Continued-Fraction Representation of Propagator Functions in a Bethe-Salpeter Model

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Using the well-known relation between the vertex function and the Bethe-Salpeter amplitude and knowledge of the bound-state energy eigenvalues of the Bethe-Salpeter equation, a continued fraction representation for the modified meson propagator D'_F is obtained. The Bethe-Salpeter equation for the nucleon-antinucleon problem with a massless–pseudoscalar-meson coupling is solved in a certain approximation, and the corresponding energy eigenvalues are determined through a continued-fraction technique. We have considered the nucleon both as a Dirac particle and also as a scalar particle. The analytic properties of the continued fraction are discussed and the existence of a Lehmann spectral-function representation for the D'_F obtained in the approximation is shown.

I. INTRODUCTION

It is well known that the integral equation for the vertex function in field theory is closely related to the Bethe-Salpeter equation, which provides an energy eigenvalue equation for the bound state[1] of two relativistic particles. In particular if $\psi(p)$ describes the wave function in momentum space for a bound state of two scalar particles interacting via a scalar particle, where p is the relative four-momentum between them, the energy eigenvalue can be expressed as

$$1=\frac{-2ig^2}{(2\pi)^4}\,\frac{1}{E^2}\int\psi(p)\,d^4p,$$

where E denotes the total energy of the system in the center-of-mass system. It is interesting to point out that from a knowledge of the relation between $\psi(p)$ and the vertex function Γ, one can show that $\int\psi(p)\,d^4p$ in this case reduces to the sum of all self-energy corrections $\pi^*(E)$ to the free scalar propagator function. Noting that the free propagator for the scalar particle in our frame of reference is simply $-i/E^2$, we see that the energy eigenvalue equation

$$1+\frac{2ig^2}{(2\pi)^4}\,\frac{1}{E^2}\int\psi(p)\,d^4p=0$$

reduces to $1-D_F(E)\pi^*(E)=0$. Comparing this with the expression for the exact propagator

$$D'_F(E)=D_F(E)[1-D_F(E)\pi^*(E)]^{-1},$$

we see that solving the energy eigenvalue problem is the same problem as finding all the poles of D'_F. Thus a study of the energy eigenvalue problem in the framework of relativistic field theory should lead to a calculation of the propagator function. Most of the studies of the energy eigenvalue problem using the Bethe-Salpeter equation have been confined to a special case when the total energy of the system is zero.[2]

In this paper we will consider the calculation of the meson propagator obtained from an approximate solution of the Bethe-Salpeter equation[3] for a nucleon-antinucleon system for nonvanishing E, assuming that the exchange particle has mass zero and that the interaction is of the pseudoscalar type. In the main part of the work we take the nucleon to be a Dirac particle. Assuming the most general Lorentz-invariant form for $\psi(p)$, we obtain a system of differential equations for the invariant amplitude from the relevant integral equation of the problem. An approximate scheme is then developed to obtain the $J^P=0^-$ bound-state solution of the problem.[4] The associated energy eigenvalue is then expressed as the vanishing of an infinite continued J fraction.[5] This functional equation is then used to obtain a nonperturbative expression for the propagator in the form of a continued fraction. From the analytic properties of the continued fraction, we then derive a number of interesting properties for the D'_F function. The continued fraction representing D'_F is a real J fraction, and it is well known that it represents a meromorphic function of E^2 having only simple poles with positive residues and that it has a Lehmann spectral representation.

The fact that the propagator function may be represented by a continued fraction does not ensure the existence of the propagators as well-defined functions. The problem of convergence of the continued fraction has to be dealt with. We have discussed this problem in Sec. V. The real J fraction is written as the even part of an S fraction. Such S fractions have been studied in detail by Stieltjes.[6]

It is well known that the S fraction may be expanded in a power series, and also one can associate a continued fraction with a given power series. From this point of view our continued-fraction expression for D_F' is equivalent to a series in $1/E^2$. The continued fraction is a sum of the series even when the series is divergent. In Sec. VI we have obtained a J-fraction representation for the D_F' function resulting from the scalar nucleon-antinucleon bound-state problem.

