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Multiparticle Matrix Elements of Currents at Large Mass*

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We employ a simple field-theory model to explore the extent to which kinematical light-cone dominance entails also leading light-cone singularity dominance, for higher matrix elements of current products. We retain all leading logarithmic terms in the ladder approximation, and impose the Mueller-Regge picture as a boundary condition on simple two-particle matrix elements appropriate for processes such as massive lepton-pair production and semi-inclusive electroproduction. We explore several asymptotic regimes with current mass $q^2 \to \infty$, and find that limit sequences may always be defined that become analytically equivalent to some leading-singularity limit, before any discontinuities are computed. This includes commutativity of the deep-Regge and deep-scaling limits, as suggested by Brandt and Preparata. In massive lepton-pair production the pionization (central) vertex $V_P(q^2+q_\perp^2, q^2)$ depends on the transverse momentum $q_\perp$ of the current, as well as the invariant mass q^2. In the present model, whenever $q^2 \to \infty$ with $q_\perp^2/q^4$ finite, the appropriate residue in Mellin-transform space comes in part from the leading Mellin singularity relevant for the limit $q^2+q_\perp^2 \to 0$ in V_P, for the bremsstrahlung amplitudes. This suggests that perhaps any attenuation of these amplitudes, which carry the leading light-cone singularity, will not occur with emphasis on the $q^2+q_\perp^2$ scale in V_P but rather on the second, independent q^2 scale. On the other hand, whenever $q^2 \to \infty$ with $q_\perp^2/q^4 \to \infty$, the appropriate residue in Mellin-transform space comes in part from the leading Mellin singularity relevant for the limit $q^2+q_\perp^2 \to \infty$, for the bremsstrahlung amplitudes. The bremsstrahlung amplitudes are not attenuated in this limit, but rather develop a square root of a triple-Regge behavior. This behavior obtains only for the bremsstrahlung amplitudes, with the annihilation amplitudes becoming asymptotically independent of $q^2+q_\perp^2$. The existence of interpolating limits and nonattenuation of the bremsstrahlung amplitudes allows us to conclude that in the model no natural mechanism precludes dominance by the leading light-cone singularities, in any kinematically light-cone-dominated regime. In semi-inclusive electroproduction, the mechanism which generally links light-cone dominance with dominance by the leading light-cone singularity is graphically equivalent to the fixed-pole mechanism operating in inclusive electroproduction.

I. INTRODUCTION

This is the second of two papers[1] in which we investigate light-cone limit sequences for inclusive processes which involve weak currents at large mass. In this paper we analyze multiparticle matrix elements of current products in various asymptotic, light-cone-dominated domains. Some of the results have been described briefly elsewhere.[2]

If for a product of operators there exists an expansion in terms of c-number functions, ordered by singularity strength on the light cone, with nonsingular operator-valued coefficients, valid in all light-cone-dominated limits for arbitrary matrix elements, then this operator expansion is said to exist in the strong or operator sense.[3] Such an invariant characterization of operator products would provide a powerful framework for investigating and classifying processes which, through kinematics, become light-cone-dominated. Of especial contemporary interest is the possibility of such a classification for current products.[4,5] Then, one might expect among other things, some modified form of asymptotic scale invariance and fixed dependence on the masses of the currents to obtain subsequently in any kinematic limit which becomes light-cone-dominated.

In general, higher hadronic matrix elements of current products introduce additional dimensional parameters which can become large, corresponding to the subenergies that can be formed among the hadrons which define the matrix element. This extra energy dependence would likely preclude full scale invariance.[6] Moreover, in some kinematic regions these additional hadronic scales conceivably could destroy the connection[1] between light-cone dominance and leading-singularity dominance.[7,8,5]

We employ a simple perturbation-theory model, the $\lambda\phi^3$ model, to determine in what kinematic domains light-cone dominance implies leading-singularity dominance for two-particle matrix elements of the product of two weak currents. The $\lambda\phi^3$ model (almost[1]) conserves canonical light-cone structure. In addition, it manifests Regge behavior, which is suspected[9] to be present in multiparticle inclusive production amplitudes at high energies. In its simplicity, the $\lambda\phi^3$ model can lead to a deeper understanding of many questions, as well as define further questions.

For brevity, we report the details of our analysis primarily for the simple model involving scalar "photons" (weak currents) and scalar hadrons. The scalar photon field is coupled weakly to its hadronic current $J(x)$ through the local density $H=e:A(x)J(x):$, where $J(x)=:\phi(x)\phi(x):$. The complete analysis with vector photons, which we have done, introduces nothing new (aside from logarithms coming from the usual singular configurations[1]). We outline in the Appendix the details of a typical amplitude involving vector photons.

We consider in detail, then, the two-scalar hadron matrix element of two scalar weak currents,

$$4p_0p_0'\,{}_{\text{in}}\langle pp'|J(x)J(0)|pp'\rangle_{\text{in}} \equiv \tilde{W}(x^2, x\cdot p, x\cdot p', s)\,, \tag{1}$$

where $s=(p+p')^2$. We introduce the Fourier transform of $\tilde{W}$,

$$W(q^2,\nu,\nu',s)=\int d^4x\, e^{iq\cdot x}\,\tilde{W}(x^2, x\cdot p, x\cdot p', s)\,, \tag{2}$$

with $\nu=q\cdot p$ and $\nu'=q\cdot p'$. From the work by Mueller[9] we know W is a discontinuity of the causal amplitude

$$T(q^2,\nu,\nu',s)=4p_0p_0' i\int d^4x\, e^{iq\cdot x} \times {}_{\text{in}}\langle pp'|\theta(x_0)[J(x),J(0)]|pp'\rangle_{\text{in}} \tag{3}$$

for the forward scattering process scalar hadron (p) + scalar hadron (p') + scalar photon $(q)\to p+p'+q$. If we define $M^2=(p+p'+q)^2$, the connection is given by[10]

$$W=\frac{1}{\pi}[T(M^2+i\epsilon)-T(M^2-i\epsilon)]\,. \tag{4}$$

Through crossing,[10] $q\to -q$, this discontinuity gives the inclusive cross section for the process $p+p'\to q$ + anything. If we take the photon to be highly virtual, $q^2\to\infty$, it may then decay into a lepton pair. The above discontinuity then gives the cross section for the process of massive lepton-pair production.[3] An alternative crossing, $p'\to -p'$, and the discontinuity in M^2 give the inclusive cross section for the process $p+q\to p'$ + anything. This is essentially the cross section for semi-inclusive electroproduction; electron + hadron $(p)\to$ electron + hadron (p') + anything. Therefore, in order to study the two-particle matrix element of the current products, i.e., W, it is sufficient to elaborate the asymptotic properties of T.

The behavior of T in asymptotic domains depends generally on the paths in the multivariable space along which the limits are taken. For the inclusive production processes, there are for T two Regge limits which Mueller showed to be especially interesting,[9] namely, the single-Regge (fragmentation) limit, and the double-Regge (central region) limit. The dominant amplitudes are distinguished by the specific asymptotic limit. Work done in the context of dual resonance theory gives theoretical support to Mueller's conjectures.[11]

In our field-theory model, considered to all leading orders in the ladder approximation, a number of distinct multi-Regge amplitudes appear in the asymptotic domains. The model does not determine the relative importance to be assigned to these distinct Regge forms. In the dual resonance calculations,[11] the roles of crossing symmetry and crossed-channel resonance are crucial in making the correct assignment of specific Regge forms to a particular limit according to the Mueller procedure. Our concern here is with the question of leading-singularity dominance in T (W) at large values of the hadronic subenergy s. In order to investigate this question, it is sufficient to let *all* subenergies become large. Upon determining the behavior at large energies of the sum of dominant contributions to T (W), we will then take the Mueller assignment, appropriate to the particular process under consideration, as a boundary condition. In particular, we focus on the double-Regge behaviors appropriate to the central regions, as illustrated in Fig. 1.

It remains to specify the details of light-cone–dominated limits for the six-point amplitude T. For convenience of calculation, we work with the q-space representation T of $\tilde{T}$. Of especial interest in light-cone–dominated limits are the dimensionless variables

$$\omega=2\nu/q^2,\quad \omega'=2\nu'/q^2,\quad \tau=s/q^2\,.$$

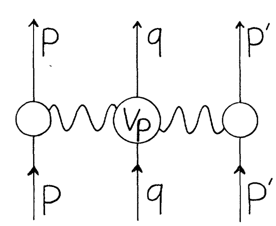

FIG. 1. Double-Regge exchange for production of particle q, from the "pionization vertex" V_P, of the central region, according to the Mueller conjecture.

Central to our interest is the light-cone–dominated limit which, according to standard arguments,[5] becomes also leading-singularity–dominated. This limit, which we will refer to as the A limit, is

$$\lim_{A} = \lim_{\substack{q^2\to\infty;\\ \omega,\,\omega',\,s\text{ fixed}}} .$$

In this limit, the dominant contributions to T are from those regions of configuration space where $x^2 \to 0$ and $|x_0|, |x_3| \leq \omega/m$ $(|x_0'|, |x_3'| \leq \omega'/m)$ in the rest frame of hadron p (p'). We consider also the A' limit, defined by taking subsequently ω, ω', and s to infinity.

An alternative kinematic route to the light cone passes through the Regge (R) limit

$$\lim_{R} = \lim_{\substack{\nu,\,\nu',\,s\to\infty;\\ q^2\text{ fixed}}} .$$

The route now takes $q^2 \to \infty$, defining thereby the R' limit. This is a light-cone–dominated limit,[5] saturated by contributions from $|x^2| \leq 1/q^2 \to 0$ as in electroproduction. Further, this limit respects the physical-region constraints in the problem of lepton-pair production. However, there is no *a priori* reason to believe the leading light-cone singularity remains important in this limit.[7]

Finally, we define the scaling (S) limit

$$\lim_{S} = \lim_{\substack{\nu,\,\nu',\,s,\,q^2\to\infty;\\ \omega,\,\omega',\,\tau\text{ fixed}}} ,$$

and the S' limit which takes now $\omega, \omega', \tau \to \infty$. Again this limit is light-cone–dominated kinematically,[5] but of course the question of leading-singularity dominance remains.

The organization of the paper is the following. In Sec. II we summarize and discuss the results obtained in the perturbation-theory model. In Sec. III we review our approximations and the devices which render possible the degree of accuracy necessary to make our investigation meaningful.

Sections IV–VIII deal with the problem of massive lepton-pair production. A detailed study is made particularly of the properties of the leading-singularity, bremsstrahlung amplitudes, and their place in the Mueller-Regge framework. In Sec. IV we consider the Regge limit at finite current mass. There we distinguish specific scales, which determine the behavior of the R' limit, that may be set by the hadronic interactions. In Sec. V we study the A and A' limits. In Sec. VI we obtain the R' limits and establish their analytic equivalence with certain A' limits. In Sec. VII we extend our analysis to the S limits, and again make a connection with an A' limit. In Sec. VIII we outline the analysis for a less singular contribution, the annihilation amplitude. Section IX summarizes the analysis for amplitudes appropriate to the central region of semi-inclusive electroproduction processes.

