PHYSICAL REVIEW D
PARTICLES AND FIELDS

THIRD SERIES, VOL. 8, NO. 2 — 15 July 1973

Possible Improvements of Gravitational Antennas

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(Received 27 November 1972; revised manuscript received 28 February 1973)

The optimization of a gravitational antenna is studied in this paper. It is shown that the gravitational cross section of a conical antenna is 20% larger than that of a cylindrical one. A conical antenna is less suitable for use with ceramic transducers; however, when capacitative detectors are employed, its sensitivity can be improved by a factor of about 5 with respect to cylindrical antennas.

I. INTRODUCTION

Many authors have studied gravitational antennas. Historically, the first antenna was the classical cylinder used by Weber.[1] A dumbbell was proposed by Tyson[2] for low frequencies and its properties were compared to Weber's antenna by Rasband *et al.*[3] Braginskii suggested that a double pendulum be used to detect gravitational radiation emitted by the Crab nebula pulsar.[4] In this paper, we shall find what shape the antenna must be in order to have the highest sensitivity.

II. OPTIMIZATION OF THE ANTENNA SHAPE

We shall start from the following assumptions:

(1) All dependence of the elastic oscillations on directions perpendicular to the horizontal symmetry axes of the antenna is ignored. This is equivalent to setting the Poisson ratio equal to zero, and is quite reasonable for *thin* antennas.

(2) The center of mass remains stationary.

(3) The stress vanishes at the ends of the antenna.

(4) The antenna is axially symmetric and is symmetric with respect to a plane.

(5) The antenna is homogeneous.

Let $y = y(Z)$ be the shape of the antenna (Z being the symmetry axis). Using assumption (1), the equation for longitudinal oscillations can be found in a straightforward way.

Let $\Phi(Z)$ be a deformation of the antenna. The elastic energy due to the deformation is

$$\epsilon = \tfrac{1}{2}\int\int\int_V E\left(\frac{\partial\Phi}{\partial Z}\right)^2 dV\,,$$

where V is the volume of the antenna and E the elastic coefficient. By virtue of hypotheses (4) and (5), this reduces to

$$\epsilon = \tfrac{1}{2}\pi E\int_{-L/2}^{L/2} y^2(Z)\left(\frac{\partial\Phi}{\partial Z}\right)^2 dZ\,,$$

where L is the length of the antenna. The kinetic energy T is obtained in the same way:

$$T = \tfrac{1}{2}\pi\rho\int_{-L/2}^{L/2} y^2(Z)\left(\frac{\partial\Phi}{\partial t}\right)^2 dZ\,,$$

where ρ is the density. Then, the Lagrangian is

$$L = T - \epsilon$$

and the equation of motion is written as

$$\rho y^2(Z)\frac{\partial^2\Phi}{\partial t^2} - Ey^2(Z)\frac{\partial^2\Phi}{\partial Z^2} - 2Ey'y\frac{\partial\Phi}{\partial Z} = 0\ ,$$

where $y' = dy/dZ$.

If we include a dissipative term η, this equation becomes

$$\rho y^2(Z)\frac{\partial^2\Phi}{\partial t^2} + \eta y^2(Z)\frac{\partial\Phi}{\partial t} - Ey^2(Z)\frac{\partial^2\Phi}{\partial Z^2} - 2Eyy'\frac{\partial\Phi}{\partial Z} = 0\,,$$

whence

$$\frac{1}{Vs^2}\frac{\partial^2\Phi}{\partial t^2}+b\frac{\partial\Phi}{\partial t}-\frac{\partial^2\Phi}{\partial Z^2}-\frac{2y'}{y}\frac{\partial\Phi}{\partial Z}=0\ , \qquad (1)$$

where $Vs^2=E/\rho$, $b=\eta/E$.