II. EIGENVALUE CONDITION AND THE POLES OF D_F'

We consider[1] an interaction Hamiltonian of the form $\lambda\phi^2(x)A(x)$, where $\phi(x)$ is a neutral scalar nucleon field and A is a massless scalar field. The Bethe-Salpeter (B-S) equation incorporating the normal exchange diagram and also the pair-annihilation diagram is

$$f(p)=\frac{-4ig^2}{(2\pi)^4}\frac{1}{(\frac{1}{2}E+p)^2+m^2}\frac{1}{(\frac{1}{2}E-p)^2+m^2}\times\int d^4k f(k)\left[\frac{1}{(p-k)^2}+\frac{1}{2E^2}\right], \qquad (2.1)$$

where E denotes the energy-momentum four-vector for the center-of-mass motion, and $f(p)$ is the wave function for relative momentum p. Following Okubo and Feldman,[1] we put

$$f(p)=\left[\frac{-2ig^2}{(2\pi)^4}\frac{1}{E^2}\int d^4k f(k)\right]\psi(p). \qquad (2.2)$$

We then obtain from (2.1) the equation for $\psi(p)$:

$$\psi(p)=\frac{1}{(\frac{1}{2}E+p)^2+m^2}\frac{1}{(\frac{1}{2}E-p)^2+m^2}\times\left[1-\frac{4ig^2}{(2\pi)^4}\int\frac{d^4k\,\psi(k)}{(p-k)^2}\right]. \qquad (2.3)$$

Let us now see the connection between $\psi(p)$ and the generalized vertex operator Γ. If we use the ladder approximation, then Γ satisfies

$$\Gamma(p+\tfrac{1}{2}E,p-\tfrac{1}{2}E)=1-\frac{4ig^2}{(2\pi)^4}\int d^4k\frac{1}{(p-k)^2}\frac{1}{(\frac{1}{2}E+k)^2+m^2}\frac{1}{(\frac{1}{2}E-k)^2+m^2}\Gamma(k+\tfrac{1}{2}E,\ k-\tfrac{1}{2}E). \qquad (2.4)$$

Comparing this with (2.3) we find

$$\psi(p)=\frac{1}{(\frac{1}{2}E+p)^2+m^2}\frac{1}{(\frac{1}{2}E-p)^2+m^2}\times\Gamma(p+\tfrac{1}{2}E,\ p-\tfrac{1}{2}E). \qquad (2.5)$$

A functional equation for the energy eigenvalue condition follows from (2.2) by integrating both sides with respect to p, and it is given by

$$1-\frac{2g^2}{(2\pi)^4}D_F(E^2)\int\psi(p)d^4p=0. \qquad (2.6)$$

The analogous condition for particles with spin is

$$1-\frac{g^2}{(2\pi)^4}D_F(E^2)\int\mathrm{Tr}[\gamma_5\psi(p)]d^4p=0. \qquad (2.7)$$

It is interesting to note that by using (2.4) and (2.5) we can easily show that $\int\psi(p)d^4p$ corresponds to all closed-loop corrections to the free propagator $D_F(E^2)$. Thus, writing $\pi^*(E)=[2g^2/(2\pi)^4]\times\int\psi(p)d^4p$ and taking the usual definition of D_F' as $D_F'(E)=D_F(E)[1-D_F(E)\pi^*(E)]^{-1}$, we have

$$D_F'(E)=D_F(E)\Big/\left[1-D_F(E)\frac{2g^2}{(2\pi)^4}\int\psi(p)d^4p\right].$$

It is thus evident that the eigenvalue condition (2.6) determines the poles of D_F'.

In Sec. III we will formulate our B-S equation for the spinor nucleon-antinucleon case and relate the functional equation for the energy eigenvalue to a continued fraction.

III. FORMULATION OF THE PROBLEM

In the ladder approximation, the Bethe-Salpeter equation for a spin-$\frac{1}{2}$ fermion and spin-$\frac{1}{2}$ antifermion, each of mass m, for a bound state of total momentum $P=(\vec{P},iE)$ is

$$[i\gamma\cdot(p+\tfrac{1}{2}P)+m]\Phi(p)[i\gamma\cdot(p-\tfrac{1}{2}P)+m]=\frac{ig^2}{(2\pi)^4}\int\frac{d^4k}{(p-k)^2}\gamma_5\Phi(k)\gamma_5, \qquad (3.1)$$

where Φ is a 4×4 matrix, and p is the relative four-momentum of the system.