In the Appendix, we outline the investigation for a typical amplitude involving vector currents.

II. RESULTS AND DISCUSSION

We employ the superrenormalizable $\lambda\phi^3$ perturbation-theory model to study the extent to which light-cone expansions of operator products and ordering of contributions strictly in terms of their light-cone singularities might be valid for higher hadronic matrix elements of weak currents. We consider in detail the prototypal two-scalar boson matrix element of two scalar weak currents. The case of vector currents is outlined briefly in the Appendix, where we illustrate that, aside from a single power of current mass q^2, essential details of the results are the same as for the scalar currents. Our analysis accounts for all leading logarithmic terms in the ladder approximation. In order to structure our investigations according to production phenomenology, we arbitrarily impose the Mueller-Regge viewpoint[9] as a boundary condition on the amplitudes in the deep asymptotic regimes. This does not affect our conclusions concerning leading-singularity dominance, but merely the final Regge forms which are assumed to be relevant.

We distinguish among and evaluate a number of limit sequences in which, excepting for pathologies which do not appear in the present model, T becomes dominated by light-cone contributions to the amplitudes. One of these, the A limit, holds fixed the state $|pp'\rangle$ of the matrix element, and so by standard arguments[5] T becomes dominated by the leading light-cone singularity. Other (R', S, and S') asymptotic limits are evaluated, and we show that each of these becomes analytically equivalent to some A' limit. Thus, we conclude that, within the model, light-cone dominance is generally equivalent to dominance by the leading-light-cone singularity.

Note added in proof. We do not intend to suggest that the Regge structure present in (42) should actually characterize the S limit, since here ω, ω', and τ are finite. The point is the following: If one considers only a single bremsstrahlung amplitude, in the S limit it has an effectively less singular light-cone behavior than that revealed in the A limit, in that, up to powers of logarithms, the q^2 behavior is weaker by one power.[2] However, the subset of diagrams whose leading terms sum to (42) manifest canonical q^2 behavior when all $\ln q^2$ terms of leading order are included in the sum.

In the S' limit, this particular subset is just that which gives the correct asymptotic behavior according to our boundary conditions. I thank Dr. Robert Jaffe for discussions related to this point.

The question arises as to whether one may conclude the same for the discontinuity W, since W actually vanishes in the A limit for physical lepton-pair production.[12] However, we note that, having identified the dominant presence of the leading singularity in T, its presence is sustained as we continue the invariants to values which are physical for the R' and S' regimes, where the discontinuities do not vanish.

It is of course not necessary that analytic interpolations exist in order for leading-singularity dominance to obtain, although it is sufficient. However, such interpolations do simplify our task of distinguishing q^2 dependences brought in through kinematic constraints [e.g., see Eq. (34)] from the q^2 dependence coming simply from the Fourier transform of an algebraic singularity. Such smooth analytic interrelatedness is of interest in itself in any case.

Also of some interest, because it raises further questions, is the possibility in massive lepton-pair production of employing the $q^2\to\infty$ limit to study the region where $q^2+q_\perp{}^2\equiv M_\perp{}^2$ $(=4\nu\nu'/s$ in the central region$)\to 0$. This region is not accessible in the production of finite-mass particles and the corresponding scale is unknown. However, there is no reason to think the vertex $V_P(M_\perp{}^2, q^2)$ should vanish for $M_\perp{}^2\to 0$. Thus, since in our model whenever $q^2\to\infty$ while $M_\perp{}^2/q^4$ is held finite we must take residues in Mellin space which correspond to the region $M_\perp{}^2\to 0$, we suggest that any attenuation of the bremsstrahlung amplitudes may occur with emphasis on the scale for the q^2 dependence in the second argument of $V_P(M_\perp{}^2, q^2)$. This result is evidently model-dependent, i.e., dependent on the features of the singularities in Mellin space. On the other hand, if $q^2\to\infty$ in such a way that $M_\perp{}^2/q^4\to\infty$, then the appropriate residues in Mellin space correspond to the region where $M_\perp{}^2\to\infty$. In a more realistic model, one would expect $V_P(M_\perp{}^2, q^2)$ to vanish in this region. In the present model, as $q^2\to\infty$ in this latter fashion, the dependence on $M_\perp{}^2$ in the bremsstrahlung amplitudes is such that a remarkable square root of a triple-Regge amplitude develops. This feature raises several interesting questions, and it seems of interest to investigate it in other models.

As a typical example of a contribution to lepton-pair production which is less singular on the light cone, we survey the details of the annihilation amplitude. In limits which measure the scale for $M_\perp{}^2\to\infty$, this amplitude becomes asymptotically independent of $M_\perp{}^2$. No root-triple-Regge behavior develops, and the double-Regge form of the amplitude in the central region is preserved. This is merely a consequence of the particular singularity structure of the amplitude in the Mellin transform space, which structure would essentially obtain for the annihilation amplitudes of any field theory model.

Finally, we study kinematic limits and amplitudes appropriate for the central regions of semi-inclusive electroproduction, and find that leading-singularity dominance again obtains generally in all domains at large current mass. The mechanism is quite simple, and in perturbation theory is equivalent to the fixed-pole mechanism operating in inclusive electroproduction.

We should point up the fact that most of our results are quite model-dependent, especially the sustained dominance by the leading light-cone singularities. As has been often pointed out,[3,7,2] the question of leading-singularity dominance is a dynamical question, which must eventually be answered by experiment. However, we feel that our results are quite suggestive, and at least raise further interesting questions to be explored in other models.

III. $\lambda\phi^3$ PERTURBATION THEORY

The classes of amplitudes which are analyzed in detail here correspond to the Feynman diagrams illustrated in Fig. 2. After correct crossing, these give nonzero contributions to W, and exemplify planar contributions which dominate in the deep asymptotic regions. Diagrams obtained by various permutations of the external lines are either negligible in the deep asymptotic domains or introduce no features which change the results

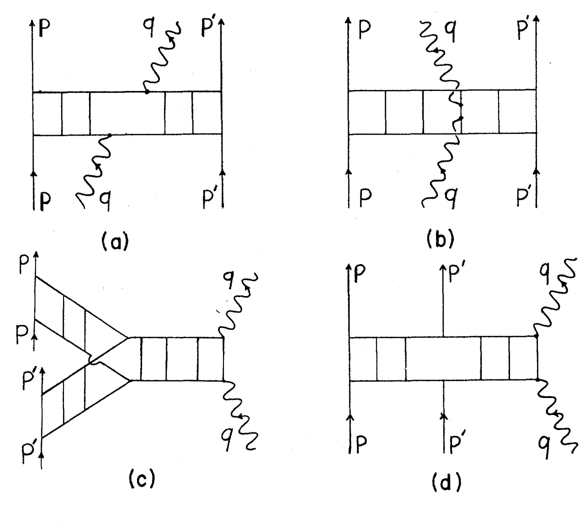

FIG. 2. Classes of $\lambda\phi^3$ diagrams, which remain important at large current mass.

of our analysis. Figure 2(a) corresponds to covariant annihilation production of the photon q in lepton-pair production, a noncovariant component of which has been argued by Drell and Yan[7] to dominate the bremsstrahlung contributions, Figs. 2(b), 2(c), and 2(d). This is the central issue here also, of course, since it is the bremsstrahlung contribution which carries the leading singularities on the light cone. We refer to Figs. 2(b), 2(c), and 2(d) generically as bremsstrahlung diagrams, and to Fig. 2(a) as annihilation diagrams.

The diagrams in Fig. 2(b) are actually special members of the class c. It is the diagrams of class c that sum in some limits to a root of a triple-Regge behavior. In the central regions for massive lepton-pair production, it is the contributions of classes b and c which sum to the appropriate double-Regge behavior. In the central regions for the process of semi-inclusive electroproduction, the classes c and d contribute the correct Regge behavior, according to Mueller's assignments.[9]

In the case of electroproduction,[1] when $q^2 \to \infty$ the leading contributions come from the amplitudes in which the currents have minimal separation. The same considerations apply here, and we need to retain *ab initio* only contributions from minimally separated insertions. Thus, our investigation is simplified essentially to asking whether anything can happen in the model to destroy the connection between light-cone dominance and leading-singularity dominance *within the bremsstrahlung contributions themselves*. As an example of a lower-singularity contribution, we consider in Sec. VIII the contributions of the annihilation amplitudes.

We make extensive use of the Mellin transform techniques which have repeatedly been found useful in investigating asymptotic features of scattering amplitudes.[1] That is, we perform a Mellin transformation of the amplitudes with respect to the variables which are to be taken large, identify the leading contributions according to their singularities in the transform space, sum these, and then invert to obtain the actual asymptotic behavior of the amplitudes. The usual analyses[13] retain only selected leading-order contributions, whereas for our investigation it is necessary to retain *all* leading orders. This seemingly prohibitive requirement is met almost trivially, through use of an observation made by Halliday,[14] and used by us in the study of electroproduction.[1] For details of the observation, we refer to the literature.[1,14]

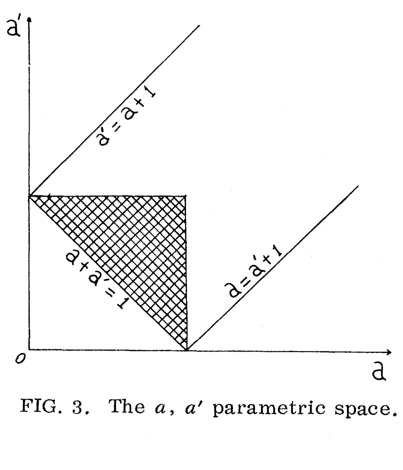

FIG. 3. The a, a' parametric space.

IV. R LIMIT FOR MASSIVE LEPTON-PAIR PRODUCTION

Leading-singularity dominance for massive lepton-pair production is perhaps particularly questionable. This is because the physical region for this process is quite remote from the kinematical regions in which light-cone dominance guarantees also leading-singularity dominance, i.e., the A limit regions.

The physical-region constraints for this process, in which only the four-momentum q of the massive virtual photon (lepton-pair) is measured, $p + p' \to q$ + anything, are readily established. Translation invariance and the timelike nature of the photon give

$$s > \tfrac{1}{2}(s + q^2) > \tfrac{1}{2}(s + q^2 - 4m^2) > \nu + \nu' > \sqrt{sq^2} > q^2 . \tag{5}$$

Thus, in the physical region we must have $s > q^2$ strictly. It is just this feature that could make irrelevant the contribution from the leading singularities, the relevance of which is more secure when $q^2/s \to \infty$.

There is, however, yet another constraint deriving from essentially geometric considerations, which further delimits the physical region. This relation is, at large ν, ν', and s,

$$4\nu\nu' = s(q^2 + q_\perp{}^2) , \tag{6}$$

where $q_\perp{}^2$ is the square of the component of the three-momentum q which is transverse to the three-momentum p in the p, p' center-of-momentum frame. This relation is readily obtained in this frame. In terms of parameters a, a', letting $\nu = s^a$ and $\nu' = s^{a'}$, for physical values of q^2 and $q_\perp{}^2$, this constraint becomes $a + a' \geq 1$. Also, from (5) we have $q^2, q_\perp{}^2 \leq q_0{}^2 < s$ so that $a, a' \leq 1$ define further boundaries. In the a, a' parametric space, then, the physical region is defined by the cross-hatched area shown in Fig. 3. The regions $a \simeq a' \simeq 1$ are accessible only at very large q^2 and/or $q_\perp{}^2$.

