^2 \approx \frac{(\text{typical charge of a particle})^2}{(\text{typical mass})(\text{typical separation of neighboring particles})} \gtrsim \frac{e_k^2}{m_k|\vec{x}_{ik}|} . \tag{12b}$$

Thus, without serious error, we may treat both terms on the right-hand sides of Eqs. (12a) and (12b) as $O(v^2)$ when ordering the terms in the expansion.

Besides the dimensionless quantity v^2 in which we *do* expand, and the dimensionless quantity U in which we *do not* expand, there is a third, less obvious dimensionless quantity:

$$gs \equiv |\vec{g}|(\text{size of test body}) \gtrsim |\vec{g}||\vec{x}_k| . \tag{13}$$

We shall expand in this quantity—independently of the v^2 expansion—but, in practice, by examining powers of g rather than gs.

Now, if $\vec{A}_{\text{c.m.}}$ is to be body-independent in general, it must be so for each order in v^2 and each order in g, independently. Surprisingly, perhaps, it will be sufficient to work to first order in v^2 and to first order in g. The imposition of WEP at this order will force the dynamical equations (1)–(3) to take on metric form, thereby guaranteeing that EEP *(and hence* WEP *a fortiori)* is satisfied at all orders.

To first order in v^2 and g, after solving Eqs. (1)–(3) for $\vec{a}_k$ and substitution into Eq. (11), we find (details given in Sec. III)

$$\vec{A}_{\text{c.m.}} = -\tfrac{1}{2}\vec{g}(T_0' H_0^{-1}) + \vec{g} M_0^{-1}\Big[\tfrac{1}{2}(H_0' H_0^{-1})\sum_i m_{0i} v_i^2 + \bar{\eta}\sum_{i,k}\eta_{ik}\Big] + M_0^{-1}\bar{\omega}\sum_{i,k}\vec{\omega}_{ik} + M_0^{-1}\theta\sum_i m_{0i}(\vec{g}\cdot\vec{v}_i)\vec{v}_i , \tag{14}$$

where

$$M_0 \equiv \sum_i m_{0i} , \tag{15a}$$

$$\bar{\eta} \equiv (T_0^{1/2} H_0^{-1})(\tfrac{1}{2}\epsilon_0'\epsilon_0^{-2} + \tfrac{1}{4} T_0' \mu_0 H_0^{-1}) , \tag{15b}$$

$$\bar{\omega} \equiv \tfrac{1}{2}(T_0^{1/2}H_0^{-1})(\tfrac{1}{2}T_0' H_0^{-1}\mu_0 + \tfrac{1}{2}T_0' T_0^{-1}\epsilon_0^{-1} - H_0' H_0^{-1}\epsilon_0^{-1}) + (1+F_0)^{-1}[F_0' T_0^{1/2} H_0^{-1}\epsilon_0^{-1} - \tfrac{1}{2}(1+G_0) T_0' T_0^{1/2} H_0^{-2}\epsilon_0^{-1}] , \tag{15c}$$

$$\theta = T_0' T_0^{-1} - H_0' H_0^{-1} + 2(1+F_0)^{-1}[F_0' - \tfrac{1}{2}(1+G_0) T_0' H_0^{-1}] , \tag{15d}$$

$$\eta_{ik} \equiv e_i e_k |\vec{x}_{ik}|^{-1} , \tag{15e}$$

$$\vec{\omega}_{ik} \equiv e_i e_k (\vec{g}\cdot\vec{x}_{ik})|\vec{x}_{ik}|^{-3}\vec{x}_{ik} . \tag{15f}$$

Equation (14) becomes much simplified when we use some gravitationally modified virial relations (see Sec. III C for details):

$$\Big\langle \sum_i m_{0i} v_i^m v_i^p + \tfrac{1}{2}(T_0^{1/2} H_0^{-1}\epsilon_0^{-1})\sum_{i,k} e_i e_k x_{ik}^m x_{ik}^p |x_{ik}|^{-3}\Big\rangle = O(M_0 v^2 gs) , \tag{16}$$

where m, p refer to to components of the appropriate vectors and $\langle\,\rangle$ denotes the usual time average. Using Eq. (16), Eq. (14) becomes

$$\langle \vec{A}_{\text{c.m.}} \rangle = -\tfrac{1}{2}\vec{g}(T_0' H_0^{-1}) - \tfrac{1}{4}\vec{g} M_0^{-1}(T_0^{1/2} H_0^{-1}\epsilon_0^{-1})(H_0' H_0^{-1} - 2\epsilon_0' \epsilon_0^{-1} - T_0' \epsilon_0 \mu_0 H_0^{-1})\left\langle \sum_{ik} \eta_{ik} \right\rangle$$
$$- \tfrac{1}{4} M_0^{-1}(T_0' T_0^{1/2} H_0^{-2}\epsilon_0^{-1})(H_0 T_0^{-1} - \epsilon_0 \mu_0)\left\langle \sum_{i,k} \vec{\omega}_{ik} \right\rangle . \tag{17}$$

The first term of this acceleration is body-independent (satisfies WEP); the second term depends on the body's self-electromagnetic energy; the third term depends on the electromagnetic energy, the shape of the body, and the orientation of the body with respect to the gravitational field gradient. Thus $\langle \vec{A}_{\text{c.m.}} \rangle$ will always be body-independent only if the second and third terms always vanish, i.e.,

$$H_0'/H_0 - 2\epsilon_0'/\epsilon_0 - T_0'\epsilon_0 \mu_0/H_0 = 0 , \tag{18a}$$

$$H_0/T_0 - \epsilon_0 \mu_0 = 0 \tag{18b}$$

(the other factors in the body-dependent terms must be nonzero for correct Newtonian and special relativistic limits), or equivalently,

$$\epsilon_0'/\epsilon_0 = \tfrac{1}{2}(H_0'/H_0 - T_0'/T_0) , \tag{19a}$$

$$\mu_0 = H_0/(T_0 \epsilon_0) . \tag{19b}$$

Since we have not specified the initial location of our test body with respect to the external gravitating source, and Eqs. (19) should be satisfied at any point we choose to deposit the body, the naught subscript can be removed from quantities in those equations, yielding, upon integration,

$$\epsilon = C(H/T)^{1/2} , \tag{20a}$$

$$\mu = C^{-1}(H/T)^{1/2} , \tag{20b}$$

where C is a constant. Since, "in the absence of gravity,"[1] we must have $\epsilon = H = T = 1$, C must also be unity. Therefore we finally obtain, as a necessary condition for our electromagnetic test body to fall with a composition-independent acceleration:

$$\epsilon = \mu = (H/T)^{1/2} . \tag{21}$$

It is worth noting that, using heuristic arguments (see, e.g., Ref. 15) about the electromagnetic energy content of atoms and the expression for the fine-structure "constant" α in a dielectric medium

$$\alpha = (\epsilon \mu)^{1/2} e^2/(\epsilon \hbar)$$

one can see why WEP should require constancy of the ratio (ϵ/μ).