In writing Eq. (3.1) we have assumed that the nucleons interact via pseudoscalar coupling with a massless neutral pseudoscalar meson field. Equation (3.1) may be written in terms of a new wave function $\psi(p)$, where

$$\psi(p)=[i\gamma\cdot(p+\tfrac{1}{2}P)+m]\Phi(p)[i\gamma\cdot(p-\tfrac{1}{2}P)+m]. \qquad (3.2)$$

We now decompose $\psi(p)$ into appropriate Lorentz-invariant amplitudes and write $\psi(p)$ as[1]

$$\psi(p)=\gamma_5\psi_1(p)+\gamma_5(\gamma\cdot P)\psi_2(p)+\gamma_5(\gamma\cdot p)\psi_3(p)+\gamma_5[(\gamma\cdot p)(\gamma\cdot P)-(\gamma\cdot P)(\gamma\cdot p)]\psi_4(p). \qquad (3.3)$$

Making a Wick rotation[7] and applying the property[8]

$$\Box_p{}^2\frac{1}{(p-k)^2}=-4\pi^2\delta^4(p-k), \qquad (3.4)$$

where

$$\Box_p{}^2 \equiv \frac{\partial^2}{\partial p_\nu \partial p_\nu},$$

the integral equation (3.1) reduces to the following differential equation for $\psi(p)$:

$$\Box_p{}^2\psi(p) = \frac{g^2}{4\pi^2} G(p)\gamma_5[i\gamma\cdot(p+\tfrac{1}{2}P)-m] \times\psi(p)[i\gamma\cdot(p-\tfrac{1}{2}P)-m]\gamma_5, \tag{3.5}$$

where

$$G(p) = \{[(p+\tfrac{1}{2}P)^2+m^2][(p-\tfrac{1}{2}P)^2+m^2]\}^{-1}. \tag{3.6}$$

Substituting (3.3) for $\psi(p)$ into (3.5) and carrying out the spin algebra, the resulting coupled differential equations for the $\psi_i(p)$ are

$$\Box_p{}^2\psi_1(p) = \frac{g^2}{4\pi^2} G(p)\{(p^2-\tfrac{1}{4}P^2+m^2)\psi_1(p)+imP^2\psi_2(p)+im(P\cdot p)\psi_3(p)+2[(P\cdot p)^2-P^2p^2]\psi_4(p)\}, \tag{3.7}$$

$$\begin{aligned} P^2\Box_p{}^2\psi_2(p)+2P_\nu\Box_p\psi_3(p)+(P\cdot p)\Box_p{}^2\psi_3(p) &= -\frac{g^2}{4\pi^2}G(p)\{imP^2\psi_1(p)+[2(P\cdot p)^2-P^2(p^2+\tfrac{1}{4}P^2-m^2)]\psi_2(p)+(P\cdot p)\psi_3(p)(p^2-\tfrac{1}{4}P^2-m^2) \\ &\quad +4im[p^2P^2-(P\cdot p)^2]\psi_4(p)\}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} (P\cdot p)\Box_p{}^2\psi_2(p)+2p_\nu\Box_p\psi_3(p)+p^2\Box_p{}^2\psi_3(p) &= -\frac{g^2}{4\pi^2}G(p)\{im(P\cdot p)\psi_1(p)+(P\cdot p)(p^2-\tfrac{1}{4}P^2-m^2)\psi_2(p)+[p^2(p^2+\tfrac{1}{4}P^2-m^2)-\tfrac{1}{2}(P\cdot p)^2]\psi_3(p)\}, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} (P\cdot p)\Box_p{}^2\psi_1(p)+4[(P\cdot p)P_\nu-p_\nu P^2]\Box_p\psi_4(p)+2[(P\cdot p)^2-p^2P^2]\Box_p{}^2\psi_4(p) &= \frac{g^2}{4\pi^2}G(p)\{\psi_1(p)[(P\cdot p)(p^2-\tfrac{1}{4}P^2+m^2)-(P\cdot p)^2+p^2P^2]+2im\psi_2(p)[\tfrac{1}{2}P^2(P\cdot p)+(P\cdot p)^2-p^2P^2] \\ &\quad +im(P\cdot p)\psi_3(p)+2\psi_4(p)[(P\cdot p)^3-(P\cdot p)^2(p^2-\tfrac{1}{4}P^2-m^2)+p^2P^2(p^2-m^2-\tfrac{1}{4}P^2-(P\cdot p))]\}. \end{aligned} \tag{3.10}$$

