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New Approach to the Renormalization Group*

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A new set of renormalization-group equations is presented. These equations are based on a renormalization procedure in which counterterms are calculated for zero unrenormalized mass. Unlike the Gell-Mann–Low and Callan-Symanzik equations, they can be solved for arbitrary momenta. The solutions involve a momentum-dependent effective mass as well as a momentum-dependent effective coupling constant. By studying these solutions at large momenta, it can be shown that the nonleading terms discarded by previous authors do, in fact, remain negligible when the perturbation series is summed to all orders if, and only if, the effective mass vanishes at large momentum, which will be the case if a certain anomalous dimension is less than unity, as it is in asymptotically free theories. In this case, the new renormalization-group equations can be used at large momentum to derive not only the leading term, but the first three terms in an asymptotic expansion of any Green's function. These results are also applied to Wilson coefficient functions, and an important cancellation of anomalous dimensions is noted.

I. INTRODUCTION

The renormalization-group equations of Gell-Mann and Low[1] and the closely related equations of Callan and Symanzik[2] have been widely used in studies of asymptotic behavior at large momenta.[3] Recently they have been brought into even greater prominence through the discovery by Gross and Wilczek[4] and Politzer[5] that non-Abelian gauge theories can exhibit free-field asymptotic behavior. However, useful as these equations are, there has always been some doubt[6] as to the nature and the truth of the assumptions that need to be made in deriving their large-momentum limit. This is not a mere matter of mathematical rigor, but a serious problem of whether or not terms which are suppressed by inverse powers of momentum in each order of perturbation theory remain asymptotically negligible when the perturbation series is summed. In addition, and even more important from a practical point of view, it is difficult to use the Gell-Mann– Low or Callan-Symanzik equations to obtain asymptotically *nonleading* terms, which play an important role in calculations of weak and electromagnetic corrections to hadronic symmetries.[7]

This paper will present a set of "new renormalization-group equations," which share the good features of the equations of Callan and Symanzik and Gell-Mann and Low, and have the following additional advantages:

(i) The new renormalization-group equations can be solved *before* passing to the high-energy limit. This is in contrast to the Gell-Mann–Low equations and the Callan-Symanzik equations, where terms which vanish in perturbation theory like powers of the ratio of mass to momentum must be discarded before a useful solution can be obtained.

(ii) The solutions of the new renormalization group equations for general nonasymptotic momenta are similar to the high-momentum solutions of the Gell-Mann–Low or Callan-Symanzik equations, except that in addition to a momentum-dependent "effective coupling constant", they also involve a momentum-dependent "effective mass". Thus, the question of the validity of the high-momentum solutions of the Gell-Mann–Low or Callan-Symanzik equations hinges on whether or not the effective mass vanishes at large momentum. We shall see that it does vanish if a certain anomalous dimension is less than unity; in particular, this is the case in theories with free-field asymptotic behavior.

(iii) By expanding in powers of the effective mass, we can easily use the new renormalization-group equations to derive not only an asymptotic limit but *the first three terms in an asymptotic expansion* for general Green's functions at large momenta.

The derivation of the new renormalization group equations depends on the use of a "zero mass" renormalization procedure, which resembles the Gell-Mann–Low procedure in that renormalized charges and fields are defined in terms of Green's functions at arbitrary nonzero momenta, but differs from it in that these Green's functions are evaluated at zero unrenormalized mass. Also, the ratio of the renormalized to the unrenormalized mass is defined in terms of the value of a vertex function at a partly arbitrary renormalization point, again evaluated at zero unrenormalized mass. In a sense, the zero-mass renormalization procedure is the opposite of the well-known Bogoliubov-Parasiuk-Hepp procedure,[8] which is based on calculations at zero momenta and nonzero mass. It is much closer in spirit to the "intermediate renormalization technique" used by Lee and Gervais[9] in their treatment of the σ model; in particular, the zero-mass renormalization procedure respects any symmetries which are broken in the Lagrangian only by mass terms.

It is essential to our derivations that this zero-mass renormalization procedure should actually work, that is, that when an unrenormalized Green's function is expressed in terms of renormalized coupling constants and masses defined through calculations at zero mass, and is multiplied with appropriate field renormalization constants defined at zero mass, then all cutoff dependence is removed. This is certainly not true for theories involving scalar fields; in this case there are quadratic divergences in the scalar self-energies which would not be removed by this renormalization procedure. However, the zero-mass procedure does seem to work in any theory which is "strictly renormalizable" in the limit of zero unrenormalized mass; that is, in theories which in the limit of zero mass do not have divergences which require mass counterterms. Thus, although I will not attempt a rigorous proof here, the considerations of this paper are intended to apply to quantum electrodynamics, and also to any renormalizable gauge theory involving only spin-1 gauge fields and spin-$\frac{1}{2}$ fermion fields.[10] Additional work would be needed before this discussion could be extended to theories involving spin-0 fields.

The zero-mass renormalization procedure is described more fully in Sec. II. Then, in Sec. III the new renormalization-group equations are derived and solved, using methods that share some elements of both the Callan-Symanzik and Gell-Mann–Low approach. The solutions are used in Sec. IV to derive an asymptotic expansion for the Green's functions at large momenta. These results are compared in Sec. V with the asymptotic formulas provided by the Gell-Mann–Low and Callan-Symanzik methods. Finally, the new formalism is applied to the coefficient functions of the Wilson operator-product expansion[11] in Sec. VI. Attention is drawn to an important cancellation of anomalous dimensions which occurs when the operator appearing in the Wilson expansion is the same as the mass operator in the Lagrangian. (This is the cancellation mentioned in Ref. 7.) Section VII deals with the constraints imposed by the new renormalization group equations on the perturbation series for general amplitudes.

II. RENORMALIZATION AT ZERO MASS

We shall consider a renormalizable field theory characterized by a single dimensionless unrenormalized coupling constant g and a single unrenormalized mass m. This would include quantum electrodynamics or any quark-gluon model based on a nonchiral non-Abelian simple gauge group with a single irreducible fermion multiplet; in these examples g would be the charge or the gauge coupling constant and m would be the fermion mass. The restriction to theories with a single coupling constant and mass is made here purely for the sake of simplicity; our discussion would apply equally well to theories with several m's or g's, such as gauge theories with reducible fermion multiplets or gauge theories based on nonsimple gauge groups. However, as mentioned in Sec. I, major changes would be needed to deal with scalar fields.

The analysis presented here depends on the introduction of a renormalization procedure which, unlike more familiar procedures, is based on calculations carried out in the limit $m\to 0$. Of course, in this limit the unrenormalized Green's functions would contain infrared divergences if evaluated at zero four-momenta. Thus, in order to avoid these infrared divergences, it will be necessary to base our renormalization procedure on prescriptions for the value of certain selected Green's functions for $m=0$ at certain selected nonzero four-momenta. We shall introduce a single arbitrary scale parameter μ with the dimensions of mass, which will characterize the values of all the momenta at these renormalization points.

In particular, the renormalized coupling constant g_R may be defined in terms of the value of some Green's function at momenta characterized by the scale μ in the limit $m=0$. Then g_R will be a function of the unrenormalized coupling g, of μ, and of the ultraviolet cutoff Λ:

$$g_R = g_R(g, \Lambda/\mu). \tag{2.1}$$

For instance, in quantum electrodynamics the renormalized charge may be defined as

$$e_R = e[\mu^2 \mathfrak{D}(\mu^2, e, 0, \Lambda)]^{1/2},$$

where $\mathfrak{D}(q^2, e, m, \Lambda)$ is the coefficient of $g_{\mu\nu}$ in the unrenormalized photon propagator at four-momentum q. In non-Abelian gauge theories g_R may be defined[12] as the value of the quark-quark-gluon or gluon-gluon-gluon vertex (including square roots of propagators on external lines) for $m=0$, evaluated at some renormalization point characterized by the parameter μ, such as the point $P_1^2 = P_2^2 = P_3^2 = \mu^2$.