Comparison of Eqs. (21) and (1)–(3) with the discussion in Sec. III E reveals that Eq. (21) is a necessary and sufficient condition for the dynamical equations (1)–(3) to take on the familiar metric form

$$\bar{L} = \sum_k \int -m_{0k} ds_k + e_k A_\mu dx_k^\mu , \tag{22}$$

$$F^{\alpha\beta}{}_{;\beta} = 4\pi J^\alpha . \tag{23}$$

In this metric form

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta , \tag{24a}$$

$$g_{00} = T , \tag{24b}$$

$$g_{ij} = -\delta_{ij} H \text{ (spherical coordinates turn out to be "isotropic")}, \tag{24c}$$

; denotes the covariant derivative with respect to $g_{\alpha\beta}$,

$$F^{\alpha\beta} = g^{\alpha\tau} g^{\beta\mu}(A_{\mu,\tau} - A_{\tau,\mu}) , \tag{24d}$$

$$J^\alpha \equiv \sum_k \int e_k \delta^4(\underline{x} - \underline{z}(s))(dx_k^\alpha/ds)(-g)^{-1/2} ds_k . \tag{24e}$$

Note that all dependence on the arbitrary functions used in the center-of-mass definition, Eq. (7), has vanished by the time one reaches Eq. (17).

Higher-order calculations [v^4 or $(gs)^2$, for example] could only yield results consistent with Eq. (21), since WEP *at first order implies that gravity has a metric-theory description (automatically satisfying* WEP*) to all orders.*

Our theoretical results can be summarized by the following statement: *Consider the class of gravitation theories that possesses a mathematical representation of the form of Eqs. (1)–(3). For that class, with each theory written in that representation,*

(WEP) $\Longleftrightarrow$ [Eq. (21)] $\Longleftrightarrow$ [the theory is metric with the metric given by Eqs. (24b)–(24c)].

III. DETAILS OF THE CALCULATION

A. Single-Particle Equations of Motion

Variation of Eq. (1) with respect to the coordinates of particle k yields

$$(H W^{-1})\vec{a}_k + \vec{v}_k \frac{d(H W^{-1})}{dt} + \tfrac{1}{2} W^{-1} \vec{\nabla}(T - H v_k^2) = \vec{A}_L(\vec{x}_k) , \tag{25}$$

where

$$W \equiv (T - H v_k^2)^{1/2} , \tag{26a}$$

$$\vec{A}_L(\vec{x}_k) \equiv \text{Lorentz acceleration of particle } k = (e_k/m_{0k})\left\{-\vec{\nabla}\varphi(\vec{x}_k) + \vec{\nabla}[\vec{v}_k\cdot\vec{A}(\vec{x}_k)] - \frac{d}{dt}\vec{A}(\vec{x}_k)\right\}, \tag{26b}$$

and all functions of U are evaluated on the particle's world line, e.g., $H \equiv H(U[\vec{x}_k(t)])$. Using Eqs. (5)–(6) and the discussion following Eqs. (13), we can write, to the order of our calculation,

$$\vec{\nabla}H = H_0'\vec{g}, \text{ etc}. \tag{27}$$

We shall regard $\vec{g}$ as spatially constant [see discussion following Eq. (6)]. Equation (25) can then be written as

$$\vec{a}_k = \tfrac{1}{2}\vec{g}(H_0'v_k^2 - T_0')H_0^{-1} - \vec{v}_k(\vec{v}_k\cdot\vec{g})[H_0'H_0^{-1} - \tfrac{1}{2}(T_0' - v_k^2H_0')W^{-2}] - \vec{v}_k(\vec{v}_k\cdot\vec{a}_k)HW^{-2} + (WH^{-1})\vec{A}_L. \tag{28}$$

Note that whenever functions like H, T, ϵ, etc. occur in terms multiplied by $\vec{g}$, we may evaluate them at naught, i.e.,

$$H\vec{g} \to H_0\vec{g},$$

because we work only to first order in g.

We further expand W in a power series in v^2 and, since we are only working to $O(v^2)$, we can set $W = T^{1/2}$ in Eq. (28). This follows from the fact that $\vec{A}_L \sim O(v^2)$ and from the explicit velocity dependence of other terms in Eq. (28). [It should be mentioned that when a term is considered $O(v^2)$, it is not necessarily intended that the term is dimensionless, but only that v^2 (or the expression in Eq. (12b)) is a multiplicative factor in the term. The same applies to the notation $O(g)$.]

By dotting $\vec{v}_k$ into both sides of Eq. (28), solving for $(\vec{a}_k\cdot\vec{v}_k)$, and substituting the result back into Eq. (28), we obtain

$$\vec{a}_k = \tfrac{1}{2}\vec{g}(H_0'v_k^2 - T_0')H_0^{-1} + \vec{v}_k(\vec{v}_k\cdot\vec{g})(T_0'T_0^{-1} - H_0'H_0^{-1}) + (T^{1/2}H^{-1})\vec{A}_L + O(v^4) + O(g^2). \tag{29}$$

B. The Gravitationally Modified Maxwell Equations

We must now solve Maxwell's equations and compute the quantity $\vec{A}_L$ which occurs in Eq. (29). If Eqs. (4c) and (4d) are substituted into Eqs. (2) and (3) and one uses the gauge

$$(\epsilon\mu)\frac{\partial\varphi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0, \tag{30}$$

the result is

$$\nabla^2\varphi = \epsilon\mu\frac{\partial^2\varphi}{\partial t^2} - 4\pi\rho\epsilon^{-1} - \epsilon^{-1}\vec{\nabla}\epsilon\cdot\left(\vec{\nabla}\varphi + \frac{\partial\vec{A}}{\partial t}\right), \tag{31a}$$

$$\nabla^2\vec{A} = \epsilon\mu\frac{\partial^2\vec{A}}{\partial t^2} - 4\pi\mu\vec{J} + (\epsilon\mu)^{-1}(\vec{\nabla}\cdot\vec{A})\vec{\nabla}(\epsilon\mu) + \mu^{-1}(\vec{\nabla}\times\vec{A})\times\vec{\nabla}\mu. \tag{31b}$$

We can now do a perturbation solution of these equations by expanding simultaneously in powers of v^2 and g, treating *formally* $v^2 \sim g$:

$$\varphi = \varphi_0 + \varphi_1 + \varphi_2 + \cdots, \tag{32a}$$

$$\vec{A} = \vec{A}_0 + \vec{A}_1 + \vec{A}_2 + \cdots, \tag{32b}$$

$$\nabla^2\varphi_0 = -4\pi\epsilon^{-1}\rho, \tag{33a}$$

$$\nabla^2\varphi_1 = \epsilon\mu(\partial^2\varphi_0/\partial t^2) - \epsilon_0^{-1}\epsilon_0'[\vec{g}\cdot(\vec{\nabla}\varphi_0 + \partial\vec{A}_0/\partial t)], \tag{33b}$$

$$\nabla^2\varphi_2 = \epsilon\mu(\partial^2\varphi_1/\partial t^2) - \epsilon_0^{-1}\epsilon_0'[\vec{g}\cdot(\vec{\nabla}\varphi_1 + \partial\vec{A}_1/\partial t)], \quad \text{etc.}, \tag{33c}$$

$$\nabla^2\vec{A}_0 = -4\pi\mu\vec{J}, \tag{34a}$$

$$\nabla^2\vec{A}_1 = \epsilon\mu(\partial^2\vec{A}_0/\partial t^2) + (\epsilon\mu)_0^{-1}(\vec{\nabla}\cdot\vec{A}_0)\vec{\nabla}(\epsilon\mu) + \mu_0^{-1}\mu_0'(\vec{\nabla}\times\vec{A}_0)\times\vec{g}, \quad \text{etc.} \tag{34b}$$

(One should not confuse the perturbation order of $\vec{A}, \vec{A}_k$, with the kth component of the vector A_k.)

