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Motion of an Electron in a Homogeneous Magnetic Field—Modified Propagation Function and Synchrotron Radiation*

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The method presented by Schwinger to obtain an exact solution (to order α) of the inverse modified electron propagation function in a homogeneous magnetic field is reviewed. Applications of his general result to obtain the magnetic susceptibility, the anomalous magnetic moment, the decay rate, and the power spectrum of radiation are discussed.

I. INTRODUCTION

The origin of nonthermal galactic radiation is believed to be due to the synchrotron radiation of charged cosmic particles moving in magnetic fields.[1] Recent interest in pulsars suggested that magnetic fields of the order 10^{12}–10^{14} gauss probably exist in neutron stars and their illumination might be attributed to the synchrotron radiation.[2] Thus, an investigation of radiation from charged particles at such high magnetic fields will be of great importance to astrophysics. Historically, the study of synchrotron radiation is intimately connected with the development of cyclotrons and storage rings.[3] Furthermore, the study of the motion of charged particles in a constant magnetic field has been of theoretical interest because such motion can be solved exactly[4,5] and because investigations in this area might lead to an understanding of strong coupling calculations. New interest in studying synchrotron radiation arises from the proposed experiment[6] at the National Accelerator Laboratory by the combination of megagauss magnetic fields[7] (1.5–3.0 MG) with high-energy electrons (150–300 BeV).

In the conventional treatment of the synchrotron-radiation problem,[3,8] the wave functions for an electron in a homogeneous magnetic field are utilized to calculate the transition amplitudes for one-photon emission. The spectral and angular distributions are then obtained by squaring the amplitudes and summing over the final states. The algebra involved is quite complicated[9] and so far only the weak-field cases have been discussed. Other related problems are the calculations of the magnetic susceptibility[10–12] and the magnetic moment of electron.[10,11,13,14] These are obtained by studying the modified propagation function in a magnetic field, either by power-series expansion in H (to order H^3) (see Ref. 10) or by a summation over the intermediate states through the use of electron wave functions.[14] In the latter method, a closed form is obtained only for the weak- and extremely high-magnetic-field cases.

One might ask whether it is possible to bypass the use of the electron wave function and to solve for the modified propagation function in a homogeneous magnetic field exactly, so that it provides not only the mass shifts and the anomalous magnetic moment but also the general expression for the power spectrum of radiation. The answer to this question has been given affirmatively by Schwinger for both the spin-0 (Ref. 15) and the spin-$\frac{1}{2}$ (Ref. 16) cases through the use of the proper-time method[4,17] and the replacement of the photon-momentum integration by an algebraic procedure.[17]

The purpose of this paper is to illustrate how to apply the exact result of Ref. 16 to some specific problems. Because it will be some time before Ref. 16 becomes available to the reader, we will first, in Sec. II, review Schwinger's calculation of the lowest-order radiative corrections to the electron propagation function in a constant field. Applications of the general result to the magnetic moment, magnetic susceptibility, and decay rate are discussed in Sec. III. In Sec. IV, by a slight modification of the calculational method given in Sec. II, the power spectrum of radiation is obtained exactly (to order α) and an application of this result to the synchrotron radiation (high-energy particles in weak magnetic field)[3,8,18–20] is also presented in this section. The evaluation of various expectation values and energy eigenvalues are presented in the Appendix.

II. METHOD OF APPROACH

In this section, we review Schwinger's calculation of the modified electron propagation function in a constant field given in Ref. 16. The starting point is the additional action term associated with single-photon exchange[21]:

$$-\tfrac{1}{2}\int (dx)(dx')\psi(x)\gamma^0 M(x,x')\psi(x'), \tag{1}$$

a contribution that alters the spin-$\frac{1}{2}$ Green's function expression into

$$(m+\gamma\Pi)\overline{G}_+(x,x')+\int (dx'')M(x,x'')\overline{G}_+(x'',x') = \delta(x-x'). \tag{2}$$

Here, in a matrix notation, we have

$$M = ie^2\int \frac{(dk)}{(2\pi)^4}\gamma^\mu \frac{1}{k^2}\frac{1}{m+\gamma(\Pi-k)}\gamma_\mu + \text{c.t.}, \tag{3}$$

with

$$\Pi_\mu = \frac{1}{i}\partial_\mu - eqA_\mu \;. \tag{4}$$

The contact terms (c.t.), which will be determined later, are linear functions of $\gamma\Pi$ that are designed to satisfy the propagation-function normalization conditions.

The program involves constructing $M(x,x')$ exactly. This is made possible by the application of two devices[15–17]: the proper-time technique and the replacement of the photon-momentum integration by an algebraic procedure. The first will become apparent when one uses an exponential representation for the combination of two propagation functions:

$$\frac{1}{k^2}\frac{1}{(\Pi-k)^2+m^2-eq\sigma F} = -\int_0^\infty s\,ds\int_0^1 du\, e^{-is\chi(u)}, \tag{5}$$

where

$$\chi(u) = (k-u\Pi)^2 + u(1-u)[m^2-(\gamma\Pi)^2] + u^2(m^2-eq\sigma F), \tag{6}$$

$$\sigma F \equiv \tfrac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}\;. \tag{7}$$

The second is the transformation of the k integration into a matrix form by the use of a variable ξ_μ that is complementary to k_μ,

$$[\xi_\mu, k_\nu] = ig_{\mu\nu}, \tag{8}$$

and by observing that (ξ' is the eigenvalue of ξ)

$$\int \frac{(dk)}{(2\pi)^4} f(k) = \langle \xi'=0|f(k)|\xi'=0\rangle. \tag{9}$$

Using these two representations [Eqs. (5) and (9)], one can then convert Eq. (2) in the form

$$M = -ie^2\int s\,ds\,du\,\gamma^\mu\langle e^{-is\chi(u)}[m-\gamma(\Pi-k)]\rangle\gamma_\mu + \text{c.t.}, \tag{10}$$

where the expectation value refers to the $\xi'=0$ state.

The evaluation of $\langle e^{-is\chi}\rangle$ can be performed by considering the analogy with the quantum-mechanical system having the variable s as the proper time and χ as the Hamiltonian. The "time" evolution of any operator, say O, in this system is then described by

$$O(s) = e^{is\chi}Oe^{-is\chi}, \tag{11}$$

which obeys the equation of motion

$$\frac{d}{ds}O(s) = \frac{1}{i}[O(s),\chi]. \tag{12}$$

Here, the full set of equations of motion to be used in our calculation is

$$\frac{d}{ds}k(s) = 0, \tag{13}$$

$$\frac{d}{ds}\xi(s) = 2[k-u\Pi(s)], \tag{14}$$

$$\frac{d}{ds}\Pi(s) = 2ueqF[\Pi(s)-k]. \tag{15}$$

Note that it is the constancy of the external field F that enables one to solve the above linear equations exactly. They are the same equations as those encountered in Ref. 15, and the solutions are (in the matrix notation)

$$k(s) = k, \tag{16}$$

$$\Pi(s) - k = (1+A)(\Pi-k), \tag{17}$$

$$eqF(\xi(s)-\xi) = Dk - A\Pi, \tag{18}$$

where we have defined

$$A = e^{2ueqFs} - 1, \tag{19}$$

$$D = A + 2(1-u)eqFs\;. \tag{20}$$

Now the expectation values, $\langle e^{-is\chi}k\rangle$ and $\langle e^{-is\chi}kk\rangle$, are related to $\langle e^{-is\chi}\rangle$ as follows: By using Eqs. (8) and (18), we have

$$\langle e^{-is\chi}k\rangle = \left\langle e^{-is\chi}\left[\left(\frac{A}{D}\Pi\right) + \frac{eqF}{D}[\xi(s)-\xi]\right]\right\rangle = \langle e^{-is\chi}\rangle\left(\frac{A}{D}\Pi\right) \tag{21}$$

and

$$\begin{aligned}\langle e^{-is\chi}k_\mu k_\nu\rangle &= \left\langle e^{-is\chi}\left(\frac{A}{D}\Pi + \frac{eqF}{D}[\xi(s)-\xi]\right)_\mu k_\nu\right\rangle \\ &= \langle e^{-is\chi}k_\nu\rangle\left(\frac{A}{D}\Pi\right)_\mu - \langle e^{-is\chi}\rangle i\left(\frac{eqF}{D}\right)_{\mu\nu} \\ &= \langle e^{-is\chi}\rangle\left[\left(\frac{A}{D}\Pi\right)_\nu\left(\frac{A}{D}\Pi\right)_\mu - i\left(\frac{eqF}{D}\right)_{\mu\nu}\right],\end{aligned} \tag{22}$$

where we have used the relation [cf. Eq. (11)]

$$e^{-is\chi}\,\xi(s)=\xi e^{-is\chi}.$$

The main task of evaluating $\langle e^{-is\chi}\rangle$ can now be performed by constructing a differential equation

$$i\frac{\partial}{\partial s}\langle e^{-is\chi}\rangle=\langle e^{-is\chi}\chi\rangle =\langle e^{-is\chi}k^2\rangle-2u\langle e^{-is\chi}k\rangle\Pi +\langle e^{-is\chi}\rangle u(m^2+\Pi^2-eq\sigma F)\,. \tag{23}$$