It is also necessary to carry out a renormalization of fields. Again, this may be based on the values of Green's functions for $m=0$ at momenta characterized by the parameter μ. For instance, in quantum electrodynamics we may define field renormalization constants

$$Z_2(e,\Lambda/\mu)=\mu^2\mathcal{S}_1(\mu^2,e,0,\Lambda),$$

$$Z_3(e,\Lambda/\mu)=\mu^2\mathfrak{D}(\mu^2,e,0,\Lambda) = e_R{}^2/e^2,$$

where $\mathcal{S}_1(p^2,g,m,\Lambda)$ is the coefficient of $ip^\mu\gamma_\mu$ in the unrenormalized electron propagator at four-momentum p. Similar prescriptions apply in non-Abelian gauge theories.[12]

Finally, since we are really interested here in the case of nonvanishing mass, we must provide for mass renormalization. Let us suppose that m appears in the Lagrangian multiplying an operator Θ. (In quantum electrodynamics, Θ is just $\bar\psi\psi$). The infinities associated with the mass dependence of the unrenormalized Green's functions then take the same form as the infinities associated with insertion of Θ vertices in the graphs for the Green's functions in the theory with $m=0$. We can define a renormalized Θ operator:

$$\Theta_R\equiv Z_\Theta(g,\Lambda/\mu)\Theta, \tag{2.2}$$

where Z_Θ is the infinite factor needed to remove infinities associated with insertion of Θ vertices in Green's functions with $m=0$. For instance, in quantum electrodynamics the only subgraph which remains superficially divergent after insertion of a $\bar\psi\psi$ vertex is the electron self-energy, and we may define

$$Z_{\bar\psi\psi}{}^{-1}\equiv\Gamma_{e,e,\bar\psi\psi}(\mu^2,e,0,\Lambda)Z_2,$$

where $\Gamma_{e,e,\bar\psi\psi}(p^2,e,m,\Lambda)$ is the complete one-particle-irreducible vertex for an incoming electron with momentum p, an outgoing electron with momentum p, and a $\bar\psi\psi$ insertion with zero momentum. (In the limit $m\to 0$ this amplitude becomes proportional to the unit Dirac matrix; otherwise $Z_{\bar\psi\psi}$ could be defined in terms of the part of the amplitude which commutes with γ_5.) With this definition of $Z_{\bar\psi\psi}$, we can define a renormalized operator

$$(\bar\psi\psi)_R\equiv(\bar\psi\psi)Z_{\bar\psi\psi},$$

whose matrix elements are cutoff-dependent, at least in the theory with $m=0$.

The renormalized mass m_R will in general be defined as

$$m_R=m\,Z_\Theta{}^{-1}(g,\Lambda/\mu). \tag{2.3}$$

Of course this is not a mass of any direct physical significance, just as g_R is not necessarily a directly observable coupling constant. However, m_R and g_R are parameters which can be used to characterize a physical theory as well as any other, and as we shall see, all physical quantities can be expressed as finite functions of m_R and g_R.

The fundamental result on which our work in this paper is based is that all unrenormalized Green's functions become cutoff-independent if we express m and g in terms of m_R and g_R and multiply the amplitude by a suitable Z factor, with m_R, g_R and the Z factors defined as above. That is, given any unrenormalized Green's function $\Gamma(p,g,m,\Lambda)$, we may form a Λ-independent renormalized amplitude

$$\Gamma_R(p,g_R,m_R,\mu)=Z_\Gamma(g,\Lambda/\mu)\,\Gamma(p,g,m,\Lambda), \tag{2.4}$$

where Z_Γ depends on the number and types of the external lines[13] and p labels all the components of all their various four-momenta. (For instance, in quantum electrodynamics Γ might be an amputated Green's function with n_e external electron or positron lines and n_γ external photon lines, and then Z_Γ would have to be taken as

$$Z_\Gamma=Z_2{}^{n_e/2}Z_3{}^{n_\gamma/2},$$

with Z_2 and Z_3 defined for $m=0$.)

This result is certainly not true in theories involving scalar fields. In this case there is a quadratic divergence in the scalar self-energy, which does not go away when we let the unrenormalized mass vanish. This divergence is not removed by coupling-constant and field renormalization, so the expression (2.4) is not Λ-independent even in the limit $m\to 0$.

On the other hand, in quantum electrodynamics and other gauge theories, there is a symmetry, γ_5 reflection, which prevents the appearance of a divergent fermion self-mass (at least in perturbation theory) when the unrenormalized fermion mass vanishes. Of course, gauge invariance does the same for the gauge field self-mass. Thus, for gauge theories without scalar fields, Eq. (2.4) does give a Λ-independent amplitude for $m=0$, this being just the statement that gauge theories with zero fermion mass are renormalizable. The question then is whether this remains true when we turn on the unrenormalized fermion mass m, while continuing to use the $m=0$ definitions of g_R, m_R/m, and Z_Γ.

To construct a proof that this is the case, we might proceed inductively. If all divergences in Feynman graphs of order $n-1$ or less are eliminated by the $m=0$ renormalization procedure, then the only Feynman graphs of order n which could be divergent are those which are superficially divergent, in the sense of ordinary power counting.[14]

By using gauge invariance and Lorentz invariance, we find in the usual way that these are all logarithmically divergent, except for the fermion self-mass, which is superficially linearly divergent. Now, because we are using the $m=0$ renormalization procedure to eliminate divergences up to order $n-1$, the derivative with respect to m_R lowers the over-all superficial divergence by one unit. (This is not true with conventional renormalization procedures, where the renormalization counter-terms would depend on m_R.) In addition this differentiation eliminates all overlapping divergences, so that by the induction hypothesis all subintegrations are now convergent. We now see that the mass derivative of the logarithmically divergent Green's functions is superficially convergent, and all its subintegrations are superficially convergent, so it is actually convergent[14]; thus, the logarithmic divergences are all mass-independent, and are therefore removed by the order-n coupling-constant and field-renormalization counterterms evaluated at $m=0$. This leaves only the linear divergence which occurs in the term of the fermion self-energy which does not have a factor $\gamma_\mu p^\mu$. By γ_5-bookkeeping, we see that this term is odd in m, so it takes the form of m times a logarithmically divergent function. The above arguments show that the logarithmic divergence in this function is m-independent, so it is removed by the term of order n in the counterterm $(Z_\Theta-1)m_R$.

The above line of argument is reasonably persuasive, but not at all rigorous. To do better, we would have to replace the ordinary ultraviolet cutoff used here with a gauge-invariant regularization procedure. One approach would be to introduce the regulator fields of Lee and Zinn-Justin,[12] in which case Λ in Eq. (2.4) would presumably be replaced with a very large regulator mass. Alternatively, we may apply the 't Hooft-Veltman[15] technique of dimensional regularization. This latter approach does not introduce any free parameter like Λ with the dimensions of mass, so we would have to return to the spirit of the original Gell-Mann–Low analysis,[1] and use the renormalization-point scale parameter μ in place of a cutoff. Instead of taking Eq. (2.4) as a basis for the new renormalization group equations, we would then have to make direct use of the equation relating Green's functions defined with two different renormalization prescriptions, characterized by the scale factors μ and μ':

$$\Gamma_R(p,g_R',m_R',\mu')=\mathfrak{z}_\Gamma\Gamma_R(p,g_R,m_R,\mu)\,, \qquad (2.5)$$

where g_R', m_R'/m_R, and $\mathfrak{z}_\Gamma$ are functions of g_R and μ'/μ, but not of m_R. All the results derived below from Eq. (2.4) could equally well be derived from Eq. (2.5).

III. NEW RENORMALIZATION-GROUP EQUATIONS

We can derive the new renormalization-group equations for our renormalized Green's functions by simply differentiating Eq. (2.4) with respect to μ. Recalling that m_R and g_R depend on μ, while the unrenormalized amplitude does not, we find immediately that

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g_R)\frac{\partial}{\partial g_R}-\gamma_\Theta(g_R)m_R\frac{\partial}{\partial m_R}-\gamma_\Gamma(g_R)\right]\Gamma_R=0\,, \qquad (3.1)$$

where

$$\beta(g_R)\equiv\mu\frac{\partial}{\partial\mu}g_R(g,\Lambda/\mu)\,, \qquad (3.2)$$

$$\gamma_\Theta(g_R)\equiv\mu\frac{\partial}{\partial\mu}\ln Z_\Theta(g,\Lambda/\mu)\,, \qquad (3.3)$$

$$\gamma_\Gamma(g_R)\equiv\mu\frac{\partial}{\partial\mu}\ln Z_\Gamma(g,\Lambda/\mu)\,. \qquad (3.4)$$

Note that the coefficients (3.2)–(3.4) must be Λ-independent, because they appear in a differential equation for a renormalized amplitude. Since these coefficients are dimensionless, they must also be μ-independent.[16] (It was precisely in order to eliminate the μ dependence of β and the γ's that we were careful in Sec. II to perform our renormalization in an m-independent manner.)