Here the ψ_i's are functions of p^2, P^2, and $p\cdot P$. The angular dependence of the wave functions is projected out by making use of Gegenbauer polynomials.[9] We expand the wave functions in terms of a complete set of Gegenbauer polynomials[10,11]:

$$\psi_i(p^2,P^2,p\cdot P) = \sum_{n=0}^{\infty} C_n^1(\cos\theta)\psi_i^{(n)}(p^2,P^2), \quad i=1,2,3,4. \tag{3.11}$$

Confining our attention to S wave only and ignoring coupling between the various radial functions, we get in this approximation the following set of uncoupled radial differential equations:

$$s(s-c)(s-d)\psi_1''(s)+2(s-c)(s-d)\psi_1'(s)-\frac{g^2}{16\pi^2}(s-\tfrac{1}{4}P^2+m^2)\psi_1(s)=0, \tag{3.12}$$

$$s(s-c)(s-d)\psi_2''(s)+2(s-c)(s-d)\psi_2'(s)+\frac{g^2}{16\pi^2}(m^2-\tfrac{1}{4}P^2)\psi_2(s)=0, \tag{3.13}$$

$$s(s-c)(s-d)\psi_3''(s)+3(s-c)(s-d)\psi_3'(s)+\frac{g^2}{16\pi^2}(s-m^2)\psi_3(s)=0, \tag{3.14}$$

and

$$s(s-c)(s-d)\psi_4''(s)+3(s-c)(s-d)\psi_4'(s)+\frac{g^2}{16\pi^2}(s-\tfrac{1}{4}P^2-m^2)\psi_4(s)=0, \tag{3.15}$$

where prime denotes differentiation with respect to s, and for simplicity we have written $\psi_i(s)$ for $\psi_i^{(0)}(s)$. Here s stands for p^2, and c and d are the roots of the quadratic equation

$$x^2+2m^2x+(\tfrac{1}{4}P^2+m^2)^2=0.$$

These are our basic equations and are the well-known Heun's equations[12] with four singularities. Equations (3.12), (3.14), and (3.15) are similar and it is quite sufficient to discuss only one, namely, Eq. (3.12), which contributes to the $J^P = 0^-$ state[4] of the nucleon-antinucleon system.

Since we are concerned here with the $J^P = 0^-$ meson state of the nucleon and antinucleon states, the invariant amplitudes that contribute to this state are given by the combination $\psi = \gamma_5(\psi_1 - \psi_4[\gamma\cdot p, \frac{1}{2}\gamma\cdot P])$, where ψ_1 and ψ_4 are functions of p^2, P^2, and $p\cdot P$ only. Hence, in this particular state we need only consider the coupled equations between ψ_1 and ψ_4 which can be trivially obtained from set (3.10) and by setting ψ_2 and ψ_3 equal to zero. Furthermore, if we consider the weak-binding limit of the nucleon-antinucleon system (we assume the coupling is weak), then in the limit of $P\to 0$ we may confine our attention to the case of ψ_1 only. It is worthwhile to point out that in this limit ψ_4, however, does not possess any singularity. To be precise, the relevant equation for ψ_4 in the weak-binding limit is

$$s^2(s+2m^2)\psi_4'' + 3s(s+2m^2)\psi_4' + \lambda\psi_4(s-mW) = 0, \tag{3.16}$$

where W is the binding energy defined in Sec. IV and $\lambda = g^2/16\pi^2$.