The substitution of Eqs. (21) and (22) into Eq. (23) then yields

$$i\frac{\partial}{\partial s}\ln\langle e^{-is\chi}\rangle=\Pi\left(\frac{A}{D}\right)^T\left(\frac{A}{D}\right)\Pi-2u\Pi\left(\frac{A}{D}\right)^T\Pi -i\,\mathrm{tr}\left(\frac{eqF}{D}\right)+u(\Pi^2+m^2-eq\sigma F), \tag{24}$$

which is also the differential form for the spin-0 case except the $u(m^2-eq\sigma F)$ term. Following the arguments given in Ref. 15, one has

$$\langle e^{-is\chi}\rangle=-\frac{i}{(4\pi)^2}\frac{1}{s^2}\left(\det\frac{2eqFs}{D}\right)^{1/2}e^{-is\Phi}, \tag{25}$$

where

$$\Phi=u(\Pi^2+m^2-eq\sigma F)+\Pi\left[-\frac{1}{2eqFs}\ln\left(-\frac{D}{D^T}\right)\right]\Pi \equiv\Phi_1-ueq\sigma F. \tag{26}$$

By substituting Eq. (25) together with Eq. (21) into Eq. (10), we obtain

$$M=-\frac{\alpha}{4\pi}\int\frac{ds}{s}du\left(\det\frac{2eqFs}{D}\right)^{1/2}e^{-is\Phi_1}\gamma^\mu e^{isueq\sigma F}\times\left(m-\gamma\frac{2(1-u)eqFs}{D}\Pi\right)\gamma_\mu+\text{c.t.} \tag{27}$$

To simplify the Dirac algebra, we apply the proper-time technique to γ_μ by defining

$$\gamma_\mu(s)=e^{-isueq\sigma F}\gamma_\mu e^{isueq\sigma F}, \tag{28}$$

which satisfies the equation of motion

$$\frac{d}{ds}\gamma(s)=2ueqF\gamma(s), \tag{29}$$

since

$$[\gamma,\sigma F]=-2iF\gamma\,. \tag{30}$$

The solution of Eq. (29) is

$$\gamma(s)=(1+A)\gamma =\gamma(1+A^T) \tag{31}$$

which implies that

$$\gamma e^{isueq\sigma F}=e^{isueq\sigma F}\gamma(1+A^T)\,. \tag{32}$$

When we combine this with the rearrangement

$$\left(m-\gamma\frac{2(1-u)eqFs}{D}\Pi\right)\gamma=\gamma\left(m+\gamma\frac{2(1-u)eqFs}{D}\Pi\right) +2\frac{2(1-u)eqFs}{D}\Pi\,, \tag{33}$$

Eq. (27) can be simplified to the form (with $\sigma A\equiv\frac{1}{2}\sigma_{\mu\nu}A^{\mu\nu}$)

$$M=-\frac{\alpha}{4\pi}\int\frac{ds}{s}du\left(\det\frac{2eqFs}{D}\right)^{1/2}e^{-is\Phi}\left[(-4-\mathrm{tr}A+2i\sigma A)\left(m+\gamma\frac{2(1-u)eqFs}{D}\Pi\right)+2\gamma(1+A^T)\frac{2(1-u)eqFs}{D}\Pi\right]+\text{c.t.} \tag{34}$$

The contact terms are now to be determined by the physical normalization conditions that, as $F=0$, M and its first derivative with respect to $\gamma\Pi$ must vanish at $\gamma\Pi=-m$. Accordingly, one has

$$\text{c.t.}=-m_c-\zeta_c(m+\gamma\Pi)\,, \tag{35}$$

where

$$m_c=\frac{\alpha}{2\pi}m\int\frac{ds}{s}du(1+u)e^{-ism^2u^2}\,, \tag{36}$$

and

$$\zeta_c=\frac{\alpha}{2\pi}\int\frac{ds}{s}du(1-u)e^{-ism^2u^2}-i\frac{\alpha}{\pi}m^2\int ds\,du\,u(1-u^2)e^{-ism^2u^2}\,. \tag{37}$$

The combination of Eq. (34) together with Eqs. (35)–(37) then gives an exact construction of M for a spin-$\frac{1}{2}$ charged particle in a constant external field.

If one specializes to a pure magnetic field which is chosen to be in the z direction,

$$F_{12}=-F_{21}=H\,, \tag{38}$$

then Eq. (34) can be further simplified by using the following reductions (in the following, without loss of generality, we will assume that $\Pi_3 = 0$):

$$\Delta \equiv \det\left(\frac{D}{2eqFs}\right) = \left(1 - u + u\frac{\sin x}{x}e^{ix}\right)\left(1 - u + u\frac{\sin x}{x}e^{-ix}\right)$$

$$= (1-u)^2 + 2u(1-u)\frac{\sin 2x}{2x} + u^2\left(\frac{\sin x}{x}\right)^2 , \tag{39}$$

$$-4 - \mathrm{tr}A + 2i\sigma A = -4\cos x\, e^{-i\zeta x} , \tag{40}$$

$$\gamma\frac{2(1-u)eqFs}{D}\Pi = (1-u)\gamma\Pi + (1-u)\left(\frac{1-u+u[(\sin x)/x]e^{-i\zeta x}}{\Delta} - 1\right)\vec{\gamma}\cdot\vec{\Pi}_\perp , \tag{41}$$

$$\gamma(1+A^T)\frac{2(1-u)eqFs}{D}\Pi = (1-u)\gamma\Pi + (1-u)\left[\frac{1-u+u[(\sin x)/x]e^{-i\zeta x}}{\Delta}e^{-2i\zeta x} - 1\right]\vec{\gamma}\cdot\vec{\Pi}_\perp , \tag{42}$$

where

$$\zeta = q\sigma_3 , \quad x = eHus , \quad \vec{\gamma}\cdot\vec{\Pi}_\perp = \gamma_1\Pi_1 + \gamma_2\Pi_2 , \tag{43}$$

and we have used the identity

$$eq\gamma F\Pi = i\zeta eH\vec{\gamma}\cdot\vec{\Pi}_\perp . \tag{44}$$

As for Φ, we have

$$\Phi = u(1-u)[m^2 - (\gamma\Pi)^2] + u^2(m^2 - eq\sigma F) + \Pi\left[-\frac{1}{2eqFs}\ln\left(-\frac{D}{D^T}\right) + u^2\right]\Pi$$

$$= u(1-u)[m^2 - (\gamma\Pi)^2] + u^2(m^2 - eq\sigma F) + \frac{u}{x}[\beta - (1-u)x]\vec{\Pi}_\perp^{\,2}$$

$$\equiv u^2m^2 + \varphi , \tag{45}$$

where

$$\tan\beta = \frac{(1-u)\sin x}{(1-u)\cos x + u(\sin x)/x} , \tag{46}$$

$$\varphi = u(1-u)[m^2 - (\gamma\Pi)^2] + \frac{u}{x}[\beta - (1-u)x]\vec{\Pi}_\perp^{\,2} - u^2eq\sigma F . \tag{47}$$

The simplified version of Eq. (34) is

$$M = \frac{\alpha}{2\pi}m\int_0^\infty \frac{dx}{x}\int_0^1 du\, \exp\left(-i\frac{m^2}{eH}ux\right)\Bigg\{\frac{e^{-is\varphi}}{\Delta^{1/2}}\left[1 + e^{-2i\zeta x}\left(1 + (1-u)\frac{\gamma\Pi}{m}\right) + \frac{1-u}{m}\left(\frac{1-u}{\Delta} + \frac{u}{\Delta}\frac{\sin x}{x}e^{-i\zeta x} - e^{-2i\zeta x}\right)\vec{\gamma}\cdot\vec{\Pi}_\perp\right]$$

$$- (1+u) - \left(1 + \frac{\gamma\Pi}{m}\right)(1-u)[1 - 2im^2su(1+u)]\Bigg\} , \tag{48}$$

with Δ and φ given in Eqs. (39) and (47). Note that Eq. (48) is the exact solution obtained without making the usual approximation that the wave function ψ in Eq. (1) satisfies the equation $(m + \gamma\Pi)\psi = 0$.