We wish to use (3.1) to learn about the momentum dependence of the Green's function. Let us suppose that all the momentum components vary together with fixed ratio, so that $p=\kappa p_0$, where p_0 is a set of fixed momenta and κ is a momentum scale variable. If Γ has the dimensions (in the sense of ordinary dimensional analysis) of mass to the power D_Γ, then

$$\left[\mu\frac{\partial}{\partial\mu}+m_R\frac{\partial}{\partial m_R}+\kappa\frac{\partial}{\partial\kappa}\right]\Gamma_R=D_\Gamma\Gamma_R\,, \qquad (3.5)$$

so (3.1) may be rewritten as

$$\left\{\kappa\frac{\partial}{\partial\kappa}-\beta(g_R)\frac{\partial}{\partial g_R}+[1+\gamma_\Theta(g_R)]m_R\frac{\partial}{\partial m_R}-D_\Gamma+\gamma_\Gamma(g_R)\right\}\times\Gamma_R(\kappa p_0,g_R,m_R,\mu)=0 \qquad (3.6)$$

This is our new equation in its final form.

The solution of this kind of equation is well known.[17] Define a κ-dependent effective coupling and mass through the differential equations

$$\kappa \frac{d}{d\kappa} g(\kappa) = \beta(g(\kappa)), \tag{3.7}$$

$$\kappa \frac{d}{d\kappa} m(\kappa) = -[1+\gamma_\Theta(g(\kappa))]m(\kappa) \tag{3.8}$$

and the initial conditions

$$g(1)=g_R, \quad m(1)=m_R. \tag{3.9}$$

Then (3.6) has the solution

$$\Gamma_R(\kappa p_0, g_R, m_R, \mu) = \kappa^{D_\Gamma}\Gamma_R(p_0, g(\kappa), m(\kappa), \mu) \times \exp\left[-\int_1^\kappa \gamma_\Gamma(g(\kappa'))\frac{d\kappa'}{\kappa'}\right], \tag{3.10}$$

valid for all κ.

It is worth mentioning in passing that the β and γ coefficients may be written as logarithmic derivatives with respect to the cutoff:

$$\beta(g_R) = -\Lambda \frac{\partial}{\partial\Lambda} g_R(g, \Lambda/\mu), \tag{3.11}$$

$$\gamma_\Theta(g_R) = -\Lambda \frac{\partial}{\partial\Lambda} \ln Z_\Theta(g, \Lambda/\mu), \tag{3.12}$$

$$\gamma_\Gamma(g_R) = -\Lambda \frac{\partial}{\partial\Lambda} \ln Z_\Gamma(g, \Lambda/\mu). \tag{3.13}$$

The renormalization-point parameter μ plays a purely passive role in Eqs. (3.10)–(3.13).

IV. ASYMPTOTIC ANALYSIS AT LARGE MOMENTA

Up to now, our analysis has dealt with arbitrary momenta. We now turn to the high momentum limit, in which the momentum scale κ tends to infinity.

The asymptotic behaviour of the Green's functions Γ_R depends on the asymptotic behaviour of the effective coupling $g(\kappa)$ and effective mass $m(\kappa)$. If the anomalous dimension $\gamma_\Theta(g(\kappa))$ stays sufficiently above the value −1, then (3.8) shows that $m(\kappa)$ vanishes as $\kappa\to\infty$. Also if, Γ_R and its first N derivatives are finite at $m_R=0$, then Eq. (3.10) yields the asymptotic expression

$$\Gamma_R(\kappa p_0', g_R, m_R, \mu) \sim \kappa^{D_\Gamma}\exp\left[-\int_1^\kappa \gamma_\Gamma(g(\kappa'))\frac{d\kappa'}{\kappa'}\right] \times \sum_{n=0}^{N}\frac{m(\kappa)^n}{n!}\Gamma_R^{(n)}(p_0, g(\kappa), 0, \mu), \tag{4.1}$$

where $\Gamma_R^{(n)}$ denotes the nth derivative of Γ_R with respect to m_R. We shall see in Sec. V that the leading ($n=0$) term here gives just the same asymptotic behavior as the Gell-Mann–Low[1] and Callan-Symanzik[2] methods. Thus a crucial task for us will be to determine whether or not $m(\kappa)$ actually does vanish as $\kappa\to\infty$. This depends critically on the behavior of $g(\kappa)$, which, of course, also enters directly in (4.1), so we will have to proceed by first cataloging the possible ways that $g(\kappa)$ could behave as $\kappa\to\infty$, and then working out the asymptotic behavior of $m(\kappa)$ for each case. Afterwards, in order to judge the validity of Eq. (4.1), and to choose N, we will have to return to the question of the existence and the differentiability of Γ_R at $m_R=0$.

According to Eq. (3.7), the effective coupling $g(\kappa)$ must tend as $\kappa\to\infty$ to a "fixed point", which may be either the point at infinity, or any zero of the function β, including the zero at the origin. We may therefore distinguish three qualitatively different cases:

(1) If β at g_R has opposite sign to g_R, and has no zeroes between g_R and the origin, then $|g(\kappa)|$ must decrease from $|g_R|$ as κ increases, approaching the origin for $\kappa\to\infty$. Such theories are called "asymptotically free". In the usual case, perturbation theory gives [18]

$$\beta(g_R) = g_R^3 \times (\text{power series in } g_R^2). \tag{4.2}$$

(Here and below, "power series in ..." indicates a power series, with a constant term presumed *not* to vanish. The series may actually converge, or may be merely an asymptotic expansion for small values of the argument.) The solution of Eq. (3.7) then takes the form

$$g^2(\kappa) \propto (\ln\kappa)^{-1} + O\left(\frac{\ln\ln\kappa}{(\ln\kappa)^2}\right). \tag{4.3}$$

The anomalous dimensions of interest here usually take the form[18]

$$\gamma(g_R) = g_R^2 \times (\text{power series in } g_R^2), \tag{4.4}$$

so that

$$\gamma(g(\kappa)) \propto (\ln\kappa)^{-1} + O\left(\frac{\ln\ln\kappa}{(\ln\kappa)^2}\right) \tag{4.5}$$

and

$$\int_1^\kappa \gamma(g(\kappa'))\frac{d\kappa'}{\kappa'} = L\ln\ln\kappa + O(1), \tag{4.6}$$

where L is the calculable numerical constant

$$L = -\frac{3}{2}\frac{\gamma''(0)}{\beta'''(0)}.$$

In particular, the solution of Eq. (3.8) is

$$m(\kappa) \underset{\kappa\to\infty}{\propto} \kappa^{-1}(\ln\kappa)^{-L_\Theta}, \tag{4.7}$$

so $m(\kappa)$ definitely vanishes as $\kappa\to\infty$ in all asymptotically free theories. We can therefore use

Eq. (4.1), which here reads

$$\Gamma_R(\kappa p_0, g_R, m_R, \mu) \propto \kappa^{D_\Gamma}(\ln\kappa)^{-L_\Gamma} \times \sum_{n=0}^{N} \frac{m(\kappa)^n}{n!} \Gamma_R^{(n)}(p_0, g(\kappa), 0, \mu), \tag{4.8}$$

with $m(\kappa)$ and $g(\kappa)$ given by (4.7) and (4.3), respectively. If we are willing to neglect powers of κ^{-1} but wish to keep all powers of $(\ln\kappa)^{-1}$, then we would usually drop all but the term of zeroth order in $m(\kappa)$, and obtain the familiar result [4,5]

$$\Gamma_R(\kappa p_0, g_R, m_R, \mu) \sim \kappa^{D_\Gamma}(\ln\kappa)^{-L_\Gamma} \times [\text{power series in } (\ln\kappa)^{-1}]. \tag{4.9}$$

However, it sometimes happens that the zeroth-order term in $m(\kappa)$ is absent, because there is some symmetry of the Lagrangian, broken only by the term $m\Theta$, which if unbroken would force the amplitude Γ to vanish. In this case, the asymptotic behavior may be dominated by the term of first order in $m(\kappa)$, so that in place of Eq. (4.9), we would have

$$\Gamma_R(\kappa p_0, g_R, m_R, \mu) \sim \kappa^{D_\Gamma - 1}(\ln\kappa)^{-L_\Gamma - L_\Theta} \times [\text{power series in } (\ln\kappa)^{-1}]. \tag{4.10}$$

We see that the extraction of a factor m_R from Γ_R not only reduces the asymptotic limit of Γ_R by one factor of κ, but also changes the power of $\ln\kappa$. In any case, it is noteworthy that all coefficients in the power series appearing in (4.9) or (4.10) may be determined (up to an over-all multiplicative constant) from the perturbative calculations of β, γ_Γ, γ_Θ, and Γ_R. (See Sec. VII.)