The solution of these equations is far simpler if we remember from the beginning that since the particle acceleration is required only to $O(v^2)$ and $O(g)$ we need $\vec{A}_L$ only to the same order. Remember also that $\vec{a}_k = O(v^2) + O(g)$ whenever the solution of Eqs. (33)–(34) requires a particle acceleration as a source term (right-hand side of equations).

We solve the equations for $\vec{A}$ first. Clearly, from the expression for $\vec{J}$ [cf. Eq. (4a)],

$$\vec{A}_0(\vec{x}_k) = \sum_i e_i\vec{v}_i\mu(\vec{x}_i)|\vec{x}_{ki}|^{-1}. \tag{35}$$

Equation (35) gives the lowest-order vector potential at particle k due to all other particles ($i \neq k$). Note that $\mu(\vec{x}_i)$ is considered to be a constant with respect to the d'Alembertian operator acting on functions of $\vec{x}_k$. The above $\vec{A}_0$ can produce terms of the desired order in $\vec{A}_L$. For example,

$$e_k\frac{d}{dt}\vec{A}_0(x_k) = \sum_i e_ie_k\vec{a}_i\mu(\vec{x}_i)|\vec{x}_{ki}|^{-1} + \cdots \tag{36a}$$

$$= \sum_i e_ie_k\vec{g}\mu(\vec{x}_i)|\vec{x}_{ki}|^{-1} + \cdots, \tag{36b}$$

where we have substituted $\vec{a}_i = \vec{g} + O(v^2) + O(g^2)$. The indicated term in Eq. (36b) is bilinear in v^2 and g and is therefore acceptable. However it can be shown that no higher orders of $\vec{A}$ after $\vec{A}_0$ can contribute. For example, the second source term on the right-hand side of Eq. (34b) makes the contribution

$$\vec{A}_1 \sim O(g)\vec{A}_0 \sim O(g)O(v^3),$$

$$\vec{A}_L \sim \frac{d\vec{A}_1}{dt} + \vec{\nabla}(\vec{v}\cdot\vec{A}_1) = O(gv^4) + O(g^2v^2)\,.$$

From the expression for ρ [cf. Eq. (4b)], we can write down the lowest-order solution for the scalar electromagnetic potential:

$$\varphi_0(\vec{x}_k) = \sum_i e_i \epsilon^{-1}(\vec{x}_i)|\vec{x}_{ki}|^{-1}\,. \tag{37}$$

The source term proportional to $\partial\vec{A}_0/\partial t$ in Eq. (33b) doesn't contribute to our order of calculation. Now, define a "superpotential" χ by the equation

$$\nabla^2\chi = \varphi_0\,. \tag{38}$$

Using χ we can write Eq. (33b) as, to appropriate order,

$$\nabla^2\varphi_1 = \nabla^2\left(\epsilon\mu\frac{\partial^2\chi}{\partial t^2}\right) - 2\vec{\nabla}(\epsilon\mu)\cdot\vec{\nabla}\left(\frac{\partial^2\chi}{\partial t^2}\right) - \nabla^2[\epsilon_0^{-1}\epsilon_0'(\vec{g}\cdot\vec{\nabla}\chi)]\,. \tag{39}$$

Using Eqs. (37) and (38), we obtain

$$\chi(\vec{x}_k) = \tfrac{1}{2}\sum_i e_i \epsilon^{-1}(\vec{x}_i)|\vec{x}_{ki}|\,, \tag{40a}$$

$$\frac{\partial\chi}{\partial t} = -\tfrac{1}{2}\sum_i e_i(\vec{v}_i\cdot\vec{x}_{ki})\epsilon^{-1}(\vec{x}_i)|\vec{x}_{ki}|^{-1} - \tfrac{1}{2}\sum_i e_i(\vec{g}\cdot\vec{v}_i)\epsilon^{-2}(\vec{x}_i)\epsilon_0'|\vec{x}_{ki}|\,, \tag{40b}$$

$$\frac{\partial^2\chi}{\partial t^2} = -\tfrac{1}{2}\sum_i e_i(\vec{a}_i\cdot\vec{x}_{ki})\epsilon^{-1}(\vec{x}_i)|\vec{x}_{ki}|^{-1} + O(v^4)\,, \tag{40c}$$

where we have carefully interpreted the partial time derivative on functions of $\vec{x}_k$ as acting on coordinates of particles labeled i with $i \neq k$. From Eq. (40c) it is clear that the second source term in Eq. (39) does not contribute and the remaining equation is trivially integrated to yield

$$\varphi_1 = \epsilon\mu\frac{\partial^2\chi}{\partial t^2} - \epsilon_0^{-1}\epsilon_0'(\vec{g}\cdot\vec{\nabla}\chi)\,. \tag{41}$$

Using Eqs. (40a) and (40c), Eq. (41) becomes

$$\varphi_1 = -\tfrac{1}{2}\mu_0\sum_i e_i(\vec{a}_i\cdot\vec{x}_{ki})|\vec{x}_{ki}|^{-1} - \tfrac{1}{2}\epsilon_0^{-2}\epsilon_0'\sum_i e_i(\vec{g}\cdot\vec{x}_{ki})|\vec{x}_{ki}|^{-1} \tag{42}$$

and, using Eq. (29) for $\vec{a}_i$,

$$\varphi_1 = (\tfrac{1}{4}T_0'\mu_0 H_0^{-1} - \tfrac{1}{2}\epsilon_0^{-2}\epsilon_0')\sum_i e_i(\vec{g}\cdot\vec{x}_{ki})|\vec{x}_{ki}|^{-1}\,. \tag{43}$$

In the same manner as with the vector potential, one can show that φ_2, φ_3, etc. do not contribute to the Lorentz acceleration at the desired order. Using Eqs. (26b), (35), (37), (43), one obtains

$$\vec{A}_L(\vec{x}_k) = (e_k/m_{0k})\sum_i\{[\vec{x}_{ki}\epsilon^{-1}(\vec{x}_i)|\vec{x}_{ki}|^{-3} - \vec{a}_i\mu(\vec{x}_i)|\vec{x}_{ki}|^{-1}]e_i\} + \tfrac{1}{2}(\tfrac{1}{2}T_0'\mu_0H_0^{-1} - \epsilon_0^{-2}\epsilon_0')\sum_i[\vec{\omega}_{ki} - (e_k/m_{0k})e_i\vec{g}|\vec{x}_{ki}|^{-1}]\,, \tag{44}$$

where $\vec{\omega}_{ki}$ is as defined in Eq. (15f). From Eqs. (29) and (6) we obtain the relations

$$\vec{a}_i = -\tfrac{1}{2}(T_0'H_0^{-1})\vec{g} + O(v^2)\,, \tag{45a}$$

$$\epsilon(\vec{x}_i) = \epsilon_0 + (\vec{g}\cdot\vec{x}_i)\epsilon_0'\,, \tag{45b}$$

which, when substituted into Eq. (44), yield

$$\vec{A}_L(\vec{x}_k) = \sum_i(e_ie_k/m_{0k})\left[\frac{\vec{x}_{ki}\epsilon_0^{-1}}{|\vec{x}_{ki}|^3} - \frac{\epsilon_0'\epsilon_0^{-2}(\vec{g}\cdot\vec{x}_i)\vec{x}_{ki}}{|\vec{x}_{ki}|^3} + \frac{1}{2}\frac{(\epsilon_0^{-2}\epsilon_0' + \frac{1}{2}T_0'\mu_0H_0^{-1})}{|\vec{x}_{ki}|}\vec{g}\right] + \tfrac{1}{2}(\tfrac{1}{2}T_0'\mu_0H_0^{-1} - \epsilon_0^{-2}\epsilon_0')\sum_i\vec{\omega}_{ki}\,. \tag{46}$$