III. MAGNETIC MOMENT, MAGNETIC SUSCEPTIBILITY, AND DECAY RATE

For most applications, to an accuracy of order α, we may approximate M by its expectation value taken between the particle field ψ obeying the equation

$$(m + \gamma\Pi)\psi = 0 , \quad \text{or} \quad (\Pi^2 + m^2 - eq\vec{\sigma}\cdot\vec{H})\psi = 0 . \tag{49}$$

With this approximation, Eq. (48) becomes

$$M = \frac{\alpha}{2\pi}m\int\frac{dx}{x}du\,\exp\left(-i\frac{m^2}{eH}ux\right)\Bigg\{\Delta^{-1/2}\exp\left(-i\frac{us}{x}[\beta - (1-u)x]\vec{\Pi}_\perp^{\,2}\right)e^{i\zeta ux}$$

$$\times\left[1 + ue^{-2i\zeta x} + \frac{1-u}{m}\left(\frac{1-u}{\Delta} + \frac{u}{\Delta}\frac{\sin x}{x}e^{-i\zeta x} - e^{-2i\zeta x}\right)\vec{\gamma}\cdot\vec{\Pi}_\perp\right] - (1+u)\Bigg\} . \tag{50}$$

Further simplification requires a knowledge of the expectation values of the operators, γ^0, ζ, $\gamma^0\zeta$, $\vec{\gamma}\cdot\vec{\Pi}_\perp$, $\vec{\Pi}_\perp{}^2$, and $\zeta\vec{\gamma}\cdot\vec{\Pi}_\perp$, between states satisfying Eq. (49). The details of these evaluations will be given in the Appendix. Here we only remark that the various operators in Eq. (50) can be effectively replaced by their corresponding expectation values as follows:

$$\gamma^0 \to \frac{E}{m}, \quad \gamma^0\zeta \to \zeta', \tag{51}$$

$$\zeta \to \frac{E}{m}\zeta', \tag{52}$$

$$\vec{\gamma}\cdot\vec{\Pi}_\perp \to (2n+1-\zeta')\frac{eH}{m}, \tag{53}$$

$$\zeta\vec{\gamma}\cdot\vec{\Pi}_\perp = \frac{1}{2eH}[eq\vec{\sigma}\cdot\vec{H}, m+\gamma\Pi] \to 0, \tag{54}$$

$$\vec{\Pi}_\perp{}^2 \to (2n+1-\zeta'+\zeta)eH, \tag{55}$$

where $E^2 = m^2 + (2n+1-\zeta')eH$ is the energy eigenvalue of Eq. (49), $\zeta' = \pm 1$ is the eigenvalue of $\gamma^0\zeta$, and $n = 0, 1, 2, \ldots, .$

Introducing these simplifications, Eq. (50) now becomes (to order α)

$$M = \frac{\alpha}{2\pi}m\int_0^\infty \frac{dx}{x}\int_0^1 du \exp\left(-i\frac{m^2}{eH}ux\right)\Big\{\Delta^{-1/2}\exp\{-i[\beta-(1-u)x](2n+1-\zeta')\} \times\left[e^{-i\zeta(\beta-x)}(1+ue^{-2i\zeta x}) + (1-u)(2n+1-\zeta')\frac{eH}{m^2} \times\left(\frac{1-u}{\Delta}\cos(\beta-x) + \frac{u}{\Delta}\frac{\sin x}{x}\cos\beta - \cos(\beta+x)\right)\right] - (1+u)\Big\}, \tag{56}$$

where, for convenience, we have used both $\zeta' = \pm 1$ and $\zeta = q\sigma_3$ representations. Some applications of this result will now be discussed below.[22]

A. Ground State

By specifying Eq. (56) to the ground state with $n=0$, $\zeta' = +1$, $E = m$, and $\zeta \to \zeta' = +1$, we have

$$M^{(0)} = \frac{\alpha}{2\pi}m\int_0^\infty \frac{dx}{x}\int_0^1 du \exp\left(-i\frac{m^2}{eH}ux\right) \times\left(\frac{1+ue^{-2ix}}{1-u+u[(\sin x)/x]e^{-ix}} - 1 - u\right), \tag{57}$$

since [by using Eqs. (39) and (46)]

$$\Delta^{-1/2}e^{-i(\beta-x)} = \frac{1}{\Delta}\left(1-u+u\frac{\sin x}{x}e^{ix}\right) = \left(1-u+u\frac{\sin x}{x}e^{-ix}\right)^{-1}. \tag{58}$$

We can also express Eq. (57) in another form by rotating the integration path from the positive x axis to the lower imaginary axis to yield

$$M^{(0)} = \frac{\alpha}{2\pi}m\int_0^\infty \frac{dy}{y}\int_0^1 du \exp\left(-\frac{m^2}{eH}uy\right) \times\left(\frac{1+ue^{-2y}}{1-u+u[(\sinh y)/y]e^{-y}} - 1 - u\right). \tag{59}$$

This is the known result of Demeur.[13] The justification of the rotation of axis involves the absence of a singularity at the origin and throughout the quadrant

$$x = \xi - i\eta, \quad \xi > 0, \ \eta > 0. \tag{60}$$

In particular, a zero of the denominator in Eq. (57) would require that

$$2(1-u)\eta + u(1-e^{-2\eta}\cos 2\xi) = 0, \tag{61}$$

which obviously cannot be satisfied for $\eta > 0$. The reality of M thus made explicit was to be expected since the ground state is stable against radiative decay.

1. Strong Magnetic Field

The strong magnetic field limit of Eq. (59) has been evaluated by several authors[23,24] recently in connection with the claim of possibly large radiative corrections.[25,26] The main contribution comes from the region $y \to \infty$ and $u \to 1$, and has the leading behavior

$$M^{(0)} \approx \frac{\alpha}{4\pi}m\left(\ln\frac{2eH}{m^2}\right)^2. \tag{62}$$

In terms of the ground-state energy, we have

$$E = m\left[1 + \frac{\alpha}{4\pi}\left(\ln\frac{2eH}{m^2}\right)^2 + \cdots\right]. \tag{63}$$

Note that, even when $2eH/m^2 \sim 100$ or $H \sim 2\times 10^{14}$ G, the radiative correction is only about 1% of the rest energy. As $(\alpha/4\pi)[\ln(2eH/m^2)]^2$ becomes large compared to unity, the approximation of replacing M by its expectation value between the particle fields obeying Eq. (49) is no longer valid, and we must go back to examine Eq. (48).

2. Weak Field

For weak fields, which are characterized by the condition $eH/m^2 \ll 1$, we wish to evaluate Eq. (59) to order $(eH/m^2)^3$. This can be achieved by dividing the u integration into two regions: $0 \leq u < u_0$ and $u_0 < u \leq 1$, with $1 \gg u_0 \gg eH/m^2$.

In the region $1 \geq u > u_0$, only small values of y contribute, and we may expand the parenthesis in Eq. (59) as

$$(\;) \approx -u(1-u)y + (\tfrac{4}{3} - \tfrac{5}{3}u + u^2)uy^2 - (1-u)(1-\tfrac{4}{3}u + u^2)uy^3 + \cdots , \tag{64}$$

and obtain

$$M_>^{(0)} = \frac{\alpha}{2\pi} m \int_{u_0}^1 du\, u \int_0^\infty dy \exp\left(-\frac{m^2}{eH}uy\right)\left[-(1-u) + (\tfrac{4}{3} - \tfrac{5}{3}u + u^2)y - (1-u)(1-\tfrac{4}{3}u + u^2)y^2\right]$$

$$= \frac{\alpha}{2\pi} m \left[-\frac{eH}{2m^2} + \left(\frac{eH}{m^2}\right)^2\left(\frac{4}{3}\ln\frac{1}{u_0} - \frac{7}{6}\right) + \left(\frac{eH}{m^2}\right)^3\left(\frac{14}{3}\ln\frac{1}{u_0} - \frac{5}{3}\right)\right], \tag{65}$$

where we have omitted terms which are power series in u_0 or $1/u_0$, since they will be canceled by the corresponding terms in the other region. The final result should not depend on u_0.

In the region $0 \leq u < u_0$, we expand the parenthesis in power series of u and obtain

$$M_<^{(0)} = \frac{\alpha}{2\pi} m \int_0^{u_0} du\, u \int_0^\infty \frac{dy}{y} \exp\left(-\frac{m^2}{eH}uy\right)\left[\left(e^{-2y} - \frac{1-e^{-2y}}{2y}\right) + u\left(1 - \frac{1-e^{-2y}}{2y}\right)\left(1 + e^{-2y} - \frac{1-e^{-2y}}{2y}\right)\right]$$

$$= \frac{\alpha}{2\pi} m \int_0^{u_0} du\, u \left\{\left[\frac{2m^2}{eH}u \ln\left(1 + \frac{2eH}{m^2 u}\right) - 1\right]\right.$$

$$\left. + \left[-\frac{3}{2} + \left(1 + \frac{3m^2}{2eH}u + \frac{m^4}{4e^2H^2}u^2\right)\ln\left(1 + \frac{2eH}{m^2u}\right) - \frac{m^2u}{8eH}\left(\frac{m^2u}{eH} + 4\right)\ln\left(1 + \frac{4eH}{m^2u}\right)\right]\right\}$$

$$= \frac{\alpha}{2\pi} m \left[\left(\frac{eH}{m^2}\right)^2\left(\frac{4}{3}\ln\frac{m^2u_0}{2eH} + \frac{4}{9}\right) + \left(\frac{eH}{m^2}\right)^3\left(\frac{233}{90} - \frac{32}{5}\ln 2 + \frac{14}{3}\ln\frac{m^2u_0}{2eH}\right)\right], \tag{66}$$

where, in arriving at the last form, we have used the condition $u_0 \gg eH/m^2$ to expand the integrated form into power series of $eH/(m^2u_0)$, and again we have omitted terms which are power series in u_0 or $1/u_0$.