(2) If β at g_R has the same sign as g_R, and if there are no zeroes of β between g_R and $+\infty$ or $-\infty$ (for $g_R > 0$ or $g_R < 0$), then $|g(\kappa)|$ must increase from $|g_R|$ as κ increases, approaching infinity for $\kappa \to \infty$. We know approximately nothing about the behavior of $\gamma(g(\kappa))$ or $m(\kappa)$ in this case. However, if $-\gamma_\Theta(g_R)$ is less than some quantity ϵ for all sufficiently large values of g_R, then (3.8) shows that

$$m(\kappa) = O(\kappa^{-1+\epsilon}). \tag{4.11}$$

Thus $m(\kappa)$ will vanish as $\kappa \to \infty$, provided $\epsilon < 1$. The Gell-Mann–Low or Callan-Symanzik equations apply in this case, even though they tell us very little about the behavior of Green's functions for $\kappa \to \infty$.

(3) If β has zeroes, and if the first zero of β encountered as its argument increases [for $\beta(g_R) > 0$] or decreases [for $\beta(g_R) < 0$] from g_R is at a finite point $g_\infty \neq 0$, then (3.7) shows that $g(\kappa)$ will increase or decrease to g_∞ as $\kappa \to \infty$. Suppose that g_∞ is a simple zero of β, so that

$$\beta(g_R) = (g_R - g_\infty) \times [\text{power series in } (g_R - g_\infty)]. \tag{4.12}$$

The solution of Eq. (3.7) is then of the form

$$g(\kappa) - g_\infty = \kappa^{-\nu} \times (\text{power series in } \kappa^{-\nu}), \tag{4.13}$$

where ν is the positive quantity

$$\nu = |\beta'(g_\infty)|. \tag{4.14}$$

We expect the anomalous dimensions γ to be regular at g_∞, so that

$$\gamma(g(\kappa)) - \gamma(g_\infty) = \kappa^{-\nu} \times (\text{power series in } \kappa^{-\nu}) \tag{4.15}$$

and

$$\int_1^\kappa \gamma(g(\kappa')) \frac{d\kappa'}{\kappa'} = \gamma(g_\infty)\ln\kappa + (\text{power series in } \kappa^{-\nu}). \tag{4.16}$$

In particular, the solution of Eq. (3.8) is

$$m(\kappa) = \kappa^{-1-\gamma_\Theta(g_\infty)} \times (\text{power series in } \kappa^{-\nu}). \tag{4.17}$$

We see that $m(\kappa)$ will vanish as $\kappa \to \infty$ if (and usually only if) the anomalous dimension γ_Θ satisfies the inequality

$$-\gamma_\Theta(g_\infty) < 1. \tag{4.18}$$

In this case, we can use Eq. (4.1), which here reads

$$\Gamma_R(\kappa p_0, g_R, m_R, \mu) \sim \kappa^{D_\Gamma - \gamma_\Gamma(g_\infty)} \times (\text{power series in } \kappa^{-\nu}) \times \sum_{n=0}^{N} \frac{m(\kappa)^n}{n!} \Gamma_R^{(n)}(p_0, g(\kappa), 0, \mu), \tag{4.19}$$

with $m(\kappa)$ and $g(\kappa)$ given by (4.17) and (4.13), respectively. If Γ_R is nonzero at $m_R = 0$, then the leading term is the term of zeroth order in $m(\kappa)$, which for $\kappa \to \infty$ gives

$$\Gamma_R(\kappa p_0, g_R, m_R, \mu) \propto \kappa^{D_\Gamma - \gamma_\Gamma(g_\infty)}. \tag{4.20}$$

On the other hand, if there is some symmetry principle which makes Γ_R vanish for $m_R = 0$, then Γ_R is asymptotically dominated by the term of first order in $m(\kappa)$, which for $\kappa \to \infty$ gives[19]

$$\Gamma_R(\kappa p_0, g_R, m_R, \mu) \propto \kappa^{D_\Gamma - 1 - \gamma_\Gamma(g_\infty) - \gamma_\Theta(g_\infty)}. \tag{4.21}$$

We see that the extraction of a factor m_R from Γ_R reduces its asymptotic behavior, not by a factor κ^{-1}, but by a factor

$$\kappa^{-1-\gamma_\Theta(g_\infty)}\,. \tag{4.22}$$

The neglected terms in (4.20) and (4.21) are smaller than the leading term by powers of κ. In particular, if ν is less than $1+\gamma_\Theta(g_\infty)$, then the leading corrections to (4.20) or (4.21) arise from the κ dependence of $g(\kappa)$ and [for Eq. (4.21)] $m(\kappa)$, and are smaller than (4.20) or (4.21) by a factor $\kappa^{-\nu}$. On the other hand, if ν is greater than $1+\gamma_\Theta(g_\infty)$, then the leading corrections to (4.20) or (4.21) may arise from terms of higher order in $m(\kappa)$, and are smaller than (4.20) or (4.21) by the factor (4.22).

So far, we have only considered the powers of κ and $\ln\kappa$ which appear in the asymptotic Green's function. However, in some cases it is possible to determine the constant factors as well. The effective mass $m(\kappa)$ takes the form

$$m(\kappa)=\kappa^{-1}m_R\exp\left[-\int_1^\kappa \gamma_\Theta(g(\kappa'))\frac{d\kappa'}{\kappa'}\right], \tag{4.23}$$

and so according to Eq. (4.1), the $(n+1)$th term in the asymptotic expansion of Γ_R is

$$\Gamma_{Rn}(\kappa p_0, g_R, m_R, \mu)=\frac{1}{n!}\kappa^{D_\Gamma-n}m_R{}^n \times\exp\left[-\int_1^\kappa \gamma_\Gamma^{(n)}(g(\kappa'))\frac{d\kappa'}{\kappa'}\right] \times\Gamma_R^{(n)}(p_0, g(\kappa), 0, \mu)\,, \tag{4.24}$$

where $\gamma_\Gamma^{(n)}$ is an effective anomalous dimension

$$\gamma_\Gamma^{(n)}\equiv\gamma_\Gamma+n\gamma_\Theta\,. \tag{4.25}$$

In general Γ_{Rn} contains an unknown multiplicative factor, because whether or not $g(\kappa)$ vanishes as $\kappa\to\infty$, the exponential in (4.24) receives contributions from κ' values where $g(\kappa')$ is not small. This factor is absent when the effective anomalous dimension (4.25) vanishes, so that in this case

$$\Gamma_{Rn}(\kappa p_0, g_R, m_R, \mu)=\frac{1}{n!}\kappa^{D_\Gamma-n}m_R{}^n \times\Gamma_R^{(n)}(p_0, g(\kappa), 0, \mu)\,. \tag{4.26}$$

We shall see in Sec. VI how this can happen for the $n=1$ term in a Wilson coefficient function. A result like (4.26), of course, finds its most interesting application in an asymptotically free theory, where $g(\infty)$ is zero, so that the right-hand-side of (4.26) is given by the Born approximation.