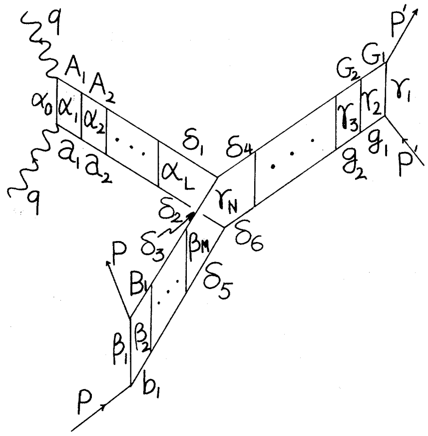

FIG. 4. Feynman parameterization of triple-ladder diagram.

The amplitudes which, according to Mueller,[9] are appropriate for the production of q, in the regions where $s, \nu, \nu' \gg q^2$, are those which build up the "p ionization" vertex V_p of the central-region double-Regge amplitude illustrated in Fig. 1. V_p thus comprises amplitudes a, b, and c of Fig. 2, subject to crossing, etc., and as $q^2 \to \infty$ may be saturated by contributions from classes b and c. Let us therefore describe the R limit for these latter diagrams.

We recall the representation which can be given for scalar field-theory amplitudes,[13]

$$T = \int d\xi \frac{e^{D/\Delta}}{\Delta^2} , \tag{7}$$

where the discriminant D and determinant Δ are functions of the Feynman parameters ξ which are assigned to the propagator lines of the diagrams. [We do not retain in (7) the inessential multiplicative factors of coupling constants, powers of π, etc., that are properly present.[13]] According to familiar considerations,[13] the present three-to-three amplitudes for forward scattering, D, will be of the form

$$D(q^2, \nu, \nu', s, \xi) = q^2 \mathfrak{F}(\xi) + 2\nu g(\xi) + 2\nu' g_1(\xi) + sF(\xi) - \Delta J. \tag{8}$$

For the amplitudes under consideration, a suitable parameterization is given in Fig. 4.

Let Δ_M be the Δ function corresponding to the M-rung ladder to which the momenta p are attached, after this ladder has been severed from the others at the δ_3 and δ_5 lines. Let Δ_{M-1} be the Δ function

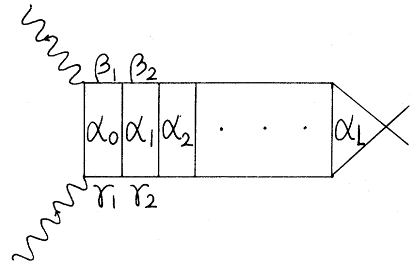

FIG. 5. Contributions to C_L.

for this same ladder segment after the removal of the Mth rung, etc., down to $\Delta_2 = \beta_1 + \beta_2 + b_1 + B_1$. In a similar fashion we define $\Delta_L, \Delta_{L-1}, \ldots, \Delta_1$, and $\Delta_N, \Delta_{N-1}, \ldots, \Delta_2$, etc. Further, let

$$\Pi_\beta = \prod_{j=1}^{M} \beta_j, \quad \Pi_\alpha = \prod_{j=0}^{L} \alpha_j, \quad \Pi_\gamma = \prod_{j=1}^{N} \gamma_j . \tag{9.1}$$

Then some analysis gives

$$g = \Pi_\alpha \Pi_\beta [(\delta_4 + \delta_6)\Delta_N + \gamma_N (g_N + G_N)\Delta_{N-1} + \cdots + \Pi_\gamma] \equiv \Pi_\alpha \Pi_\beta G, \tag{9.2}$$

$$g_1 = \Pi_\alpha \Pi_\gamma [(\delta_3 + \delta_5)\Delta_M + \beta_M (b_M + B_M)\Delta_{M-1} + \cdots + \Pi_\beta] \equiv \Pi_\alpha \Pi_\gamma B, \tag{9.3}$$

$$F = \Pi_\beta \Pi_\gamma [(\delta_1 + \delta_2)\Delta_L + \alpha_L (a_L + A_L)\Delta_{L-1} + \cdots + \Pi_\alpha] \equiv \Pi_\beta \Pi_\gamma A, \tag{9.4}$$

$$\mathfrak{F} = \alpha_0 \Delta^0, \quad \Delta^0 \equiv \Delta|_{\alpha_0 = 0}, \tag{9.5}$$

identifying thereby the most important functions.

In order to obtain the R limit we employ the Mellin transform analysis.[1] We take for the moment $q^2 < 0$ and fixed, while $-\nu$, $-\nu'$, and $-s$ are all >0. Then D will be negative-definite and normal-threshold cuts will be avoided in the transform.[13] The results are always to be continued back into the physical region before inclusive cross section discontinuities are taken. The Mellin transform with respect to -2ν, $-2\nu'$, and $-s$ is[15]

$$\tilde{T}(q^2, \beta, \beta', \sigma) = \Gamma(-\sigma)\Gamma(-\beta)\Gamma(-\beta') \times \int d\xi\, g^\beta g_1{}^{\beta'} F^\sigma \frac{\exp[(q^2 \mathfrak{F} - \Delta J)/\Delta]}{\Delta^{\beta+\beta'+\sigma+2}}, \tag{10}$$

where β, β', and σ are conjugate to -2ν, $-2\nu'$, and $-s$, respectively. We follow now the usual procedure for obtaining the Regge structure of the amplitude.[14,1] In the transform space this leads to

$$\tilde{T}(q^2, \beta, \beta', \sigma) = \sum_{L,M,N} \tilde{T}^{L,M,N}(q^2, \beta, \beta', \sigma) = \frac{\tilde{C}(\beta+\beta', q^2)\mathcal{G}(\beta+\sigma, 0)\mathcal{G}(\beta'+\sigma, 0)\tilde{V}(\beta, \beta', \sigma)\Gamma(-\beta)\Gamma(-\beta')\Gamma(-\sigma)}{[\beta+\sigma-\alpha_1(0)][\beta+\beta'-\alpha_2(0)][\beta'+\sigma-\alpha_3(0)]}, \tag{11}$$

where $\alpha_i(t_i)$ is the usual[1] scalar $\lambda\phi^3$ trajectory and $\mathcal{G}(x,t)$ is the usual[1] scalar $\lambda\phi^3$ residue factor. For $\tilde{C}$ we have

$$\tilde{C}(\beta+\beta',q^2)=\sum_{L=1}^{\infty}C_L(\beta+\beta',q^2), \tag{12}$$

with

$$\tilde{C}_L(\beta+\beta',q^2)=\int d\xi\,\alpha_0^{\beta+\beta'}\prod_{i=1}^{L}\frac{\alpha_i^{\beta+\beta'+1}-1}{\beta+\beta'+1}\frac{\partial}{\partial\alpha_i}\frac{\exp[(\mathcal{F}q^2-\Delta J)/\Delta]}{\Delta^{\beta+\beta'+2}}\;. \tag{13}$$

In (13) the functions $\mathcal{F}$, Δ, J, etc., are to be computed for the $(L+1)$-rung amplitude illustrated in Fig. 5. Finally,

$$\tilde{V}(\beta,\beta',\sigma)=\sum_{L,M,N=1}^{\infty}\tilde{V}^{L,M,N}(\beta,\beta',\sigma), \tag{14}$$

where

$$\tilde{V}^{L,M,N}(\beta,\beta',\sigma)=\int d\xi\prod_{i=1}^{L}\frac{\alpha_i^{\beta+\beta'+1}-1}{\beta+\beta'+1}\frac{\partial}{\partial\alpha_i}\prod_{j=1}^{M}\frac{\beta_j^{\beta+\sigma+1}-1}{\beta+\sigma+1}\frac{\partial}{\partial\beta_j}\prod_{k=1}^{L}\frac{\gamma_k^{\beta'+\sigma+1}-1}{\beta'+\sigma+1}\frac{\partial}{\partial\gamma_k}\frac{G^{\beta}B^{\beta'}A^{\sigma}e^{-J}}{\Delta^{\beta+\beta'+\sigma+2}}\,, \tag{15}$$

and the functions G, B, A [see Eq. (9)], J, and Δ are to be computed for the contracted triple-ladder vertex shown in Fig. 6.

The full amplitude $T(q^2,\nu,\nu',s)$ may be obtained by inverting (11), which gives

$$T(q^2,\nu,\nu',s)=\frac{1}{(2\pi i)^3}\int_{\Omega}d\beta\,d\beta'\,d\sigma\,\tilde{T}(q^2,\beta,\beta',\sigma)\times(-2\nu)^{\beta}(-2\nu')^{\beta'}(-s)^{\sigma}\,. \tag{16}$$

The surface Ω of integration is constrained to pass to the left of the imaginary axes in each of β, β', and σ, because of the Γ function poles, and to the right of the Regge zeros in the denominator of $\tilde{T}$. Thus, Ω must lie within the real subspace of the enclosure defined by these singularity surfaces, as illustrated in Fig. 7.

The central R limit takes the parameters a and a' both greater than zero. However, holding both q^2 and $q_\perp{}^2$ fixed, the constraint (6) requires $a+a'=1$. Let us introduce the variable $M_\perp{}^2=q^2+q_\perp{}^2$, the longitudinal mass squared for the photon. The limit with $a+a'=1$ is then taken at fixed $M_\perp{}^2$, and we denote it as the $R(M_\perp)$ limit. In the $R(M_\perp)$ limit, then, we may write

$$T(q^2,\nu,\nu',s)=\frac{1}{(2\pi i)^3}\int_{\Omega}d\beta\,d\beta'\,d\sigma\,\tilde{T}(q^2,\beta,\beta',\sigma)(-s)^{\Sigma}, \tag{17}$$

where $\Sigma=\sigma+a\beta+a'\beta'$. The leading behavior in the $R(M_\perp)$ limit is thus determined by the rightmost singularities in the left half of the Σ plane. The direction of most rapid decrease in Σ inside the real subspace of Fig. 7 is along the vector $-\vec{\nabla}\Sigma$, which for $a+a'=1$ is readily seen to be always orthogonal to the line of intersection of the singularity surfaces at α_1 and α_3. Thus the most remote value Σ can attain, without our being forced into a singularity surface, is at the intersection of the planes $\beta+\alpha=\alpha_1$ and $\beta'+\sigma=\alpha_3$. The singularity surface $\beta+\beta'=\alpha_2$ thus does not participate in the $R(M_\perp)$ limit, and it is sufficient here to displace Ω across the surfaces at α_1 and α_3. The residue at these surfaces then gives the leading behavior in the $R(M_\perp)$ limit.[16] We obtain, finally,

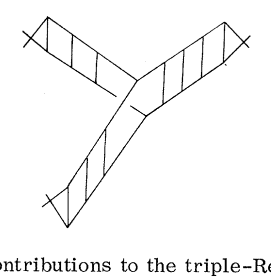

FIG. 6. Contributions to the triple-Regge vertex.