C. Virial Conditions

We now have enough information to derive some useful virial conditions. Substitution of the expression for $\vec{A}_L$ [cf. Eq. (46)] into Eq. (29) reveals

$$m_{0k}(a_k)^p = T_0^{1/2}H_0^{-1}\epsilon_0^{-1}\sum_i e_ie_k(x_{ki})^p|\vec{x}_{ki}|^{-3} + O(g)\,, \tag{47}$$

where p denotes a particular vector component. Multiplication of both sides of Eq. (47) with $(x_k)^l$ yields

$$m_{0k}x_k^la_k^p = m_{0k}\frac{d(x_k^lv_k^p)}{dt} - m_{0k}v_k^lv_k^p = T_0^{1/2}H_0^{-1}\epsilon_0^{-1}\sum_i e_ie_k(x_{ki})^px_k^l|x_{ki}|^{-3}\,. \tag{48}$$

If we sum Eq. (48) over the index k, use the antisymmetry of $\vec{x}_{ki}$, and take a time average, the result is Eq. (16). Summing Eq. (16) on l and p produces another useful virial relation:

$$\left\langle \sum_k m_{0k} v_k{}^2 + \tfrac{1}{2} T_0{}^{1/2} H_0{}^{-1} \epsilon_0{}^{-1} \sum_{i,k} e_i e_k |\vec{x}_{ik}|^{-1} \right\rangle = 0 + O(g) . \tag{49}$$

D. Center-of-Mass Acceleration

We now have all of the necessary tools at our disposal for calculating the test-body acceleration. We begin with Eq. (7). To the required order

$$\dot{m}_k = m_{0k}[F_0'(\vec{g}\cdot\vec{v}_k) + (1+G)(\vec{a}_k\cdot\vec{v}_k) + \tfrac{1}{2} v_k{}^2 G_0'(\vec{g}\cdot\vec{v}_k)] + \tfrac{1}{2}\sum_i e_i e_k \{K_0'(\vec{g}\cdot\vec{v}_i) + S_0'(\vec{g}\cdot\vec{v}_k) - [K(\vec{x}_i) + S(\vec{x}_k)](\vec{x}_{ik}\cdot\vec{v}_{ik})|\vec{x}_{ik}|^{-2}\}|\vec{x}_{ik}|^{-1} , \tag{50}$$

$$\ddot{m}_k = m_{0k}[F_0'(\vec{g}\cdot\vec{a}_k) + (\vec{a}_k\cdot\vec{a}_k)(1+G)] . \tag{51}$$

In obtaining Eqs. (50)–(51) we have, as before, used the fact that $\vec{a}_k \sim O(g) + O(v^2)$. To be exact, Eqs. (29) and (46) show that

$$\vec{a}_k = -\tfrac{1}{2}\vec{g}(T_0' H_0{}^{-1}) + T_0{}^{1/2} H_0{}^{-1}\epsilon_0{}^{-1}\sum_i (e_i e_k/m_{0k})\vec{x}_{ki}|\vec{x}_{ki}|^{-3} + O(gv^2) . \tag{52}$$

Using Eqs. (50)–(52), the first two terms in the expression for $\vec{A}_{\text{c.m.}}$ [cf. Eq. (11)] become

$$M^{-1}\sum_k \ddot{m}_k \vec{x}_k = \tfrac{1}{2} M^{-1} H_0{}^{-1}\epsilon_0{}^{-1} T_0{}^{1/2}[F_0' - (1+G_0)T_0' H_0{}^{-1}]\sum_{i,k}\vec{\omega}_{ki} , \tag{53a}$$

$$2M^{-1}\sum_k \dot{m}_k \dot{\vec{x}}_k = 2M^{-1}[F_0' - \tfrac{1}{2}(1+G_0)T_0' H_0{}^{-1}]\sum_k m_{0k}(\vec{v}_k\cdot\vec{g})\vec{v}_k . \tag{53b}$$

Again using Eqs. (29) and (46) to get the $O(gv^2)$ contribution to $\vec{a}_k$ [cf. Eq. (52)], the third and last term contributing to $\vec{A}_{\text{c.m.}}$ is

$$M^{-1}\sum_k m_k \vec{a}_k = M^{-1}\vec{g}\left\{-\tfrac{1}{2} M_0 T_0'(1+F_0)H_0{}^{-1} + \tfrac{1}{2} H_0{}^{-1}[H_0'(1+F_0) - \tfrac{1}{2}T_0'(1+G_0)]\sum_k m_{0k} v_k{}^2 + \tfrac{1}{2}\tau_1 \sum_{i,k}\eta_{ik}\right\} + (1+F_0)(T_0' T_0{}^{-1} - H_0' H_0{}^{-1}) M^{-1}\sum_k m_{0k}(\vec{v}_k\cdot\vec{g})\vec{v}_k + \tfrac{1}{2}\tau_2 M^{-1}\sum_{i,k}\vec{\omega}_{ik} , \tag{54}$$

where

$$\tau_1 \equiv T_0{}^{1/2} H_0{}^{-1}(1+F_0)(\epsilon_0{}^{-2}\epsilon_0' + \tfrac{1}{2}T_0'\mu_0 H_0{}^{-1}) - \tfrac{1}{2}T_0' H_0{}^{-1}(1+K_0+S_0) , \tag{55a}$$

$$\tau_2 \equiv T_0{}^{1/2} H_0{}^{-1}[(1+F_0)H_0{}^{-1}(\tfrac{1}{2}T_0'\mu_0 - \epsilon_0{}^{-1}H_0') + \epsilon_0{}^{-1}F_0' + \tfrac{1}{2}(1+F_0)\epsilon_0{}^{-1}T_0{}^{-1}T_0'] , \tag{55b}$$

with M_0, η_{ik}, $\vec{\omega}_{ik}$ defined in Eqs. (15).

Now, expand the expression for M^{-1} using Eqs. (7) and (8):

$$M^{-1} = M_0{}^{-1}(1+F_0)^{-1}\left[1 - \frac{1}{2}\frac{(1+G_0)}{M_0(1+F_0)}\sum_k m_{0k} v_k{}^2 - \frac{1}{2}\frac{1+K_0+S_0}{M_0(1+F_0)}\sum_{i,k}\eta_{ik}\right] + O(v^4) + O(g) . \tag{56}$$

With Eqs. (53)–(56), the expression for $\vec{A}_{\text{c.m.}}$, Eq. (11), becomes that given in Eq. (14). Use of Eqs. (16) and (49) then yields Eq. (17), and subsequently Eq. (21).