Now let us return to the question of the existence and differentiability of Γ_R at $m_R=0$. Any singularity of Γ_R at $m_R=0$ can arise only from the poles in propagators at zero momentum, appearing either in the graphs for Γ_R itself, or in the renormalization counterterms used to calculate Γ_R. As long as we stay away from exceptional momenta, only one internal line in each loop can have zero momentum at a time. (Strictly speaking, this is true only if by "internal line" we mean the fully dressed internal line, with all self-energy insertions. It is tacitly assumed below that the momentum-space integral of the full propagator is no more singular at zero mass than is the integral of the bare propagator, as is the case in perturbation theory.) With our renormalization procedure, the renormalization counterterms are also defined at momenta which are m_R independent and nonexceptional, so in evaluating these counterterms also, only one internal line in each loop can have zero momentum at a time. Hence Γ_R behaves as $m_R\to0$ like a sum of powers of integrals of *single* propagators, such as

$$\int_0 (i\gamma_\mu p^\mu+m_R)^{-1}d^4p\,, \tag{4.27}$$

where 0 indicates that the integral is taken over some finite neighborhood of the origin. This integral is not infrared-divergent at $m_R=0$, and its first and second derivatives with respect to m_R are not infrared-divergent at $m_R=0$, but its *third* derivative is logarithmically divergent for $m_R=0$. Hence we expect Γ_R, in general, to be twice but not thrice differentiable at $m_R=0$. This is, we expect that the strongest singularity of Γ_R at $m_R=0$ to be of the form[20]

$$m_R{}^3\ln m_R{}^2\,. \tag{4.28}$$

For theories in which $m(\kappa)$ vanishes as $\kappa\to\infty$, we can use (4.1) with $N=2$, but usually not with any larger value of N.

It is worth emphasizing that the $m_R{}^3\ln m_R{}^2$ singularity found here is much weaker than would be encountered if we used the conventional definition of renormalized mass, as was done by Gell-Mann and Low[1] and by Callan and Symanzik.[2] In the conventional procedure, mass-renormalization counterterms are defined in terms of self-energy integrals evaluated at the renormalized mass, so that more than one propagator in each loop can have a pole at zero momentum at the same time in the limit of zero unrenormalized mass. The self-energy counterterms are still not infrared-divergent in this limit, but they do have singularities of the form $m\ln m^2$ and $m^2\ln m^2$ in terms odd or even in m. Thus, with a conventional definition of renormalized mass, general Green's functions would be finite but not differentiable at zero mass. If such singularities had occurred in the "zero mass" renormalization procedure used here, we would have been able to derive only the leading term in Γ_R for $\kappa\to\infty$, not the first three terms of an asymptotic expansion. The virtue of the zero-mass renormalization procedure is that mass here plays only the role of a coupling constant, while in the conventional renormalization procedure it also determines the location of all renormalization points, and even in the Gell-Mann–Low procedure[1] it determines the self-energy renormalization point.

V. COMPARISON WITH OTHER APPROACHES

At this point we pause, in order to compare the above results with those obtained by the Gell-Mann–Low and Callan-Symanzik approaches.

The renormalization-group method of Gell-Mann and Low[1] differs from the present method in that the renormalized charge and Z factors depend on the bare mass m as well as on g, μ, and Λ, while the renormalized mass is taken as the true position of the pole in the propagator, and therefore depends on m, g, and Λ, but not on the renormalization-point scale parameter μ. Hence the Gell-Mann–Low equation corresponding to Eq. (3.1) would read

$$\left[\mu\frac{\partial}{\partial\mu}+\beta^*(g_R^*,m_R^*/\mu)\frac{\partial}{\partial g_R^*}-\gamma_\Gamma^*(g_R^*,m_R^*/\mu)\right]\times\Gamma_R^*(p,g_R^*,m_R^*,\mu)=0\,, \quad (5.1)$$

where

$$\beta^*(g_R^*,m_R^*/\mu)\equiv\mu\frac{\partial}{\partial\mu}g_R^*(\mu,g,m,\Lambda)\,, \quad (5.2)$$

$$\gamma_\Gamma^*(g_R^*,m_R^*/\mu)\equiv\mu\frac{\partial}{\partial\mu}\ln Z_\Gamma^*(\mu,g,m,\Lambda)\,, \quad (5.3)$$

the asterisk here indicating the use of the Gell-Mann–Low renormalization prescriptions. Again, β^* and γ_Γ^* must be Λ-independent because they appear in the differential equation for a Λ-independent amplitude, but now they may depend on the renormalized mass m_R^*, and therefore, even though dimensionless, they may depend on μ through the ratio m_R^*/μ. The μ dependence of β^* and γ^* prevents us from being able to find useful solutions of (5.1), even though this equation appears simpler than (3.6) in so far as the mass-derivative term is missing. It is usual to seek a solution by taking both μ and p very large compared with m_R^*. Under the assumption that in this limit m_R^* may be neglected altogether, we then have

$$\Gamma_R^*(p,g_R^*,m_R^*,\mu)\to\Gamma_R(p,g_R^*,0,\mu) \quad (5.4)$$

for $p\gg m_R^*$ and $\mu\gg m_R^*$. Ordinary dimensional analysis then gives

$$\left[\mu\frac{\partial}{\partial\mu}+\kappa\frac{\partial}{\partial\kappa}-D_\Gamma\right]\Gamma_R^*(p,g_R^*,0,\mu)=0\,, \quad (5.5)$$

where κ again is the momentum scale variable, with

$$p=\kappa p_0\,. \quad (5.6)$$

Thus Eq. (5.1) may now be written

$$\left[\kappa\frac{\partial}{\partial\kappa}-\beta^*(g_R^*,0)\frac{\partial}{\partial g_R^*}-D_\Gamma+\gamma_\Gamma(g_R^*,0)\right]\times\Gamma_R^*(\kappa p_0,g_R^*,0,\mu)=0\,. \quad (5.7)$$

The solution is well known, and yields the result

$$\Gamma_R^*(\kappa p_0,g_R^*,m_R^*,\mu)\sim\kappa^{D_\Gamma}\exp\left[-\int_1^\kappa\gamma_\Gamma^*(g^*(\kappa'),0)\frac{d\kappa'}{\kappa'}\right]\times\Gamma_R^*(p_0,g^*(\kappa),0,\mu) \quad (5.8)$$

for $p_0\kappa\gg m_R^*$ and $\mu\gg m_R^*$. But Γ_R^* and our previously defined amplitudes Γ_R should differ only by a constant factor, so comparing (5.8) with (4.1) for $\kappa\to\infty$, we see that the Gell-Mann–Low results agree with the results of our present analysis if, and only if, $m(\kappa)$ vanishes as $\kappa\to\infty$. In this case, the β and γ coefficients are simply related by

$$\beta^*(g_R^*,0)=\beta(g_R)\frac{\partial g_R^*}{\partial g_R}\,, \quad (5.9)$$

$$\gamma_\Gamma^*(g_R^*,0)=\gamma_\Gamma(g_R)\,. \quad (5.10)$$

However, if $-\gamma_\Theta$ becomes too large, then $m(\kappa)$ may not vanish as $\kappa\to\infty$, in which case the Gell-Mann–Low approach would fail. This failure can occur even though Γ_R^* may have a well-defined limit at $m_R^*=0$, because this limit is not *uniform* in the ratio p/μ, and we are interested in the case where $\mu\gg m_R^*$ and $p\gg\mu$.

In the Callan-Symanzik approach,[2] a conventional renormalization procedure is used, so that no arbitrary renormalization scale parameter μ need be introduced. In place of Eqs. (3.6) or (5.7), the momentum dependence of the renormalized Green's functions is governed by the equation

$$\left[\kappa\frac{\partial}{\partial\kappa}-\beta^\sim(g_R^\sim)\frac{\partial}{\partial g_R^\sim}+\gamma_\Gamma^\sim(g_R^\sim)-D_\Gamma\right]\Gamma_R^\sim(\kappa p_0,g_R^\sim,m_R^\sim)=\Gamma_{R\Theta}^\sim(\kappa p_0,g_R^\sim,m_R^\sim)\,, \quad (5.11)$$

with a tilde indicating the use of a conventional renormalization procedure. The amplitude $\Gamma_{\Theta R}^\sim$ is the renormalized Green's functions related to $\Gamma_R^\sim$ by the insertion of a zero-momentum Θ vertex. (Recall that Θ is the operator, usually $\bar\psi\psi$, which appears in the mass term of the Lagrangian.) The coefficients $\beta^\sim$ and $\gamma^\sim$ are Λ-independent and dimensionless, and therefore independent of renormalized mass, so here it is the extra term on the right-hand side of Eq. (5.11) that prevents a useful general solution. In order to derive a solution, it is usual[2] to argue that the insertion of a Θ vertex lowers the asymptotic behavior of the right-hand side in such a way that this term becomes negligible for $\kappa\to\infty$. In this case, Eq. (5.11) becomes

$$\left[\kappa\frac{\partial}{\partial\kappa}-\beta^{\sim}(g_R^{\sim})\frac{\partial}{\partial g_R^{\sim}}+\gamma_\Gamma^{\sim}(g_R^{\sim})-D_\Gamma\right]\Gamma_R^{\sim}(\kappa p_0,g_R^{\sim},m_R^{\sim})\to 0\,. \tag{5.12}$$