Summation over the contributions from both regions, Eqs. (65) and (66), yields the result

$$M^{(0)} = \frac{\alpha}{2\pi} m \left[-\frac{eH}{2m^2} + \left(\frac{eH}{m^2}\right)^2\left(\frac{4}{3}\ln\frac{m^2}{2eH} - \frac{13}{18}\right) + \left(\frac{eH}{m^2}\right)^3\left(\frac{14}{3}\ln\frac{m^2}{2eH} - \frac{32}{5}\ln 2 + \frac{83}{90}\right)\right], \tag{67}$$

which is the same as that obtained in Refs. 10 and 14.

B. Arbitrary State

For this situation, we go back to the general expression, Eq. (56). Note that the rotation of integration path, $x \to -iy$, cannot be used here, as expected from the radiative instability of all the level above the ground state. Some special cases will now be discussed.

1. Low Energy $(E \sim m)$ and Weak-Field $(eH/m^2 \ll 1)$ Case

Here we will evaluate Eq. (56) to cubic terms in H. As in Sec. III A 2, we divide the u integration into two regions: $0 \leq u < u_0$ and $u_0 < u \leq 1$, with $1 \gg u_0 \gg eH/m^2$.

In the region $u_0 < u \leq 1$, small values of x still dominate and the quantity in large braces in Eq. (56) can be expanded into power series of x, which yields

$$\{\;\} \simeq -i\zeta u(1-u)x - (\tfrac{4}{3} - \tfrac{5}{3}u + u^2)ux^2 + \frac{eH}{m^2}(2n+1-\zeta')(2 - \tfrac{10}{3}u + \tfrac{4}{3}u^2)x^2 - \tfrac{1}{3}i(2n+1-\zeta')u(1+u)(1-u)^2x^3 + i\zeta u(1-u)(1-\tfrac{4}{3}u + u^2)x^3 . \tag{68}$$

Here, in obtaining Eq. (68), we have used the following expansions:

$$\Delta^{-1} \approx 1 + (\tfrac{4}{3}u - u^2)x^2 + \cdots , \tag{69}$$

$$\beta \approx (1-u)x + \tfrac{1}{3}u(1-u)^2 x^3 + \cdots \,, \tag{70}$$

$$\exp\{-i[\beta-(1-u)x](2n+1-\zeta')\} \approx 1-\tfrac{1}{3}i(2n+1-\zeta')u(1-u)^2x^3+\cdots \,, \tag{71}$$

$$\frac{1-u}{\Delta}\cos(\beta-x)+\frac{u}{\Delta}\frac{\sin x}{x}\cos\beta-\cos(\beta+x) \approx (2-\tfrac{4}{3}u)x^2+\cdots \,, \tag{72}$$

$$e^{-i\zeta(\beta-x)}(1+ue^{-2i\zeta x}) \approx 1+u-i\zeta u(1-u)x-(2-\tfrac{3}{2}u+\tfrac{1}{2}u^2)ux^2 + i\zeta(1-\tfrac{5}{3}u+\tfrac{7}{6}u^2-\tfrac{1}{2}u^3)ux^3+\cdots \,. \tag{73}$$

The integrations over x and u are simple and straightforward, and we find

$$M_> = \frac{\alpha}{2\pi}m\left\{-\zeta\frac{eH}{2m^2}+\left(\frac{eH}{m^2}\right)^2\left(\frac{4}{3}\ln\frac{1}{u_0}-\frac{7}{6}\right) +\left(\frac{eH}{m^2}\right)^3\left[(2n+1-\zeta')\left(\frac{8}{3}\ln\frac{1}{u_0}-\frac{1}{3}\right) +\zeta\left(\frac{14}{3}\ln\frac{1}{u_0}-\frac{5}{3}\right)\right]\right\}, \tag{74}$$

which reduces to Eq. (65) in the ground state. Again in writing the above equation, we omit terms that are power series in u_0 and $1/u_0$. We also note that Eq. (74) is purely real, i.e., to order H^3, the imaginary part comes solely from the region $0 \leq u < u_0$.

In the region $0 \leq u < u_0$, we expand the terms in the brace of Eq. (56) into power series of u:

$$\Delta^{-1} \approx 1+2u\left(1-\frac{\sin 2x}{2x}\right)+u^2\left(3-3\frac{\sin 2x}{x}+\frac{\sin^2 2x-\sin^2 x}{x^2}\right), \tag{75}$$

$$\sin(\beta-x)\approx -u\frac{\sin^2 x}{x}+O(u^3), \quad \cos(\beta-x)\approx 1+O(u^2), \tag{76}$$

$$e^{-i\zeta(\beta-x)}(1+ue^{-2i\zeta x}) \approx 1+u\left[\cos 2x+i\zeta\left(\frac{\sin^2 x}{x}-\sin 2x\right)\right] +\frac{u^2}{2x}\left[\sin 2x-\tfrac{1}{2}\sin 4x-\frac{1}{x}\sin^4 x+i\zeta\sin 2x\left(\sin 2x-\frac{\sin^2 x}{x}\right)\right], \tag{77}$$

$$\frac{1-u}{\Delta}\cos(\beta-x)+\frac{u}{\Delta}\frac{\sin x}{x}\cos\beta-\cos(\beta+x)\approx(1-\cos 2x)+u\left(1-\frac{\sin 2x}{2x}-\frac{\sin^2 x}{x}\sin 2x\right), \tag{78}$$

$$\Delta^{-1/2}\exp\{-i[\beta-(1-u)x](2n+1-\zeta')\}\approx 1+u\left(1-\frac{\sin 2x}{2x}\right)-i(2n+1-\zeta')ux\left(1-\frac{\sin^2 x}{x^2}\right) + u^2\left[-\tfrac{1}{2}(2n+1-\zeta')^2\left(x-\frac{\sin^2 x}{x}\right)^2-i(2n+1-\zeta')\left(x-\frac{2\sin^2 x}{x}\right)\left(1-\frac{\sin 2x}{2x}\right) -\frac{\sin^4 x}{2x^2}+\left(1-\frac{\sin 2x}{2x}\right)^2\right]. \tag{79}$$

The resulting M is

$$M_< \approx \frac{\alpha}{2\pi}m\int_0^{u_0}du\int_0^\infty\frac{dx}{x}\exp\left(-i\frac{m^2}{eH}ux\right)\left\{\left[uB_0+(2n+1-\zeta')\frac{eH}{m^2}(1-\cos 2x)\right]+\frac{eH}{m^2}(2n+1-\zeta')uB_1+u^2B_2\right\}, \tag{80}$$

where

$$B_0=\left(\cos 2x-\frac{\sin 2x}{2x}\right)+i\left[\zeta\left(\frac{\sin^2 x}{x}-\sin 2x\right)-(2n+1-\zeta')\left(x-\frac{\sin^2 x}{x}\right)\right], \tag{81}$$

$$B_1=\left(1-\frac{3}{2}\frac{\sin 2x}{x}+\frac{\sin 4x}{2x}\right)-i(2n+1-\zeta')(1-\cos 2x)\left(x-\frac{\sin^2 x}{x}\right), \tag{82}$$

$$\mathrm{Re}B_2=\left(1+\cos 2x-\frac{1}{2x}\sin 2x-\frac{1}{2x}\sin 4x+\frac{\sin^2 x\cos 2x}{x^2}\right) +\zeta(2n+1-\zeta')\left(x-\frac{\sin^2 x}{x}\right)\left(\frac{\sin^2 x}{x}-\sin 2x\right)-\tfrac{1}{2}(2n+1-\zeta')^2\left(x-\frac{\sin^2 x}{x}\right)^2, \tag{83}$$

$$-\mathrm{Im}B_2=\zeta\left(\sin 2x-\frac{\sin^2 x}{x}\right)\left(1-\frac{\sin 2x}{x}\right)+(2n+1-\zeta')\left[\left(x-\frac{2\sin^2 x}{x}\right)\left(1-\frac{\sin 2x}{2x}\right)+\left(x-\frac{\sin^2 x}{x}\right)\cos 2x\right]. \tag{84}$$