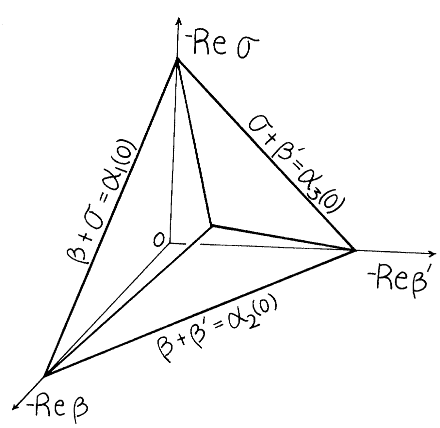

FIG. 7. The projection onto the real Mellin subspaces of the volume containing the contour Ω.

$$\lim_{R(M_\perp)} T(q^2,\nu,\nu',s) = \mathcal{G}(\alpha_1,0)\mathcal{G}(\alpha_3,0)V_P(q^2,M_\perp{}^2)\,(-2\nu)^{\alpha_1}(-2\nu')^{\alpha_3}, \tag{18}$$

with [17]

$$V_P(q^2,M_\perp{}^2) = \frac{1}{2\pi i}\int_{C_\sigma}\tilde{V}(\beta=-\sigma+\alpha_1,\beta'=-\sigma+\alpha_3,\sigma)\tilde{C}(-2\sigma+\alpha_3+\alpha_1,q^2)\frac{\Gamma(\sigma-\alpha_1)\Gamma(\sigma-\alpha_3)\Gamma(-\sigma)(-M_\perp{}^2)^{-\sigma}}{-2\sigma+\alpha_1+\alpha_3-\alpha_2}, \tag{19}$$

where the contour C_σ passes to the right of the poles in $\Gamma(\sigma-\alpha_1)$ and $\Gamma(\sigma-\alpha_2)$, and to the left of the pole at $\sigma=(\alpha_1+\alpha_3-\alpha_2)/2$.

Formally, the result (18) is simply the double-Regge behavior suggested by Mueller [9] to dominate the central region. In purely strong-interaction physics, where $q^2=M_\pi{}^2$, say, the scale governing the $M_\perp{}^2$ behavior rapidly attenuates the amplitude [18] as $M_\perp{}^2(\simeq q_\perp{}^2)\to\infty$. In the processes under consideration, q^2 also becomes large and, in the first argument of V_P, a second scale manifests itself which we refer to as the q^2 scale. Of course, it is possible that through the role of either of these scales the leading light-cone singularities may actually become unimportant in some light-cone regions.[5]

The two scales which control the light-cone-dominated, $q^2\to\infty$ regimes have now been identified. It remains to determine the actual features of the A and A' limits, and to identify leading-singularity contributions. We then return to the R limit and identify the roles of the above scales in our specific dynamical model, particularly in regard to how they may affect leading-singularity dominance.

V. A AND A' LIMITS FOR MASSIVE LEPTON-PAIR PRODUCTION

The A limit, by familiar arguments,[15] should project the leading light-cone singularity, which, as has been pointed out, comes from the bremsstrahlung amplitudes, of class c in Fig. 2. These amplitudes have the representation (7), with D specified in (8).

In order to obtain the A limit we take the Mellin transform of (7) with respect to $-q^2$ at fixed s, ω, and ω'. For a typical amplitude of class c, parameterized as in Fig. 4, we obtain

$$\tilde{T}^{L,M,N}(\chi,\omega,\omega',s) = \Gamma(-\chi)\int d\xi\left(\frac{\mathfrak{F}+\omega g+\omega' g_1}{\Delta}\right)^{\chi} \times\frac{\exp[(Fs-\Delta J)/\Delta]}{\Delta^2}, \tag{20}$$

where χ is the Mellin variable conjugate to $-q^2$. T is obtained by inversion:

$$\tilde{T}^{L,M,N}(q^2,\omega,\omega',s) = \frac{1}{2\pi i}\int_{-i\infty-\epsilon}^{+i\infty-\epsilon}\tilde{T}^{L,M,N}(\chi,\omega,\omega',s)\times(-q^2)^{\chi}d\chi\,. \tag{21}$$

The leading behavior as $-q^2\to\infty$ is obtained by taking the residue at the rightmost singularity in the left half χ plane. For the amplitudes of class c we find that the coefficient of q^2 in D, i.e., $\mathfrak{F}+\omega g+\omega' g_1$, has an over-all factor of α_0. Thus we may write this coefficient as $\alpha_0(\bar{\mathfrak{F}}+\omega\bar{g}+\omega'\bar{g}_1)$, defining thereby the barred functions. An integration by parts in α_0, as usual,[1] exposes the simple pole at $\chi=-1$ in $\tilde{T}^{L,M,N}$. The residue at this pole term defines the A limit of $T^{L,M,N}$. We find

$$\lim_{A} T^{L,M,N}(q^2,\omega,\omega',s) = -\frac{1}{q^2}T_R^{L,M,N}(\omega,\omega',s), \tag{22}$$

where $T_R^{L,M,N}$ is finite, and is given by

$$T_R^{L,M,N}(\omega,\omega',s) = \int\frac{d\xi}{d\alpha_0}\frac{\Delta^{-1}}{\bar{F}+\omega\bar{g}+\omega'\bar{g}_1}\times\exp[(Fs-\Delta J)/\Delta]. \tag{23}$$

This identifies the leading-singularity, $1/q^2$ behavior, characterizing $\delta(x^2)$ behavior in configuration space. That is, the leading singularity is indeed present in $T^{L,M,N}$ and is unaltered by the interactions. Further, as usual,[1] any contributions where the current insertions are separated by more than a single line are less important by powers of q^2.

We now wish to compute the A' limit, in order eventually to make connection with the R' limit. We may choose an arbitrary parametric limit of the form $\omega=(s)^a$ and $\omega'=(s)^{a'}$ [more properly, $\omega=(-s)^a$, $\omega'=(-s)^{a'}$ with $-s>0$]. The constraints (5) and (6) again require a and a' to lie inside the (closed) region illustrated in Fig. 3. It is for our purposes sufficient to take again $a+a'=1$, with both $a,a'>0$. The computation of the A' limit exactly parallels that for the R limit. The same approximations lead, upon summing all M by N by $(L+1)$ rung ladders,[16] to the result

$$\lim_{A'(a+a'=1)} T(q^2,\omega,\omega',s) = -\mathcal{G}(\alpha_1,0)\mathcal{G}(\alpha_3,0)V_P(-M_\perp{}^2/q^4)\frac{\omega^{\alpha_1}\omega'^{\alpha_3}}{q^2}, \tag{24}$$

where

$$V_P(-M_\perp{}^2/q^4) = \frac{1}{2\pi i}\int_{C_\sigma} \tilde{V}(\beta = -\sigma + \alpha_1, \beta' = -\sigma + \alpha_3, \sigma)\tilde{C}(-2\sigma + \alpha_3 + \alpha_1)\frac{\Gamma(\sigma - \alpha_1)\Gamma(\sigma - \alpha_3)\Gamma(-\sigma)}{-2\sigma + \alpha_1 + \alpha_3 - \alpha_2}(-M_\perp{}^2/q^4)^{-\sigma} \tag{25}$$

and

$$\tilde{C}(-2\sigma + \alpha_3 + \alpha_1) = \sum_{L=1}^{\infty} \tilde{C}_L(-2\sigma + \alpha_3 + \alpha_1), \tag{26}$$

with

$$\tilde{C}_L(-2\sigma + \alpha_3 + \alpha_1) = \int \frac{d\xi}{d\alpha_0} \prod_{i=1}^{L} \frac{\alpha_i{}^{-2\sigma+\alpha_1+\alpha_3} - 1}{-2\sigma + \alpha_1 + \alpha_3 + 1} \frac{\partial}{\partial \alpha_i} \frac{e^{-J}}{\Delta^{-2\sigma+\alpha_1+\alpha_3+1}}\Bigg|_{\alpha_0=0}. \tag{27}$$

In (27) the constituent functions Δ and J under the integral are to be computed as for the L-rung, contracted diagram illustrated in Fig. 8. The contour C_σ in (25) goes between the singularities under the integral in the same way as C_σ in (19).

This identifies the behavior of the leading-singularity contribution, including its Regge structure. We now establish its connection with the physical regions for massive lepton-pair production.

VI. R' LIMITS FOR MASSIVE-PAIR PRODUCTION

We recall the result (18), for the $R(M_\perp)$ limit. We seek to determine the behavior of $V_P(q^2, M_\perp{}^2)$ as $q^2 \to \infty$, particularly in regard to whether the result may in any sense be regarded as coming from the leading $[\delta(x^2)]$ light-cone singularity.

We first rewrite (19) as the inversion of its Mellin transform with respect to $-q^2$. This gives

$$V_P(q^2, M_\perp{}^2) = \frac{1}{(2\pi i)^2}\int_S d\sigma d\chi\, \tilde{V}(\chi, \sigma)(-q^2)^\chi(-M_\perp{}^2)^{-\sigma}, \tag{28}$$

where we have

$$\tilde{V}(\chi, \sigma) = \tilde{V}(\beta = -\sigma + \alpha_1, \beta' = -\sigma + \alpha_2, \sigma)\overline{C}(-2\sigma + \alpha_1 + \alpha_3, \chi)\frac{\Gamma(\sigma - \alpha_1)\Gamma(\sigma - \alpha_3)\Gamma(-\sigma)\Gamma(-\chi)}{-2\sigma + \alpha_1 + \alpha_3 - \alpha_2} \tag{29}$$

and

$$\overline{C}(-2\sigma + \alpha_1 + \alpha_3, \chi) = \sum_{L=1}^{\infty} \overline{C}_L(-2\sigma + \alpha_1 + \alpha_3, \chi), \tag{30}$$

with

$$\overline{C}_L(-2\sigma + \alpha_1 + \alpha_3, \chi) = \int d\xi\, \alpha_0{}^{\chi-2\sigma+\alpha_1+\alpha_3} \prod_{i=1}^{L} \frac{\alpha_i{}^{-2\sigma+\alpha_1+\alpha_3+1} - 1}{-2\sigma + \alpha_1 + \alpha_3 + 1} \frac{\partial}{\partial\alpha_i}\left(\frac{\overline{\mathfrak{F}}}{\Delta}\right)^\chi \frac{e^{-J}}{\Delta^{-2\sigma+\alpha_1+\alpha_3+2}}, \tag{31}$$

again letting $\mathfrak{F} = \alpha_0\overline{\mathfrak{F}}$. In (31) we see, through integration by parts in α_0, that there is a simple pole in χ at $2\sigma - \alpha_1 - \alpha_3 - 1$. The inversion hypersurface S in (28) must then be confined in that region of the real subspace of σ and χ illustrated in Fig. 9.

