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# Potential Scattering and Galilei-Invariant Expansions of Scattering Amplitudes*

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Previously derived Galilei-group expansions for the four-particle nonrelativistic scattering amplitude are applied to potential scattering and the expansion coefficients (or Galilei amplitudes) are related in the first Born approximation to the potential. For the spherical partial-wave expansion the coefficients require a knowledge of the Clebsch-Gordan coefficients of E(3), and for the cylindrical eikonal expansion they are simply related to the usual eikonal function. A model amplitude containing Breit-Wigner resonances and other $k$-plane singularities, having correct threshold and reasonable asymptotic behavior, is analyzed in detail. It is shown that poles of partial-wave amplitudes $a_l(k)$ in the $k$ plane correspond to exponential-type asymptotics in the Galilei amplitudes. Specific models, in particular the Bargmann and separable potentials, are examined and their Galilei amplitudes calculated. A Schwinger-type variational principle is given for the Galilei amplitudes.

## I. INTRODUCTION

In a previous article,[1] hereafter quoted as I, we have presented two-variable expansions of nonrelativistic scattering amplitudes. The expansions are written in terms of basis functions of the homogeneous Galilei groups, isomorphic to the three-dimensional Euclidean group E(3), and they are the nonrelativistic limits (obtained when the velocity of light $c\to\infty$) of Lorentz-group two-variable expansions of relativistic amplitudes, considered previously.[2-6] The essential property of both the relativistic and nonrelativistic expansions is that for reactions of the type $1+2\to 3+4$ (and also $1\to 2+3+4$) they completely display the dependence on both kinematic variables (e.g., energy and scattering angle). These variables are contained only in known special functions, provided by the representation theory of the Lorentz group, or the Galilei group, and thus reflect some of the kinematic symmetries of the problem (or at least of the space-time framework in which the reaction occurs). The dynamics is completely represented by the expansion coefficients, which we call the "Lorentz amplitudes" in the relativistic case or the "Galilei amplitudes" in the nonrelativistic one. The expansion coefficients themselves can be investigated by using the general principles of scattering theory, by performing phenomenological fits to experimental data, or by considering specific models. This last approach is particularly straightforward in the nonrelativistic case, where a ready-made model exists, namely, potential scattering. The purpose of this article is to investigate some general properties of the Galilei amplitudes for potential scattering and their relation to the potential.

The actual form of the expansion depends not only on the group under consideration, but also on the choice of a specific basis for the representation theory of the group and also on the choice of a frame of reference for scattering. In I we have considered all bases, corresponding to the reduction of E(3) to subgroups (it should be mentioned that nonsubgroup type bases are also of interest[3,7,8]) and have noted that expansions of scattering amplitudes in terms of two of these bases are of direct physical interest. These are the following.

(i) The partial-wave type, or spherical expansion, corresponding to the reduction $E(3)\supset O(3)\supset O(2)$:

$$F(k,\cos\theta)=\sum_{l=0}^{\infty}(2l+1)\int_{\beta-i\infty}^{\beta+i\infty}r^2dr\,B_l(r)\zeta_l(kr)\times P_l(\cos\theta),\quad \beta>0 \tag{1a}$$

$$B_l(r)=(\pi^2 i)^{-1}\int_0^\infty k^2dk\int_0^\pi \sin\theta\,d\theta\,F(k,\cos\theta)\times\kappa_l(kr)P_l(\cos\theta), \tag{1b}$$

where

$$\zeta_l(z)=(\pi/2z)^{1/2}I_{l+1/2}(z),$$
$$\kappa_l(z)=(\pi/2z)^{1/2}K_{l+1/2}(z)$$

[$I_\nu(z)$ and $K_\nu(z)$ are cylindrical functions of an imaginary argument,[9] $P_l(z)$ is a Legendre polynomial]. $B_l(r)$ is the Galilei amplitude of interest. The scattering amplitude $F(k,\cos\theta)$ is given as a function of the energy and scattering angle in the center-of-mass frame. Expansion (1) is written in terms of nonunitary infinite-dimensional representations of E(3) and converges for a very large class of scattering amplitudes (they may increase as exponentials $e^{\tilde\beta k}$ for $k\to\infty$ with $\tilde\beta<\beta$). If the function $F(k,\theta)$ is square-integrable, i.e., decreases as $k^{-3/2-\epsilon}$, $\epsilon>0$ for $k\to\infty$, we can expand in terms of unitary representations, i.e., take $\beta\to0$ in (1). The expansion then takes a more familiar form, involving spherical Bessel functions:

$$F(k,\cos\theta)=\sum_{l=0}^{\infty}(2l+1)\int_0^\infty r^2dr\,A_l(r)j_l(kr)\times P_l(\cos\theta), \tag{2a}$$

$$A_l(r)=B_l(-ir)e^{-i\pi(l+1)/2}-B_l(ir)e^{i\pi(l+1)/2}. \tag{2b}$$

(ii) The eikonal type or cylindrical expansion, corresponding to the group reduction $E(3)\supset E(2)\times T_\perp$ [where E(2) is the Euclidean group of a plane and $T_\perp$ are translations perpendicular to that plane]:

$$F(k_\perp,k_\parallel)=\int_{\beta-i\infty}^{\beta+i\infty}b\,db\int_{\gamma-i\infty}^{\gamma+i\infty}dz\,B(b,z)I_0(bk_\parallel)e^{k_\perp z},\quad \beta>0,\ \gamma>0 \tag{3a}$$

$$B(b,z)=-(2\pi^2)^{-1}\int_0^\infty k_\parallel dk_\parallel\int_0^\infty dk_\perp F(k_\perp,k_\parallel)\times K_0(bk_\parallel)e^{-k_\perp z}, \tag{3b}$$

where $I_0(z)$ and $K_0(z)$ are again cylindrical functions and $B(b,z)$ is the Galilei amplitude. The variables $k_\perp$ and $k_\parallel$ are two orthogonal components of the momentum of one of the scattering particles (in the scattering plane). In particular, if we consider the scattering of equal-mass particles in the brick-wall (or Breit) frame of reference, we have

$$k_\parallel=2k\sin\tfrac12\theta,\quad k_\perp=k\cos\tfrac12\theta. \tag{4}$$

Expansion (3a) is again written in terms of nonunitary representations of E(3), which makes it possible to expand exponentially increasing functions. If the scattering amplitude decreases as $k_\perp\to\infty$ and/or $k_\parallel\to\infty$ fast enough for $F(k_\perp,k_\parallel)$ to be square-integrable, then we can expand in terms of unitary representations, taking $\beta\to0$ and $\gamma\to0$ in (3). We obtain

$$F(k_\perp,k_\parallel)=\int_0^\infty b\,db\int_{-\infty}^{\infty}dz\,A(b,z)e^{ik_\perp z}J_0(bk_\parallel), \tag{5a}$$

with

$$A(b,z)=-i[B(ib,iz)-B(-ib,-iz)]. \tag{5b}$$

The expansion (5) is readily recognized as the eikonal, or impact-parameter expansion,[10-12] supplemented by a Fourier transform of the eikonal amplitude

$$a(b,k_\perp)=\int_{-\infty}^{\infty}dz\,A(b,z)e^{ik_\perp z}. \tag{6}$$

Expansion (5) can thus be viewed as a far-going generalization of the usual eikonal expansion, being applicable for arbitrary directions (not only close to forward scattering), for square-integrable and non-square-integrable amplitudes (exponentially bounded amplitudes) and being a two-variable expansion rather than a single variable one.