This has the solution

$$\Gamma_R^{\sim}(\kappa p_0,g_R^{\sim},m_R^{\sim})\sim\kappa^{D_\Gamma}\exp\left[-\int_1^\kappa\gamma_\Gamma^{\sim}(g^{\sim}(\kappa'))\frac{d\kappa'}{\kappa'}\right]\times\Gamma_R^{\sim}(p_0,g^{\sim}(\kappa'),m_R^{\sim})\,. \tag{5.13}$$

Again, $\Gamma_R^{\sim}$ should differ from Γ_R only by a constant factor, so by comparing (5.13) with (4.1), we see that the Callan-Symanzik results will agree with our present analysis if, and only if, $m(\kappa)$ vanishes as $\kappa\to\infty$. In this case, the β and γ coefficients are simply related by

$$\beta^{\sim}(g_R^{\sim})=\beta(g_R)\frac{\partial g_R^{\sim}}{\partial g_R}\,, \tag{5.14}$$

$$\gamma_\Gamma^{\sim}(g_R^{\sim})=\gamma_\Gamma(g_R)\,. \tag{5.15}$$

If $m(\kappa)$ does not vanish as $\kappa\to\infty$, then the Callan-Symanzik solution (5.13) must fail. Such a failure can occur even though $\Gamma_{\Theta R}$ is negligible for $\kappa\to\infty$ in each order of perturbation theory,[14] if it is not asymptotically negligible when the perturbation series is summed.

We see that the precise condition, which ensures that the zero-mass limit has the uniformity properties assumed in the Gell-Mann–Low approach, and also ensures that the right-hand side of the Callan-Symanzik equations remains asymptotically negligible when summed to all orders in perturbation theory, is just that $m(\kappa)$ should vanish as $\kappa\to\infty$. This will depend on the magnitude and sign of the anomalous dimension γ_Θ, and, in particular, is always satisfied in asymptotically free theories.

However, even when $m(\kappa)$ does vanish, it is very difficult to use the Gell-Mann–Low or Callan-Symanzik methods to derive the terms of higher order in $m(\kappa)$. In making use of the Callan-Symanzik equations this problem arises when $\Gamma_R^{\sim}$ is anomalously small as $\kappa\to\infty$, say, because it contains a factor of mass, so that even though the right-hand side of the Callan-Symanzik equations behaves just as expected from perturbation theory, it is not asymptotically negligible. In this case we must use the Callan-Symanzik equation for $\Gamma_{\Theta R}^{\sim}$, which takes the form[21]

$$\left[\kappa\frac{\partial}{\partial\kappa}-\beta^{\sim}(g_R^{\sim})\frac{\partial}{\partial g_R^{\sim}}+\gamma_\Gamma^{\sim}(g_R^{\sim})+\gamma_\Theta^{\sim}(g_R^{\sim})-D_\Gamma\right]\times\Gamma_{\Theta R}^{\sim}(\kappa p_0,g_R^{\sim},m_R^{\sim})=\Gamma_{\Theta\Theta R}^{\sim}(\kappa p_0,g_R^{\sim},m_R^{\sim})\,.$$

This can easily be solved if $\Gamma_{\Theta\Theta R}^{\sim}$ is asymptotically negligible for $\kappa\to\infty$. However, it is not so clear how to use this solution to determine the asymptotic behavior of $\Gamma_R^{\sim}$ itself. (As indicated in Sec. IV, the use of a conventional mass-renormalization procedure gives $\Gamma_R^{\sim}$ a singularity of form $m_R^{\sim}\ln m_R^{\sim}$ at $m_R^{\sim}=0$, so $\Gamma_R^{\sim}$ is *not* given as $\kappa\to\infty$ by even the first few terms of a Taylor series in $m_R^{\sim}$.) The problem of deriving an asymptotic expansion seems even more difficult in the Gell-Mann–Low approach, and I have no idea how this could be done.

Because the β and γ functions depend on the renormalization prescription adopted, here we could not make direct use of the perturbative calculations of β and γ in Refs. 4 and 5, where prescriptions different from ours were used. However, all definitions of the renormalized coupling constants agree in lowest order, so the partial derivatives in (5.9) and (5.14) are equal to unity in lowest order, and therefore the lowest-order terms in β do not depend on the renormalization prescription used. This is of some importance, because it is the sign of the lowest-order term in β that determines whether or not a theory can be asymptotically free.

VI. WILSON COEFFICIENT FUNCTIONS

So far, we have dealt only with the asymptotic behavior of Green's functions when all external momenta are taken to infinity. In physical applications, it is often more interesting to consider the behavior of Green's functions when some subset of momenta go to infinity, the others remaining fixed. The asymptotic behavior in such cases is described by the coefficient functions in a Wilson operator-product expansion.[11] We shall now apply the new renormalization-group approach to these coefficient functions.

Let us divide the external lines of a Green's function Γ into two sets, labeled A and B, and use k and q to label all the components of all the four-momenta of these two sets, respectively. The Wilson operator-product expansion states that when the various components of k tend to infinity with fixed ratios, with q fixed, these Green's functions have the asymptotic expansion

$$\Gamma_{AB,R}(k,q,g_R,m_R,\mu)\sim\sum_O U_{AO}(k,g_R,m_R,\mu)\times\Gamma_{BO,R}(q,g_R,m_R,\mu)\,, \tag{6.1}$$

where O runs over all local renormalized operators (say, evaluated at $x=0$); $\Gamma_{BO,R}$ denotes the renormalized Green's function for the external lines B with four-momenta labeled q, with an extra zero-momentum O vertex; and U_{AO} is a finite coefficient function. This expansion will apply to whatever renormalization procedure we use for Γ, since the difference is only a multiplicative constant.

The simplest way to derive the new renormalization-group equation for the coefficient functions U_{AO} is to write down the renormalization group equations analogous to (3.1) for $\Gamma_{AB,R}$ and $\Gamma_{BO,R}$:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g_R)\frac{\partial}{\partial g_R}-\gamma_\Theta(g_R)m_R\frac{\partial}{\partial m_R}-\gamma_A(g_R)-\gamma_B(g_R)\right]\Gamma_{AB,R}=0\,,$$

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g_R)\frac{\partial}{\partial g_R}-\gamma_\Theta(g_R)m_R\frac{\partial}{\partial m_R}-\gamma_B(g_R)-\gamma_O(g_R)\right]\Gamma_{BO,R}=0\,,$$

where γ_A, γ_B, and γ_O are anomalous dimensions associated with the external-line sets A and B and the operator O. In order for these equations to hold for all O, the coefficient functions U_{AO} must satisfy the differential equation [22]:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g_R)\frac{\partial}{\partial g_R}-\gamma_\Theta(g_R)m_R\frac{\partial}{\partial m_R}-\gamma_A(g_R)+\gamma_O(g_R)\right]U_{AO}=0\,. \quad (6.2)$$

(This can also be derived by observing that the μ dependence of U_{AO} arises only from a factor Z_A, a factor $Z_O{}^{-1}$, and from the μ dependence of g_R and m_R.) This equation is very much like the renormalization-group equations satisfied by ordinary Green's functions, but with a crucial change of sign in the γ_O term.

Once again, we can easily convert this into an equation governing the momentum dependence of U_{AO}. If U_{AO} has dimensions D_{AO}, and if we set k equal to a constant k_0 times a scale factor κ, then ordinary dimensional analysis gives

$$\left[\kappa\frac{\partial}{\partial\kappa}+\mu\frac{\partial}{\partial\mu}+m_R\frac{\partial}{\partial m_R}-D_{AO}\right]U_{AO}=0\,, \quad (6.3)$$

and therefore (6.2) becomes

$$\left\{\kappa\frac{\partial}{\partial\kappa}-\beta(g_R)\frac{\partial}{\partial g_R}+[1+\gamma_\Theta(g_R)]\,m_R\frac{\partial}{\partial m_R}-D_{AO}+\gamma_A(g_R)-\gamma_O(g_R)\right\}U_{AO}=0\,. \quad (6.4)$$

The solution takes the form

$$U_{AO}(\kappa k_0,g_R,m_R,\mu)=\kappa^{D_{AO}}\exp\left\{\int_1^\kappa[\gamma_O(g(\kappa'))-\gamma_A(g(\kappa'))]\frac{d\kappa'}{\kappa'}\right\}U_{AO}(k_0,g(\kappa),m(\kappa),\mu)\,, \quad (6.5)$$

with $g(\kappa)$ and $m(\kappa)$ given by Eqs. (3.7)–(3.9).