Consider first the limit $-q^2 \to \infty$ with $-M_\perp{}^2/q^4$ fixed. We have to consider the behavior of

$$V_P(q^2, M_\perp{}^2) = \frac{1}{(2\pi i)^2}\int_S d\sigma d\chi\, \tilde{V}(\chi, \sigma)(-q^2)^{\chi-2\sigma} \times (-M_\perp{}^2/q^4)^{-\sigma}. \tag{32}$$

The most rapid decrease in the variable $\chi - 2\sigma$ is easily seen to be orthogonal to the line of singularity $\chi = 2\sigma - \alpha_1 - \alpha_3 - 1$. The leading behavior in the present limit is then given by the residue at this line, so we find

$$\lim_{\substack{-q^2\to\infty \\ -M_\perp{}^2/q^4 \text{ fixed}}} V_P(q^2, M_\perp{}^2) = (-q^2)^{-\alpha_1-\alpha_3-1} \times V_P(-M_\perp{}^2/q^4), \tag{33}$$

where $V_P(-M_\perp{}^2/q^4)$ is exactly the same as in (24). Using the result (33) in (18), we see that the analytic form of (24) is reproduced exactly, in-

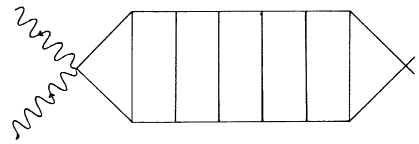

FIG. 8. Doubly contracted vertex contributions to C_L.

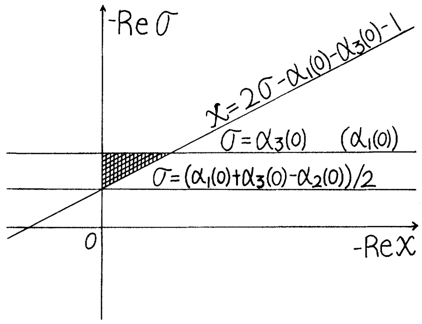

FIG. 9. The projection onto the real Mellin subspaces of the region containing the contour S.

clusive of *all* constant factors. That is, leading-singularity dominance is present in the above $R(M_\perp)$ limit. Further, we note that this particular limit measures only the q^2 scale, and is insensitive to the $M_\perp{}^2$ scale.

Let us consider now an R' limit defined by $-q^2 \to \infty$, with $-M_\perp{}^2/q^4 \to 0$ at an arbitrary rate. The R' limit which Brandt[5] takes to commute with our S' limit, for instance, is of this form, with $M_\perp{}^2/q^2$ fixed. In this limit, the leading behavior is determined in any parameterization by the residue at the intersection of singularities $\chi = 2\sigma - \alpha_1 - \alpha_3 - 1$ and $\sigma = \alpha_3$, say (or $\sigma = \alpha_1$, whichever is greater; for simplicity we assume here $\alpha_1 \neq \alpha_3$). In the present model, the double residue gives the limit as

$$\lim_{\substack{-q^2\to\infty \\ M_\perp{}^2/q^4\to 0}} : \lim_{R(M_\perp)} T(q^2,\nu,\nu',s) = -\frac{\omega^{\alpha_1}\omega'^{\alpha_3}}{q^2}\left(\frac{-M_\perp{}^2}{q^4}\right)^{-\alpha_3} V_C \tag{34}$$

with

$$V_C = \tilde{V}(\chi = 2\sigma - \alpha_1 - \alpha_3 - 1, \sigma = \alpha_3). \tag{35}$$

Thus, this limit measures also the scale relevant in the limit $M_\perp{}^2 \to 0$, which is not directly accessible in finite-mass processes. That this limit, that is, the amplitude in (34), is dominated by the leading light-cone singularity is readily established; take the limit $-M_\perp{}^2/q^4 \to 0$ in (25). We obtain immediately the result (34) since the leading singularity in the σ plane comes again from the function $\Gamma(\sigma - \alpha_3)$ [or $\Gamma(\sigma - \alpha_1)$].

We see that a wide range of limits is saturated in the model by the leading light-cone singularity [$O(1/x^2)$, here]. Moreover, any limit $q^2 \to \infty$ with $M_\perp{}^2/q^4 \to 0$ measures also the scale which governs the regime $M_\perp{}^2 \to 0$. As there is no reason to believe that the amplitudes should be drastically attenuated in this regime (attenuation occurs as $M_\perp{}^2 \to \infty$, at least for finite-mass processes[18]), this leads one to believe that any attenuation of the bremsstrahlung amplitudes as $q^2 \to \infty$ must come from the q^2 scale, *not* from the $M_\perp{}^2$ scale.

Finally, let us consider the limit where $q^2 \to \infty$, but $M_\perp{}^2/q^4 \to \infty$, corresponding to an extremely high $q_\perp{}^2$. Now the relevant singularities in σ and χ come from the lines $-2\sigma + \alpha_1 + \alpha_3 - \alpha_2 = 0$ and $\chi = 2\sigma - \alpha_1 - \alpha_3 - 1$. This leads to the behavior

$$\begin{aligned}\lim_{\substack{-q^2\to\infty \\ -M_\perp{}^2/q^4\to\infty}} : \lim_{R(M_\perp)} T(q^2,\nu,\nu',s) = &-\frac{1}{q^2} C(\alpha_2, 0)\mathcal{G}(\alpha_1, 0) \\ &\times \mathcal{G}(\alpha_3, 0) V(\alpha_1, \alpha_2, \alpha_3)\Gamma\left(\frac{\alpha_3 - \alpha_1 - \alpha_2}{2}\right)\Gamma\left(\frac{\alpha_1 - \alpha_2 - \alpha_3}{2}\right)\Gamma\left(\frac{\alpha_2 - \alpha_1 - \alpha_3}{2}\right) \\ &\times \Gamma(1 - \alpha_2)\omega^{(\alpha_1+\alpha_2-\alpha_3)/2}\omega'^{(\alpha_2+\alpha_3-\alpha_1)/2}(-s)^{(\alpha_1+\alpha_3-\alpha_2)/2},\end{aligned} \tag{36}$$

where

$$V(\alpha_1, \alpha_2, \alpha_3) = \tilde{V}\left(\beta = \frac{\alpha_1 + \alpha_2 - \alpha_3}{2}, \beta' = \frac{\alpha_2 + \alpha_3 - \alpha_1}{2}, \sigma = \frac{\alpha_1 + \alpha_3 - \alpha_2}{2}\right). \tag{37}$$

Once again, rather than a rapid attenuation of the amplitudes, we find a finite limit, now characterized by an unusual square root of a triple-Regge amplitude. If we take the limit $-M_\perp{}^2/q^4 \to \infty$ in (24), we obtain exactly the result (36). We conclude that this limit again is dominated by the leading-singularity contributions. Finally, we should point out that the result (36) and the $-M_\perp{}^2/q^4 \to \infty$ limit of (24) are reproduced analytically in any a, a' parameterization with $a + a' > 1$ and $a, a' > 0$.

VII. *S* LIMITS AND COMMUTATIVITY

So far we have established the primary importance of the leading light-cone singularity in any of the regions where first s goes to infinity and then q^2 becomes infinite. It would be a peculiar state of affairs to find in the S' limit that leading-singularity dominance could somehow fail. However, if one takes an arbitrary M by N by $(L+1)$-rung ladder and computes the S' limit, one finds, indeed, that up to $\ln q^2$ terms, the leading behavior is weaker, by powers of q^2, than the A-limit behavior.[2] The problem, of course, is to determine the effect of the $\ln q^2$ factors when they are summed in all leading powers of q^2. It is in regard to this problem that the result (11), so easily obtained with Halliday's observations, becomes so important. That is, since (11) preserves all the leading q^2 dependence complete with manifest Regge structure, computing the S' limit is now almost trivial.

To find the S limit, we perform a Mellin transform of (11) with respect to $-q^2$. In order to retrieve the amplitude $T(q^2, \nu, \nu', s)$, we take the four-dimensional inversion. Thus,

$$T(q^2,\nu,\nu',s) = \frac{1}{(2\pi i)^4}\int d\beta\, d\beta' d\sigma\, d\chi\, (-2\nu)^{\beta} \times (-2\nu')^{\beta'}(-s)^{\sigma}(-q^2)^{\chi} \times \tilde{T}(\chi,\beta,\beta',\sigma)\,. \tag{38}$$

$\tilde{T}(\chi,\beta,\beta',\sigma)$ is obtained by replacing $\tilde{C}_L(\beta+\beta', q^2)$ in (13) by

$$\tilde{C}_L(\beta+\beta',\chi) = \int d\xi\, \alpha_0^{\beta+\beta'+\chi} \times \prod_{i=1}^{L} \frac{\alpha_i^{\beta+\beta'+1}-1}{\beta+\beta'+1}\, \frac{\partial}{\partial\alpha_i}\left(\frac{\overline{\mathfrak{F}}}{\Delta}\right)^{\chi} \frac{e^{-J}}{\Delta^{\beta+\beta'+2}}\,, \tag{39}$$

where $\alpha_0\overline{\mathfrak{F}} = \mathfrak{F}$ as before. Now write (37) in terms of scaling variables:

$$T(q^2,\omega,\omega',\tau) = \frac{1}{(2\pi i)^4}\int d\beta\, d\beta' d\chi\, d\sigma (-q^2)^{\chi+\beta+\beta'+\sigma} \times \omega^{\beta}\omega'^{\beta'}\tau^{\sigma}\tilde{T}(\chi,\beta,\beta',\sigma)\,. \tag{40}$$

The leading behavior as $-q^2\to\infty$ is given by the leading singularity in the left half plane of the variable $\chi+\beta+\beta'+\sigma$. This leading singularity occurs at the intersection of the singularity surface $\chi+\beta+\beta'+1=0$ in (39), with the rightmost singularity in the left half of the σ plane. This latter singularity depends on β', and in the $\beta'\otimes\sigma$ space occurs at $\beta'+\sigma=\alpha_3$ and $\beta'=0$. We find

$$\lim_{S} T(q^2,\omega,\omega',\tau) = -\frac{1}{q^2}(-q^2)^{\alpha_3}\tau^{\alpha_3}h(\omega)\,, \tag{41}$$

with $h(\omega)$ of the form

$$h(\omega) = \frac{\Gamma(-\alpha_3)}{2\pi i}\int_{C_\beta} d\beta\, \frac{\Gamma(-\beta)\Gamma(\beta+1)}{(\beta+\alpha_3-\alpha_1)(\beta-\alpha_2)}\,\tilde{h}(\beta)\,. \tag{42}$$

$\tilde{h}(\beta)$ is smooth in the neighborhood of the contour C_β, where C_β passes to the left of the pole at zero in $\Gamma(-\beta)$ and to the right of the poles $\beta=\alpha_1-\alpha_3$, $\beta=\alpha_2$, and $\beta=-1$. The S' limit is now readily obtained, this limit being given by the singularity at $\beta=\alpha_1-\alpha_3$ in (42). Thus, we find

$$\lim_{S'} T(q^2,\omega,\omega',\tau) = -\frac{1}{q^2}(-q^2)^{\alpha_3}\omega^{\alpha_1-\alpha_3}\tau^{\alpha_3}V_C\,, \tag{43}$$

where V_C is the same constant as that which appears in (34). Using $\omega\omega' q^2 = \tau M_\perp^2$, (43) becomes

$$\lim_{S'} T(q^2,\omega,\omega',\tau) = -\frac{\omega^{\alpha_1}\omega'^{\alpha_3}}{q^2}\left(\frac{-M_\perp^2}{q^4}\right)^{-\alpha_3} V_C\,, \tag{44}$$

which exactly reproduces the analytic form of (34), with equality in the constant factors. Thus the commutativity assumptions made by Brandt and Preparata are indeed valid in the present model.