The solution of (3.16) is of the form

$$\psi_4 \sim s^k\, {}_2F_1\left[k+1+(1-\lambda)^{1/2},\ k+1-(1-\lambda)^{1/2}, 2k+3; \frac{-s}{2m^2}\right], \tag{3.17}$$

which (for $\lambda<1$) does not produce any singularity for $P^2\to 0$ [for $P^2\to 0$, k tends to $\frac{1}{4}\lambda$].

In Sec. IV we study the functional equation for the energy eigenvalues, and in Sec. V we use this equation to write down a continued-fraction representation for the meson propagator function.

IV. EIGENVALUE EQUATION

Equation (3.12) is a particular form of Heun's equation. Heun's equation with four singularities has the form[12]

$$x(x-1)(x-a)\frac{d^2y}{dx^2} + [\gamma(x-1)(x-a)+\delta x(x-a)+\epsilon x(x-1)]\frac{dy}{dx} + \alpha\beta(x-h)y = 0, \tag{4.1}$$

where the exponents are assumed to be connected by Riemann's relation

$$\alpha+\beta-\gamma-\delta-\epsilon+1=0. \tag{4.2}$$

The general solution of (4.1) is of the form

$$y = \sum_{j=0}^{\infty}\sum_{l=1}^{6} C_j(a,h;\ \alpha,\beta,\gamma,\delta,\epsilon)\,\phi_j^l(\alpha,\beta;\ \delta,x). \tag{4.3}$$

The summation over l results from the fact that there are six different fundamental branches of ϕ which contribute to the solutions for certain domains of the variable x. The ϕ's are connected with the hypergeometric functions.

The C's in (4.3) satisfy the following difference equation:

$$M_k C_{k+1} + L_k C_k + K_k C_{k-1} = 0, \tag{4.4a}$$

where

$$K_{k+1} = \frac{a(\alpha+k)(\beta+k)(\epsilon+k)(\alpha+\beta-\delta+k)}{(\alpha+\beta-\delta+2k)(\alpha+\beta-\delta+2k+1)}, \tag{4.4b}$$

$$L_k = ak(\gamma+k-1)\left[\frac{(\alpha+k)(\alpha-\delta+k+1)+(\beta+k)(\beta-\delta+k+1)}{(\alpha+\beta-\delta+2k-1)(\alpha+\beta-\delta+2k+1)} - \frac{1}{\alpha+\beta-\delta+2k-1}\right] - k(\alpha+\beta-\delta+k) - \alpha\beta h + a\frac{\alpha\beta(\gamma+2k)-\epsilon k(\delta-k-1)}{(\alpha+\beta-\delta+2k+1)}, \tag{4.4c}$$

and

$$M_{k-1} = \frac{a(\alpha-\delta+k)(\beta-\delta+k)k(\gamma+k-1)}{(\alpha+\beta-\delta+2k-1)(\alpha+\beta-\delta+2k)}. \tag{4.4d}$$

For Eq. (3.12) we have

$$a=\frac{d}{c},\quad \gamma=2,\quad \delta=\epsilon=0;\quad x=\frac{s}{c},$$
$$\alpha+\beta=1,\quad \alpha\beta=\frac{-g^2}{16\pi^2},\quad h=\frac{1}{c}(\tfrac{1}{4}P^2-m^2). \tag{4.5}$$

The nontrivial solution of (4.4a) exists if and only if the following functional equation is satisfied[13]:

$$\frac{L_1}{M_1} = \frac{K_2/M_2}{L_2/M_2 - \frac{K_3/M_3}{L_3/M_3 - \frac{K_4/M_4}{L_4/M_4 - \cdots}}}. \tag{4.6}$$

Equation (4.6) is also equivalent to

$$L_1 = \frac{K_2M_1}{L_2 - \frac{K_3M_2}{L_3 - \frac{K_4M_3}{L_4 - \cdots}}}. \tag{4.7}$$