E. The "ϵ-μ" Formulation for Metric Theories

In any static, spherically symmetric, locally Lorentz manifold with metric, one can introduce "spatially isotropic coordinates," for which

$$g_{00} = g_{00}(r) , \tag{57a}$$

$$g_{0k} = 0 , \tag{57b}$$

$$g_{ij} = -\delta_{ij} f(r) ,$$

$$r \equiv [(x^1 - x_s^1)^2 + (x^2 - x_s{}^2) + (x^3 - x_s{}^3)^2]^{1/2} . \tag{57c}$$

(For proof, see any standard textbook on general relativity.) For the problem at hand we can regard g_{00} and f as functions of $U = M_s/r$ rather than as functions of r. In such a coordinate system, the standard metric-theory Lagrangian for the motion of charged particles reduces to

$$L = \sum_k \left[-m_{0k}\int (g_{\alpha\beta}dx_k^\alpha dx_k^\beta)^{1/2} + e_k \int A_\mu dx_k^\mu\right] = \sum_k \int [-m_{0k}(g_{00} - f\vec{v}_k{}^2)^{1/2} + e_k A_\mu v_k^\mu]dt , \tag{58}$$

and the metric-theory Maxwell equations read

$$F^{\alpha\beta}{}_{;\beta} = (-g)^{-1/2}[F^{\alpha\beta}(-g)^{1/2}]_{,\beta} = 4\pi J^{\alpha}, \qquad (59a)$$

where

$$J^{\alpha} \equiv \sum_k e_k \int (dx_k^{\alpha}/ds_k)\delta^4(\underline{x} - \underline{x}_k)(-g)^{-1/2}ds_k$$
$$= \sum_k e_k(-g)^{-1/2}\delta^3(\vec{x} - \vec{x}_k)(dx_k^{\alpha}/dt). \qquad (59b)$$

Here $g \equiv$ determinant of $g_{\alpha\beta}$, and commas and semicolons denote partial and covariant differentiation, respectively. Combining Eqs. (59) gives

$$[g^{\alpha\tau}g^{\beta\mu}F_{\tau\mu}(-g)^{1/2}]_{,\beta} = 4\pi\sum_k e_k\delta^3(\vec{x} - \vec{x}_k)(dx_k^{\alpha}/dt). \qquad (60)$$

Equation (60), when written out for the diagonal, spatially isotropic metric of Eq. (57), has the "ϵ-μ" form of Eqs. (2) and (3), with

$E_i = F_{0i}$, etc.

and

$$\epsilon = \mu = (f/g_{00})^{1/2}. \qquad (61)$$

Conversely, for a theory with GMM equations of the form of Eqs. (2) and (3) and with

$$\epsilon = \mu \qquad (62)$$

one can define an "effective electromagnetic metric" by

$$g_{00} = \Psi, \qquad (63a)$$

$$g_{ij} = -\epsilon^2\Psi\delta_{ij}; \qquad (63b)$$

then the GMM equations will take on metric-theory form. In Eqs. (63) Ψ is an arbitrary function and reflects the well-known conformal invariance of Maxwell's equations. If, in addition to satisfying Eq. (62), the effective metric determined by Eqs. (63) is correctly related to the functions appearing in the particle Lagrangian [cf. Eqs. (57)–(58)], then the entire theory of particles and electromagnetic fields can be consistently put into metric form.

IV. CONCLUSIONS AND APPLICATIONS

A. Theoretical Implications of the Results

We have shown that, in a spherically symmetric gravitational field, a theory of gravity described by Eqs. (1)–(4) can be put into metric form (with respect to the dynamical equations for particles and electromagnetic fields) if and only if it satisfies the weak equivalence principle.[16] Equivalently, if such a theory is nonmetric then Eq. (21) will not be satisfied, the acceleration of test bodies will have body-dependent contributions [cf. Eq. (17)], and WEP will be violated. The result has far-reaching consequences if one accepts WEP as a valid principle: Having proved, from WEP, the metric nature of the GMM equations inside of an electromagnetic test body, one knows how to describe all gravitational-electromagnetic phenomena—e.g., the bending of light by the sun, electromagnetic radiation in a gravitational field, etc.

There are two potential weaknesses of our calculation. First we have assumed a spherically symmetric gravitational field. Now, it is conceivable that a theory could be of "metric form" for spherically symmetric gravitational fields, but nonmetric in other cases. Such theories would have to be analyzed on an individual basis, to see whether their non-SSS fields violated WEP. However, we feel that such a theory would be difficult to formulate and, in fact, have seen no examples in the literature. In practical applications, one considers a particular nonmetric theory, solves the spherically symmetric problem, and finds that Eq. (21) is not satisfied, thus constituting a violation of WEP at some order. Examples will be given below.

A second possible weakness, discussed previously, is the limitation to the types of equations discussed in the beginning of Sec. II. However, except for the Naida-Capella nonmetric theory, discussed below, Eqs. (1)–(4) appear to be quite general among "complete" theories. (There are many theories which are not explicit as to the formulation of the GMM equations, and we must require that such theories be completed before given further consideration.)

Finally, we point out that WEP and Eq. (21) demand that the center-of-mass acceleration be body-independent at each order in the external gravitational potential U. As will be seen below, a given theory violating the WEP will do so at some order of U. To be more explicit, suppose that one expands the functions H, T, μ, ϵ appearing in Eq. (17) in a power series in U, i.e.,

$$H = 1 + 2\gamma U + \tfrac{3}{2}\delta U^2 + \cdots, \qquad (64a)$$

$$T = 1 - 2\alpha U + 2\beta U^2 + \cdots, \qquad (64b)$$

$$\epsilon = 1 + \epsilon_1 U + \epsilon_2 U^2 + \cdots, \qquad (64c)$$

$$\mu = 1 + \mu_1 U + \mu_2 U^2 + \cdots. \qquad (64d)$$

Then, Eq. (17) can be written in the form

$$\langle\vec{A}_{\text{c.m.}}\rangle = -\tfrac{1}{2}\vec{g}(T_0' H_0^{-1}) - \tfrac{1}{2}\vec{g}M_0^{-1}\left\langle\sum_{i,k}\eta_{ik}\right\rangle \times(\Gamma_0 + \Gamma_1 U_0 + \Gamma_2 U_0^2 + \cdots) + \frac{\alpha}{2}M_0^{-1}\left\langle\sum_{i,k}\vec{\omega}_{ik}\right\rangle \times(\Upsilon_0 + \Upsilon_1 U_0 + \Upsilon_2 U_0^2 + \cdots), \qquad (65)$$

where

$$\Gamma_0 \equiv \gamma - \epsilon_1 + \alpha\,, \tag{66a}$$

$$\Upsilon_0 = 0\,, \tag{66b}$$

$$\Gamma_1 \equiv 2(\tfrac{3}{4}\delta - 2\gamma^2 - \epsilon_2 - \beta + \epsilon_1{}^2) + \gamma\epsilon_1 + \alpha(\mu_1 - 5\gamma + \epsilon_1 - \alpha)\,, \tag{66c}$$

$$\Upsilon_1 \equiv 2\gamma + 2\alpha - \epsilon_1 - \mu_1\,, \tag{66d}$$

etc.

(For the correct Newtonian limit, one must require that $\alpha = 1$, but we leave α arbitrary here.) Each theory will yield certain values for the Γ's and Υ's. We have shown that nonmetric theories must have some of the Γ's or Υ's nonzero—the first nonzero Γ or Υ determines the order at which the theory violates WEP.

B. Experimental Verification of WEP and Applications of Our Calculations

Thus far, our results have been completely within a theoretical context. We now investigate the experimental and practical applications.