To evaluate Eq. (80), we first separate it into two parts:

$$M_< = M_<^{(0)} + M_<' , \tag{85}$$

the first part $M_<^{(0)}$, which is obtained from $M_<$ by setting $n=0$, ζ, $\zeta'=+1$, has already been evaluated in Eq. (66), and is purely real. The remaining part is

$$M_<' = \frac{\alpha}{2\pi} m \int_0^{u_0} du \int_0^\infty \frac{dx}{x} \exp\left(-i\frac{m^2}{eH}ux\right)\left\{\left[-iuB_0' + (2n+1-\zeta')\frac{eH}{m^2}(1-\cos 2x)\right] + \frac{eH}{m^2}(2n+1-\zeta')uB_1 + u^2 B_2'\right\}, \tag{86}$$

where

$$B_0' = (2n+1-\zeta')\left(x - \frac{\sin^2 x}{x}\right) + (1-\zeta)\left(\frac{\sin^2 x}{x} - \sin 2x\right), \tag{87}$$

$$\mathrm{Re}B_2' = \zeta(2n+1-\zeta')\left(x-\frac{\sin^2 x}{x}\right)\left(\frac{\sin^2 x}{x} - \sin 2x\right) - \tfrac{1}{2}(2n+1-\zeta')^2\left(x - \frac{\sin^2 x}{x}\right)^2, \tag{88}$$

$$-\mathrm{Im}B_2' = (1-\zeta)\left(\frac{\sin^2 x}{x} - \sin 2x\right)\left(1 - \frac{\sin 2x}{x}\right) + (2n+1-\zeta')\left[\left(x - \frac{2\sin^2 x}{x}\right)\left(1 - \frac{\sin 2x}{2x}\right) + \left(x - \frac{\sin^2 x}{x}\right)\cos 2x\right]. \tag{89}$$

The integrations over u and x are then performed by considering the real and imaginary parts separately. For the real part, we have

$$\mathrm{Re}M_<' = \frac{\alpha}{2\pi} m \int_0^{u_0} du \int_0^\infty \frac{dx}{x}\left\{\left[-uB_0' \sin\lambda x + (2n+1-\zeta')\frac{eH}{m^2}(1-\cos 2x)\cos\lambda x\right]\right.$$
$$\left. + u(2n+1-\zeta')\frac{eH}{m^2}(\mathrm{Re}B_1 \cos\lambda x + \mathrm{Im}B_1 \sin\lambda x) + u^2(\mathrm{Re}B_2' \cos\lambda x + \mathrm{Im}B_2' \sin\lambda x)\right\}, \tag{90}$$

with

$$\lambda \equiv \frac{m^2 u}{eH}. \tag{91}$$

Some of the integrals are effectively given by (omitting terms which are polynomials in u_0 or $1/u_0$)

$$(2n+1-\zeta')\frac{eH}{m^2}\int_0^{u_0} du\, u \int_0^\infty \frac{dx}{x}\mathrm{Re}B_1\cos\lambda x = (2n+1-\zeta')\left(\frac{eH}{m^2}\right)^3\left(\frac{25}{9} - \frac{16}{3}\ln 2 + \frac{10}{3}\ln\frac{m^2 u_0}{2eH}\right), \tag{92}$$

$$\int_0^{u_0} u^2\, du \int_0^\infty \frac{dx}{x}\mathrm{Im}B_2' \sin\lambda x = \left(\frac{eH}{m^2}\right)^3\left[(2n+1-\zeta')\left(\frac{73}{90} - \frac{16}{15}\ln 2 - \frac{2}{3}\ln\frac{m^2u_0}{2eH}\right) + (1-\zeta)\left(-\frac{233}{90} + \frac{32}{5}\ln 2 - \frac{14}{3}\ln\frac{m^2 u_0}{2eH}\right)\right], \tag{93}$$

while all other integrals are effectively zero. Therefore, we obtain

$$\mathrm{Re}M_<' = \frac{\alpha}{2\pi} m\left(\frac{eH}{m^2}\right)^3\left[(2n+1-\zeta')\left(\frac{323}{90} - \frac{32}{5}\ln 2 + \frac{8}{3}\ln\frac{m^2u_0}{2eH}\right) + (1-\zeta)\left(-\frac{233}{90} + \frac{32}{5}\ln 2 - \frac{14}{3}\ln\frac{m^2u_0}{2eH}\right)\right], \tag{94}$$

which is to be combined with Eqs. (66) and (74) to give the total real part

$$\mathrm{Re}M = \frac{\alpha}{2\pi} m\left\{-\zeta\frac{eH}{2m^2} + \left(\frac{eH}{m^2}\right)^2\left(\frac{4}{3}\ln\frac{m^2}{2eH} - \frac{13}{18}\right)\right.$$
$$\left. + \left(\frac{eH}{m^2}\right)^3\left[(2n+1-\zeta')\left(\frac{8}{3}\ln\frac{m^2}{2eH} - \frac{32}{5}\ln 2 + \frac{293}{90}\right) + \zeta\left(\frac{14}{3}\ln\frac{m^2}{2eH} - \frac{32}{5}\ln 2 + \frac{83}{90}\right)\right]\right\}. \tag{95}$$

This is precisely the same result obtained by Newton.[10] The parts that depend explicitly on ζ are usually referred to as the magnetic-moment terms, which have also been reproduced recently.[11,14] The remaining parts are referred to as the mass-shift terms and have not been confirmed since.

The imaginary part of M (to order H^3) comes only from Eq. (86) since both $M_>$ and $M_<^{(0)}$ are purely real. We have

$$-\mathrm{Im}M=\frac{\alpha}{2\pi}m\int_0^{u_0}du\int_0^{\infty}\frac{dx}{x}\left\{\left[uB_0'\cos\lambda x+(2n+1-\zeta')\frac{eH}{m^2}(1-\cos 2x)\sin\lambda x\right.\right.$$
$$\left.\left.+\frac{eH}{m^2}(2n+1-\zeta')u(\mathrm{Re}B_1\sin\lambda x-\mathrm{Im}B_1\cos\lambda x)+u^2(\mathrm{Re}B_2'\sin\lambda x-\mathrm{Im}B_2'\cos\lambda x)\right]\right\}. \tag{96}$$

The integrals can be easily evaluated:

$$\int_0^{u_0}du\int_0^{\infty}\frac{dx}{x}\left[uB_0'\cos\lambda x+(2n+1-\zeta')\frac{eH}{m^2}(1-\cos 2x)\sin\lambda x\right]=\tfrac{2}{3}\pi\left(\frac{eH}{m^2}\right)^2[(2n+1-\zeta')-(1-\zeta)]$$
$$\cong\tfrac{4}{3}\pi\left(\frac{eH}{m^2}\right)^2n+\tfrac{1}{3}\pi\zeta'(2n+1-\zeta')\left(\frac{eH}{m^2}\right)^3, \tag{97}$$

$$(2n+1-\zeta')\frac{eH}{m^2}\int_0^{u_0}u\,du\int_0^{\infty}\frac{dx}{x}[\mathrm{Re}B_1\sin\lambda x-\mathrm{Im}B_1\cos\lambda x]=-\tfrac{1}{3}\pi(2n+1-\zeta')\left(\frac{eH}{m^2}\right)^3[(2n+1-\zeta')+5], \tag{98}$$

$$\int_0^{u_0}u^2du\int_0^{\infty}\frac{dx}{x}[\mathrm{Re}B_2'\sin\lambda x-\mathrm{Im}B_2'\cos\lambda x]=\pi\left(\frac{eH}{m^2}\right)^3[\tfrac{1}{5}(2n+1-\zeta')^2+\tfrac{1}{15}\zeta'(2n+1-\zeta')+\tfrac{1}{3}(2n+1-\zeta')+\tfrac{7}{3}(1-\zeta')], \tag{99}$$

where we have used Eq. (52). The result is

$$-\mathrm{Im}M=\tfrac{1}{2}\alpha m\left\{\tfrac{4}{3}n\left(\frac{eH}{m^2}\right)^2+\left(\frac{eH}{m^2}\right)^3[-\tfrac{2}{15}(2n+1-\zeta')^2+\tfrac{2}{5}\zeta'(2n+1-\zeta')-\tfrac{4}{3}(2n+1-\zeta')+\tfrac{7}{3}(1-\zeta')]\right\}. \tag{100}$$

which has not been previously presented.

Finally, consider the eigenvalue equation

$$(m+\gamma\Pi+M)\psi=0, \tag{101}$$

which implies the energy eigenvalues (to order α) (see Appendix)

$$E'=E+\frac{m}{E}M. \tag{102}$$

The energy shift due to radiative corrections is

$$\Delta E=\frac{m}{E}\mathrm{Re}M$$
$$=\frac{\alpha}{2\pi}m\left\{-\zeta'\frac{eH}{2m^2}+\left(\frac{eH}{m^2}\right)^2\left(\frac{4}{3}\ln\frac{m^2}{2eH}-\frac{8}{3}\right)\right.$$
$$\left.+\left(\frac{eH}{m^2}\right)^3\left[(2n+1-\zeta')\left(2\ln\frac{m^2}{2eH}-\frac{32}{5}\ln 2+\frac{217}{60}\right)+\zeta'\left(\frac{14}{3}\ln\frac{m^2}{2eH}-\frac{32}{5}\ln 2+\frac{83}{90}\right)\right]\right\}, \tag{103}$$

while the decay rate γ is defined by

$$\tfrac{1}{2}\gamma=-\mathrm{Im}E'$$
$$=-\frac{m}{E}\mathrm{Im}M, \tag{104}$$

or

$$\gamma=\tfrac{4}{3}\alpha m\left(\frac{eH}{m^2}\right)^2n$$
$$+\alpha m\left(\frac{eH}{m^2}\right)^3[-\tfrac{7}{15}(2n+1-\zeta')^2$$
$$+(\tfrac{1}{15}\zeta'-1)(2n+1-\zeta')+\tfrac{7}{3}(1-\zeta')]. \tag{105}$$

As a check of our result, we note that the H^2 term can also be obtained from a semiclassical or elementary quantum-mechanical consideration of the decay rate due to electric dipole radiation, while the H^3 terms (specializing to the $n=0$ and $\zeta'=-1$ state) agree with the expected rate of magnetic dipole radiation (since only a spin-flip transition is possible to the ground state).