The rest of this article is devoted to an investigation of the Galilei amplitudes (1b) and (3b). In Sec. II we relate these Galilei amplitudes to the potential in the first Born approximation. For the spherical amplitude (1b) this relation involves the Clebsch-Gordan coefficients of O(3); the cylindrical amplitude (3b) is shown to be related to the usual eikonal function. In Sec. III we study a model partial-wave amplitude, having correct threshold behavior and containing a finite number of resonances and bound states. The corresponding spherical Galilei amplitude $B_l(r)$ is calculated explicitly and its characteristic features are analyzed. In particular, poles in the partial-wave amplitude $a_l(k)$ are related to exponential-type asymptotics of $B_l(r)$ for $\mathrm{Im}\,r \to \pm\infty$. In Sec. IV we discuss the Galilei amplitudes for specific solvable potentials, in particular the Bargmann potentials,[13] for which $a_l(k)$ has a finite number of singularities in the $k$ plane, and other potentials, for which $a_l(k)$ has infinitely many singularities. Certain nonlocal separable potentials, for which $A_l(r)$ can be calculated explicitly, are also considered. In Sec. V we derive a variational principle for the Galilei amplitude, analogous to the Schwinger variational principle[14] for the usual partial-wave amplitude $a_l(k)$. In Sec. VI we discuss our results and their possible further development.

## II. GALILEI AMPLITUDES IN BORN APPROXIMATION

The nonrelativistic scattering amplitude can be expressed in terms of a local potential $V(\vec{r})$ in first Born approximation as

$$F(k,\cos\theta) = -\frac{\mu}{2\pi\hbar^2}\int e^{i(\vec{k}_i-\vec{k}_f)\cdot\vec{r}}V(\vec{r})d\vec{r}, \tag{7}$$

where $\mu$ is the particle mass (or the reduced mass for two-body scattering); $k^2 = 2\mu E/\hbar^2$, where $E$ is the energy of the relative motion; $\vec{k}_i$ and $\vec{k}_f$ are the initial and final relative momenta; and $\theta$ is the c.m. scattering angle.[15] We shall now use formula (7) to calculate the Galilei amplitudes.

### A. The Spherical Galilei Amplitudes

For simplicity, let us limit ourselves to an expansion in terms of unitary representations of E(3). The Galilei amplitude $A_l(r)$ of (2) is

$$A_l(r) = \pi^{-1}\int_0^\infty k^2dk\int_0^\pi \sin\theta\, d\theta\, F(k,\cos\theta)j_l(kr) \times P_l(\cos\theta). \tag{8}$$

We now substitute the first Born approximation (7) for $F(k,\cos\theta)$ and assume that the potential is spherically symmetric: $V(\vec{r}) = V(r)$. Expanding the exponentials in terms of spherical harmonics and performing the angular integrations, we find

$$A_l(r) = -\frac{4\mu}{\pi\hbar^2}\int_0^\infty r'^2dr'V(r')\begin{Bmatrix} r' & r' & r \\ l & l & l\end{Bmatrix}, \tag{9}$$

where

$$\begin{Bmatrix} r' & r' & r \\ l & l & l\end{Bmatrix} = \int_0^\infty k^2dk\, j_l(kr')j_l(kr')j_l(kr). \tag{10}$$

The integral (10) is directly related to the Clebsch-Gordan coefficients of E(3) and has been calculated explicitly in I (at least for $l$ even). In particular the integral is zero unless

$$0 \le r \le 2r'. \tag{11}$$

Expressions (9) and (10) throw some light on the physical meaning of $A_l(r)$. First of all, the connection with the potential is quite straightforward and simple. The selection rule (11) shows that for a given value of $r$ the Galilei amplitude $A_l(r)$ receives contributions only from that part of the potential for which $r' \ge \frac{1}{2}r$, where $r'$ is the distance from the scattering center. Thus, $A_l(0)$ is sensitive to the entire potential; $A_l(r)$ for $r\to\infty$ is only sensitive to the asymptotic tail of the potential. In particular if the potential has a finite range $R$, we can write

$$A_l(r) = -\frac{4\mu}{\pi\hbar^2}\int_{r/2}^R r'^2dr'V(r')\begin{Bmatrix} r' & r' & r \\ l & l & l\end{Bmatrix}, \tag{12}$$

so that

$$A_l(r) = 0 \quad \text{for } r \ge 2R\,. \tag{13}$$

The above results are only valid in the first Born approximation and apply only to square-integrable amplitudes. We do however expect the Born approximation to reproduce some of the essential features of the problem, in particular the connection between the finite range of a potential and the amplitudes $A_l(r)$ [see (13)]. The case of non-square-integrable amplitudes can be treated analogously, starting from formula (2) for $B_l(r)$. We obtain a formula analogous to (9), but $j_l(kr)$ in (10) must be replaced by $\kappa_l(kr)$. A group theoretic treatment of this integral involves a study of nonunitary representations of E(3) and will not be presented here.

### B. The Cylindrical Galilei Amplitudes

Let us now consider scattering in the brick-wall frame of Fig. 1. First we consider the case of square-integrable amplitudes, so that the cylindrical Galilei amplitude of (5) is

$$A(b,z)=\frac{1}{2\pi}\int_0^\infty k_\parallel dk_\parallel \int_{-\infty}^{\infty} dk_\perp F(k_\perp,k_\parallel) \times J_0(k_\parallel b)e^{-ik_\perp z}. \quad (14)$$

In the first Born approximation the scattering amplitude $F(k_\perp, k_\parallel)$ is given by formula (7). We substitute (7) into (14), interchange the order of integration, put $\vec{r}\,'=(\rho'\cos\phi', \rho'\sin\phi', z')$ and $\vec{k}_i-\vec{k}_f=(k_\parallel,0,0)$, with $k_\parallel=2k\sin\frac{1}{2}\theta$ and use the expansion

$$e^{i(\vec{k}_i-\vec{k}_f)\cdot\vec{r}\,'}=\sum_{n=-\infty}^{\infty} i^n e^{in\phi'}J_n(k_\parallel\rho'). \quad (15)$$

Finally we obtain

$$A(b,z)=-\frac{\mu}{\hbar^2}\delta(z)\int_{-\infty}^{\infty} V((b^2+z'^2)^{1/2})dz'. \quad (16)$$