We could now proceed to discuss the asymptotic behavior of U_{AO}, but the analysis runs along just the same lines as in Sec. IV, and there is no point in going into details again here. However, one particular case is worthy of special notice. Suppose that the set A consists only of conserved or partially conserved vector or axial-vector currents, so that [4,5]

$$\gamma_A=0\,.$$

Suppose also that O is the mass operator Θ, so that

$$\gamma_O=\gamma_\Theta\,.$$

In spinor gauge theories Θ is the operator $\bar\psi\psi$, and γ_5–bookkeeping forces U_{AO} to be odd in m_R, so that if $m(\kappa)$ vanishes as $\kappa\to\infty$, U_{AO} will be dominated by the term of *first* order in $m(\kappa)$:

$$U_{A\Theta}(k_0,g(\kappa),m(\kappa),\mu)\sim m(\kappa)U_{A\Theta}^{(1)}(k_0,g(\kappa),0,\mu)\,.$$

If we recall the formula for $m(\kappa)$:

$$m(\kappa)=\kappa^{-1}m_R\exp\left[-\int_1^\kappa\gamma_\Theta(g(\kappa'))\frac{d\kappa'}{\kappa'}\right]\,,$$

we see that *all γ terms cancel in the exponential*, so that

$$U_{A\Theta}(\kappa k_0,g_R,m_R,\mu)\sim\kappa^{D_{A\Theta}-1}m_R U_{A\Theta}^{(1)}(k_0,g(\kappa),0,\mu)\,, \quad (6.6)$$

as in Eq. (4.26). In particular, in asymptotically free theories $g(\kappa)$ vanishes as $\kappa\to\infty$, so $U_{A\Theta}$ is given in this case by the zeroth-order Born approximation. The particular circumstances described here actually occur when we calculate the weak corrections of order α to natural strong-interaction symmetries, [7] and our result shows that the strong interactions may be disregarded in such calculations.

VII. CONSTRAINTS IN PERTURBATION THEORY

Our emphasis here has been on the use of the new renormalization-group equations to study the asymptotic behavior of the exact Green's functions and Wilson coefficient functions at large momenta. However, the renormalization-group approach also imposes interesting constraints on the structure of the perturbation series for these amplitudes. Such constraints are useful aids in actual perturbative calculations, and by checking that these constraints are indeed satisfied, we can verify the validity of our approach.

We suppose that the function $\beta(g_R)$ is calculated up to some finite order in g_R. In electrodynamics and other gauge theories, this series takes the form

$$\beta(u)=b_1u^3+b_2u^5+b_3u^7+\cdots . \tag{7.1}$$

It is straightforward to check that (3.7) and (3.9) have the solution

$$g^2(\kappa)=g_R^{\,2}+2g_R^{\,4}b_1\ln\kappa + 2g_R^{\,6}(2b_1^{\,2}\ln^2\kappa+b_2\ln\kappa)+\cdots . \tag{7.2}$$

In gauge theories the various anomalous dimensions $\gamma(g_R)$ will usually have a power series of the form

$$\gamma(u)=c_1u^2+c_2u^4+\cdots . \tag{7.3}$$

Using (7.2) in (7.3), we find

$$\exp\left[-\int_1^\kappa \gamma(g(\kappa'))\frac{d\kappa'}{\kappa'}\right] = 1-g_R^{\,2}c_1\ln\kappa+g_R^{\,4}[(\tfrac{1}{2}c_1^{\,2}-b_1c_1)\ln^2\kappa-c_2\ln\kappa]+\cdots . \tag{7.4}$$

In particular, the effective mass defined by (3.8) and (3.9) is given by

$$m(\kappa)=\kappa^{-1}m_R\{1-g_R^{\,2}c_{1\Theta}\ln\kappa + g_R^{\,4}[(\tfrac{1}{2}c_{1\Theta}^{\,2}-b_1c_{1\Theta})\ln^2\kappa-c_{2\Theta}\ln\kappa] +\cdots\} , \tag{7.5}$$

where $c_{n\Theta}$ are the c_n coefficients for the particular anomalous dimension γ_Θ.

Now let us consider some renormalized amplitude Γ_R with dimensionality D_Γ. The Gell-Mann–Low and Callan-Symanzik methods deal with an "asymptotic" amplitude Γ_{RO}, defined by keeping only those terms in each order of perturbation theory which contain the maximum number D_Γ of powers of momentum, times any powers of logarithms of momentum. We shall take advantage here of our capability for dealing with nonleading terms, and consider an amplitude Γ_{Rn}, defined by keeping all terms in Γ_R in each order of perturbation theory which have the asymptotic behavior

$$\Gamma_{Rn}\propto\kappa^{D_\Gamma-n}\times(\text{powers of }\ln\kappa)\,, \tag{7.6}$$

where $p=\kappa p_0$ with $\kappa\to\infty$. Inspection of Eq. (7.5) shows that such terms are given by the terms in Eq. (3.10) of nth order in $m(\kappa)$:

$$\Gamma_{Rn}(\kappa p_0,g_R,m_R,\mu)=\kappa^{D_\Gamma}\exp\left[-\int_1^\kappa\gamma_\Gamma(g(\kappa'))\frac{d\kappa'}{\kappa'}\right]\times\frac{1}{n!}m(\kappa)^n\Gamma_R^{(n)}(p_0,g(\kappa),0,\mu)\,. \tag{7.7}$$

We can usually expand the $\Gamma_R^{(n)}(p,g_R,0,\mu)$ in a power series in $g_R^{\,2}$:

$$\Gamma_R^{(n)}(p,g_R,0,\mu)=f^{(n0)}(p,\mu)+g_R^{\,2}f^{(n1)}(p,\mu) +g_R^{\,4}f^{(n2)}(p,\mu)+\cdots . \tag{7.8}$$

Using (7.8), (7.2), (7.4), and (7.5) in (7.7) yields the power series for Γ_{Rn}:

$$\begin{aligned}\Gamma_{Rn}(\kappa p_0,g_R,m_R,\mu)=\frac{1}{n!}\kappa^{D_\Gamma-n}m_R^{\,n}\{&f^{(n0)}(p_0,\mu)+g_R^{\,2}[f^{(n1)}(p_0,\mu)-C_{1\Gamma}^{(n)}\ln\kappa f^{(n0)}(p_0,\mu)]\\&+g_R^{\,4}[f^{(n2)}(p_0,\mu)+(2b_1-C_{1\Gamma}^{(n)})\ln\kappa f^{(n1)}(p_0,\mu)\\&+((\tfrac{1}{2}C_{1\Gamma}^{(n)2}-b_1C_{1\Gamma}^{(n)})\ln^2\kappa-C_{2\Gamma}^{(n)}\ln\kappa)f^{(n0)}(p_0,\mu)]+\cdots\}\,,\end{aligned} \tag{7.9}$$

where

$$C_{m\Gamma}^{(n)}\equiv C_{m\Gamma}+nC_{m\Theta}\,. \tag{7.10}$$

For example, if we calculate Γ_{Rn} up to first order in $g_R^{\,2}$ we can determine the quantities $f^{(n0)}$, $f^{(n1)}$, and $C_{1\Gamma}^{(n)}$, and then (7.9) immediately yields the coefficient of $\ln^2\kappa$ in the term of second order in $g_R^{\,2}$.

The result (7.9) becomes particularly useful in cases where the effective anomalous dimension $\gamma_\Gamma+n\gamma_\Theta$ vanishes. We then have $C_{m\Gamma}^{(n)}=0$ for all m, so that (7.9) simplifies to

$$\Gamma_{Rn}(\kappa p_0,g_R,m_R,\mu)=\frac{1}{n!}\kappa^{D_\Gamma-n}m_R^{\,n}\{f^{(n0)}(p_0,\mu)+g_R^{\,2}f^{(n1)}(p_0,\mu)+g_R^{\,4}[f^{(n2)}(p_0,\mu)+2b_1\ln\kappa f^{(n1)}(p_0,\mu)]+\cdots\}\,. \tag{7.11}$$

We see that there are no $\ln\kappa$ terms in order g_R^2, no $(\ln\kappa)^2$ terms in order g_R^4, and so on.