We introduce the relations

$$P^2 = -E^2 < 0, \tag{4.8}$$

and

$$\omega^2 = \tfrac{1}{4}P^2 + m^2$$
$$= m^2 - \tfrac{1}{4}E^2 > 0.$$

Here E is the rest mass of the bound system and is therefore also the total energy in the c.m. system. For bound states we must have $E < 2m$, where $\omega^2 > 0$. The relation between ω and the binding energy W is given by[1]

$$\omega^2 = W(m - \tfrac{1}{4}W). \tag{4.9}$$

We consider the case where the binding energy is quite small compared with the nucleon mass. For small binding energy the solution of Eq. (3.12) exists if

$$2+\lambda\left(1-\frac{W}{m}\right)=\cfrac{\frac{1}{3}\times\frac{3}{5}(2-\lambda)(6-\lambda)}{6+\lambda\left(1-\frac{W}{m}\right)-\cfrac{\frac{2}{5}\times\frac{4}{7}(6-\lambda)(12-\lambda)}{12+\lambda\left(1-\frac{W}{m}\right)-\cfrac{\frac{3}{7}\times\frac{5}{9}(12-\lambda)(20-\lambda)}{20+\lambda\left(1-\frac{W}{m}\right)-\cdots}}}, \tag{4.10}$$

where

$$\lambda = g^2/16\pi^2.$$

The eigenvalue equation (4.10) for our problem has the form of an infinite continued fraction. Now we discuss two cases of Eq. (4.10):

(a) The infinite continued fraction (4.10) terminates[14] when

$$\lambda = 2, 6, 12, \ldots,$$
$$\text{or} \tag{4.11}$$
$$\lambda = k(k+1);\quad k = 1, 2, 3, \ldots .$$

The termination of the continued fraction (4.10) means that the solution of (3.12) is a finite series of hypergeometric functions. Remembering that $\lambda = g^2/16\pi^2$, where g is the pseudoscalar coupling constant, we see that for the indicated values of λ the nucleon-antinucleon system does possess bound states. For example, for $\lambda = 2, 6, 12$, etc., we have $W = 2m$, $W = \frac{4}{3}m$, $W = m$, $\frac{5}{3}m$, etc. It is interesting to note that with increasing strength we find a spectrum of new bound states when the coupling constant reaches a certain critical value. This result is quite different from that which one expects in standard perturbation theory.

(b) Obviously our J fraction (4.10) is nonterminating unless we choose special values of λ. In this case we have to discuss the convergence of the J fraction. We will consider this in Sec. V in connection with the continued-fraction representation of the propagator function.

V. REPRESENTATION OF THE PROPAGATOR FUNCTION BY A CONTINUED FRACTION AND ITS ANALYTIC PROPERTIES

It was pointed out in Sec. II that the bound states are the poles of the meson propagator. To see the connection between the energy eigenvalues of the system and the zeros of the inverse of the propagator function, we consider in the Appendix an exactly solvable one-dimensional model Lagrangian. In the present case we can represent the D_F' function by the following continued fraction:

$$\frac{1}{x}D_F' = \cfrac{1}{E^2+2x-\cfrac{a_1x^2}{E^2+6x-\cfrac{a_2x^2}{E^2+12x-\cdots}}}, \tag{5.1}$$

where $x = 4m^2/\lambda$,

$$a_k = \frac{k(k+2)}{(2k+1)(2k+3)}\{k(k+1)-\lambda\}$$
$$\times\{(k+1)(k+2)-\lambda\},\quad k = 1, 2, 3, \ldots .$$

In writing Eq. (5.1) we have utilized Eq. (4.10) for the energy eigenvalue problem.

Assuming $0 < \lambda < 2$, we see that (5.1) is a real J fraction[15] in which the constants a_k's are positive. We then know that it represents a meromorphic function of E^2 having only simple poles with positive residues[16] and that it has a Lehmann spectral representation of the Stieltjes form[17]:

$$D_F'(E^2) \sim \int_{-\infty}^{\infty}\frac{d\varphi(s)}{E^2-s}, \tag{5.2}$$

where $\varphi(s)$ is bounded and nondecreasing.

It is further known that the zeros of the denominator of the various approximants of D_F' are all real, showing the reality of the energy eigenvalues of the corresponding Bethe-Salpeter equation. From the convergence property[18] one can show that if D_F' converges for single nonreal values of E^2, then it converges uniformly over every finite closed region whose distance from the real axis is positive, and its value in each of the lower and upper half planes is an analytic function of E^2 in that plane.