2. *Total Decay Rate (for $E^2/m^2\gg 1$ and $eH/m^2\ll 1$)*

The radiation from a high-energy electron in a weak magnetic field is the usual situation encountered in connection with the operation and construction of cyclotrons. It has been extensively discussed in the literature.[3] However, because the conventional approach to calculate the decay rate is quite long and complicated, we will here demonstrate how simply it can be obtained from Eq. (56).

Under the high-energy and weak-field circumstances expressed by

$$\frac{eH}{m^2} \ll 1$$

and

$$\frac{E^2}{m^2} = (2n+1-\zeta')\frac{eH}{m^2}+1 \quad \approx \frac{2neH}{m^2} \gg 1\,, \tag{106}$$

the parameter that characterizes the quantum correction[18-20] is

$$\Upsilon = \frac{3}{2}\frac{E}{m}\frac{eH}{m^2}\,, \tag{107}$$

which, for the usual synchrotron operation condition, is negligibly small. However, it has the magnitude ~0.03 for the proposed experiment[6] at the National Accelerator Laboratory (with $E \approx 150$–300 BeV and $H \approx 1.5$–3 G). To present a more general situation, we wish here to calculate the total decay rate of radiation by keeping Υ as an arbitrary parameter and omitting all terms of the order m^2/E^2 or eH/m^2, and higher.

In order to proceed, let us return to Eq. (56). To evaluate the imaginary part of M, which is our concern here, we divide the u integration into three regions: (i) $0 \leq u < u_0$, with $1 \gg u_0 \gg eH/m^2$, (ii) $u_0 < u < 1-\epsilon$, with $\epsilon \gg m/E$, $1-\epsilon \approx 1$, and (iii) $1-\epsilon < u \leq 1$. It can be shown that the contributions from regions (i) and (iii) are negligible. For region (ii), we examine the exponential structure

$$\exp\left[-i\left(ux + [\beta - (1-u)x]\frac{eH}{m^2}\right)\frac{m^2}{eH}\right], \tag{108}$$

which reduces to [cf. Eq. (71)]

$$\exp\left\{-i\left[1+\tfrac{1}{3}(1-u)^2x^2\frac{E^2}{m^2}\right]ux\frac{m^2}{eH}\right\}, \tag{109}$$

which is dominated by values of x such that

$$x \sim \frac{eH}{m^2 u} \ll 1\,. \tag{110}$$

The important range of x occurs when the two arguments are roughly comparable, i.e.,

$$(1-u)^2x^2\frac{E^2}{m^2} \sim 1$$

or

$$x \sim \frac{1}{1-u}\frac{m}{E}\,, \tag{111}$$

which is indeed small compared to unity as required by Eq. (110). Accordingly, by using Eqs. (69)–(73), we retain only the leading terms in an x expansion of the quantity in square brackets in Eq. (56):

$$[\ \ ] \approx 1+u-i\zeta u(1-u)x + \frac{E^2}{m^2}(1-u)(2-\tfrac{4}{3}u)x^2$$
$$= 1+u-i\zeta' uz + (2-\tfrac{4}{3}u)(1-u)^{-1}z^2\,, \tag{112}$$

where, in writing the last form, we have introduced a new variable

$$x = \frac{1}{1-u}\frac{m}{E}z\,, \tag{113}$$

and used Eq. (52). The use of z variable then converts Eq. (109) into the form

$$\exp[-i\tfrac{3}{2}\xi(z+\tfrac{1}{3}z^3)]\,, \tag{114}$$

where

$$\xi = \frac{2}{3}\frac{m^2}{eH}\frac{m}{E}\frac{u}{1-u}$$
$$= \frac{1}{\Upsilon}\frac{u}{1-u}\,, \tag{115}$$

and Υ is defined in Eq. (107).

Substituting Eqs. (112) and (114) into Eq. (65), we we obtain, for the imaginary part of M,

$$\mathrm{Im}M = \frac{\alpha}{2\pi}m\,\mathrm{Im}\left\{\int_0^1 du\int_0^\infty \frac{dz}{z}[(1+u-i\zeta' uz+(2-\tfrac{4}{3}u)(1-u)^{-1}z^2)e^{-i\Theta} - (1+u)e^{-i3\xi z/2}]\right\}, \tag{116}$$

where we have extrapolated the region of u integration from region (ii) to cover all three regions, and defined

$$\Theta = \tfrac{3}{2}\xi(z+\tfrac{1}{3}z^3)\,. \tag{117}$$

In terms of the decay rate γ defined in Eq. (104), we have

$$\gamma = \frac{\alpha}{\pi}\frac{m^2}{E}\int_0^1 du\int_0^\infty \frac{dz}{z}\{[1+u+(2-\tfrac{4}{3}u)(1-u)^{-1}z^2]\sin\Theta - (1-u)\sin\tfrac{3}{2}\xi z + \zeta' uz\cos\Theta\}$$
$$= \frac{\alpha}{\sqrt{3}\,\pi}\frac{m^2}{E}\int_0^1 du\left[(1+u)\int_\xi^\infty K_{5/3}(\eta)d\eta + \tfrac{2}{3}u(3u-2)(1-u)^{-1}K_{2/3}(\xi) + \zeta' uK_{1/3}(\xi)\right]. \tag{118}$$

In obtaining Eq. (118), we have used the following Airy's integrals[19,27]

$$\int_0^\infty \cos\Theta\, dz = (\tfrac{1}{3})^{1/2} K_{1/3}(\xi)\,, \tag{119}$$

$$\int_0^\infty z \sin\Theta\, dz = (\tfrac{1}{3})^{1/2} K_{2/3}(\xi)\,, \tag{120}$$

$$\int_0^\infty \frac{1+2z^2}{z} \sin\Theta\, dz - \tfrac{1}{2}\pi = (\tfrac{1}{3})^{1/2} \int_\xi^\infty K_{5/3}(\eta)\, d\eta\,, \tag{121}$$

where $K_\nu(x)$ is the modified Bessel's function of the second kind. Here we observe that γ depends on the eigenvalue ζ': The decay rate is larger for the $\zeta' = +1$ state. For the unpolarized electron, we average over the initial states and obtain

$$\gamma_{\text{unpol}} = \frac{\alpha}{\sqrt{3}\,\pi} \frac{m^2}{E} \int_\xi^1 du \left[(1+u) \int_\xi^\infty K_{5/3}(\eta)\, d\eta + \tfrac{2}{3} u (3u-2)(1-u)^{-1} K_{2/3}(\xi) \right], \tag{122}$$

which is to be compared with the corresponding result of Refs. 3 and 20:

$$\gamma_{\text{unpol}} = \frac{\alpha}{\sqrt{3}\,\pi} \frac{m^2}{E} \Upsilon \int_0^1 \frac{d\xi}{(1+\Upsilon\xi)^2} \left[\int_\xi^\infty K_{5/3}(\eta)\, d\eta + \frac{\Upsilon^2 \xi^2}{1+\Upsilon\xi} K_{2/3}(\xi) \right]. \tag{123}$$

One notes that even though they appear to be quite different, they are actually identical to each other. This can be easily proved by a change of variable

$$u = \frac{\Upsilon\xi}{1+\Upsilon\xi}\,, \tag{124}$$

and by the use of the differential equation

$$\left(\frac{d}{d\xi} - \frac{2}{3\xi} \right) K_{2/3}(\xi) = -K_{5/3}(\xi)\,, \tag{125}$$

and the asymptotic behavior

$$K_\nu(\xi) \approx \left(\frac{\pi}{2\xi} \right)^{1/2} e^{-\xi}\,, \quad \text{as } \xi \to \infty. \tag{126}$$

In the limit $\Upsilon \ll 1$, since the functions $K_\nu(\xi)$ of direct concern decrease exponentially for $\xi \gg 1$, the important values of u are such that [cf. Eq. (115)]

$$u \sim \Upsilon \ll 1\,. \tag{127}$$

Accordingly, we obtain

$$\gamma = \frac{5\alpha}{3\sqrt{3}} \frac{m^2}{E} \Upsilon \left(1 - \frac{16}{15\sqrt{3}} \Upsilon + \frac{14}{9} \Upsilon^2 + \frac{1}{5} \zeta' \Upsilon - \frac{8}{5\sqrt{3}} \zeta' \Upsilon^2 \right), \tag{128}$$

where we have used the integral[27]

$$\int_0^\infty d\xi\, \xi^{\mu-1} K_\nu(\xi) = 2^{\mu-2} \Gamma(\tfrac{1}{2}(\mu+\nu))\, \Gamma(\tfrac{1}{2}(\mu-\nu)). \tag{129}$$

In the other limit $\Upsilon \gg 1$, we may use the expansion

$$K_\nu(\xi) \approx \frac{2^{\nu-1}\Gamma(\nu)}{\xi^\nu}\,, \quad \text{as } \xi \to 0 \tag{130}$$

and the β-function integral

$$\int_0^1 du\, u^{p-1} (1-u)^{q-1} = B(p, q)\,, \tag{131}$$

to yield

$$\gamma \approx \frac{m^2}{E} \frac{\alpha}{\sqrt{3}\,\pi} \Upsilon \left[\frac{7\pi}{3\sqrt{3}} \left(\frac{4}{\Upsilon} \right)^{1/3} \Gamma(\tfrac{5}{3}) \right]. \tag{132}$$

IV. POWER SPECTRUM OF RADIATION

In Sec. III, the total decay rate is obtained from the imaginary part of M through the definition of Eq. (104). This is to be compared with the conventional approach,[3,20] which utilizes the electron wave function to obtain first the power spectrum of radiation, $P(\omega)$, and thereby the decay rate by the relation

$$\gamma = \int_0^E \frac{d\omega}{\omega} P(\omega)\,. \tag{133}$$

One might ask whether our method discussed in the previous sections can also be used to obtain $P(\omega)$? The answer is that one cannot extract $P(\omega)$ from γ, since there are many equivalent integral forms which represent γ [such as Eqs. (122) and (123)]; each of these leads to different power spectrum and different power of radiation:

$$I = \int_0^E d\omega\, P(\omega)\,. \tag{134}$$

However, by a slight modification of the computational method discussed in Sec. II, one can unambiguously identify $P(\omega)$ before performing the photon-momentum integration. The resulting form is quite similar to those of Sec. II and can be evaluated in an analogous manner. In this way, $P(\omega)$ can be calculated exactly.

In the expression for M, Eq. (3) or subsequent forms, we insert a unit factor[22]

$$1 = \int_{-\infty}^\infty d\omega\, \delta(\omega - k^0) = \int_{-\infty}^\infty d\omega \int_{-\infty}^\infty \frac{d\tau}{2\pi} e^{i(\omega - k^0)\tau}\,, \tag{135}$$

and obtain

$$M=-ie^2\int d\omega\int\frac{d\tau}{2\pi}e^{i\omega\tau}\int s\,ds\,du\,\gamma^\mu\langle e^{-is\chi}e^{-ik^0\tau}[m-\gamma(\Pi-k)]\rangle\gamma_\mu+\text{c.t.} \tag{136}$$

When applied to $\text{Im}M$, which arises only from the real particle exchange process, the inferred spectral distribution in ω will be that of the radiated photon energy and will supply the desired photon spectrum without ambiguity. The power spectrum of radiation, $P(\omega)$, is then identified [from Eqs. (104) and (133)] as

$$P(\omega)=-2m\frac{\omega}{E}\text{Im}\int_{-\infty}^{\infty}\frac{d\tau}{2\pi}e^{i\omega\tau}M', \tag{137}$$

where

$$M'=-ie^2\int s\,ds\,du\,\gamma^\mu\langle e^{-is\chi}e^{-ik^0\tau}[m-\gamma(\Pi-k)]\rangle\gamma_\mu+\text{c.t.} \tag{138}$$

Here we note that the k integration symbolized by $\langle\ \rangle$ is now modified by the presence of the factor

$$e^{-ik^0\tau}=e^{ik\rho},\quad \rho^0=\tau,\ \vec{\rho}=0 \tag{139}$$

and thus the substitution

$$(k-u\Pi)^2\to(k-u\Pi)^2-\frac{k\rho}{s}=\left(k-u\Pi-\frac{\rho}{2s}\right)^2-\frac{u}{s}(\rho\Pi)-\frac{\rho^2}{4s} \tag{140}$$

takes place in $\chi(u)$, Eq. (6). The subsequent k integration

$$k\to k+\frac{\rho}{2s} \tag{141}$$

together with the last two ρ-dependent terms of Eq. (140) then modified Eq. (27) by the additional factor

$$e^{iu\rho\Pi+i\rho^2/4s}=e^{-iuE\tau}e^{-i\tau^2/4s}, \tag{142}$$

and by the substitution

$$2\frac{(1-u)eqFs}{D}\Pi\to2\frac{(1-u)eqFs}{D}\Pi-\frac{\rho}{2s}. \tag{143}$$

After all these modifications, we obtain [cf. Eq. (56)]

$$\begin{aligned}P(\omega)=-2m\frac{\omega}{E}\text{Im}\bigg(\frac{\alpha}{2\pi}m\int_0^\infty\frac{ds}{s}\int_0^1du\,e^{-ism^2u^2}\\
&\times\bigg\{\Delta^{-1/2}\exp\{-i[\beta-(1-u)x](2n+1-\zeta')\}\\
&\times\bigg[e^{-i\zeta(\beta-x)}(1+ue^{-2i\zeta x})\\
&+(2n+1-\zeta')\frac{eH}{m^2}(1-u)\left(\frac{1-u}{\Delta}\cos(\beta-x)+\frac{u}{\Delta}\frac{\sin x}{x}\cos\beta-\cos(\beta+x)\right)+\frac{i}{2m^2s}e^{-i\zeta(\beta+x)}\frac{d}{du}\bigg]\\
&-\left(1+u+\frac{i}{2m^2s}\frac{d}{du}\right)\bigg\}\int_{-\infty}^{\infty}\frac{d\tau}{2\pi}e^{i(\omega-uE)\tau}e^{-i\tau^2/4s}\bigg).\end{aligned} \tag{144}$$

This is the general expression (to order α) of $P(\omega)$.

The τ-dependent part of Eq. (144) is

$$\int_{-\infty}^{\infty}\frac{d\tau}{2\pi}e^{i(\omega-uE)\tau}\exp\left[-\frac{i}{6}\frac{(\tau uE)^2}{z\xi}\left(\frac{m}{E}\right)^2\right], \tag{145}$$

where we have used the variable z of Eq. (113) and ξ of Eq. (115). In the high-energy and weak-field limit, the several variables are $z\sim1$, $\xi\sim1$, $\tau uE\sim1$. Therefore, the Gaussian function is close to unity and the integral of Eq. (145) is reduced to $\delta(\omega-uE)$. Now the quantity in curly brackets in Eq. (144) can be expanded in the same ways as those given in Sec. III B 2. The subsequent u integration can be easily performed, and we obtain

$$P(\omega) = -\frac{\alpha}{\pi}\frac{m^2}{E}\frac{\omega}{E}\,\mathrm{Im}\left(\int_0^\infty \frac{dx'}{x'}\exp\left(-i\frac{m^2}{eH}\frac{\omega}{E}x'\right)\left\{\exp\left[-\frac{i}{3}\frac{\omega}{E}\left(1-\frac{\omega}{E}\right)^2\frac{E^2}{eH}x'^3\right]\left[1-i\zeta\frac{\omega}{E}\left(1-\frac{\omega}{E}\right)x' + \frac{E^2}{m^2}\left(1-\frac{\omega}{E}\right)\left(2-\frac{2\omega}{E}+\frac{\omega^2}{E^2}\right)\right]-1\right\}\right)$$

$$= \frac{\alpha}{\pi}\frac{m^2}{E}\frac{\omega}{E}\int_0^\infty \frac{dz'}{z'}\left\{\left[1+2z'^2+\left(\frac{\omega}{E}\right)^2\left(1-\frac{\omega}{E}\right)^{-1}z'^2\right]\sin\Theta' - \sin\tfrac{3}{2}\xi' z' + \zeta'\frac{\omega}{E}z'\cos\Theta'\right\}, \tag{146}$$

where we have defined the variables analogously to Eqs. (43), (113), (115), and (117) with u replaced by ω/E, i.e.,

$$x' = \frac{\omega}{E}seH = \left(1-\frac{\omega}{E}\right)^{-1}\frac{m}{E}z', \tag{147}$$

$$\xi' = \frac{1}{\Upsilon}\frac{\omega}{E}\left(1-\frac{\omega}{E}\right)^{-1}, \tag{148}$$

$$\Theta' = \tfrac{3}{2}\xi'(z'+\tfrac{1}{3}z'^3). \tag{149}$$

The integration over x' is immediate [cf. Eqs. (119)–(121)], and the result is

$$P(\omega) = \frac{\alpha}{\sqrt{3}\,\pi}\frac{m^2}{E}\frac{\omega}{E}\left[\int_{\xi'}^\infty K_{5/3}(\eta)d\eta + \left(\frac{\omega}{E}\right)^2\left(1-\frac{\omega}{E}\right)^{-1}K_{2/3}(\xi') + \zeta'\frac{\omega}{E}K_{1/3}(\xi')\right], \tag{150}$$

which depends on ζ' as in Eq. (118). The total energy radiated per unit time is then

$$I = \int_0^E P(\omega)d\omega = \frac{\alpha}{\sqrt{3}\,\pi}m^2\int_0^E \frac{d\omega}{E}\frac{\omega}{E}\left[\int_{\xi'}^\infty K_{5/3}(\eta)d\eta + \left(\frac{\omega}{E}\right)^2\left(1-\frac{\omega}{E}\right)^{-1}K_{2/3}(\xi') + \zeta'\frac{\omega}{E}K_{1/3}(\xi')\right], \tag{151}$$

which, when averaged over the initial states, is precisely the same result as that of Refs. 3, 8, and 20.