In the presence of a gravitational wave, we have

$$\frac{1}{Vs^2}\frac{\partial^2\Phi}{\partial t^2}+b\frac{\partial\Phi}{\partial t}-\frac{\partial^2\Phi}{\partial Z^2}-\frac{2y'}{y}\frac{\partial\Phi}{\partial Z}=c^2R^0_{zz0}Z\ . \qquad (2)$$

Let us put

$$R^0_{zz0}=\int_{-\infty}^{\infty}A(\omega)e^{i\omega t}d\omega\ ,$$

$$\Phi=(\Phi_1+i\Phi_2)e^{i\omega t}\ .$$

Equation (1) can be written in the form

$$-\frac{1}{Vs^2}\omega^2\Phi_1-b\omega\Phi_2-\frac{\partial^2\Phi_1}{\partial Z^2}-\frac{2y'}{y}\frac{\partial\Phi_1}{\partial Z}=c^2\mathrm{Re}(A)Z\ ,$$

$$(3)$$

$$-\frac{1}{Vs^2}\omega^2\Phi_2+b\omega\Phi_1-\frac{\partial^2\Phi_2}{\partial Z^2}-\frac{2y'}{y}\frac{\partial\Phi_2}{\partial Z}=c^2\mathrm{Im}(A)Z\ ,$$

with the boundary conditions

$$(\partial\Phi_1/\partial Z)_{Z=L/2}=(\partial\Phi_2/\partial Z)_{Z=L/2}=0\ . \qquad (4)$$

III. OPTIMIZATION OF THE SINGAL-TO-NOISE RATIO

Let $V(Z,t)=V(Z)e^{i\omega_0 t}$ be a solution of Eq. (1) for the fundamental mode. Using the equipartition theorem, we can compute the mean stress

$$\frac{\Phi_N{}^2(0)}{\eta^2}=\left\langle\left(E\frac{d}{dZ}V(Z)\right)^2\Big|_{Z=0}\right\rangle$$

due to thermal noise fluctuations at a given temperature T. In the same way, we can compute the maximum stress

$$\Phi_s(0)=E\left(\frac{\partial\Phi_1}{\partial Z}\right)^2\Big|_{Z=0}+\left(\frac{\partial\Phi_2}{\partial Z}\right)^2\Big|_{Z=0}$$

at the center of the antenna due to a gravitational wave.

Consider the functional $N(y(Z))=\Phi_s(0)/\Phi_N(0)$. We must find a value of $y(Z)$ which maximizes N for given R^0_{zz0}, mass, and frequency. This problem has no simple analytic solutions, and so we have chosen to solve it numerically by approximating $y(Z)$ with the following functions:

$$y_1=a_0-a_1Z^2-a_2Z^4-a_3Z^6-a_4Z^8-a_5Z^{10}\ , \qquad (5a)$$

$$y_2=d_0(1-\alpha_1Z-\alpha_2Z^2-\alpha_3Z^3-\alpha_4Z^4)^\gamma\ . \qquad (5b)$$

In these expansions,

$$\left(\frac{\partial y_1}{\partial Z}\right)\Big|_{Z=0}=0\ ,$$

$$\left(\frac{\partial y_2}{\partial Z}\right)\Big|_{Z=0}\neq 0\ .$$

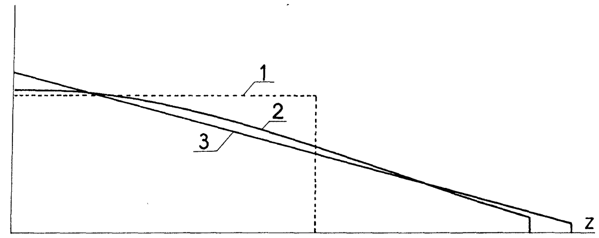

FIG. 1. Longitudinal sections through the symmetry axis Z, showing one quadrant of the antenna profiles: (1), (2), and (3) represent, respectively, Weber's antenna, the antenna described by Eq. (5a), and the one described by Eq. (5b).

It is useful to calculate the N(antenna): N(cylinder) ratio Γ:

$$\Gamma=\frac{N(y(Z))_{\text{ant}}}{N(y(Z))_{\text{cylinder}}}=\frac{[\Phi_s(0)/\Phi_N(0)]_{\text{ant}}}{[\Phi_s(0)/\Phi_N(0)]_{\text{cylinder}}}\ .$$

For the classes of functions (5a) and (5b), it was found that $\Gamma_{1\max}=1.09$ and $\Gamma_{2\max}=1.11$, respectively. In the latter case, the values of the coefficients were $\alpha_1=\frac{1}{2}$, $\alpha_2=0.0083$, $\alpha_3=\alpha_4=0$, $\gamma=1.04$, which corresponds to an approximately conical antenna.