Formula (16) is to be compared to standard expressions used in the eikonal approximation[10-12]:

$$F(k_\perp,k_\parallel)=-ik\int_0^\infty bdb\,(e^{2i\chi(k_\perp,b)}-1)J_0(2kb\sin\tfrac{1}{2}\theta), \quad (17)$$

$$\chi(k_\perp,b)=-\frac{k}{4E}\int_{-\infty}^{\infty}V(b,z)dz. \quad (18)$$

For large $E$ we put

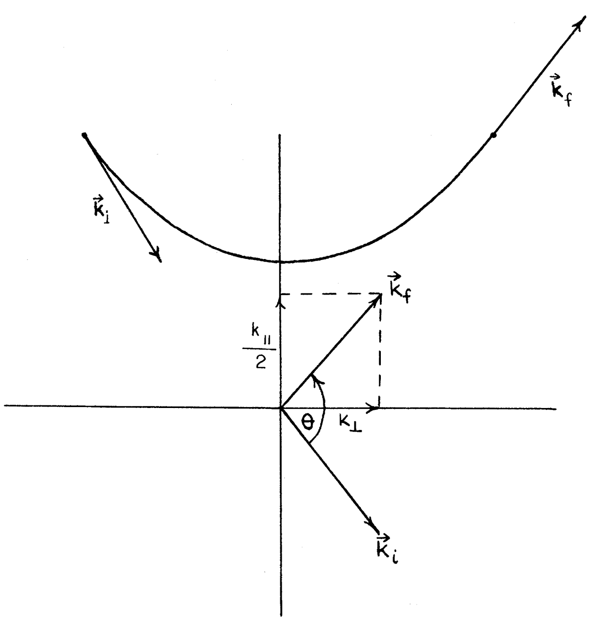

FIG. 1. Brick-wall frame for potential scattering.

$$-ik[e^{2i\chi(k_\perp,b)}-1]\approx 2k\chi(k_\perp,b)=-\mu\int_{-\infty}^{\infty}V(b,z)dz. \quad (19)$$

Comparing (16) and (19), we obtain the Born approximation relation between the cylindrical Galilei amplitude and the usual eikonal, namely,

$$A(b,z)\approx\frac{2k}{\hbar^2}\delta(z)\chi(k_\perp,b) \quad (20)$$

(note that in this approximation $k\chi(k_\perp,b)$ depends only on the impact parameter $b$).

Following I we can also write a "backward eikonal expansion" which coincides with (5) except that $k_\perp$ and $k_\parallel$ are replaced by

$$q_\perp=k\sin\tfrac{1}{2}\theta, \quad q_\parallel=2k\cos\tfrac{1}{2}\theta. \quad (21)$$

This time we find that the Galilei amplitude is

$$A(b,z)=-\frac{\mu}{\hbar^2}\frac{\delta(b)}{b}\int_0^\infty \rho d\rho\, V(\rho,z). \quad (22)$$

Thus we again have a very simple relation between the Galilei amplitudes and the potential. The generalization to non-square-integrable amplitudes is quite straightforward in this case, but we shall not dwell upon it here.

## III. MODEL AMPLITUDES

In paper I we have investigated some general features of the two-variable expansions, in particular the appearance of bound states and resonances, threshold behavior, asymptotic behavior, etc. In particular it was shown that simple poles of the partial-wave amplitudes $a_l(k)$ in the complex $k$ plane are generated by exponential-type asymptotic behavior of the Galilei amplitudes $A_l(r)$ and $B_l(r)$ (for $|r|\to\infty$). Let us now construct some physically reasonable partial-wave amplitudes $a_l(k)$ and calculate the corresponding unitary and nonunitary spherical Galilei amplitudes.

### A. Square-Integrable Partial-Wave Amplitudes

For square-integrable partial-wave amplitudes we use the unitary expansion (2a) and have

$$A_l(r)=\frac{2}{\pi}\int_0^\infty k^2 a_l(k)j_l(kr)dk, \quad (23)$$

and in particular for the $s$ wave

$$A_0(k)=\frac{2}{\pi r}\int_0^\infty ka_0(k)\sin kr\,dk. \quad (24)$$

Let us first consider some simple models for the $s$ wave.

*(i) Simple resonance or bound state.*

$$a_0(k)=\frac{A}{k^2+k_0^{\,2}}, \tag{25}$$

where $A$ and $k_0$ are complex constants; the Galilei amplitude is (see Ref. 9)

$$A_0(r)=\frac{A}{r}e^{-rk_0}, \quad \mathrm{Re}k_0>0. \tag{26}$$

*(ii) Smooth-background term exponentially damped for* $k\to\infty$.

$$a_0(k)=bk^n e^{-\alpha k}, \quad \mathrm{Re}\alpha>0. \tag{27}$$

We have

$$A_0(r)=\frac{2b}{\pi}\frac{\Gamma(n+2)}{r(\alpha^2+r^2)^{n/2+1}}\sin\left[(n+2)\arctan\frac{r}{\alpha}\right]. \tag{28}$$

*(iii) Resonance or bound state damped exponentially for* $k\to\infty$.

$$a_0(k)=\frac{A}{k^2+k_0^{\,2}}K_2(\alpha[k^2+k_0^{\,2}]^{1/2}), \quad \alpha>0, \quad \mathrm{Re}k_0>0 \tag{29}$$

where $A$ and $k_0$ are complex and $\alpha$ is real. For $k\to\infty$, the Macdonald function $K_2(\alpha[k^2+k_0^{\,2}]^{1/2})$ behaves like $\exp[-\alpha(k^2+k_0^{\,2})^{1/2}]$. This function was inserted in (29), rather than an exponential, simply to facilitate the integration. The Galilei amplitude is (see Ref. 9, formula 6.596.7)

$$A_l(r)=\frac{A}{\alpha^2}\frac{\exp[-k_0(\alpha^2+r^2)^{1/2}]}{k_0}. \tag{30}$$

Let us now consider general partial waves.

*(a) Resonance or bound-state term, damped exponentially for* $k\to\infty$ *and having "reasonable" threshold behavior.*

$$a_l(k)=\frac{Ak^l}{k^2+k_0^{\,2}}K_2(\alpha[k^2+k_0^{\,2}]^{1/2}), \quad \alpha>0, \quad \mathrm{Re}k_0>0; \tag{31}$$

we find that

$$A_l(r)=A\left(\frac{2}{\pi}\right)^{1/2}\frac{r^l}{\alpha^2}\frac{k_0^{\,l-1/2}}{(\alpha^2+r^2)^{(1/2)(l+1/2)}}\times K_{1/2-l}(k_0[\alpha^2+r^2]^{1/2}). \tag{32}$$

*(b) Smooth exponentially damped background term.*

$$a_l(k)=bk^{n+l}e^{-\alpha k}, \quad \alpha>0. \tag{33}$$

The Galilei amplitude can, after some manipulations of the integral, be written as

$$A_l(r)=b(i)^{l+1/2}2^{l+1}\Gamma(l+1)\Gamma(n+2)\times\frac{r^l}{(r^2+\alpha^2)^{l+n/2+3/2}}\times C_{n+1}^{l+1}(\alpha(\alpha^2+r^2)^{-1/2}), \tag{34}$$

where $C_{n+1}^{l+1}(z)$ is a Gegenbauer polynomial.

We see that for the unitary expansions a pole of $a_l(k)$ in the complex plane at $k=+ik_0$ corresponds directly and simply to an exponential term of the type (26) in the Galilei amplitude $A_l(r)$ (note that we have the behavior

$$A_l(r)\to\frac{A}{\alpha^2}k_0^{\,l-3/2}\frac{1}{r}e^{-k_0r}$$

for large $r$).

### B. Non-Square-Integrable Partial-Wave Amplitudes

We use expansion (1) in terms of non-unitary representations of E(3) and write the Galilei amplitude as

$$B_l(r)=\frac{2}{\pi^2 i}\int_0^\infty k^2a_l(k)\kappa_l(kr)dk, \quad r=\beta+iq, \quad \beta>0, \tag{35}$$

which for the $s$ wave reduces to

$$B_0(r)=\frac{1}{\pi ir}\int_0^\infty ka_0(k)e^{-kr}dk. \tag{36}$$

Let us again first consider specific models for the $s$ wave.

*(i) A resonance or bound-state term.*

$$a_0(k)=\frac{Ae^{-\alpha k}}{k^2+k_0^{\,2}}, \quad \mathrm{Re}\alpha>-\beta. \tag{37}$$

Using formula 3.354.2 of Ref. 9, we find that

$$B_0(r)=-\frac{1}{\pi ir}\{\mathrm{ci}(\alpha+r)k_0\cos(\alpha+r)k_0+\mathrm{si}(\alpha+r)k_0\sin(\alpha+r)k_0\}, \tag{38}$$

where $\mathrm{ci}(x)$ and $\mathrm{si}(x)$ are the cosine and sine integrals, respectively.[9]

In the determination of the asymptotic behavior of (38) we proceed as follows: It is known that the upper half of the complex $k$ plane corresponds to the physical sheet of the scattering amplitude and that the dynamically significant singularities such as resonances, virtual, and antibound states occur on the second sheet, i.e., the lower half of the complex $k$ plane.[15] (Note: This is not true for bound states, which we exclude from our considerations here.) The condition that (37) have a "dynamical pole" at $k=ik_0$ is then that

$$\tfrac{1}{2}\pi<\arg k_0<\tfrac{3}{2}\pi.$$

So we see that asymptotically

$$\arg(\alpha + r)k_0 \to \tfrac{1}{2}\pi + \arg k_0 > \pi$$

for $q \to \infty$. Thus, for $q$ sufficiently large we cannot use the conventional asymptotic expansions for (38) which are valid only in the region $|\arg(\alpha + r)k_0| < \pi$. In order to apply the asymptotic formulas correctly we make use of the formulas[9]

$$\begin{aligned} \mathrm{si}(\chi e^{-i\pi}) &= -\pi - \mathrm{si}\chi\,, \\ \mathrm{ci}(\chi e^{-i\pi}) &= -i\pi + \mathrm{ci}\chi\,. \end{aligned} \tag{39}$$

We have, for sufficiently large $q$,

$$\begin{aligned} B_0(r) = -\frac{e^{-(\alpha+r)k_0}}{r} & - \frac{1}{\pi i r}\{\mathrm{ci}[e^{-i\pi}(\alpha + r)k_0]\cos[e^{-i\pi}(\alpha + r)k_0] \\ & + \mathrm{si}[e^{-i\pi}(\alpha + r)k_0] \\ & \times \sin[e^{-i\pi}(\alpha + r)k_0]\}\,, \end{aligned} \tag{40}$$

where now

$$-\pi < \arg[e^{-i\pi}(\alpha + r)k_0] < \pi\,.$$

We now obtain the exponential term we expect with the remaining term in the expression (40) having the asymptotic behavior $i[\pi r(\alpha + r)^2 k_0^2]^{-1}$ for large $q$.

*(ii) A smooth background term of type (27) with* $Re\alpha > -\beta$. This gives

$$B_0(r) = \frac{b}{\pi i r}\,\frac{\Gamma(n+2)}{(r+\alpha)^{n+3}}\,. \tag{41}$$

Let us now consider general partial waves: a resonance or bound state with reasonable threshold behavior

$$a_l(k) = \frac{k^l}{k^2 + k_0^2}\,. \tag{42}$$

The Galilei amplitude is then

$$\begin{aligned} B_l(r) = [\pi r]^{-1/2}\,\frac{1}{\pi i}\,2^{l+1}\Gamma(l+\tfrac{3}{2})k_0^{\,l+1/2} \\ \times S_{-l-3/2,\,l+1/2}(k_0 r)\,, \end{aligned} \tag{43}$$

where $S_{\mu,\nu}(z)$ is a Lommel function.[9,16] In particular for $l=0$, we have

$$\begin{aligned} S_{-3/2,1/2}(k_0 r) = -\frac{1}{(k_0 r)^{1/2}}[\sin(k_0 r)\mathrm{si}(k_0 r) \\ + \cos(k_0 r)\mathrm{ci}(k_0 r)]\,, \end{aligned}$$

i.e., the $s$-wave result (38). By considerations exactly analogous to those used in the analysis of the $s$ wave we can asymptotically obtain the required exponential term.

### C. Comments on the Galilei Amplitudes

The main point that we wish to abstract both from the general consideration of I and from the examples discussed at length above is the following: A physically "reasonable" partial-wave amplitude, containing a finite number of resonances (and/or bound states) and a smooth background, behaving reasonably at the threshold and at asymptotic energies, can be written as

$$a_l(k) = k^l\left(\sum_{i=1}^{N}\frac{A_i}{k^2 + k_i^{\,2}}\,e^{-\alpha_i k} + \sum_{j=0}^{M} b_j k^j e^{-\beta_j k}\right)\,. \tag{44}$$

We have shown above that the corresponding spherical Galilei amplitudes $A_l(r)$ and $B_l(r)$ can be readily computed. The important thing to realize is that a resonance term of the type $(k^2 + k_0^2)^{-1}$ in $a_l(k)$ corresponds to a term of the type $e^{-k_0 r}/r$ in the Galilei amplitudes. Similarly, a background term behaving as $k^n$ in $a_l(k)$ corresponds to a term like $r^{-n-3}$ in the Galilei amplitudes. This in turn suggests a method of parametrizing the Galilei amplitudes $A_l(r)$ and $B_l(r)$, that is quite simple and does reproduce the essential features of the partial-wave amplitude (44), namely,

$$\left.\begin{matrix} A_l(r) \\ B_l(r) \end{matrix}\right\} = \sum_{i=1}^{N} B_i\,\frac{e^{-k_i r}}{r} + \sum_{j=1}^{M} C_j r^{-j}\,, \tag{45}$$

where $B_i$, $C_j$, and $k_i$ are complex constants. An ansatz like (45) would then seem to be a reasonable starting point for a phenomenological analysis of scattering data, simultaneously for all energies $0 \le k < \infty$ and all angles $0 \le \theta \le \pi$. We shall return to this problem of energy-dependent partial-wave analysis separately in the context of relativistic two-variable expansions.

Finally, let us note that an infinite number of resonances or bound states in a given partial wave can be incorporated by letting $N \to \infty$ in (45). Alternatively, we can construct model Galilei amplitudes, generating infinitely many poles of $a_l(k)$ in the $k$ plane. A class of examples that springs to mind is provided by the generalized hypergeometric functions[17,18]:

$$B_l(r) = \frac{1}{r}\;{}_pF_q(\alpha_1 \cdots \alpha_p;\,\beta_1 \cdots \beta_q;\,e^{-k_0 r})\,. \tag{46}$$

## IV. EXAMPLES OF SOLVABLE POTENTIALS

In this section we shall investigate the Galilei amplitudes for specific local and nonlocal solvable potentials, for which the partial-wave amplitudes $a_l(k)$ can be obtained explicitly. These can then be substituted into expression (23) for $A_l(r)$ or (35) for $B_l(r)$, as the case may be.

Let us note that for $k\to 0$ we have

$$j_l(kr)\underset{k\to 0}{\sim}\tfrac{1}{2}\sqrt{\pi}\,\frac{1}{\Gamma(l+\frac{3}{2})}(\tfrac{1}{2}kr)^l\times\left[1-\frac{1}{l+\frac{3}{2}}(\tfrac{1}{2}kr)^2+\cdots\right], \tag{47}$$

$$\kappa_l(kr)\underset{k\to 0}{\sim}\sqrt{\pi}\,2^{l-1}\Gamma(l+\tfrac{1}{2})\frac{e^{-kr}}{(kr)^{l+1}}(1+kr+\cdots). \tag{48}$$

For $k\to\infty$, on the other hand, we have

$$j_l(kr)\underset{k\to\infty}{\sim}\frac{1}{kr}\left[\sin(kr-\tfrac{1}{2}\pi l)+l(l+1)\cos(kr-\tfrac{1}{2}\pi l)\frac{1}{2kr}+\cdots\right], \tag{49}$$

$$\kappa_l(kr)\underset{k\to\infty}{\sim}\frac{\pi}{2kr}e^{-kr}\left[1+\frac{l(l+1)}{2kr}+\cdots\right]. \tag{50}$$

Equations (47)–(50) show that the expression (23) for the unitary Galilei amplitude $A_l(r)$ will converge at $k=0$, as long as $a_l(k)$ is not too singular for $k\to 0$. The corresponding integral will diverge for $k\to\infty$, unless $a_l(k)$ vanishes at least as $1/k$. The integral (35) for the nonunitary amplitude $B_l(r)$ will, on the other hand, diverge at $k\to 0$ unless the partial wave vanishes at least as $a_l(k)\sim k^{l-2+\epsilon}$, $\epsilon>0$ for $k\to 0$. For $k\to\infty$ the integral (35) will converge even if $a_l(k)$ increases exponentially.

Let us now consider specific examples.

### A. Local Solvable Potentials

Let us first consider a class of potentials, introduced by Bargmann,[13] of such a nature that the corresponding Jost functions[15] for the $s$-wave partial-wave amplitude can be written explicitly, it has a finite number of simple poles in the complex $k$ plane, and the corresponding Galilei amplitudes can readily be computed. Indeed, Bargmann has proposed many different potentials, or classes of potentials, for which the $s$-wave partial-wave amplitude can be written as

$$a_0(k)=\frac{2(\lambda+\nu)}{(2k-i\nu)(2k-i\lambda)}=2i\,\frac{\lambda+\nu}{\lambda-\nu}\left(\frac{1}{2k-i\nu}-\frac{1}{2k-i\lambda}\right). \tag{51}$$

Examples of such potentials are, e.g.,

$$V_1(r)=\frac{2}{(r+\alpha)^2},\qquad V_2(r)=-2\beta\lambda^2\frac{e^{-\lambda r}}{(\beta e^{-\lambda r}+1)^2} \tag{52}$$

[for the connection between the constants in (51) and (52), see the original article[13]].

Unless $\nu=0$ or $\lambda=0$, the integral for the unitary Galilei amplitude $A_0(r)$ will diverge. The nonunitary one is obtained by substituting (51) into (36). Performing the integration we find

$$B_0(r)=\frac{i(\lambda+\nu)}{\pi r(\lambda-\nu)}\left[\lambda e^{-i\lambda r/2}\,\mathrm{Ei}(\tfrac{1}{2}i\lambda r)-\nu e^{-i\nu r/2}\,\mathrm{Ei}(\tfrac{1}{2}i\nu r)\right]. \tag{53}$$

Thus, the Galilei amplitude again involves exponential integrals and the asymptotic exponential behavior for $|r|\to\infty$ can be extracted as in Sec. III [see Eq. (38) and below].

Of obvious interest are those potentials for which the partial-wave amplitude $a_l(k)$ can be calculated for all values of $l$, namely the Coulomb potential[19] $1/r$, the square-well potential,[20,21] the delta[21,22] potential $g\delta(r-a)$ and the repulsive core.[15,21]

The threshold and asymptotic behavior of the Coulomb partial-wave amplitude is such that the integrals for both $A_l(r)$ and $B_l(r)$ diverge. All the other potentials mentioned above give amplitudes $a_l(k)$ that can be expanded in terms of nonunitary representations, since the correct threshold behavior $k^{2l}$ ensures the convergence of the integral (35) for $k\to 0$. For the $\delta$-function potential the integral (23) for the unitary amplitude $A_l(r)$ will also converge. The simplest example to consider is that of the repulsive core, so let us concentrate on it:

$$V(r)=\begin{cases}+\infty, & r\leqslant R\\ 0, & r>R\end{cases} \tag{54}$$

[this can be considered to be the limit of the potential $V(r)=g\delta(r-R)$ for $g\to\infty$]. The partial-wave amplitude is

$$a_l(k)=-\frac{J_{l+1/2}(kR)}{ikH^{(1)}_{l+1/2}(kR)}. \tag{55}$$

The expressions for the general Galilei amplitudes are quite complicated, so let us again just consider $l=0$. We have

$$a_0(k)=-\frac{1}{k}\sin kR\,e^{-ikR}, \tag{56}$$

so that

$$B_0(r)=-\frac{1}{\pi i r}\int_0^\infty \sin kR\,e^{-k(r+iR)}dk=\frac{iR}{\pi r^2(r+2iR)}. \tag{57}$$

To conclude this subsection, let us make a few comments on the Galilei amplitudes for the δ-shell potential. For

$$V(r) = -g\delta(r-a)$$

we have

$$a_l(k) = \frac{1}{k}\,\frac{\pi g a J_{l+1/2}{}^2(ka)}{1 - i\pi g a J_{l+1/2}(ka) H^{(1)}_{l+1/2}(ka)}\,. \qquad (58)$$

Putting $\hbar^2 = \mu = 1$ in (12) we find that in the Born approximation for the δ shell we have

$$A_l^B(r) = \frac{4ga^2}{\pi}\begin{Bmatrix} a & a & r \\ l & l & l \end{Bmatrix} = \frac{4ga^2}{\pi}\int_0^\infty k^2 dk\, j_l{}^2(ka) j_l(kr)\,. \qquad (59)$$

The exact Galilei amplitude is, however, obtained by substituting (58) into (23). Let us split the corresponding integral into two parts—from 0 to $K$, and from $K$ to $\infty$. If $K \gg 2g$, then the second term in the denominator of (58) can be dropped in the second integral and we obtain

$$A_l(r) \approx ga\left(\frac{2\pi}{r}\right)^{1/2}\left[\int_0^K \frac{J_{l+1/2}{}^2(ka) J_{l+1/2}(kr)\sqrt{k}\, dk}{1 - i\pi g a J_{l+1/2}(ka) H^{(1)}_{l+1/2}(ka)} + \int_K^\infty J_{l+1/2}{}^2(ka) J_{l+1/2}(kr)\sqrt{k}\, dk\right]. \qquad (60)$$

After trivial transformations we can rewrite (60) as

$$A_l(r) \approx i\pi g^2 a^2\left(\frac{2\pi}{r}\right)^{1/2}\int_0^K J_{l+1/2}{}^3(ka) J_{l+1/2}(kr) H^{(1)}_{l+1/2}(ka)\sqrt{k}\, dk + \frac{4ga^2}{\pi}\int_0^\infty j_l^2(ka) j_l(kr) k^2 dk. \qquad (61)$$

Thus, we have written $A_l(r)$ as an integral over a finite region, plus the Born term.

### B. Nonlocal Separable Potentials

Separable potentials are of interest, because the corresponding Schrödinger equation can be solved explicitly[23,24] and they have been used with some success in the treatment of low-energy nucleon-nucleon scattering.[25,26]

A separable potential is nonlocal, i.e., nondiagonal in the position representation. Its simplicity derives from the fact that its position-matrix elements have the factored form $\langle \vec{r}\,'|V|\vec{r}\rangle = V(\vec{r}\,')V(\vec{r})$; its momentum-matrix elements factor in a similar way. We will consider a potential which is a rotationally invariant sum of such terms:

$$\langle \vec{r}\,'|V|\vec{r}\rangle = -\sum_l \frac{\lambda_l}{2l+1} v_l(r') v_l(r) P_l(\hat{r}\,'\cdot\hat{r})\,. \qquad (62)$$

In the momentum representation the potential is

$$\langle \vec{k}\,'|V|\vec{k}\rangle = -\frac{1}{2\pi^2}\sum_l \lambda_l u_l(k') u_l(k) P_l(\hat{k}\,'\cdot\hat{k})\,, \qquad (63)$$

where

$$u_l(k) = \left(\frac{4\pi}{2l+1}\right)^{1/2}\int_0^\infty dr\, r^2 j_l(kr) v_l(r)\,. \qquad (64)$$

Then the $l$th partial-wave scattering amplitude can be shown to be

$$a_l(k) = \lambda_l u_l(k)^2 \bigg/ \left(1 + \frac{2\lambda_l}{\pi}\int_0^\infty dq\, \frac{q^2 u_l(q)^2}{k^2 - q^2 + i\epsilon}\right). \qquad (65)$$

There is a (single) bound state in the $l$th wave if

$$\lambda_l \int_0^\infty dq\, u_l(q)^2 > \tfrac{1}{2}\pi\,. \qquad (66)$$

Let us consider a few simple examples. For simplicity we suppress $l$ as a subscript on λ and other parameters of the potential.

*Example 1.*

$$u_l(k) = k^{-\alpha}, \quad \tfrac{1}{2} < \alpha < \tfrac{3}{2}\,. \qquad (67)$$

This corresponds to a spatial "potential"

$$v_l(r) = C/r^{3-\alpha}, \qquad C = \frac{(2l+1)^{1/2} 2^{1-\alpha}\Gamma(\frac{1}{2}l - \frac{1}{2}\alpha + \frac{3}{2})}{\pi\Gamma(\frac{1}{2}l + \frac{1}{2}\alpha)}\,. \qquad (68)$$

For $\lambda > 0$ there is a bound state in the $l$th partial wave at $E = k_B{}^2/2m$, where

$$k_B = i(-\lambda \sec\pi\alpha)^{1/(2\alpha-1)}\,. \qquad (69)$$

The partial-wave amplitude is

$$a_l(k) = \frac{\lambda}{k(k^{2\alpha-1} + \lambda\tan\pi\alpha - i\lambda)}\,. \qquad (70)$$

We examine this for two particular values of $\alpha$.

*(a)* $\alpha = 1$.

$$a_l(k) = \frac{\lambda}{k(k - i\lambda)}\,. \qquad (71)$$

The unitary Galilei amplitude is

$$A_l(r) = \frac{2\lambda}{\pi}\int_0^\infty dk\, \frac{k j_l(kr)}{k - i\lambda}\,. \qquad (72)$$

The integral can be evaluated with the help of a formula given by Watson[27] together with some rearrangement of terms; the required formula is

$$\int_0^\infty dk \frac{k^\rho j_l(kr)}{k-i\lambda} = \frac{e^{-i\pi(\rho/2-1/4)}2^{\rho-1}\pi^{1/2}\lambda^{\rho-1/2}}{r^2} \times \left[\frac{\Gamma(\frac{1}{2}\rho+\frac{1}{2}l+\frac{1}{2})}{\Gamma(-\frac{1}{2}\rho+\frac{1}{2}l+1)} S_{-\rho+1/2,\,l+1/2}(-ir\lambda) - \frac{2\Gamma(\frac{1}{2}\rho+\frac{1}{2}l+1)}{\Gamma(-\frac{1}{2}\rho+\frac{1}{2}l+\frac{1}{2})} S_{-\rho-1/2,\,l+1/2}(-ir\lambda)\right], \quad (73)$$

where $S_{\mu,\nu}(z)$ is a Lommel function.[9,27] Thus, our Galilei amplitude is

$$A_l(r) = \frac{e^{-i\pi/4}2\lambda^{3/2}}{\pi^{1/2}r^{1/2}} \times \left[\frac{\Gamma(\frac{1}{2}l+1)}{\Gamma(\frac{1}{2}l+\frac{1}{2})} S_{-1/2,\,l+1/2}(-ir\lambda) - \frac{2\Gamma(\frac{1}{2}l+\frac{3}{2})}{\Gamma(\frac{1}{2}l)} S_{-3/2,\,l+1/2}(-ir\lambda)\right]. \quad (74)$$

It is of interest to consider the limit of $A_l(r)$ as $r \to \infty$. According to Watson, $S_{\mu,\nu}(z) \to z^{\mu-1}$. The first term of (74) dominates and we have

$$A_l(r) \sim 2i\pi^{-1/2} \frac{\Gamma(\frac{1}{2}l+1)}{\Gamma(\frac{1}{2}l+\frac{1}{2})} \frac{1}{r^2}. \quad (75)$$

We wish to examine the behavior of $A_l(r)$ when $r$ is continued to negative values by passing below the origin, $r \to re^{-i\pi}$; this necessitates passing through the cut in $S_{\mu,\nu}(z)$ which lies along the negative $z$ axis. The continuation can be accomplished by noticing that the integral in (73), apart from a factor $r^{-\rho}$, depends on $r$ and $\lambda$ only through $r\lambda$. So we can achieve the continuation by letting $\lambda \to \lambda e^{-i\pi}$ and multiplying by $e^{i\pi\rho}$. It is evident that the continuation $\lambda \to \lambda e^{-i\pi}$ of the integral in (73) is equivalent to substituting $\lambda \to \lambda e^{i\pi}$ and adding

$$2\pi e^{i(1-\rho)\pi/2}\lambda^\rho j_l(\lambda r e^{-i\pi/2}), \quad (76)$$

whose asymptotic value as $r \to \infty$ is

$$\pi e^{i(1-\rho-l)\pi/2}\lambda^{\rho-1}r^{-1}e^{r\lambda}. \quad (77)$$

The asymptotic form of $A_l(r)$ for $r \to \infty \cdot e^{-i\pi}$ is thus

$$e^{-il\pi/2}\,2\lambda r^{-1}e^{r\lambda}. \quad (78)$$

Note that the pole of the partial-wave amplitude at $k=i\lambda$ appears as the exponent in (78) (since $e^{r\lambda} = e^{-ir(i\lambda)}$) in the usual manner.

*(b)* $\alpha = \frac{3}{4}$.

$$a_l = \frac{\lambda}{k[k^{1/2}-\lambda(1+i)]} = \frac{\lambda[k^{1/2}+2^{1/2}e^{i\pi/4}\lambda]}{k(k-2i\lambda^2)}. \quad (79)$$

The Galilei amplitude can be evaluated in the same way as before with the result

$$A_l(r) = \frac{2\lambda^3}{\pi^{1/2}r^{1/2}}\left[\frac{2\Gamma(\frac{1}{2}l+1)}{\Gamma(\frac{1}{2}l+\frac{1}{2})} S_{-1/2,\,l+1/2}(-2ir\lambda^2) - i2^{-1/2}(2l+1)S_{-1,\,l+1/2}(-2ir\lambda^2) - \frac{4\Gamma(\frac{1}{2}l+\frac{3}{2})}{\Gamma(\frac{1}{2}l)} S_{-3/2,\,l+1/2}(-2ir\lambda^2) + i2^{-3/2}(2l-1)(2l+3)S_{-2,\,l+1/2}(-2ir\lambda^2)\right]. \quad (80)$$

For $r \to \infty$ the first term in (80) dominates and we find that

$$A_l(r) \sim 2^{1/2}\frac{e^{i3\pi/4}\Gamma(\frac{1}{2}l+1)}{\pi^{1/2}\Gamma(\frac{1}{2}l+\frac{1}{2})r^2}. \quad (81)$$

For $r \to \infty\, e^{-i\pi}$ we find that

$$A_l(r) \sim 4e^{-il\pi/2}\lambda^2 r^{-1}e^{2r\lambda^2}.$$

*Example 2.*

$$u_l(k) = (k^2+\beta^2)^{-1}. \quad (82)$$

The spatial "potential" can be evaluated and is expressible in terms of Lommel functions. For $l=0$ it is of Yukawa type

$$v_0(r) = (4\pi)^{-1/2}\frac{e^{-\beta r}}{r}. \quad (83)$$

Write

$$\alpha = \left(\frac{\lambda}{2\beta}\right)^{1/2}. \quad (84)$$

Then if $\alpha > \beta$, as we suppose, there is a bound state at

$$k_B = i(\alpha-\beta). \quad (85)$$

The scattering amplitude is

$$a_l(k) = \frac{\lambda}{(k-i\beta)^2[k+i(\alpha+\beta)][k-i(\alpha-\beta)]}. \quad (86)$$

To evaluate the Galilei amplitude we break (86) into partial fractions and use Eq. (73) to determine the contribution of each term. In the case of the $(k-i\beta)^2$ term it is necessary to differentiate Eq. (73) with respect to $\lambda$.

The Galilei amplitude works out to be

$$A_l(r) = \frac{e^{-i3\pi/4}4}{\pi^{1/2}r^{1/2}} \Big\{ A e^{i3\pi/2}(\alpha+\beta)^{3/2} F(ir(\alpha+\beta)) + B(\alpha-\beta)^{3/2}F(-ir(\alpha-\beta)) + C\beta^{3/2}F(-ir\beta) - iD\frac{\partial}{\partial\beta}[\beta^{3/2}F(-ir\beta)] \Big\}, \quad (87)$$

where

$$A = -\frac{i}{2\alpha(\alpha+2\beta)^2}, \quad C = \frac{4i\beta}{(\alpha+2\beta)^2(\alpha-2\beta)^2}, \quad B = \frac{i}{2\alpha(\alpha-2\beta)^2}, \quad D = \frac{1}{(\alpha+2\beta)(\alpha-2\beta)}, \quad (88)$$

and

$$F(z) = \frac{\Gamma(\frac{1}{2}l+\frac{3}{2})}{\Gamma(\frac{1}{2}l)} S_{-3/2,l+1/2}(z) - \frac{2\Gamma(\frac{1}{2}l+2)}{\Gamma(\frac{1}{2}l-\frac{1}{2})} S_{-5/2,l+1/2}(z). \quad (89)$$

For $r\to\infty$ the $S_{-3/2,l+1/2}$ terms dominate, except for $l=0$, where its coefficient vanishes. Asymptotically,

$$A_l(r) \sim \frac{4i\Gamma(\frac{1}{2}l+\frac{3}{2})}{\pi^{1/2}\Gamma(\frac{1}{2}l)r^3}\left[-\frac{A}{\alpha+\beta}+\frac{B}{\alpha-\beta}+\frac{C}{\beta}+\frac{iD}{\beta^2}\right], \quad l\neq 0 \quad (90)$$

and

$$A_0(r) \sim \frac{1}{r^4}\left[\frac{A}{(\alpha+\beta)^2}+\frac{B}{(\alpha-\beta)^2}+\frac{C}{\beta^2}+\frac{2iD}{\beta^3}\right]. \quad (91)$$

For $r\to\infty\, e^{-i\pi}$,

$$A_l(r) \sim -ie^{-il\pi/2}2\left[\frac{B(\alpha-\beta)}{r}e^{(\alpha-\beta)r} - iD\beta^2 e^{\beta r}\right]. \quad (92)$$

*Example 3.*

$$u_l(k) = (k^2+\beta^2)^{-1/2}. \quad (93)$$

In order that various integrals be well-defined, a convergence factor $\lim_{\epsilon\to 0} e^{-\epsilon k}$ or $\lim_{\epsilon\to 0} k^{-\epsilon}$ should be included on the right-hand side of Eq. (72). The spatial "potential" $v(r)$ is expressible in terms of hypergeometric functions.

If, as we suppose, $\lambda>\beta$, there is a bound state at $k_B = i(\lambda-\beta)$. The scattering amplitude is

$$a_l(k) = \frac{\lambda}{(k-i\beta)[k-i(\lambda-\beta)]} \quad (94)$$

and the Galilei amplitude is

$$A_l(r) = \frac{e^{-i\pi/4}4\lambda}{\pi^{1/2}(\lambda-2\beta)r^{1/2}} \times [\beta^{3/2}F(-ir\beta) - (\lambda-\beta)^{3/2}F(-ir(\lambda-\beta))], \quad (95)$$

where $F(z)$ is defined by Eq. (89). The asymptotic form of (95) for $r\to\infty$ is

$$A_l(r) \sim \frac{4\Gamma(\frac{1}{2}l+\frac{3}{2})}{\pi^{1/2}\beta(\lambda-\beta)r^3\Gamma(\frac{1}{2}l)}, \quad (96)$$

except for the $s$ wave when

$$A_0(r) \sim \frac{-i4\lambda^2}{\pi\beta^2(\lambda-\beta)^2 r^4}. \quad (97)$$

For $r\to\infty e^{-i\pi}$ the asymptotic form is

$$A_l(r) \sim \frac{e^{-il\pi/2}\lambda}{(\lambda-2\beta)r}[\beta e^{\beta r} - (\lambda-\beta)e^{(\lambda-\beta)r}]. \quad (98)$$

These few simple examples of separable potentials could be augmented by many more. In none of the examples does a nonunitary Galilei amplitude exist, since the scattering amplitudes do not vanish strongly enough, or even at all, for $k\to 0$; hence we deal with unitary Galilei amplitudes.