For all presently attainable experimental arrangement, the parameter Υ is very small. In this case, we have

$$I = \frac{\alpha}{\sqrt{3}\,\pi}m^2\int_0^E \frac{d\omega}{E}\frac{\omega}{E}\left[\int_{\xi'}^\infty K_{5/3}(\eta)d\eta + \zeta'\frac{\omega}{E}K_{1/3}(\xi')\right] \approx \tfrac{8}{27}\alpha m^2\Upsilon^2(1-\tfrac{55}{24}\sqrt{3}\,\Upsilon+\zeta'\Upsilon+\cdots). \tag{152}$$

When averaged over the initial states, this is the first quantum correction obtained by Schwinger[19] in 1954.

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APPENDIX: ENERGY EIGENVALUES AND EXPECTATION VALUES

1. Energy Eigenvalues

The Dirac equation for a spin-$\frac{1}{2}$ particle in an external field,

$$(m+\gamma\Pi)\psi = 0, \tag{A1}$$

applied to a homogeneous magnetic field implies the eigenvalue equation

$$(E - m\gamma^0 - \gamma^0\vec{\gamma}\cdot\vec{\Pi}_\perp)\psi = 0, \tag{A2}$$

where, without loss of generality, we have chosen $\vec{H}$ to be in the z direction and $\Pi_B = 0$. Now let $\psi_\pm$ be the projection of the field ψ onto subspaces of intrinsic parity where $i\gamma_5$ has only unit element between them. Then we can decompose Eq. (A2) into

$$(E-m)\psi_+ = (\vec{\sigma}\cdot\vec{\Pi}_\perp)\psi_-, \tag{A3}$$

$$(E+m)\psi_- = (\vec{\sigma}\cdot\vec{\Pi}_\perp)\psi_+, \tag{A4}$$

since

$$\vec{\gamma} = \gamma^0 i\gamma_5\vec{\sigma}. \tag{A5}$$

On eliminating the fields between the equations, we obtain

$$(E^2-m^2)\psi_\pm = (\vec{\sigma}\cdot\vec{\Pi}_\perp)^2\psi_\pm = (\vec{\Pi}_\perp{}^2 - eHq\sigma_3)\psi_\pm. \tag{A6}$$

Evidently, the energy eigenvalues are obtained by assigning to $\zeta = q\sigma_3$ an eigenvalue $\zeta' = \pm 1$, and, independently, introducing an eigenvalue for $\vec{\Pi}_\perp{}^2$. The familiar one-dimensional oscillator problem provides the latter spectrum

$$(\vec{\Pi}_\perp{}^2)' = (2n+1)eH, \quad n = 0, 1, 2, \ldots \tag{A7}$$

and one infers the energy eigenvalues

$$E^2 = m^2 + (2n+1-\zeta')eH. \tag{A8}$$

Note that, for the ground state of the system which is characterized by the quantum numbers $n = 0$ and $\zeta' = +1$, we have $E = m$. All other energy levels are doubly degenerate.

2. Expectation Values

In the above discussion, no distinction has been drawn in this account between the quantum numbers assigned to ψ_+ and ψ_-. Since σ_3 anticommutes with $\vec{\sigma}\cdot\vec{\Pi}_\perp$, the eigenvalues assigned to ζ in the two subspaces for a state of given energy must be of opposite sign, with corresponding differences in the eigenvalues of $\vec{\Pi}_\perp{}^2$. Thus, a more precise description of the eigenvalues associated with the energy, Eq. (A8), is given by[28]

$$(\zeta - \zeta')\psi_+ = 0, \quad [\vec{\Pi}_\perp{}^2 - (2n+1)eH]\psi_+ = 0, \tag{A9}$$

$$(\zeta + \zeta')\psi_- = 0, \quad [\vec{\Pi}_\perp{}^2 - (2n+1-2\zeta')eH]\psi_- = 0, \tag{A10}$$

which are summarized in the following characteristics of the complete ψ field:

$$\gamma^0\zeta\psi = \zeta'\psi, \quad \vec{\Pi}_\perp{}^2\psi = (2n+1-\zeta'+\zeta)\psi. \tag{A11}$$

To evaluate $\langle\gamma^0\rangle$, the expectation value of γ^0 between states satisfying Eq. (A2), we differentiate Eq. (A1) with respect to m,

$$(m+\gamma\Pi)\frac{\partial}{\partial m}\psi + \left(1 - \gamma^0\frac{\partial E}{\partial m}\right)\psi = 0, \tag{A12}$$

which implies

$$\langle\gamma^0\rangle = \left(\frac{\partial E}{\partial m}\right)^{-1} = \frac{E}{m}. \tag{A13}$$

The expectation value of $\langle\vec{\gamma}\cdot\vec{\Pi}_\perp\rangle$ can be obtained from Eq. (A2):

$$\langle\gamma^0 E - m - \vec{\gamma}\cdot\vec{\Pi}_\perp\rangle = 0, \tag{A14}$$

or

$$\langle\vec{\gamma}\cdot\vec{\Pi}_\perp\rangle = \langle\gamma^0\rangle E - m = (2n+1-\zeta')\frac{eH}{m}. \tag{A15}$$

Finally, from Eqs. (A11) and (A13), we have

$$\langle\zeta\rangle = \langle\gamma^0(\gamma^0\zeta)\rangle = \zeta'\frac{E}{m}. \tag{A16}$$

3. Energy Eigenvalues Including Radiative Corrections.

The general eigenvalue equation including radiative corrections is

$$(m + \gamma\Pi + M)\psi = 0, \tag{A17}$$

where M is given in Eq. (48). We first note that Eq. (A17) can be rewritten in the form (A_i's are numbers)

$$E'\psi = \gamma^0(A_1 + A_2\zeta + A_3\vec{\gamma}\cdot\vec{\Pi}_\perp + A_4\zeta\vec{\gamma}\cdot\vec{\Pi}_\perp)\psi, \tag{A18}$$

since M is a function of ζ, $\gamma\Pi$, $\vec{\gamma}\cdot\vec{\Pi}_\perp$, and $\zeta\vec{\gamma}\cdot\vec{\Pi}_\perp$ only. The characteristic equation can then be obtained by the method described in subsection 1 of this appendix or by other methods.[29,30] However, to solve the energy eigenvalues from the resulting characteristic equation is not a simple matter, since the energy eigenvalues E' also appear in the exponential factor [cf. Eq. (48)].

Nevertheless, if we are content with a solution that is accurate to order α, we may first approximate the various quantities in M according to Eqs. (A8), (A11), (A13), and (A15). In this way, M is effectively replaced by its expectation value between states satisfying Eq. (A1), and is a pure number. Therefore, the energy eigenvalues are [with $m \to m + M$ in Eq. (A8)]

$$E'^2 \approx (m+M)^2 + (2n+1-\zeta')eH \approx E^2 + 2mM, \tag{A19}$$

or

$$E' = E + \frac{m}{E}M, \tag{A20}$$

where E is given in Eq. (A8).

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Magnetic Bremsstrahlung and Modified Propagation Function. Spin-0 Charged Particles in a Homogeneous Magnetic Field*

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Recently, Schwinger was able to solve exactly the modified propagation function for the spin-0 charged particle in a homogeneous magnetic field. Here, we consider some applications of his result. In particular, the energy shift, the total decay rate, and the quantum corrections to the classical radiative spectrum are obtained explicitly.

I. INTRODUCTION

Recently, through the replacement of the photon momentum integration by an algebraic procedure[1] and the use of the proper-time method,[1,2] Schwinger was able to obtain an exact expression (to order α) for the modified inverse propagation function of a charged particle in a homogeneous magnetic field.[3,4] The interest in this problem can be seen both from the methodological point of view[5] and from the applications of the result to the discussion of synchrotron radiation in the cyclotrons and the storage rings,[6] and the high-energy strong-magnetic-field bremsstrahlung spectrum, which might be of great importance to the astrophysicist.[7]

The applications of the spin-$\frac{1}{2}$ modified propagator to obtain the mass shift, magnetic moment, radiative decay rate, and the power spectrum have been considered by Yildiz and the author in the preceding paper.[8] As for the analogous spin-0 case, even though the modified propagator has been evaluated by Schwinger in Paper I, there the only application discussed is to reproduce the known classical spectrum of radiation from the quantum point of view.[3] The purpose of this paper is to extend the application of his result to include the mass shift, radiative decay rate, and the quantum corrections to the classical radiative spectrum by following closely the discussions of Paper II.

II. MODIFIED PROPAGATION FUNCTION

In this section, we will review Schwinger's calculation[3] of the modified propagator for the spin-0 charged particle in a homogeneous magnetic field. One starts from the modified action term associ-