We now discuss the convergence of the continued fraction (5.1). For this, let us consider a continued fraction[19]:

$$\frac{1}{\delta_1+\delta_2+z-\cfrac{\delta_2\delta_3}{\delta_3+\delta_4+z-\cfrac{\delta_4\delta_5}{\delta_5+\delta_6+z-\cdots}}}, \tag{5.3}$$

where $\delta_1, \delta_2, \delta_3, \ldots$ are any numbers such that

$$\delta_p - \mathrm{Re}\,\delta_p \leq \tfrac{1}{2},$$
$$\delta_{2p}\delta_{2p+1} > 0,$$
$$\mathrm{Im}(\delta_{2p-1}+\delta_{2p}) \geq 0; \quad p=1,2,3,\ldots .$$

Then the above continued fraction converges if z is outside the region bounded by the lower half of the parabola:

$$|z| + \mathrm{Re}z = 2. \tag{5.4}$$

For our J fraction representation of D_F' we find that we can identify (5.1) with (5.3) provided we choose $\delta_1=0$, $\delta_2=2x$, $\delta_3+\delta_4=6x$, $\delta_5+\delta_6=12x$, etc., and $\delta_2\delta_3=a_1x^2$, $\delta_4\delta_5=a_2x^2$, etc. We can satisfy these with real δ's, and further if $0<\lambda<2$, then $a_1, a_2, \ldots$ are all positive; thus the relevant conditions of convergence of our continued fraction (5.1) are satisfied. We thus conclude that the propagator function $D_F'(E^2)$ converges if e ($\equiv E^2$) is outside the region bounded by the lower half of the parabola:

$$|e| + \mathrm{Re}(e) = 2.$$

It is interesting to note that the continued J fraction (5.3) is an even part[20] of the Stieltjes S fraction[21]:

$$\frac{1}{k_1z+\cfrac{1}{k_2+\cfrac{1}{k_3z+\cfrac{1}{k_4+\cdots}}}}, \tag{5.5}$$

where $k_1=1$ and $k_pk_{p+1}=1/\delta_p$; $p=2,3,\ldots$. It is well known[22] that an S fraction may be expanded formally in a power series in $1/z$ and that the series is unambiguously defined by the S fraction. We can write the power series in the form

$$D_F'(e) \sim c_0 + \frac{c_1}{e} + \frac{c_2}{e^2} + \cdots . \tag{5.6}$$

If the quotient c_{n+1}/c_n goes to a finite limit l, then the series (5.6) converges for $|1/e|<l$; if, however, the quotient increases to ∞, then the series is divergent. What we are doing with the continued-fraction representation is summing the divergent expansion in $1/e$ by means of a continued-fraction technique.

VI. PROPAGATOR FUNCTION IN SCALAR NUCLEON-ANTINUCLEON CASE

We conclude this paper with a short discussion of the propagator function resulting from a scalar nucleon-antinucleon Bethe-Salpeter equation. The energy eigenvalue equation in this case can be easily obtained from the study of the difference equations[23]

$$M_0a_1 + L_0a_0 = 0,$$
$$M_\nu a_{\nu+1} + L_\nu a_\nu + K_\nu a_{\nu-1} = 0, \tag{6.1}$$

where

$$M_\nu = (\nu+1)(\nu+n+1),$$
$$L_\nu = \tfrac{1}{4}\lambda - e\nu(\nu+n) - (\nu+\tfrac{1}{2}n)[\nu+\tfrac{1}{2}(n+1)], \tag{6.2}$$
$$K_\nu = e[\nu+\tfrac{1}{2}(n-1)](\nu+\tfrac{1}{2}n-1), \quad e\equiv E^2/m^2.$$

The energy eigenvalue condition is given by the vanishing of the continued fraction:

$$L_0/M_0 - \cfrac{K_1/M_1}{L_1/M_1-\cfrac{K_2/M_2}{L_2/M_2-\cdots}}. \tag{6.3}$$