Experimental support for WEP comes from the type of experiment developed by Eötvös in the late nineteenth century, and redesigned extensively by Dicke in the 1960's.[17] The particular Eötvös-Dicke (ED) experiments of highest reported precision are the Princeton experiment of Roll, Krotkov, and Dicke,[17] and the Moscow experiment of Braginsky and Panov.[18] These experiments measure the relative acceleration toward the sun of two different substances (gold and aluminum in the Princeton experiment; platinum and aluminum in the Moscow experiment). The reported results are

$$\frac{|\langle\vec{A}_{\rm c.m.}\rangle_{\rm Al} - \langle\vec{A}_{\rm c.m.}\rangle_{\rm Au}|}{|\langle\vec{A}_{\rm c.m.}\rangle|} \approx \frac{|\langle\vec{A}_{\rm c.m.}\rangle_{\rm Al} - \langle\vec{A}_{\rm c.m.}\rangle_{\rm Au}|}{|\vec{g}|} < 10^{-11}\,, \tag{67a}$$

$$\frac{|\langle\vec{A}_{\rm c.m.}\rangle_{\rm Al} - \langle\vec{A}_{\rm c.m.}\rangle_{\rm Pt}|}{|\langle\vec{A}_{\rm c.m.}\rangle|} < 10^{-12}\,. \tag{67b}$$

Our calculation involved a test body dropped in a static field. The following argument justifies direct comparison of our calculation with the results of the above experiments:

(i) The 24-hour component of the acceleration can easily be isolated so that the sun can really be considered as the sole external source of gravitation (see page 173 of Ref. 17). To make this more clear, if one uses the 24-hour period variation to select out $\vec{g}_{\rm sun}$ from $\vec{g}_{\rm sun} + \vec{g}_{\rm earth}$, then Eq. (17) has body-dependent terms of the form

$$\langle\vec{A}_{\rm c.m.}\rangle \approx \vec{g}_{\rm sun} M_0{}^{-1}\Big\langle \sum_{i,k} \eta_{ik}\Big\rangle \times [\Gamma_0 + \Gamma_1(U_{\rm sun} + U_{\rm earth}) + \cdots] \approx \vec{g}_{\rm sun} M_0{}^{-1}\Big\langle \sum_{i,k} \eta_{ik}\Big\rangle [\Gamma_0 + \Gamma_1 U_{\rm sun} + \cdots]$$

since $U_{\rm sun} \approx 10 U_{\rm earth}$.

(ii) The fact that the earth is rotating rather than at rest can only contribute *inertial* accelerations; in particular no *relative* accelerations between the two test bodies can be introduced in this manner.

(iii) We have considered only electromagnetic test bodies; but we wish to apply our results to the actual atoms used in the experiments, atoms which have nuclear as well as electromagnetic interactions. Thus the complete equation for $\langle\vec{A}_{\rm c.m.}\rangle$ for realistic atoms has, in addition to the terms shown in Eq. (17), terms which involve nuclear energies. Is it possible that the nuclear and electromagnetic terms would cancel each other? The only mechanism by which the terms could be combined and related is through the virial relations; yet an examination of Eq. (17) reveals that μ_0 does not even occur in the electromagnetic portion of the virial relations. In particular, *given* the combined virial relations for both electromagnetic and nuclear interactions one could construct an infinity of different theories merely by changing μ (and thus changing the body-dependent terms in $\langle\vec{A}_{\rm c.m.}\rangle$). Thus there is no credible mechanism by which nuclear and electromagnetic body dependent terms could conspire to cancel each other. The "electromagnetic violation" of WEP thus constitutes a lower limit to the total violation (allowing for possible nuclear violations).

We can now ask to what order does Eq. (67) test the GMM equations of a theory. Equation (17) has the form

$$\langle\vec{A}_{\rm c.m.}\rangle \sim \vec{g}\left[\frac{\text{electromagnetic energy}}{\text{total mass}}\right] \times F(H_0, T_0, \epsilon_0, \mu_0, H_0', T_0', \epsilon_0') + \text{body-independent term}\,, \tag{68}$$

where F is a function of the indicated variables. Now, the largest contribution to the electromagnetic energy of the total atom certainly comes from the nuclear protons and for platinum or gold this amounts to, using the semiempirical mass formula,[19]

$$\left[\frac{\text{electromagnetic energy}}{\text{total mass}}\right]_{\rm Pt\ or\ Au} \approx 5\times 10^{-3}\,. \tag{69a}$$

For aluminum, the corresponding quantity is

$$\left[\frac{\text{em energy}}{\text{total mass}}\right]_{\text{Al}} \sim \frac{(Z^2A^{-4/3})_{\text{Al}}}{(Z^2A^{-4/3})_{\text{Pt or Au}}}\left[\frac{\text{em energy}}{\text{total mass}}\right]_{\text{Pt}} \approx 2\times 10^{-3}. \tag{69b}$$

Noting that U_0 has the magnitude

$U_0 \equiv$ potential of sun at earth $\sim 10^{-8}$

and using Eqs. (65) and (67), we see that current experimental accuracy bears upon the Γ_k and Υ_k only for $k \leq 1$. *The accuracy*[20] *of the experiment must go up by a factor of* 10^7 *to require that* Γ_2 *and* Υ_2 *vanish.* Equations (66) show that the experiment thus measures H, T, and ϵ to $O(U^2)$, but μ only to $O(U)$. We expect that almost all theories will do well enough to have $\Gamma_0 \equiv 0$.

Before continuing with direct applications to theories of the current experimental verification of WEP, let us return to Eq. (17) and analyze the specific way in which it constrains the GMM equations of a gravitation theory. The second body-dependent term in Eq. (17)—the "directional Coulomb energy" term—involves the GMM equations only through the product $\epsilon\mu$. This particular product is also equal to the square of the index of refraction, n^2, and is tested by light-bending and time-delay experiments (see, e.g., Ref. 21 for a discussion of these experiments—although in the context of metric theories). In fact, exploiting the "ϵ-μ" analogy for the GMM equations and taking the geometrical optics limit, one sees that the current experimental tests, with the exception of WEP, are sensitive *only* to the product $\epsilon\mu$—and only to first order in U of that quantity. On the other hand, the *first* body-independent term in Eq. (17)—the "nondirectional Coulomb energy" term—samples the GMM equations in a deeper manner, both qualitatively and quantitatively. Not only is ϵ distinguished from μ (magnetic and electric effects distinguished) but also is ϵ explored to *second* order in U (cf. the ϵ'_0) for the current experimental verification of WEP. Thus WEP is revealed as a powerful tool for probing the GMM equations—the most sensitive probe of those equations existing in 1973.

On purely theoretical grounds one can require, as we have previously remarked, that the Γ's and Υ's vanish independently. However, in practical experimental applications, the second body-dependent vector in Eq. (65) has some particular relation to the first for any given experiment. Since the nuclei of the atoms in the ED experiment are approximately spherical,

$$\left\langle \sum_{i,k} \vec{\omega}_{ik}\right\rangle \approx \tfrac{1}{3}\vec{g}\left\langle \sum_{i,k}\eta_{ik}\right\rangle. \tag{70}$$

Usings Eqs. (65)–(70), one finally obtains, for $\alpha = 1$ (correct Newtonian limit)

$$\frac{\langle \vec{A}_{\text{c.m.}}\rangle_{\text{Pt, Au}} - \langle\vec{A}_{\text{c.m.}}\rangle_{\text{Al}}}{\vec{g}} \approx -3\times 10^{-3}[\Gamma_0 + 10^{-8}(\Gamma_1 - \tfrac{1}{3}\Upsilon_1)]. \tag{71}$$

C. Applications to Specific Nonmetric Theories

In this section we discuss WEP for three particular nonmetric theories. The Belinfante-Swihart and Whitehead theories have equations of the form of Eqs. (1)–(3). As an illustration of the formalism of Sec. IV A and IV B, the WEP violation is calculated explicitly in the case of the Belinfante-Swihart theory. The Naida-Capella theory, which is an apparently rare example of a theory *not* having a particle Lagrangian of the form of Eq. (1) in the SSS limit, is treated on an individual basis, using the techniques developed in Secs. II and III.