Our antenna is taken to be initially at rest. Therefore, it is reasonable to apply to its interaction with a gravitational wave the notion of *cross section* (Braginskii's criticisms would not seem to apply here), and the cross section of the conical antenna is seen to be 22% higher than that of the cylindrical. Note that for both cases (Fig. 1), the central stress in the cone is 35% lower than in the cylinder, while the end displacement $\Phi(\frac{1}{2}L)$ is twice as large.

IV. CONCLUSION

Weber's cylindrical antenna is quite a good compromise insofar as the cross section is concerned. In fact, a conical antenna would be more expensive because its vacuum chamber would have to be twice as large and its lower stress at the center would inhibit the effective use of detectors based on piezoelectric transducers. On the other hand, when the end displacement of the antenna is used to activate a capacitative detection device, a conical antenna would seem to be optimal.

In fact, in a Weber-type device, the sensitivity is limited by the noise of the piezoelectric transducers,[5,6] and by the low Q of the antenna due to mechanical losses in the ceramics.

Now, a capacitative device is more effective in the detection of small deformations of the antenna. The sensitivity is then limited by the noise of the electronics [phase noise of the high-frequency generator in the active method of detection,[7] amplifier noise in the passive method (in preparation)].

In conclusion, it would appear that a conical antenna can improve the sensitivity of gravitational energy detection by a factor 4.8. The fourfold contribution arises from the square dependence on end displacement, while an improvement of 1.2 is obtained as a consequence of larger cross section.

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Geodesic Motions in the Tomimatsu-Sato Metric

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A solution for the geodesic motions in the gravitational field of a rotating body due to Tomimatsu and Sato is studied.

Recently a new solution for the gravitational field of a rotating body according to general relativity has been found by Tomimatsu and Sato.[1] In this note we study the geodesic motions appropriate to this solution. Our investigation is restricted to the equatorial plane. After obtaining the first integrals of the equation of motion, we construct the equations for radial and angular velocities of test bodies and light rays. We then discuss a number of applications. These include circular orbits in two selected cases, the gravitational deflection of light, and the motion of a test body which, at infinity, is nonrelativistic and has zero angular momentum.

We start with the metric given by Tomimatsu and Sato:

$$d\tau^2 = f(dt - \omega d\phi)^2 - f^{-1}[e^{2\gamma}(dz^2 + d\rho^2) + \rho^2 d\phi^2]. \quad (1)$$

The functions f, γ, and ω depend on the variables ρ and z only, and are given in Ref. 1. The cylindrical coordinates, ρ, z, and ϕ, are related to the Schwarzschild polar coordinates, r, θ, and ϕ, as follows:

$$\rho = (r^2 - 2mr + m^2q^2)^{1/2}\sin\theta, \quad z = (r - m)\cos\theta. \quad (2)$$

The equatorial plane, to which we will confine our attention throughout this paper, is defined by $\theta = \frac{1}{2}\pi$, $z = 0$. The metric functions, f, γ, and ω, on the $z = 0$ plane, are given by the following:

$$f = \frac{A}{B}, \quad \omega = 2mq\frac{C}{A}, \quad e^{2\gamma} = \frac{A}{p^4x^8}, \quad (3)$$

when

$$A = [p^2(x^2-1)^2 + q^2]^2 - 4p^2q^2(x^2-1)x^4, \quad (4)$$

$$B = (p^2x^4 + 2px^3 - 2px - 1)^2, \quad (5)$$

$$C = -p^2(x^2-1)[4x^4 - x^2 + 1] - p^3x(x^2-1)[2x^4 + x^2 + 1] + q^2(1 + px), \quad (6)$$

and

$$x = \frac{1}{mp}(4\rho^2 + m^2p^2)^{1/2}. \quad (7)$$

In the above $p^2 + q^2 = 1$, and q is related to the angular momentum J of the rotating body as $q = J/m^2$; m is the mass energy of the body.

Let us now consider the geodesic motions on the $z = 0$ plane. We are not interested in the equations of motion as such, but only in their first integrals. Applying a standard argument[2] to the metric given by Eq. (1), two of these can be immediately written down: