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Covariant Harmonic Oscillators and the Quark Model*

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An attempt is made to give a physical interpretation to the phenomenological wave function of Yukawa, which gives a correct nucleon form factor in the symmetric quark model. This wave function is first compared with the Bethe-Salpeter wave function. It is shown that they have similar Lorentz-contraction properties in the high-momentum limit. A hyperplane harmonic oscillator is then introduced. It is shown that the Yukawa wave function, which is defined over the entire four-dimensional Euclidean space, can be interpreted in terms of the three-dimensional hyperplane oscillators. It is shown further that this wave function satisfies a Lorentz-invariant differential equation from which excited harmonic-oscillator states can be constructed, and from which a gauge-invariant electromagnetic interaction can be generated.

I. INTRODUCTION

The quark,[1,2] which was originally introduced to explain SU(3) symmetry and its consequences, has gained considerable ground as a fundamental constituent particle in hadrons. The inventors of the quark did not make any commitment to its existence.[1] In spite of this and other well-known difficulties, model calculations based on this constituent particle are producing increasingly encouraging results.[3]

In both the successful and the disappointing features of the quark model, there seems to be one crucial question: What "forces" are responsible for making quarks stay together in hadrons?[4] In the early days of the quark model, quarks were put into an infinite potential purely for convenience,[2,5] and no attempts were made to assert that these simplified forces were of fundamental importance. In the symmetric quark model, for instance, Greenberg used the harmonic-oscillator potential in order to borrow the well-known formalism from the nuclear shell model.

As the resonance spectrum became richer, the search for quantum numbers that correspond to binding forces continued.[6] It was Kim and Noz[7] who established the existence of harmonic-oscillator-like radial modes for nonstrange baryon resonances for which there is barely enough experimental data to test the linearity of the three lowest energy levels.

There are numerous calculations of decay rates in the harmonic-oscillator model.[8] A more important analysis seems to be that of elastic form factors. The first objection to the use of the harmonic oscillator, that is the Gaussian, wave function is that the form factor decreases exponentially for large t values. This discrepancy with the real world together with our field-theoretic common sense once led us to conclude that the harmonic-oscillator potential, which is manifestly analytic at the origin, cannot be the fundamental force between the quarks.[9] However, an encouraging development was that the relativistic effect suitably applied on the Gaussian wave function eliminates this unwanted exponential decrease and gives the desired dipole effect.[10-13]

The above-mentioned relativistic models are essentially one or another form of the Gaussian wave function multiplied by a Lorentz contraction factor, and they do not necessarily represent a completely consistent picture of the relativistic bound state. The important fact, however, is that all those "wrong" models give a correct form factor. We are thus led to believe that there is some truth in the Lorentz contraction of quantum-mechanical wave functions.[14]

We realize that there are no completely consistent relativistic measurement theories and that we are not going to solve this difficult problem in this paper. For this reason, we can give relativistic interpretations only in terms of the existing languages that have been developed to answer this ultimate question. One commonly used language is the Bethe-Salpeter equation.[14,15] This equation is well known to us and has been used extensively in both low- and high-energy physics.[16]

Another important language developed for the same purpose is the hyperplane generalization of the Schrödinger equation. The concept of a spacelike hyperplane played a crucial role in the early days of quantum field theory.[17] This hyperplane technique was used recently by Fleming to understand the Newton-Wigner localization problem.[18,19] We shall use this hyperplane language in order to understand Lorentz-contracted Gaussian wave functions.

We are specifically interested in the covariant oscillator wave function first introduced by Yukawa[20] and used by Fujimura *et al.*[10,11] in their successful calculation of the nucleon form factor in the symmetric quark model. In spite of their numerical success, there does not seem to be any physical basis for the covariant differential equation from which the wave function is derived. Thus it is fair to say that the Yukawa wave function has been a purely phenomenological entity. The purpose of this paper is to give a physical meaning to this wave function in terms of the accepted relativistic languages.

In Sec. II, we compare the Yukawa wave function with the Bethe-Salpeter wave function. It is pointed out that both wave functions are to be integrated over the four-dimensional Euclidean space in the low-momentum region. We note that both the Yukawa and the Bethe-Salpeter wave functions have the same Lorentz contraction properties in the large-momentum limit. Since the Bethe-Salpeter equation is a field-theoretic model, we believe that this is the point where Yukawa's nonlocal theory makes contact with local field theory.

In Sec. III, we introduce the hyperplane technique. The nonrelativistic harmonic oscillator can be generalized to covariant hyperplanes. We present a hyperplane interpretation of the Yukawa wave function which is consistent with the Lorentz-invariant probability and the observed nucleon form factor.

In Sec. IV, we discuss a Lorentz-invariant differential equation which the Yukawa wave function satisfies. This equation can generate a gauge-invariant electromagnetic interaction. It is shown that this harmonic-oscillator differential equation can be separated in the normal coordinate variables which are Lorentz transformations of the space-time variables, and that the excited states can be constructed in this normal coordinate system. A Lorentz-invariant mass eigenvalue is given.

In Sec. V, we discuss briefly the experimental basis upon which the harmonic-oscillator quark model is built.

II. PROPERTIES OF THE YUKAWA AND THE BETHE-SALPETER WAVE FUNCTIONS

In this section, we compare the covariance properties of the Bethe-Salpeter and the Yukawa wave functions. In the early days of nonrelativistic quantum mechanics, the standing-wave properties for the square well, the harmonic oscillator, and the other bound-state potentials were described by different mathematical techniques. However, the inherent similarities enabled the creators of quantum mechanics to formulate a new concept of bound states in terms of the quantum superposition principle. By studying the properties that are common to the Bethe-Salpeter and the Yukawa wave functions, which have different mathematical forms, we expect to work toward finding a possible new form of relativistic dynamics.

Since the Bethe-Salpeter equation and its wave functions are well known,[15] we will only describe here how Yukawa arrived at his covariant harmonic-oscillator model. Yukawa noticed that Born's reciprocity relation[21] gives an oscillator-like Hamiltonian and attempted to write down a normalizable wave function in terms of the relative internal coordinates. The covariance requirement, however, forced him to introduce time-like excitations with negative energies. As a consequence, the energy levels were infinitely degenerate. In order to eliminate this undesirable feature, Yukawa introduced a coupling with an external momentum. His wave function takes the form

$$\Psi(x,p)=\exp\{-\tfrac{1}{2}\omega[\,x^2+2(p\cdot x)^2/m^2]\}\,, \tag{1}$$

where x is the relative space-time four-vector and p is the total four-momentum of the bound state. Throughout this paper we use the space-favored metric where $x^2=(\vec{x})^2-x_0^{\,2}$.

The bound-state Bethe-Salpeter Green's function takes the form[14,15]

$$G(x,p)=\left(\frac{1}{4\pi}\right)^2\int_0^1 d\alpha\cos(\tfrac{1}{2}\alpha p\cdot x)\times K_0\big(\tfrac{1}{2}(x^2)^{1/2}[4M^2-(1-\alpha^2)m^2]^{1/2}\big)\,. \tag{2}$$

This Green's function is seen to be a function of x and p as in Yukawa's function above [Eq. (1)]. The mass of the bound state is given by m. We consider here the bound state of two equal-mass particles whose individual mass is M. This Green's function is not the solution of the equation but contains most of the features of the exact wave function.[15]

We are now ready to discuss the covariance properties that are common to Eq. (1) and Eq. (2). We start from the rest frame where $\vec{p}=0$. In this system, Eq. (1) becomes a harmonic-oscillator wave function in the four-dimensional Euclidean space of $\vec{x}$ and t, and is manifestly normalizable. We can make the Bethe-Salpeter Green's function of Eq. (2) normalizable in the four-dimensional Euclidean space of $(\vec{x},t)$ by making the Wick rotation.[15]

As we increase $|\vec{p}|$, this property holds for Eq. (2) until the kinetic energy becomes larger than the binding energy.[14] For $|\vec{p}|$ larger than the binding energy, the Bethe-Salpeter wave function is no longer normalizable in the above-mentioned four-dimensional Euclidean space. The harmonic-oscillator wave function of Eq. (1) does not suffer from this effect and remains normalizable for large values of $|\vec{p}|$. This is expected because particles bound by an oscillator potential have infinite binding energy.

Let us rewrite the oscillator wave function assuming that $\vec{p}$ is in the z direction. We use E for p_0 and p for p_z. Then

$$\Psi(x,p) = \exp[-\tfrac{1}{2}\omega(x^2+y^2)] \times \exp\{(-\omega/4m^2)[(E-p)^2(t+z)^2 + (E+p)^2(t-z)^2]\}. \quad (3)$$

For large p,

$$\frac{\omega(E-p)^2}{4m^2} \to \frac{\omega}{16}\left(\frac{m}{p}\right)^2, \qquad \frac{\omega(E+p)^2}{4m^2} \to \omega\left(\frac{p}{m}\right)^2. \quad (4)$$

Thus

$$\Psi(x,p) \to \exp[-\tfrac{1}{2}\omega(x^2+y^2)] \times \exp[-\tfrac{1}{16}\omega(m/p)^2(t+z)^2] \times \exp[-\omega(p/m)^2(t-z)^2]. \quad (5)$$

The last factor becomes $(\sqrt{\pi}/\omega)(m/p)\delta(t-z)$ for large p, and the dependence on the variable $(t+z)$ becomes insensitive by the factor $(m/p)^2$. This contraction behavior is strikingly similar to that of the Bethe-Salpeter equation.[14] The Bethe-Salpeter wave function is a model derivable from field theory. The oscillator function is a phenomenological wave function giving correct form factors. It is interesting to note that these two wave functions have the same Lorentz contraction properties in the large-p limit.

We now restrict ourselves to the Yukawa wave function. Let us analyze the form factor calculation of Fujimura *et al.*[10] in the Breit system. We can sketch the initial and final "Lorentz-contracted" wave functions as in Fig. 1. The form-factor integral

$$F(q^2) = \int d^4x\, \psi_f^*(x)\psi_i(x)\exp(iq\cdot x), \quad (6)$$

where q is the momentum transfer, receives contributions only from the small overlapping region indicated in Fig. 1. This region shrinks as the momentum transfer increases, and this coherent

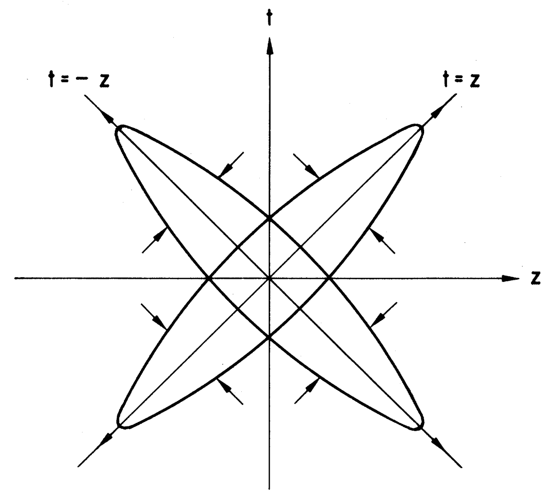

FIG. 1. Lorentz-contracted wave functions with two equal and opposite momenta. The form-factor integral of Fujimura *et al.* receives contributions primarily from the small overlapping region.

shrinkage is responsible for the nonexponential decrease of the form factor.

In Eq. (6), the integral is performed over Euclidean space-time. We know clearly the physical meaning of the probability distribution over the three-dimensional space, but we do not know what physics, if any, the timelike probability distribution corresponds to. We shall discuss this problem in Sec. III.

III. HYPERPLANE FORMALISM OF HARMONIC OSCILLATOR

Here we study Yukawa's phenomenological wave function from the point of view of the nonrelativistic-harmonic-oscillator wave function, generalized to covariant hyperplanes.

Let us start with the nonrelativistic harmonic oscillator. The Hamiltonian is separable and the wave function is Gaussian multiplied by the appropriate polynomials corresponding to excited energy levels. Because the ground-state wave function depends only on $(\vec{x})^2$ in the exponent, we can Lorentz-generalize $\vec{x}$ to the three-vector on the hyperplane which is perpendicular to the total four-momentum of the system. We follow the standard method of constructing this three-vector and

$$\tilde{x}_\mu = \left(\delta_\mu^\nu + \frac{p_\mu p^\nu}{m^2}\right)x_\nu. \quad (7)$$

When the momentum $\vec{p}$ is zero, $\tilde{x}_\mu$ becomes $\vec{x}$. For nonzero $\vec{p}$,

$$p^\mu \tilde{x}_\mu = 0$$

and

$$\tilde{x}^{\mu}\tilde{x}_{\mu}=x^{\mu}x_{\mu}+\left(\frac{p\cdot x}{m}\right)^2 . \quad (8)$$

Assume now that $\vec{p}$ is in the z direction. Using p for p_z and β for v/c, Eq. (8) becomes

$$\tilde{x}^{\mu}\tilde{x}_{\mu}=x^{\mu}x_{\mu}+(1-\beta^2)^{-1}(t-\beta z)^2 . \quad (9)$$

The three independent hyperplane coordinate variables are

$$x,\ y,\ \text{and}\ (1-\beta^2)^{-1/2}(z-\beta t) . \quad (10)$$

The hyperplane ground-state oscillator wave function then takes the form

$$\phi(x,\beta)=\exp\{-\tfrac{1}{2}\omega[x^2+(1-\beta^2)^{-1}(t-\beta z)^2]\} . \quad (11)$$

There are two important differences between the above wave function and that of Eq. (1). First, the coefficients of $(p\cdot x/m)^2$ are different. In Eq. (1), it is 2, while it is 1 in Eq. (11). Next, Eq. (1) is integrated over the entire four-space while Eq. (11) is integrated only over the three-dimensional hyperplane. The purpose of this section is to point out that we can indeed give a hyperplane interpretation to the Yukawa wave function of Eq. (1).

The wave function given in Eq. (11), which depends explicitly on β, is the ground-state wave function. We can excite the harmonic oscillator just as in the nonrelativistic case. If we multiply ϕ or its excited form by $\exp[-\frac{1}{2}\omega(1-\beta^2)^{-1}(t-\beta z)^2]$, it does not change the hyperplane oscillator because the variable $-(1-\beta^2)^{-1/2}(t-\beta z)$ is perpendicular to the three hyperplane variables given in Eq. (9). If we perform the integration over the variable $-(1-\beta^2)^{-1/2}(t-\beta z)$ after this multiplication, this certainly leaves the hyperplane oscillator intact. Therefore we can write the inner product of two wave functions belonging to the same hyperplane as

$$(\phi_n,\phi_m)=\int\exp[-\omega(1-\beta^2)^{-1}(t-\beta z)^2]\times\phi_n^*(x,\beta)\phi_m(x,\beta)d^4x . \quad (12)$$

The integration measure d^4x is invariant under Lorentz transformation. Because of this the above quantity does not depend on the hyperplane parameter β. Hence, we have introduced a multiplication factor, $\exp\{-\frac{1}{2}\omega[\frac{1}{2}(p\cdot x)]^2\}$, and an inner product of the form of Eq. (12), while leaving the hyperplane oscillator intact. By doing this we have been able to show that the hyperplane probability is Lorentz-invariant.

Let us consider next the inner product between wave functions belonging to two different hyperplanes.[22] Since nonrelativistic quantum mechanics does not say anything about Lorentz transformation, it cannot give the transition probability between two such wave functions. We believe that this is one of the most pressing problems of our time and that we can solve this problem only by building models that can produce the observed experimental data.

In order to build such a model, we go back to our original rule that $\exp[-\frac{1}{2}\omega(p\cdot x/m)^2]$ multiply each wave function and that the integral be performed over the entire four-space. Then the inner product becomes

$$(\phi_1,\phi_2)=\int\exp\left\{-\frac{\omega}{2}\left[\left(\frac{p_1\cdot x}{m_1}\right)^2+\left(\frac{p_2\cdot x}{m_2}\right)^2\right]\right\}\times\phi_1(x,\beta_1)\phi_2(x,\beta_2)d^4x . \quad (13)$$

Here again the integration measure d^4x is hyperplane-independent and is good for both the β_1 and the β_2 plane. The above expression becomes Eq. (12) when β_1 and β_2 are equal.

The next and most crucial question is whether the above inner product produces experimentally measurable effects. The answer is contained in the fact that because of the additional exponential factor, the form factor calculation with this inner product becomes exactly the phenomenological form of Fujimura *et al.* which we discussed in Sec. II. The single-oscillator ground-state form factor becomes in the Breit system

$$F(q^2)=\int d^4x\exp[i\vec{q}\cdot\vec{x}]\exp[-\omega(x^2+y^2)]\times\exp\left[-\frac{\omega}{m^2}(m^2+2q^2)(t^2+z^2)\right] . \quad (14)$$

For large q^2, the time integral is like a δ-function integral, and hence this form becomes that of Licht and Pagnamenta[12] who proposed the instant $(t=0)$ probability integral.

We have thus generalized the time-independent harmonic oscillator to covariant hyperplanes, and then introduced a covariant inner product. This operation leaves the hyperplane oscillator intact, produces the Lorentz-invariant probability for the states belonging to the same hyperplane and gives the correct nucleon form factor.

At this point, we may mention that the mathematics of the covariant harmonic oscillator is very similar to that of the quantization of the electromagnetic field. There are two well-known approaches to the electromagnetic field quantization. One uses the Lorentz gauge, and the other uses the Coulomb gauge. The Coulomb-gauge method is not manifestly covariant, but its main advantage is that we do not introduce unphysical photons and thus we can make quick references to the real world.

There have been many previous attempts to understand the covariant harmonic oscillator.[23] In this paper, we used the hyperplane coordinates to avoid timelike excitations. The advantages are similar to those in the Coulomb gauge case. By eliminating completely the burden of handling those unphysical excitations, we have been able to separate clearly what can be done and what cannot be done in the framework of nonrelativistic quantum mechanics. We emphasize here that a relativistic measurement theory has yet to be constructed.[24]

IV. COVARIANT DIFFERENTIAL EQUATION AND EXCITED STATES

In the preceding sections, we studied a possible physical interpretation of the Gaussian factor which corresponds to a ground-state harmonic oscillator. In order to construct excited states, we use the Lorentz-invariant differential equation which is needed in generating a gauge-invariant electromagnetic interaction of the harmonic-oscillator quarks.[11]

We rewrite here the ground-state solution

$$\psi_0(x,p) = \exp\left\{-\frac{\omega}{2}\left[x^2 + 2\left(\frac{x\cdot p}{m}\right)^2\right]\right\}$$

as

$$\psi_0(x,p) = \psi_0(y) = \exp[-\tfrac{1}{2}\omega(y_1^2 + y_2^2 + y_3^2 + y_0^2)]\,, \qquad (15)$$

where

$$y_1 = x_1\,, \quad y_2 = x_2\,,$$
$$y_3 = (1-\beta^2)^{-1/2}(x_3 - \beta t)\,, \qquad (16)$$
$$y_0 = (1-\beta^2)^{-1/2}(t - \beta x_3)\,.$$

The above linear transformation is a homogeneous Lorentz transformation of the original coordinate variables. Thus $\psi_0(x,p)$ satisfies the equation

$$\left\{-\nabla_y^2 + \frac{\partial^2}{\partial y_0^2} + \omega^2[(\vec{y})^2 - y_0^2]\right\}\psi_0(y) = \lambda\psi_0(y)\,. \qquad (17)$$

Since the transformation of Eq. (16) is a Lorentz transformation, we also have

$$\left\{-\nabla^2 + \frac{\partial^2}{\partial t^2} + \omega^2[(\vec{x})^2 - t^2]\right\}\psi_0(x,p) = \lambda\psi_0(x,p)\,. \qquad (18)$$

Eq. (17) and Eq. (18) represent the same equation. The form of Eq. (18) has been discussed in the literature and is used in constructing a gauge-invariant electromagnetic interaction.

The Gaussian form of Eq. (15) is not separable in the x-coordinate variables. It is separable in the y variables which contain the p dependence. Thus we have to use Eq. (17) to construct excited states. Because of the Lorentz invariance of the harmonic-oscillator operator, the excited-state wave functions also satisfy the differential equation of Eq. (18).

We now write the excited-state solution as

$$\psi_\lambda(y) = H_{n_1}(y_1)H_{n_2}(y_2)H_{n_3}(y_3)H_{n_0}(y_0) \times\exp[-\tfrac{1}{2}\omega(\vec{y}^2 + y_0^2)]\,, \qquad (19)$$

where

$$\lambda = \omega(n_1 + n_2 + n_3 - n_0 + 1)\,. \qquad (20)$$

The above solution is possible because the starting differential equation of Eq. (17) is separable and remains separable as we change the value of the total four-momentum p. The quantum numbers n_i are separation constants. Our Lorentz transformation therefore preserves this separability. Because of the minus sign in front of n_0, the eigenvalues of Eq. (20) are infinitely degenerate. In order to remove this ambiguity, we set $n_0 = 0$; the physics of this procedure has been discussed in Sec. III. Thus

$$\lambda = \omega(N+1)\,,$$

where (21)

$$N = n_1 + n_2 + n_3\,.$$

Since the separability is preserved, the $n_0 = 0$ condition is invariant under a Lorentz transformation. The covariant harmonic oscillator now has three normal excitation variables, namely, y_1, y_2, and y_3, and they are precisely the hyperplane variables mentioned in Sec. III. They are O(3)-invariant within the hyperplane and generate covariant excited-state wave functions in exactly the same way as in the nonrelativistic oscillator.

The eigenvalue λ can serve as the mass of the covariant harmonic oscillator or as its mass squared. There have been many previous attempts to express the bound-state mass as an eigenvalue of a differential equation.[8,10,20,23] In all these attempts, except possibly that of Lipes,[25] the x variables are used to excite the harmonic oscillator. Since the Gaussian factor is separable in the x variables only in the rest frame, the mass quantum numbers are good only in that frame, and an attempt to boost will bring in an infinite number of unphysical wave functions.

It has been twenty years since Yukawa introduced the Gaussian factor corresponding to the Lorentz contraction.[20] The concept of quark did not exist at that time. It was stated in Yukawa's paper that the differential equation representing the coupling of the total momentum to the internal oscillation is so complicated that the study of the interaction of the internal mode with the external field is very difficult. We have shown in this paper that the differential equation is similar to the Klein-Gordon equation, and that the interaction can be manufactured in the usual way.

V. CONCLUDING REMARKS

In this paper, we discussed, first, Lorentz contraction properties of the covarient Gaussian factor. We then proposed the use of the hyperplane technique to study possible relativistic ingredients in quantum mechanics. Finally, we introduced the normal-coordinate method in solving the covariant harmonic-oscillator equation, and showed that this method is technically equivalent to the hyperplane method.

The normal-coordinate method is the most powerful weapon in attacking harmonic-oscillator problems. It is a convenient way of describing covariantly the orbital and radial quantum numbers. Therefore we have studied in this paper a possible theoretical tool which can link the basic concepts of quantum mechanics to quantities that can be measured experimentally.

The most widely available numbers that can be both calculated and measured are decay rates.[8] Since the decay rate calculations are not sensitive to the exact shape of the wave function, the decay rate alone does not force us to accept the harmonic-oscillator model.

The form factor study such as the one discussed in this paper strengthens our assertion on the harmonic oscillator and enables us to relate the observed curve to Lorentz contractions.[26]

The most important characteristic of the harmonic oscillator is, of course, the linearity of its eigenvalues. In order to study the linearity in the observed mass spectra, we need at least three radial modes. For nonstrange baryons, we barely have these three levels, and the present authors studied this linearity.[7]

Radial quantum numbers ↓	Baryons		Mesons	
	non-strange	strange	non-strange	strange
n = 0	A	A	A	A
n = 1	A	A^-	B^-	C
n = 2	A^-	C	D	D
n = 3				

FIG. 2. Summary of the present status of the multiplet scheme in the symmetric quark model. A means "excellent", B means "good", etc.

In this paper, we have restricted ourselves to nonstrange baryons. We realize that there are some difficulties in pionic form factors.[10] As we see in the experimental summary of Fig. 2, we do not yet have enough experimental information from which a linear mass spectrum can be derived for the mesons. Therefore we cannot and do not insist on the simple harmonic oscillator for the mesons. Consequently, we do not have to explain the above-mentioned difficulty at this time.

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Potential Scattering and Galilei-Invariant Expansions of Scattering Amplitudes*

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Previously derived Galilei-group expansions for the four-particle nonrelativistic scattering amplitude are applied to potential scattering and the expansion coefficients (or Galilei amplitudes) are related in the first Born approximation to the potential. For the spherical partial-wave expansion the coefficients require a knowledge of the Clebsch-Gordan coefficients of E(3), and for the cylindrical eikonal expansion they are simply related to the usual eikonal function. A model amplitude containing Breit-Wigner resonances and other k-plane singularities, having correct threshold and reasonable asymptotic behavior, is analyzed in detail. It is shown that poles of partial-wave amplitudes $a_l(k)$ in the k plane correspond to exponential-type asymptotics in the Galilei amplitudes. Specific models, in particular the Bargmann and separable potentials, are examined and their Galilei amplitudes calculated. A Schwinger-type variational principle is given for the Galilei amplitudes.

I. INTRODUCTION

In a previous article,[1] hereafter quoted as I, we have presented two-variable expansions of nonrelativistic scattering amplitudes. The expansions are written in terms of basis functions of the homogeneous Galilei groups, isomorphic to the three-dimensional Euclidean group E(3), and they are the nonrelativistic limits (obtained when the velocity of light $c\to\infty$) of Lorentz-group two-variable expansions of relativistic amplitudes, considered previously.[2-6] The essential property of both the relativistic and nonrelativistic expansions is that for reactions of the type $1+2\to 3+4$ (and also $1\to 2+3+4$) they completely display the dependence on both kinematic variables (e.g., energy and scattering angle). These variables are contained only in known special functions, provided by the representation theory of the Lorentz group, or the Galilei group, and thus reflect some of the kinematic