The presence of a bound state at $k_B = i\alpha$, say, is reflected in each of the preceding examples by a term in the Galilei amplitude which grows like $e^{r\alpha}$ for $r\to\infty e^{-i\pi}$. For an antibound state, a pole in $a_l(k)$ on the negative imaginary axis, there is no such growing exponential for $r\to\infty e^{-i\pi}$ (but there is one for $r\to\infty e^{+i\pi}$).

### C. Comments on Partial Waves with Infinitely Many Poles

In all examples for which the Galilei amplitudes were calculated explicitly, the partial-wave amplitude had a finite number of singularities, which correspond to resonances, bound states, and virtual states. There are, however, many well-known examples for which the number of such singularities is infinite (e.g., the $s$-wave Hulthén potential as well as the square-well and $\delta$-function potentials mentioned above). These amplitudes can be readily incorporated in our scheme by using the Mittag-Leffler expansion[28] of the $S$ matrix:

$$S_l(k) = 1 + k^{2l+1}\left\{P(k) + \left[\sum_1^{N'}\frac{iR_n'}{x_n(k+ix_n)} - \sum_1^{N}\frac{i\tilde{R}_n}{\tilde{k}_n(k-i\tilde{k}_n)} + \sum_1^{\infty}\left(\frac{R_n/k_n}{k-k_n}+\frac{R_n^*/k_n^*}{k+k_n^*}\right)\right]\right\}, \quad (99)$$

where $-ix_n$ are the positions of the $N'$ virtual states, $i\tilde{k}_n$ of the $N$ bound states and $k_n$ ($\text{Re}k_n > 0$) of the resonance states; $P(k)$ is an entire function of $k$, and $R_n'$, $\tilde{R}_n$, and $R_n$ are the corresponding residues. From this representation we see that we can, in principle, treat any partial-wave amplitude corresponding to a potential with finite range by the methods we have already outlined. One example for which the singularities in the $k$ plane are known, is the $s$-wave Hulthén potential amplitude,[29] which for a repulsive potential has infinitely many integer-spaced virtual states at the points

$$k = -\frac{in}{a},$$

where the potential is

$$V(r) = \frac{v_0 e^{-r/a}}{1 - e^{-r/a}}.$$

The $s$-wave Galilei amplitude is then a semi-infinite series in the variable $e^{-r/a}$ plus a background term

$$A_0(r) = \sum_{n=1}^{\infty} a_n e^{-nr/a} + \sum_m b_m r^{-m-3}. \tag{100}$$

Generally speaking, the positions of the singularities of $a_l(k)$ are only implicitly known as is the entire function $P(k)$. From the phenomenological point of view, however, this does not matter.

## V. VARIATIONAL PRINCIPLE FOR GALILEI AMPLITUDES

It is possible to construct a stationary expression for the Galilei amplitude in potential scattering analogous to that given by Schwinger for the usual partial-wave amplitude.[14]

The Galilei amplitude may be written as

$$A_l(r) = \int_0^\infty dk\, k^2 j_l(kr) \int_0^\infty dr' r'^2 j_l(kr') V(r') \times \psi_l(k, r') = A_1\{\psi_l(k, r')\}, \tag{101}$$

where $\psi_l(k, r)$ is the radial wave function and satisfies

$$\psi_l(k, r) = j_l(kr) + \int_0^\infty dr' r'^2 g_l^k(r, r') V(r') \times \psi_l(k, r'); \tag{102}$$

$g_l^k(r, r')$ is the radial Green's function. Solving (102) for $j_l(kr)$ and substituting for $j_l(kr)$ in (101) gives a second expression for the Galilei amplitude:

$$\begin{aligned} A_l(r) &= \int_0^\infty dk\, k^2 j_l(kr) \int_0^\infty dr' r'^2\, V(r')[\psi_l(k, r')]^2 \\ &\quad - \int_0^\infty dk\, k^2 j_l(kr) \int_0^\infty dr'' r''^2\, V(r'') \psi_l(k, r'') \\ &\quad \times \int_0^\infty dr' r'^2 g_l^k(r'', r') V(r') \psi_l(k, r') \\ &= A_2\{\psi_l(k, r')\}. \end{aligned} \tag{103}$$

Then the expression $A_1^2/A_2$ is also a correct formula for the Galilei amplitude and, moreover, is stationary with respect to variations of the wave functions $\psi_l(k, r)$ considered as a function of $k$ and $r$. The functional derivative which is asserted to vanish is

$$\frac{\delta}{\delta\psi_l(k, r)} \frac{A_1^2}{A_2} = \frac{1}{A_2^2}\left[ 2A_2 A_1 \frac{\delta A_1}{\delta\psi_l(k, r)} - A_1^2 \frac{\delta A_2}{\delta\psi_l(k, r)} \right].$$

Now

$$\frac{\delta A_1}{\delta\psi_l(k, r)} = k^2 r^2 V(r) j_l(kR) j_l(kr)$$

and

$$\begin{aligned} \frac{\delta A_2}{\delta\psi_l(k, r)} &= 2k^2 r^2 V(r) j_l(kR) \psi_l(k, r) \\ &\quad - 2k^2 r^2 V(r) j_l(kR) \\ &\quad \times \int_0^\infty dr' r'^2 g_l^k(r, r') V(r') \psi_l(k, r'). \end{aligned}$$

It follows easily that

$$\frac{\delta}{\delta\psi_l(k, r)} \left( \frac{A_1^2}{A_2} \right) = 0,$$

provided that $\psi_l(k, r)$ satisfies the wave equation (102).

## VI. CONCLUSIONS

The main purpose of this article was to investigate the relation between Galilei amplitudes, partial-wave amplitudes, eikonals, and potentials for specific models. We have shown that in the Born approximation the Galilei amplitudes are related to the potential in a simple manner that has a clear physical and group-theoretical meaning. We have calculated the Galilei amplitudes in the $E(3) \supset O(3) \supset O(2)$ expansions explicitly for numerous examples and established their main features. In particular, simple poles of partial-wave amplitudes $a_l(k)$ in the complex $k$ plane are generated by terms in the Galilei amplitudes $A_l(r)$ that behave as $e^{-rk_0}/r$ for $r \to \infty$ (or $r \to e^{\pm i\pi}\infty$). Similar terms appear in the nonunitary amplitudes $B_l(r)$ for Im$r$

$\to \pm\infty$ and generate $k$-plane singularities in $a_l(k)$.

The difficulties associated with the generalization of the concept of a potential to a relativistic theory are well known. The concept of a Galilei amplitude, on the other hand, has its direct relativistic counterpart, namely the Lorentz amplitudes, figuring in O(3, 1) expansions.[2-6] It may be argued that our examples show that the Galilei amplitudes are much more complicated than the potentials themselves. They are, however, not more complicated than the corresponding partial-wave amplitudes $a_l(k)$, which are generally used to describe relativistic reactions. The Galilei amplitudes, or the Lorentz ones, in a relativistic theory, go far beyond the partial-wave amplitudes, in that they describe both the angular and energy dependence of a reaction. In other words, these Galilei and Lorentz expansions provide parametrizations of scattering amplitudes that can be used for performing energy-dependent partial-wave analysis.

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