1. *Belinfante-Swihart Theory*[4,6]

An analysis of the Belinfante-Swihart theory in Ref. 14 reveals that its *particle* Lagrangian can be put into metric form with

$$g_{\alpha\beta} = (1-Kh)^2[\eta_{\alpha\beta} + h_{\alpha\beta} + \tfrac{3}{4}h_{\alpha\tau}h_{\beta\mu}\eta^{\mu\tau} + O(h^3)], \tag{72}$$

where K is an arbitrary constant, $h \equiv \eta^{\alpha\beta}h_{\alpha\beta}$, and $\eta_{\alpha\beta}$ is the Minkowski metric. The GMM equations are of "ϵ-μ" form [i.e., have the form of Eqs. (2)–(3)], with, in the SSS limit,

$$\epsilon = [1 - \tfrac{1}{2}(h_{00} + h_{11})]^{-1}, \tag{73a}$$

$$\mu = [1 + \tfrac{1}{2}(h_{00} + h_{11})]. \tag{73b}$$

In the SSS limit, $h_{\mu\nu}$ has the form

$$h_{00} = C_0 U, \tag{74a}$$

$$h_{ij} = \delta_{ij}C_1 U, \tag{74b}$$

$$h_{0k} = 0, \tag{74c}$$

where C_0, C_1 are arbitrary constants, but with the implicit relation

$$2K(3C_1 - C_0) + C_0 - 2 = 0 \tag{75}$$

in order to satisfy the Newtonian limit ($g_{00} = -1 + 2U + \cdots$). Defining T and H by comparison of Eqs. (72), (74) with Eqs. (24) and then evaluating the various Γ_k and Υ_k [cf. Eqs. (64) and (66)], one finds

$$\Gamma_0 = 0, \tag{76a}$$

$$\Gamma_1 - \tfrac{1}{3}\Upsilon_1 = -\tfrac{1}{2}C_0(C_0+C_1) \neq 0\,. \qquad (76b)$$

In order to predict an amount of light bending and perihelion shift compatible with experiment, one must require that C_0 and C_1 satisfy

$$0.9 \leq \tfrac{1}{2}(C_0+C_1-2) \leq 1.1\,, \qquad (77a)$$

$$0.8 \leq \tfrac{1}{2}(C_0+1) \leq 1.3\,. \qquad (77b)$$

The combinations of C_0 and C_1 occurring in Eqs. (77a) and (77b) correspond to the γ and β parameters, respectively, of the "PPN formalism"[21] and the experimental limits indicated above are discussed in Ref. 21.

Using Eqs. (71) and (77), we find that the nonmetric theory of Belinfante and Swihart predicts

$$4\times10^{-11} \lesssim \left|\frac{\langle \vec{A}_{\text{c.m.}}\rangle_{\text{Pt or Au}} - \langle \vec{A}_{\text{c.m.}}\rangle_{\text{Al}}}{\langle \vec{A}_{\text{c.m.}}\rangle}\right| \lesssim 1\times10^{-10}\,. \qquad (78)$$

If one requires the light-bending and perihelion-shift predictions of the Belinfante-Swihart theory to be same as in general relativity, Eq. (78) becomes

$$\left|\frac{\langle \vec{A}_{\text{c.m.}}\rangle_{\text{Au or Pt}} - \langle \vec{A}_{\text{c.m.}}\rangle_{\text{Al}}}{\langle \vec{A}_{\text{c.m.}}\rangle}\right| \approx 6\times10^{-11}\,. \qquad (79)$$

Thus, the Belinfante-Swihart theory violates seriously both the Princeton and the Moscow versions of the ED experiment.

2. *Whitehead's Theory*[9]

Synge analyzes only the motion of uncharged particles and the sourceless GMM equations in Whitehead's theory:

$$\delta\int (g_{\alpha\beta}\,dx^\alpha dx^\beta)^{1/2} = 0 \quad \text{[Eq. (1.7) of Ref. 8]} \qquad (80a)$$

$$(g^{\alpha\mu}g^{\beta\nu}F_{\mu\nu})_{,\beta} = 0 \quad \text{[Eq. (1.9) of Ref. 8]} \qquad (80b)$$

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \quad \text{[Eq. (1.9) of Ref. 8]}. \qquad (80c)$$

A straightforward generalization of these equations to include sources shows that the GMM equations have "ϵ-μ" form in the SSS limit, with

$$\epsilon = (-g_{00}f)^{-1}\,, \qquad (81a)$$

$$\mu = f^2 \qquad (81b)$$

[in the notation of Eqs. (57)]. Using Eqs. (17), (57), and (81), one can then show that

$$\left|\frac{\langle \vec{A}_{\text{c.m.}}\rangle_{\text{Au or Pt}} - \langle \vec{A}_{\text{c.m.}}\rangle_{\text{Al}}}{\langle \vec{A}_{\text{c.m.}}\rangle}\right| \approx 10^{-3}\frac{d}{dU}[\ln(-g_{00}f^3)]\,, \qquad (82)$$

so that, for experimentally acceptable values of g_{00} and f^3, this version of Whitehead's theory violates WEP at the order of 10^{-3}. [Note that in Whitehead's theory the product $\epsilon\mu$ is the same as in metric theories, so that the coefficient of the second body-dependent term in Eq. (17) vanishes identically. In some sense one can say that, with respect to the light bending and radar time-delay experiments, Whitehead's theory is a metric theory.]

3. *Naida-Capella Theory*

The nonmetric theory of Capella[7] as completed by Naida[8] has the following Lagrangian [cf. Eq. (2.1) of Ref. 7]:

$$L = m_0\int ds[-(\eta_{\alpha\beta}u^\alpha u^\beta)^{1/2} + \chi h_{\alpha\beta}u^\alpha u^\beta(\eta_{\rho\sigma}u^\rho u^\sigma)^{-1/2}] - e\int A_\mu dx^\mu\,, \qquad (83)$$

where $\eta_{\alpha\beta}$ is the Minkowski metric and

$$ds \equiv (\eta_{\alpha\beta}\,dx^\alpha dx^\beta)^{1/2}\,,$$

$$\chi \equiv (7\pi)^{1/2}\,,$$

$$u^\alpha \equiv (dx^\alpha/ds)\,.$$

The GMM equations are of "ϵ-μ" form [cf. Eq. (3.7) of Ref. 7] with

$$\epsilon = 1 + \chi(h_{00}+h_{11})\,, \qquad (84a)$$

$$\mu = [1-\chi(h_{00}+h_{11})]^{-1}\,. \qquad (84b)$$

Solutions to the SSS gravitational field equations yield

$$h_{00} = C_0\chi^{-1}U\,, \qquad (85a)$$

$$h_{ij} = C_1\chi^{-1}U\delta_{ij}\,, \qquad (85b)$$

where C_0 and C_1 are arbitrary constants. Variation of Eq. (83) and use of Eqs. (85) gives the particle equation of motion [analog of Eq. (29)]

$$\begin{aligned}\vec{a}_k = \vec{g}[&C_0 - C_0(C_0+2C_1)U_0 + C_1v_k^2 \\ &- U_0v_k^2(2C_0C_1 + C_0^2 + 2C_1^2)] \\ &-2\vec{v}_k(\vec{v}_k\cdot\vec{g})[C_0+C_1-2C_1(C_0+C_1)U_0] \\ &+\vec{A}_L[1-U(C_0+2C_1)]\,. \end{aligned} \qquad (86)$$