This result has been checked by Duncan[23] in a detailed calculation of the Wilson function for the operator $O=\bar\psi\psi$, with the large momentum k carried by a pair of vector or axial-vector currents, using a vector-gluon theory of strong interactions. This Wilson function has dimensionality $D=-1$, but as discussed in Sec. VI, this function is odd in the fermion mass m, so the leading term in powers of $1/\kappa$ is the $n=1$ term, which in perturbation theory behaves like $1/\kappa^2$ times powers of $\ln\kappa$. It was shown in Sec. VI that γ_Γ for this Wilson function is just $-\gamma_{\bar\psi\psi}$, so that the effective anomalous dimension (4.25) for the $n=1$ terms vanishes, and therefore the perturbation series for the Wilson function should be of the form (7.11). Duncan finds that the individual graphs of second order in the gluon gauge coupling contain both κ^{-2} and $\kappa^{-2}\ln\kappa$ terms, but that all the $\kappa^{-2}\ln\kappa$ terms cancel when the graphs are added together, in agreement with Eq. (7.11).

Notes Added in Proof

(1) The cancellation of anomalous dimensions demonstrated here in Sec. VI is essentially the same as that noted earlier by K. Wilson, Phys. Rev. 179, 1499 (1969), in a study of current propagators and electromagnetic self-energies. However, Wilson's work was in a non-Lagrangian framework, and no proof was offered that these results actually hold in a field theory.

(2) Nonleading terms in an asymptotic expansion of the inverse propagator at high energy were obtained using the Callan-Symanzik approach in a ϕ^4 theory by K. Symanzik, Commun. Math. Phys. 23, 49 (1971), Sec. III.3.

(3) A condition on the anomalous dimensions γ_{ϕ^2} analogous to Eq. (4.18) was presented for a ϕ^4 theory by K. Symanzik, Commun. Math. Phys. 23, 49 (1971), Sec. IV.2. It is interesting that this condition on γ_{ϕ^2} arose from the requirement that the beta function be continuous, rather than from the requirement imposed in the present work, that the effective mass vanish at large momentum.

(4) The considerations of this article also apply when the renormalization prescriptions are defined using the actual values for all internal masses, as long as the external momenta are kept off the mass shell in all cases, even in the case of mass renormalization. The functions β, γ_Γ, and γ_Θ then depend on both γ_R and m_R/μ. However, the renormalization group equations can still be solved for arbitrary momentum, the solutions depending on an effective coupling and mass defined by a pair of *coupled* nonlinear ordinary differential equations. As long as the anomalous dimension γ_Θ stays sufficiently above the value -1, the effective mass will still vanish at large momentum, and all the usual results will follow, even for theories involving scalar fields.

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[1]M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954).

[2]C. G. Callan, Phys. Rev. D 2, 1541 (1970); K. Symanzik, Comm. Math. Phys. 18, 227 (1970).

[3]For a sample of such applications, see N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959), Chap. VIII; K. Johnson, R. Willey, and M. Baker, Phys. Rev. 163, 1699 (1967); M. Baker and K. Johnson, *ibid.* 183, 1292 (1969); S. L. Adler, Phys. Rev. D 5, 3021 (1972); K. Symanzik, Springer Tracts Phys. 57, 222 (1971); K. Wilson, Phys. Rev. D 3, 1818 (1971); etc.

[4]D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); and this issue, Phys. Rev. D 8, 3633 (1973).

[5]H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973).

[6]See, e.g., J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), p. 376.

[7]S. Weinberg, Phys. Rev. Lett. 31, 494 (1973).

[8]N. N. Bogoliubov and O. Parasiuk, Acta. Math. 97, 227 (1957); K. Hepp, Commun. Math. Phys. 2, 301 (1966).

[9]B. W. Lee, Nucl. Phys. B9, 649 (1969); J.-L. Gervais and B. W. Lee, Nucl. Phys. B12, 627 (1969).

[10]Almost all of our discussion would also apply to Abelian gauge theories in which gauge invariance is broken in the Lagrangian by a bare vector-boson mass. Of course, for non-Abelian gauge theories any breakdown of the gauge symmetry must be purely spontaneous so as not to spoil the renormalizability of the theory. It is not clear how this can happen if we exclude scalar fields from the theory. One possibility is that the symmetry breakdown occurs through dynamical effects, as discussed by R. Jackiw and K. Johnson [Phys. Rev. D 8, 2386 (1973)], and J. M. Cornwall and R. E. Norton [this issue, Phys. Rev. D 8, 3338 (1973)]. Another possibility is that the gauge symmetry is *not* broken, as suggested by S. Weinberg (Ref. 7), and D. J. Gross and F. Wilczek (unpublished). We will not need to concern ourselves with these problems in the present work.

[11]K. Wilson, Phys. Rev. 179, 1499 (1969), and unpublished report. Also see W. Zimmermann, in *Lectures*

on Elementary Particles and Quantum Field Theory (M.I.T. Press, Cambridge, Mass., 1970).

[12]See, e.g., B. W. Lee and J. Zinn-Justin, Phys. Rev. 5, 3121 (1972).

[13]In general Z_Γ might be a matrix connecting amplitudes of different types. Also, if there were more than one mass or super-renormalizable interaction in the theory, Z_Θ might be a matrix also. Such complications are ignored here for the sake of notational simplicity.

[14]S. Weinberg, Phys. Rev. 118, 838 (1960).

[15]G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972). Also see G. 't Hooft (unpublished).

[16]This is essentially the same argument as used in Ref. 2.

[17]A hydro-bacteriological analogy is given by S. Coleman, in Lectures at the 1971 Erice Summer School, to be published. Our treatment follows his closely.

[18]See Refs. 4 and 5. These authors actually use the Callan-Symanzik approach, but the results of Sec. V show that this has no effect on the *lowest*-order formulas for β and γ.

[19]An example is provided by the analysis of the electron propagator by M. Baker and K. Johnson, Phys. Rev. D 3, 2516 (1971). Baker and Johnson use a gauge in which the electron field renormalization constant Z_2 is finite; using this gauge here, we would have $\gamma_\Gamma = 0$ for the electron propagator. They find that the $\gamma_\mu p^\mu$ term of the electron self-energy has a coefficient which becomes constant at large momentum, in agreement with (4.20). Further, they find that the term of the electron self-energy which does not contain a $\gamma_\mu p^\mu$ factor (and is therefore odd in m) behaves for $p^2 \rightarrow \infty$ like a power of p^2, in agreement with (4.21). Comparing (4.21) with their results, we find

$$\gamma_\Theta(e_\infty) = \frac{3e_\infty^2}{8\pi^2} + O(e_\infty^4).$$

However, Baker and Johnson use a conventional mass renormalization prescription, so their asymptotic formula for the electron self-energy has a complicated dependence on the renormalized electron mass, not the simple linear dependence expected here.

[20]Such singularities have recently been discussed in an entirely different context by P. Langacker and H. Pagels, Phys. Rev. D (to be published).

[21]Such equations have been considered by C. Callan, Phys. Rev. D 5, 3202 (1972).

[22]This argument is similar to that used in Refs. 17 and 21 to derive the asymptotic Callan-Symanzik equations for the Wilson coefficient functions. However, Eq. (6.2) holds for all momenta, not only in the asymptotic limit.

[23]A. Duncan, private communication. Duncan uses the BPH (Bogoliubov-Parasiuk-Hepp) renormalization procedure of Ref. 8, so that his results do not involve a renormalization-point parameter μ, and have a more complicated dependence than ours on the renormalized mass. However, the Wilson functions can only differ for differerent renormalization prescriptions by at most a finite constant factor, given by a perturbation series in the square of the coupling constant, starting with zeroth-order term equal to one. Also, all definitions of the renormalized coupling constant are the same in lowest order. Hence the second-order contributions to the Wilson functions will differ for different renormalization prescriptions at most by terms proportional to the zeroth-order contributions. Since the zeroth-order contributions do not involve $\ln\kappa$, the coefficient of $\ln\kappa$ in the second-order Wilson functions must be the same for all renormalization prescriptions.