Finally, we note that, again, the S' limit also emphasizes the scale for $M_\perp^2\to 0$, so that any strong attenuation of the bremsstrahlung amplitudes in this limit would probably occur through the independent q^2 scale.

VIII. ANNIHILATION CONTRIBUTION TO MASSIVE LEPTON-PAIR PRODUCTION

We now discuss briefly the contributions of the annihilation amplitudes a in Fig. 2 to the vertex V_P in massive-pair production. The A and A' limits for any finite-order diagram may readily be computed by rescaling the appropriate subsets of Feynman parameters [1,2] as necessary. In the A and A' limits, these amplitudes are down by powers of q^2 from the bremsstrahlung contributions c, going as $1/q^6$ up to a $\ln q^2$ term. Here we outline the features of these amplitudes in the R and R' limits.

We begin with the formal representation (7) and (8). With the parameterization illustrated in Fig. 10, we identify the components of (8) as follows:

$$g = \Pi_\alpha[(\epsilon_1'+\epsilon_2')\Delta_M + \beta_M(\gamma_{M-1}'+\Gamma_{M-1}')\Delta_{M-1} + \cdots + \Pi_\beta] \equiv \Pi_\alpha A\,, \tag{45.1}$$

$$g_1 = \Pi_\beta[(\epsilon_1+\epsilon_2)\Delta_N + \alpha_N(\gamma_{N-1}+\Gamma_{N-1})\Delta_{N-1} + \cdots + \Pi_\alpha] \equiv \Pi_\beta B\,, \tag{45.2}$$

$$\mathfrak{F} = \left[(\epsilon_1+\epsilon_2)\Delta_N + \sum_{r=1}^{N-1}(\gamma_{N-r}+\Gamma_{N-r})\Delta_{N-r}\prod_{i=N-r}^{N}\alpha_i + \prod_{i=1}^{N}\alpha_i\right] \times\left[(\epsilon_1'+\epsilon_2')\Delta_M + \sum_{s=1}^{M-1}(\gamma_{M-s}'+\Gamma_{M-s}')\Delta_{M-s}\prod_{i=M-s}^{M}\beta_i + \prod_{i=1}^{M}\beta_i\right]\,, \tag{45.3}$$

$$F = \Pi_\alpha\Pi_\beta\,. \tag{45.4}$$

The functions Δ_M, $\Delta_{M-1}, \ldots, \Delta_N$, $\Delta_{N-1}, \ldots$, etc., correspond to the Δ functions for the ladder segment severed from the N by M-rung amplitude at the Mth, $(M-1)$st, ..., Nth, $(N-1)$st, ..., etc., side parameters, in exact analogy to the Δ functions in (9).

Now perform a Mellin transformation of the representation (7) with respect to -2ν, $-2\nu'$, and $-s$ at fixed q^2. This gives

$$\tilde{T}^{MN}(q^2,\beta,\beta',\sigma)=\int d\xi\, \Pi_\alpha{}^{\beta+\sigma}\Pi_\beta{}^{\beta'+\sigma} \times \frac{A^\beta B^{\beta'}\exp[(\mathfrak{F}q^2-\Delta J)/\Delta]}{\Delta^{\beta+\beta'+\sigma+2}} \times\Gamma(-\beta)\Gamma(-\beta')\Gamma(-\sigma), \tag{46}$$

with the usual conjugate variables. Familiar procedures expose the Mellin multipoles, and we may readily perform the sums over M and N and finally invert the result. We obtain finally, in the limit $-\nu$, $-\nu'$, $-s\to\infty$, with $M_\perp{}^2$ fixed, inclusive of all leading logarithmic terms,

$$\lim_{R(M_\perp)} T(q^2,\nu,\nu',s)=\mathcal{G}(\alpha_1,0)\mathcal{G}(\alpha_2,0) \times(-2\nu)^{\alpha_1}(-2\nu')^{\alpha_2}V_P(q^2,M_\perp{}^2), \tag{47}$$

where

$$V_P(q^2,M_\perp{}^2)=\sum_{M,N=1}^{\infty}V_P^{M,N}(q^2,M_\perp{}^2) \tag{48}$$

and

$$V_P^{MN}(q^2,M_\perp{}^2)=\frac{1}{2\pi i}\int_{C_\sigma}d\xi\prod_{i=1}^{N}\frac{\alpha_i{}^{\alpha_1(0)+1}-1}{\alpha_1(0)+1}\frac{\partial}{\partial\alpha_i}\prod_{j=1}^{M}\frac{\beta_j{}^{\alpha_2(0)+1}-1}{\alpha_2(0)+1}\frac{\partial}{\partial\beta_j}A^{-\sigma+\alpha_1(0)}B^{-\sigma+\alpha_2(0)} \times\Gamma(-\sigma)\Gamma(\sigma-\alpha_1(0))\Gamma(\sigma-\alpha_2(0))(-M_\perp{}^2)^{-\sigma}\frac{\exp[(\mathfrak{F}q^2-\Delta J)/\Delta]}{\Delta^{-\sigma+\alpha_1(0)+\alpha_2(0)+2}}\ . \tag{49}$$

The functions A, B, $\mathfrak{F}$, J, and Δ in (49) are to be computed for the M by N contracted vertex graph illustrated in Fig. 11. The contour C_σ passes between the σ-plane poles $\Gamma(\sigma-\alpha_1)$, $\Gamma(\sigma-\alpha_2)$ and the poles in $\Gamma(-\sigma)$. In contrast to the case for the bremsstrahlung contributions to V_P in (11), the first σ singularity to the right of C_σ is from the $\sigma=0$ pole in $\Gamma(-\sigma)$. Therefore, as $-M_\perp{}^2\to\infty$ in (49), we obtain the constant leading behavior

$$\lim_{-M_\perp{}^2\to\infty}V_P^{MN}(q^2,M_\perp{}^2)=\int d\xi\prod_{i=1}^{N}\frac{\alpha_i{}^{\alpha_1(0)+1}-1}{\alpha_1(0)+1}\frac{\partial}{\partial\alpha_i}\prod_{j=1}^{M}\frac{\beta_j{}^{\alpha_2(0)+1}-1}{\alpha_2(0)+1}\frac{\partial}{\partial\beta_j} \times A^{\alpha_1(0)}B^{\alpha_2(0)}\Gamma(-\alpha_1(0))\Gamma(-\alpha_2(0))\frac{\exp[(\mathfrak{F}q^2-\Delta J)/\Delta]}{\Delta^{\alpha_1(0)+\alpha_2(0)+2}}\ . \tag{50}$$

That is, the annihilation amplitudes remain finite and become independent of $M_\perp{}^2$ at large $M_\perp{}^2$. Thus, no root-triple-Regge behavior obtains in the large-$M_\perp{}^2$ region for the annihilation amplitudes, in marked contrast to the bremsstrahlung amplitudes.

Consider now the $-q^2\to\infty$ limit in (50). The leading behavior is obtained from the coefficient of the terms $\alpha_N/[\alpha_1(0)+1]$ and $\beta_M/[\alpha_2(0)+1]$. The integration over α_N and β_M may be done for this coefficient, and the two ladder rungs α_N,β_M get contracted, as illustrated in Fig. 12. The coefficient of q^2 in this term, $\mathfrak{F}(\alpha_N=\beta_M=0)$, has two independent minimal scaling sets,[13] each with two members: $\{\epsilon_1,\epsilon_2\}$ and $\{\epsilon_1',\epsilon_2'\}$. These combine with the factors $(\epsilon_1+\epsilon_2)^{\alpha_2(0)}$ and $(\epsilon_1'+\epsilon_2')^{\alpha_1(0)}$ in (50) [see (45)] to give $[(-q^2)^{-2-\alpha_1(0)}+(-q^2)^{-2-\alpha_2(0)}]\ln(q^2/\mu^2)$ behavior. Thus

$$\lim_{-q^2\to\infty}:\lim_{\substack{-M_\perp{}^2\to\infty\\ q^2\text{ fixed}}}T(q^2,\nu,\nu',s)=\mathcal{G}[\alpha_1(0),0]\mathcal{G}[\alpha_2(0),0]\ln(q^2/\mu^2)\left[\frac{\omega^{\alpha_1(0)}(-2\nu')^{\alpha_2(0)}}{q^4}V_1+\frac{(-2\nu)^{\alpha_1(0)}\omega'^{\alpha_2(0)}}{q^4}V_2\right], \tag{51}$$

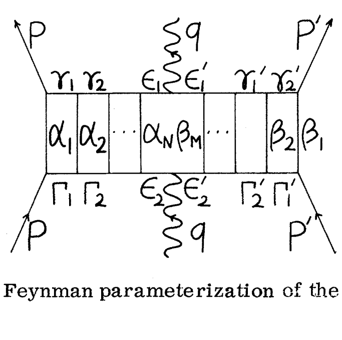

FIG. 10. Feynman parameterization of the annihilation ladder.

where V_1 and V_2 are just constants. Thus, whereas in the A limit the annihilation amplitudes have q^{-6} ($\simeq x^2$) behavior near the light cone, in the R' limit they have q^{-4} ($\simeq\ln x^2$) behavior. The effective strength of the light-cone singularity has been enhanced in the R limit. However, its net contribution is still weaker by a factor of q^2 relative to the bremsstrahlung contributions. The leading singularity, which comes from the bremsstrahlung amplitudes, indeed dominates in all the $q^2\to\infty$ limits.

IX. SEMI-INCLUSIVE ELECTROPRODUCTION

The process here is virtual photon (q) + hadron $(p) \to$ hadron (p') + anything, for which the inclusive cross section is obtained as a discontinuity in the six-point forward scattering amplitude $T(q^2, \nu, \nu', s)$. For this process, the relevance of the leading light-cone singularity is not subject to question as in massive lepton-pair production because the corresponding invariant s need not be large always. Translation invariance leads to

$$s + 2\nu - 2\nu' + q^2 \geq 0,$$

and clearly does not preclude light-cone limits at finite s.

On the other hand, it would be quite useful to know whether the leading singularity continues to dominate perhaps even at large s. Therefore we study here the possible connection between the A' limit and the R' limit for this process in the central (double-Regge) region for p' production. We imagine that p' emerges from the vertex V_P in Fig. 1, i.e., let $c \to -p'$, $p \to q$, and $p' \to p$. Then, up to crossing, which changes no essential results, the appropriate amplitudes are those of Fig. 2(d), which correspond to p' production through an annihilation, plus amplitudes (not illustrated) where p' emerges from rungs of the ladder as in the bremsstrahlung production of q in Figs. 2(b) and 2(c). The structure of V_P does not play an essential role in the present considerations, however, so that we restrict our attention to the purely annihilation contributions to production of p'.

Consider, then, the M by N-rung amplitude of Fig. 2(d). The usual formal representations (7) and (8) obtain, and for the A limit the Mellin transform is given by

$$\tilde{T}^{MN}(\chi, \omega, \omega', s) = \Gamma(-\chi) \int d\xi \left(\frac{\mathfrak{F} + \omega g + \omega' g_1}{\Delta} \right)^{\chi} \times \frac{\exp[(Fs - \Delta J)/\Delta]}{\Delta^2}. \tag{52}$$

A study of the coefficient $\mathfrak{F} + \omega g + \omega' g_1$ of q^2 reveals that there is a single over-all multiplicative Feynman parameter α_0 corresponding to the propagator line connecting the current insertions. Therefore we find

$$\lim_{A} T^{MN}(q^2, \omega, \omega', s) = -\frac{1}{q^2} T_R^{MN}(\omega, \omega', s), \tag{53}$$

with

$$T_R^{MN}(\omega, \omega', s) = \int \frac{d\xi}{d\alpha_0} \frac{\Delta^{-1}}{\mathfrak{F} + \omega g + \omega' g_1} \times \exp[(Fs - \Delta J)/\Delta] \Big|_{\alpha_0 = 0}. \tag{54}$$

The same sequence of steps which led to (24) lead now to

$$\lim_{\substack{A' \\ \omega' s/\omega \text{ fixed}}} T(q^2, \nu, \nu', s) = C(\alpha_1, 0) \mathcal{G}(\alpha_2, 0) V_P(\omega' s/\omega) \omega'^{\alpha_1} (-s)^{\alpha_2}, \tag{55}$$

where $C(\alpha_1, 0)$ is the same constant present in (35), and

$$V_P(\omega' s/\omega) = \frac{1}{2\pi i} \sum_{M,N} \int_{C_\beta} \tilde{V}^{MN}(\beta, \beta' = -\beta - \alpha_1, \sigma = -\beta - \alpha_2, \sigma)(-\omega' s/\omega)^{-\beta}, \tag{56}$$

where

$$\tilde{V}_P^{MN}(\beta, \beta', \sigma) = \int d\xi \prod_{i=1}^{N} \frac{\alpha_i^{\beta'+\beta+1} - 1}{\beta + \beta' + 1} \frac{\partial}{\partial \alpha_i} \prod_{j=1}^{M} \frac{\beta_j^{\beta+\sigma+1} - 1}{\beta + \sigma + 1} \frac{\partial}{\partial \beta_j} A^{\beta'} B^{\sigma} \Gamma(-\sigma) \Gamma(-\beta) \Gamma(-\beta') \frac{e^{-J}}{\Delta^{\beta + \beta' + \sigma + 2}}, \tag{57}$$

with definitions analogous to those in previous cases.

Consider now the R' limit for T. The calculation is not different from others, and we find

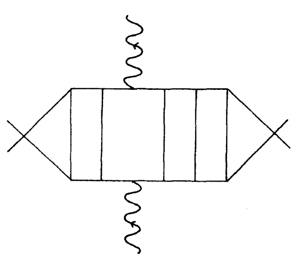

FIG. 11. Annihilation contributions to the vertex V_P.

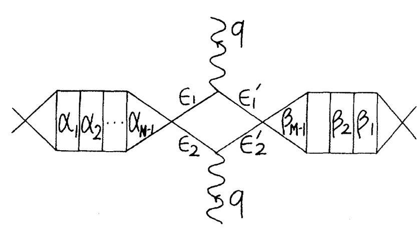

FIG. 12. Annihilation contributions, contracted for $-q^2 \to \infty$.

$$\lim_{\substack{R' \\ -\nu' s/\nu \text{ fixed}}} T(q^2,\nu,\nu',s) = \lim_{\substack{A' \\ \omega' s/\omega \text{ fixed}}} T(q^2,\nu,\nu',s)$$

$$= -C(\alpha_1, 0)\mathcal{G}(\alpha_2, 0)V_P(\omega' s/\omega)\frac{\omega'^{\alpha_1}(-s)^{\alpha_2}}{q^2} . \tag{58}$$

This is analytically equivalent to (55). The commutativity is readily traced to the singularity coming from the α_0 Mellin space pole. That is, the mechanism for commutativity is equivalent graphically to the fixed-pole mechanism of electroproduction.

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APPENDIX

We consider here the causal (covariant) tensor amplitude $T_{\mu\nu}$ for vector weak currents

$$T_{\mu\nu}(q^2,\nu,\nu',s)$$
$$= 4p_0p_0'\int d^4\chi\, e^{iq\cdot x}\,{}_{\rm in}\langle pp'|\theta(x_0)[J_\mu(x),J_\nu(0)]|pp'\rangle_{\rm in} . \tag{A1}$$

The most general form of this tensor is given by the expansion

$$T_{\mu\nu} = A_1 q_\mu q_\nu + A_2 q_\mu q_\nu + A_3 q_\mu p_\nu' + A_4 p_\mu q_\nu + A_5 p_\mu' q_\nu + A_6 p_\mu p_\nu + A_7 p_\mu p_\nu' + A_8 p_\mu' p_\nu + A_9 p_\mu' p_\nu' + A_{10} g_{\mu\nu} . \tag{A2}$$

Crossing symmetry in the photon indices reduces the number of independent amplitudes to seven, while gauge invariance brings in three further constraints. Therefore the number of independent invariants is reduced to four, and some algebra shows that we may write

$$T_{\mu\nu} = \frac{T_{2A}}{m^2}\left(p-\frac{\nu}{q^2}q\right)_\mu\left(p-\frac{\nu}{q^2}q\right)_\nu + \frac{T_{2B}}{m^2}\left(p'-\frac{\nu'}{q^2}q\right)_\mu\left(p'-\frac{\nu'}{q^2}q\right)_\nu + \frac{T_{2C}}{2m^2}\left\{\left(p-\frac{\nu}{q^2}q\right)_\mu\left(p'-\frac{\nu'}{q^2}q\right)_\nu + (p \leftrightarrow p')_{\mu\nu}\right\} + T_1\left(\frac{q_\mu q_\nu}{q^2}-g_{\mu\nu}\right) , \tag{A3}$$

where for dimensional reasons we have introduced the boson mass m^2. For a specific current (lepton-pair) polarization ϵ_μ^r, the forward amplitude becomes

$$T_r = \epsilon_\mu^r T_{\mu\nu}\epsilon_\nu^r . \tag{A4}$$

If the polarization of the current is not measured, completeness gives

$$T = \sum_r T_r = g_{\mu\nu}T_{\mu\nu} = \mathrm{Tr}T_{\mu\nu} = \frac{\nu^2}{q^2}T_{2A} + \frac{\nu'^2}{q^2}T_{2B} + \left(s - \frac{\nu\nu'}{q^2}\right)T_{2C} + 3T_1 \tag{A5}$$

whenever $s \gg m^2$.

For illustration, we analyze the M by N annihilation ladder diagram shown in Fig. 13. The usual Feynman rules give for this amplitude

$$T_{\mu\nu} = \int \prod_{i=1}^{M+N+1} d^4 z_i \prod_{j=1}^{3(M+N)-2} \frac{(2q-k)_\mu(2q-k)_\nu}{k_j^2 - \mu_j^2 + i\epsilon} , \tag{A6}$$

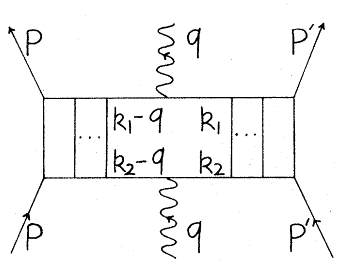

FIG. 13. Annihilation ladder for vector currents.

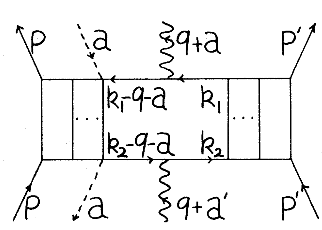

FIG. 14. Introduction of auxiliary moments.

$$T_{\mu\nu}=\int\prod_{i=1}^{M+N+1}d^4z_i\int d\xi\,(2q-k)_\mu(2q-k)_\nu\exp\Big[\sum_j\xi_j(k_j{}^2-\mu_j{}^2+i\epsilon)\Big], \tag{A7}$$

where we have introduced the same parametric set $\{\xi\}$ as given in Fig. 10. Let us introduce auxiliary momenta[19] a and a', coupled into T as illustrated in Fig. 14. Then we may write

$$T_{\mu\nu}=\int\prod_i d^4z_i\int d\xi\Big(\frac{\nabla_a}{\epsilon_1}-q\Big)_\mu\Big(\frac{\nabla_{a'}}{\epsilon_2}-q\Big)_\nu\exp\Big[\sum_j(k_j{}^2-\mu_j{}^2+i\epsilon)\Big]\Big|_{\substack{q\to q+a\\ q\to q+a'}}\Big|_{a,a'=0} \tag{A8}$$

The integrations may be done,[13] and we obtain

$$T_{\mu\nu}=\int d\xi\Big(\frac{\nabla_a}{\epsilon_1}-q\Big)_\mu\Big(\frac{\nabla_{a'}}{\epsilon_2}-q\Big)_\nu\frac{e^{D(a,a')/\Delta}}{\Delta^2}\Big|_{a=a'=0}, \tag{A9}$$

where the discriminant $D(a,a')$ and the determinant Δ are to be computed for the amplitude of Fig. 14. Letting

$$D(a,a')=D(0)+a_\mu A_\mu+a'_\nu A_\nu+a\cdot a'B+a^2C+a'^2C', \tag{A10}$$

we obtain

$$T_{\mu\nu}=\int d\xi\left[\Big(\frac{A_\mu}{\epsilon_1\Delta}-q_\mu\Big)\Big(\frac{A'_\nu}{\epsilon_2\Delta}-q_\nu\Big)+\frac{Bg_{\mu\nu}}{\epsilon_1\epsilon_2\Delta}\right]\frac{e^{D(0)/\Delta}}{\Delta^2}. \tag{A11}$$

Formally, from standard prescription,[13] we have

$$\begin{aligned}D(a,a')=&F_0(p+p'+q)^2+(F_5+F_6)(p+q)^2+(F_7+F_8)(p'+q)^2+(F_9+F_{10}+F_{11}+F_{12})q^2\\&+(F_1+F_2+F_3+F_4)m^2+(F_{13}+F_{21})a'^2+(F_{14}+F_{22})a^2+(F_{15}+F_{24})(q+a)^2+(F_{16}+F_{23})(q+a')^2\\&+F_{17}(p-a')^2+F_{18}(p-a)^2+F_{19}(p'+q+a')^2+F_{20}(p'+q+a)^2+F_{25}(a-a')^2-\Delta\sum_j\mu_j{}^2\xi_j\,.\end{aligned} \tag{A12}$$

We identify

$$A_\mu=2(F_{20}p'_\mu-F_{18}p_\mu)+2(F_{20}+F_{24}+F_{15})q_\mu\,, \tag{A13a}$$

$$A_\nu=2(F_{19}p'_\nu-F_{17}p_\nu)+2(F_{16}+F_{19}+F_{23})q_\nu\,. \tag{A13b}$$

Therefore

$$T_{2A}\propto\int\frac{F_{17}F_{18}}{\epsilon_1\epsilon_2}\frac{e^{D(0)/\Delta}}{\Delta^4}d\xi\,, \tag{A14a}$$

$$T_{2B}\propto\int\frac{F_{19}F_{20}}{\epsilon_1\epsilon_2}\frac{e^{D(0)/\Delta}}{\Delta^4}d\xi\,, \tag{A14b}$$

$$T_{2C}\propto\int\frac{F_{18}F_{19}+F_{17}F_{20}}{\epsilon_1\epsilon_2}\frac{e^{D(0)/\Delta}}{\Delta^4}d\xi\,, \tag{A14c}$$

$$T_1\propto\int\frac{F_{25}}{\epsilon_1\epsilon_2}\frac{e^{D(0)/\Delta}}{\Delta^3}d\xi\,. \tag{A14d}$$

In terms of the parameterization of Fig. 10, with functions Δ_M, Δ_{M-1}, etc., defined as before, we have

$$\begin{aligned}&F_{17}=\Pi_\alpha\Delta_M\epsilon_2\,, &&F_{20}=\Pi_\beta\Delta_N\epsilon_2\,,\\&F_{18}=\Pi_\alpha\Delta_M\epsilon_1\,, &&F_{25}=\epsilon_1\epsilon_2\Delta_M\Delta_N\,;\\&F_{19}=\Pi_\beta\Delta_N\epsilon_1\,,\end{aligned} \tag{A15}$$

when these are put into (A14) we see the T_i are symmetric between the two sets of ladder parameters. We now employ the formal representation (8) for D, and compute the Mellin transform of (A14), with respect to -2ν, $-2\nu'$, and $-s$, at fixed $q^2<0$. This gives for the M by N rung ladder

$$\tilde{T}^{MN}_{2A}(q^2,\beta,\beta',\sigma)\propto\int d\xi\,(\Pi_\alpha\Delta_M)^2g^\beta g_1^{\beta'}F^\sigma\frac{\exp[(\mathcal{F}q^2-\Delta J)/\Delta]}{\Delta^{\beta+\beta'+\sigma+4}}\Gamma(-\beta)\Gamma(-\sigma)\Gamma(-\beta')\,, \tag{A16a}$$

$$\tilde{T}^{MN}_{2B}(q^2,\beta,\beta',\sigma)\propto\int d\xi\,(\Pi_\beta\Delta_N)^2g^\beta g_1^{\beta'}F^\sigma\frac{\exp[(\mathcal{F}q^2-\Delta J)/\Delta]}{\Delta^{\beta+\beta'+\sigma+4}}\Gamma(-\beta)\Gamma(-\sigma)\Gamma(-\beta')\,, \tag{A16b}$$

$$\tilde{T}^{MN}_{2C}(q^2,\beta,\beta',\sigma)\propto\int d\xi\,\Pi_\alpha\Pi_\beta\Delta_M\Delta_N g^\beta g_1^{\beta'}F^\sigma\frac{\exp[(\mathcal{F}q^2-\Delta J)/\Delta]}{\Delta^{\beta+\beta'+\sigma+4}}\Gamma(-\beta)\Gamma(-\sigma)\Gamma(-\beta')\,, \tag{A16c}$$

$$\tilde{T}_1^{MN}(q^2,\beta,\beta',\sigma)\propto\int d\xi\,\Delta_M\Delta_N g^\beta g_1^{\beta'}F^\sigma\frac{\exp[(\mathfrak{F}q^2-\Delta J)/\Delta]}{\Delta^{\beta+\beta'+\sigma+3}}\,\Gamma(-\beta)\Gamma(-\sigma)\Gamma(-\beta'),\tag{A16d}$$

where g, g_1, and F are of the same form as in (45). The usual sums over M and N may be done and we obtain

$$\tilde{T}_I(q^2,\beta,\beta',\sigma)=\sum_{M,N}T_I^{M,N}(q^2,\beta,\beta',\sigma)$$
$$=\frac{\mathcal{G}(\beta+\sigma+S_1(I),0)\mathcal{G}(\beta'+\sigma+S_2(I),0)\tilde{V}_P(q^2,\beta,\beta',\sigma,I)}{[\sigma+\beta-\alpha_1(I)][\sigma+\beta'-\alpha_2(I)]}\,,\tag{A17}$$

where $I=2A$, $2B$, $2C$, or 1 and $\alpha_i(I)=\alpha_i-S_i(I)$, $i=1,2$ and $S_1(2A)=2$, $S_2(2A)=0$, $S_1(2B)=0$, $S_2(2B)=2$, $S_1(2C)=S_2(2C)=1$, $S_1(1)=S_2(1)=0$. Of course, α_1 and α_2 are just the usual scalar Regge trajectory.[13] $\mathcal{G}(x,t)$ is the usual scalar residue function[13] and

$$\tilde{V}_P(q^2,\beta,\beta',\sigma,I)=\sum_{M,N=1}\tilde{V}^{MN}(q^2,\beta,\beta',\sigma,I),\tag{A18}$$

with, for example,

$$\tilde{V}_P^{MN}(q^2,\beta,\beta',\sigma,2A)=\int d\xi\prod_{i=1}^{N}\frac{\alpha_i^{\sigma+\beta+3}-1}{\sigma+\beta+3}\frac{\partial}{\partial\alpha_i}\prod_{j=1}^{M}\frac{\beta_j^{\sigma+\beta'+1}-1}{\sigma+\beta'+1}\frac{\partial}{\partial\beta_j}$$
$$\times\Big\{\Delta_M^2[(\epsilon_3+\epsilon_4)\Delta_M+\beta_M(\gamma'_{M-1}+\Gamma'_{M-1})\Delta_{M-1}+\cdots+\Pi_\beta]^\beta$$
$$\times[(\epsilon_1+\epsilon_2)\Delta_N+\alpha_N(\gamma_{N-1}+\Gamma_{N-1})\Delta_{N-1}+\cdots+\Pi_\alpha]^{\beta'}$$
$$\times\Gamma(-\beta)\Gamma(-\beta')\Gamma(-\sigma)\frac{\exp[(\mathfrak{F}q^2-\Delta J)/\Delta]}{\Delta^{\beta+\beta'+\sigma+4}}\Big\},\tag{A19}$$

while the other $\tilde{V}_{(I)}^{MN}$ have similar representations. The constituent functions $\mathfrak{F}$, Δ, J, etc., for given M,N are to be computed for the contracted vertices of Fig. 11. The only new feature, we note, is the relocation of the Regge poles in (A17), due to the spin of the currents. Upon inversion, we may obtain, for example, the $R(M_\perp)$ limit

$$\lim_{R(M_\perp)}T_I(q^2,\nu,\nu',s)=(-2\nu)^{\alpha_1(I)}(-2\nu')^{\alpha_2(I)}\mathcal{G}(\alpha_1,0)\mathcal{G}(\alpha_2,0)V_P(q^2,M_\perp{}^2,I),\tag{A20}$$

with

$$V_P(q^2,M_\perp{}^2)=\int_{C_\sigma}\tilde{V}_P(q^2,\beta=\alpha_1(I)-\sigma,\beta'=\alpha_2(I)-\sigma,\sigma,I)\Gamma(\sigma-\alpha_1(I))\,\Gamma(\sigma-\alpha_2(I))\Gamma(-\sigma)(-M_\perp{}^2)^{-\sigma},\tag{A21}$$

where C_σ passes between the poles at $\sigma=\alpha_1(I)$, $\alpha_2(I)$ and those at $\sigma=0$. Inserting (A20) into (A5) we determine finally

$$\lim_{R(M_\perp)}T(q^2,\nu,\nu',s)=\mathcal{G}(\alpha_1,0)\mathcal{G}(\alpha_2,0)\Big\{(-2\nu)^{\alpha_1}(-2\nu')^{\alpha_2}[V_P(q^2,M_\perp{}^2,2A)/q^2+V_P(q^2,M_\perp{}^2,2B)/q^2+3V_P(q^2,M_\perp{}^2,1)]$$
$$+\left(s-\frac{2\nu\nu'}{q^2}\right)(-2\nu)^{\alpha_1-1}(-2\nu')^{\alpha_2-1}V_P(q^2,M_\perp{}^2,2C)\Big\}.\tag{A22}$$

If we take $-M_\perp{}^2\to\infty$ at fixed q^2 in (A20), the relevant σ singularity is the pole in $\Gamma(-\sigma)$ at $\sigma=0$. That is, this limit gives again constant leading behavior, with no root-triple-Regge behavior, in contrast to the bremsstrahlung amplitudes. Taking subsequently $-q^2\to\infty$, we obtain finally

$$\lim_{-q^2\to\infty}V_P(q^2,\infty,I)=(-q^2)^{-\alpha_i}V_P(I)\tag{A23}$$

for $I=2A$ or $2B$, with $\alpha_i=\alpha_1$ for $2A$ and $\alpha_i=\alpha_2$ for $2B$. The $V_P(I)$ are just constant residue factors. These are the leading terms in (A22), which now becomes

$$\lim_{-q^2\to\infty}:\lim_{\substack{-M_\perp{}^2\to\infty\\ q^2\text{ fixed}}}:\lim_{R(M_\perp)}T(q^2,\nu,\nu',s)=\mathcal{G}(\alpha_1,0)\mathcal{G}(\alpha_2,0)\left[\frac{\omega^{\alpha_1}(-2\nu')^{\alpha_2}}{q^2}V_P(2A)+\frac{(-2\nu)^{\alpha_1}\omega'^{\alpha_2}}{q^2}V_P(2B)\right].\tag{A24}$$

Up to $\ln q^2$ terms, we see upon comparison with (51) that the essential difference is the single power of q^2, introduced by spin factors.

A representation for the bremsstrahlung contributions to T may be obtained, and the result is similar to (A20). The limit $-q^2 \to \infty$, at any rate such that $M_\perp^2/q^4 \to 0$, again probes the $M_\perp^2 \to 0$ scale, while if $M_\perp^2/q^4 \to \infty$ we obtain again the behavior (36), up to one power of q^2. In every case, the leading light-cone singularity dominates the amplitudes in all regions at large q^2.

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[16]Actually, until we need in some asymptotic limit to take account of the Mellin singularities brought in by the $(L+1)$-rung ladder, we may just as well wait to perform the sum over L, and work at fixed L.

[17]If we work at fixed L (Ref. 16), we would not have the factorization. There is no loss in generality, however.

[18]Attenuation of the amplitudes is known to be exponential. See, e.g., E. W. Anderson *et al.*, Phys. Rev. Lett 16, 855 (1966); D. G. Crabb *et al.*, *ibid.* 21, 830 (1968); J. L. Day *et al.*, *ibid.* 23, 1055 (1969); D. B. Smith *et al.*, *ibid.* 23, 1064 (1969).

[19]I. G. Halliday, Nuovo Cimento 51A, 971 (1967).