Using Eqs. (84)–(86), the GMM equations give

$$\begin{aligned}\vec{A}_L(\vec{x}_i) = &(m_{0i})^{-1}(1-CU_0+C^2U_0^2)\sum_k e_ie_k|\vec{x}_{ik}|^{-3}\vec{x}_{ik} \\ &+\tfrac{1}{2}(m_{0i})^{-1}[C_1 - U_0(2C^2 - C_0C_1)]\vec{g}\sum_k \eta_{ik} \\ &-\tfrac{1}{2}(m_{0i})^{-1}[C_0 + C - U_0(2C^2 + C_0C_1)]\sum_k \vec{\omega}_{ik} \\ &-(m_{0i})^{-1}C(1-2CU_0)\sum_k e_ie_k(\vec{g}\cdot\vec{x}_k)|\vec{x}_{ik}|^{-3}\vec{x}_{ik}\,,\end{aligned} \qquad (87)$$

with $C \equiv C_0 + C_1$.

Using the same center-of-mass formulas as given in Eqs. (7)–(9) and the virial theorem

$$\left\langle \sum_i m_{0i}(v_i)^\alpha (v_i)^\beta + \tfrac{1}{2}[1 - U_0(3C_1 + 2C_0)]\sum_{i,k} e_i e_k (x_{ik})^\alpha (x_{ik})^\beta |x_{ik}|^{-3} \right\rangle = 0 + O(g) \tag{88}$$

one finally obtains

$$\langle \vec{A}_{\text{c.m.}}\rangle = \vec{g}C_0[1 + U_0(-2C_1 + C_0)] - \tfrac{1}{2}M_0^{-1}(C_0^2 + 3C_1^2)U_0\vec{g}\left\langle \sum_{i,k} \eta_{ik}\right\rangle + M_0^{-1}(\tfrac{1}{2} + \tfrac{3}{2}C_1 - 5C_1^2 - C_0^2 - 4C_0C_1)U_0\left\langle \sum_{i,k} \vec{\omega}_{ik}\right\rangle . \tag{89}$$

Now, with Eqs. (69)–(71) we get

$$\frac{|\langle\vec{A}_{\text{c.m.}}\rangle_{\text{Pt or Au}} - \langle\vec{A}_{\text{c.m.}}\rangle_{\text{Al}}|}{|\vec{g}|} \approx 10^{-11}(1 + 3C_1 - 19C_1^2 - 5C_0^2 - 8C_0C_1) . \tag{90}$$

The correct Newtonian and light-bending results require, respectively,

$$C_0 = 1 , \tag{91a}$$

$$0.9 \lesssim \tfrac{1}{2}(C_1 + 1) \lesssim 1.1 . \tag{91b}$$

Equations (90) and (91) indicate then the relation

$$2\times 10^{-10} \leqslant \left|\frac{\langle\vec{A}_{\text{c.m.}}\rangle_{\text{Au or Pt}} - \langle\vec{A}_{\text{c.m.}}\rangle_{\text{Al}}}{\langle\vec{A}_{\text{c.m.}}\rangle}\right| \leqslant 4\times 10^{-10} . \tag{92}$$

Thus the Naida-Capella nonmetric theory seriously violates both the Princeton and Moscow versions of the ED experiment.

ACKNOWLEDGMENTS

We wish to thank W.-T. Ni, K. Nordtvedt, Jr., K. S. Thorne, R. V. Wagoner, and C. M. Will for their helpful comments. We also thank K. S. Thorne for edition of the manuscript.

---

*Work supported in part by the National Aeronautics and Space Administration under Contract No. NGR 05-002-256 and by the National Science Foundation under Contracts No. GP-28027, GP-36687X.

†National Science Foundation Predoctoral Fellow during a portion of this work.

‡Imperial Oil Predoctoral Fellow.

[1]For a discussion of various concepts and terms used in this paper, see K. S. Thorne, D. L. Lee, and A. P. Lightman, Phys. Rev. D 7, 3563 (1973).

[2]K. S. Thorne, invited paper presented in honor of Leonard Schiff at the 1972 American Physical Society meeting in San Francisco, 1972 (unpublished).

[3]K. Nordtvedt, Jr., Int. J. Theor. Phys. 3, 133 (1970).

[4]F. J. Belinfante and J. C. Swihart, Ann. Phys. (N.Y.) 2, 196 (1957).

[5]E. J. Post, Ann. Phys. (N.Y.) 70, 507 (1972).

[6]F. J. Belinfante and J. C. Swihart, Ann. Phys. (N.Y.) 1, 168 (1957).

[7]A. Capella, Nuovo Cimento 42, 1961 (1966).

[8]O. N. Naida, Dokl. Akad. Nauk SSSR 186, 560 (1969) [Sov. Phys. Dokl. 14, 475 (1969)].

[9]We refer to the "early" version of Whitehead's theory as given by J. L. Synge, Proc. Roy. Soc. Lond. A211, 303 (1952).

[10]In this paper Greek indices take on values 0–3, Latin indices 1–3, the signature of spacetime is $(+1,-1,-1,-1)$, and we choose units such that $c = G = 1$.

[11]D. R. Brill, in *Evidence for Gravitation Theories*, edited by C. Møller (Academic, New York, 1962).

[12]We apologize to authors of theories we have overlooked.

[13]A. M. Volkov, A. A. Izmest'cv, and G. V. Krotskiĭ, Zh. Eksp. Teor. Fiz. 59, 1254 (1970) [Sov. Phys.–JETP 32, 686 (1971)].

[14]D. L. Lee and A.P. Lightman, Phys. Rev. D 7, 3578 (1973).

[15]R. H. Dicke, in *Evidence for Gravitation Theories*, edited by C. Møller (Academic, New York, 1962).

[16]We wish to point out that one should be cautious in extrapolating our results (see subsequent comments in text). The possibility always exists that one could invent a gravitation theory not fitting into any preconceived general framework. However, at the current stage of testing and analyzing gravitation theories (see Ref. 21) we feel that work such as ours is valuable as a guide post and testing ground.

[17]R. H. Dicke, in *Relativity Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964), p. 165.

[18]V. B. Braginsky and V. I. Panov, Zh. Eksp. Theor. Fiz. 61, 875 (1971) [Sov. Phys.–JETP 34, 493 (1972)].

[19]R. B. Leighton, *Principles of Modern Physics* (McGraw-Hill, New York, 1959), p. 554.

[20]It is likely that there is a limit, in principle, of measuring the acceleration to no greater accuracy than a part in 10^{23}. See V. B. Braginsky, *Physical Experiments with Test Bodies*, NASA Technical Translation (NASA TT F-672, Houston, Tex., 1972), p. 59.

[21]C. M. Will, Lectures in Proceedings of Course 56 of the International School of Physics "Enrico Fermi," edited by B. Bertotti (Academic, New York, to be published), also distributed as a Caltech Report No. OAP-289, 1972.