We can utilize the above continued fraction to write the corresponding propagator function D_F' in the form of an infinite continued J fraction:

$$D_F' \sim \cfrac{1}{a_0\lambda-b_0-ed_0-\cfrac{ec_1}{a_1\lambda-b_1-ed_1-\cfrac{ec_2}{a_2\lambda-b_2-ed_2-\cdots}}}, \tag{6.4}$$

where

$$a_\nu = \frac{1}{4(\nu+1)(\nu+n+1)},$$
$$b_\nu = \frac{(\nu+\frac{1}{2}n)[\nu+\frac{1}{2}(n+1)]}{(\nu+1)(\nu+n+1)},$$
$$c_\nu = \frac{[\nu+\frac{1}{2}(n-1)](\nu+\frac{1}{2}n-1)}{(\nu+1)(\nu+n+1)}, \tag{6.5}$$
$$d_\nu = \frac{\nu(\nu+n)}{(\nu+1)(\nu+n+1)}.$$

Analytic properties of this will be closely similar to those described in the previous sections.

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APPENDIX

We consider the Lagrangian of the form[13]

$$\mathcal{L}=\psi^\dagger\left(i\frac{\partial}{\partial t}-m_0\right)\psi+\varphi^\dagger\left(i\frac{\partial}{\partial t}-\mu\right)\varphi - g\psi^\dagger(\varphi+\varphi^\dagger)\psi\,, \quad (A1)$$

where the fields ψ and φ are functions of time alone. The canonical commutation relations are

$$[\psi,\psi^\dagger]=1 \text{ and } [\varphi,\varphi^\dagger]=1\,. \quad (A2)$$

Associating ψ with d/dx and $\psi^\dagger$ with x, and similarly φ with d/dy and $\varphi^\dagger$ with y, we have the Hamiltonian equation in function space of x and y in the following:

$$\left[m_0 x\frac{d}{dx}+\mu y\frac{d}{dy}+gx\left(y+\frac{d}{dy}\right)\frac{d}{dx}\right]f(x,y)=Ef(x,y)\,. \quad (A3)$$

Making the anstaz $f(x,y)=x\sum_n C_n y^n$, we easily find that the C_n's must satisfy

$$(m_0-E)C_n+\mu n C_n+g\left[C_{n-1}+(n+1)C_{n+1}\right]=0\,. \quad (A4)$$

The condition that this equation has a solution is[13]

$$-t-\cfrac{g^2}{1-t-\cfrac{2g^2}{2-t-\cfrac{3g^2}{3-t-\cdots}}}=0\,, \quad (A5)$$

where $\mu=1$ and $t=E-m_0$, which is the functional equation for the energy eigenvalues of this model Hamiltonian. The exact modified propagator $S'(p)$ is given by

$$[S'(p)]^{-1}=p-m_0-g\int dk_0\Delta'(k_0)\Gamma(p,p-k_0)\times S'(p-k_0)\,. \quad (A6)$$

This equation can be exactly solved from the relevant Ward Identity of the model, namely,

$$\Gamma(p,p-k)=-\frac{ig}{2\pi}\frac{1}{k}\left[S'^{-1}(p-k)-S'^{-1}(p)\right]. \quad (A7)$$

Using (A7) and the spectral representation for $S'(p)$, one obtains the following difference equation for the propagator function $S'(p)$:

$$-(t+\beta)F(t)=1-\beta F(t-1)\,, \quad (A8)$$

where

$$\beta=g^2\,,\quad t=p-m_0\,,\quad S'(p)=-F(t)\,, \quad (A9)$$

and we have used $\mu=1$.

The difference equation can be easily solved to give

$$F(t)=\frac{1}{\gamma}\,{}_1F_1(1,\gamma+1;x)\,, \quad (A10)$$

where ${}_1F_1$ is the standard hypergeometric function and we have put x for $-\beta$ and $\gamma=-t-\beta$. From the repeated use of the differential equations for the ${}_1F_1$ function we can develop the following continued-fraction representation for $F(t)$:

$$F(t)=\cfrac{1}{-t-\cfrac{g^2}{1-t-\cfrac{2g^2}{2-t-\cdots}}}\,. \quad (A11)$$

Thus S'^{-1} is related to the functional equation for the energy eigenvalue of the Hamiltonian. The convergence of this continued fraction follows directly from the well-known convergence property of the hypergeometric function.

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