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Quantization of Free Spin-Two Fields on the Light Front*

Robert G. Root†
Physics Department, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801
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We show that the free spin-two fields, both massless and massive, can be consistently quantized on a light-front surface, $x^0 + x^3 =$ constant. In this formulation, the massless theory is easily shown to be the limit of the massive theory as $m^2 \to 0$.

I. INTRODUCTION

Quantum field theories may, in principle, be quantized not only on spacelike surfaces, but also on light-front surfaces.[1,2] Several authors have thoroughly investigated this alternate scheme in quantum electrodynamics[3-5]; they find that it may be quantized consistently on light-front surfaces. Interacting scalar and Dirac fields have also been considered and they too have consistent formulations on the light front.[6] The formal equivalence of the equal-time and light-front quantizations and their S-matrix expansions have been demonstrated.[5,6] In this paper we extend the work to include free spin-two fields, our purpose being twofold: (1) to show that, despite their added complexity, we can formulate a consistent light-front quantization of free spin-two fields by using the Schwinger quantum action principle,[7] and (2) to demonstrate that the resulting theory for free massive fields leads simply to the correct free massless theory as we take the limit of vanishing mass.

We are motivated to explore the quantization of higher spins, such as spin two, because, even for free fields, they present many tests of our quantization procedure not encountered in the lower-spin theories. Not only are they algebraically more complicated, because they require many more dependent fields in order to describe the system covariantly, but also they are physically more complex, as evidenced by the fact that the Schwinger condition for the stress tensor densities is not satisfied simply, but rather requires an extra nonlocal term.[8] Moreover, there are complications which arise when we take the limit of zero mass, and generally the proper massless theory cannot be achieved by this limiting process.[9]

We use Schwinger's quantum action principle[7] to suggest the commutation relations among the independent variables on the light front, and to describe the Lorentz generators. This approach to light-front quantization has already proved successful for lower-spin theories.[5,6] It is particularly useful for finding commutation relations in theories in which some of the field variables depend upon the dynamical variables in a complicated manner. Moreover, it is useful for finding reasonable commutation relations on unequal-time surfaces where we do not have the correspondence principle to rely on. (Of course, in this particular example of a free-field theory, we could use the limit on the light front of the known covariant commutators to suggest the light-front commutation relations, but we prefer not to follow this procedure because the method is not generally applicable, it does not test the Schwinger method, and it does not indicate the form of the Lorentz generators.)

Our description of the spin-two system in terms of the Lagrange function and the field equations is presented in Sec. II. At the same time we introduce the notation and conventions to be used throughout this paper.

Section III is devoted to the massless theory. We discuss gauge invariance, the solution of the field equations in terms of the independent variables, the light-front commutation relations among the fields, and the proof of Lorentz invariance of the theory.

The massive field theory is studied in Sec. IV. We stress the importance of finding a set of independent variables for which the commutation relations are diagonal. The solutions of the field equations in terms of the dynamical variables, the commutation relations, and the verification of Lorentz invariance are all included in this section.

Having obtained consistent formulations for both the massive and massless theories, we turn in Sec. V to the transition from the massive theories to the massless formulation as $m^2 \to 0$. It is demonstrated that, with a proper choice of variables, we may easily extract in this limit the massless spin-two system of Sec. III from the massive theory of Sec. IV.

In Sec. VI we conclude with a few remarks concerning the usefulness of the light-front quantization.

II. PRELIMINARIES

We choose a particular light-front frame to quantize on, one in which $x^0 + x^3 = \text{constant}$.[10] It is convenient to use the variables previously suggested by Susskind,[11] Bardakci and Halpern,[12] and Chang and Ma.[13] For any four-vector a^μ we define

$$a^{\pm} = a^0 \pm a^3 , \tag{2.1}$$

$$\vec{a} = (a^1, a^2) = a^i . \tag{2.2}$$

In terms of these components, the light front we quantize on is defined by $x^+ = \text{constant}$.

The nonzero components of the metric tensor $g^{\mu\nu}$ are given by

$$g^{+-} = g^{-+} = 2, \quad g^{ij} = -\delta^{ij}$$

or written in covariant form,

$$g_{+-} = g_{-+} = \tfrac{1}{2}, \quad g_{ij} = -\delta_{ij} .$$

The Latin indices can assume the values 1 or 2. Henceforth we will write all tensor indices which are not contracted as contravariant indices.

Then the scalar product of any two four-vectors can be written as

$$a\cdot b = a^\mu b_\mu = \tfrac{1}{2}a^+ b^- + \tfrac{1}{2}a^- b^+ - a^i b^i . \tag{2.3}$$

Our conventions result in the following definitions for derivatives:

$$\partial^{\pm} = \frac{\partial}{\partial x_{\pm}} = \frac{1}{g_{+-}}\frac{\partial}{\partial x^{\mp}} = \frac{2\partial}{\partial x^{\mp}} , \tag{2.4}$$

$$\partial^2 \equiv \partial^\mu \partial_\mu = \partial^+\partial^- - \partial^i\partial^i \equiv \partial^+\partial^- - \nabla^2 . \tag{2.5}$$

The four-dimensional volume element is

$$d^4x = \tfrac{1}{2}dx^+ dx^- d^2\vec{x} . \tag{2.6}$$

Often we will need to integrate over only three of the variables, hence we introduce the notation

$$d\sigma_x \equiv \tfrac{1}{2}d^2\vec{x}dx^- . \tag{2.7}$$

It is also convenient to define

$$\delta^3(x-y) \equiv \delta^2(\vec{x}-\vec{y})\delta(x^- - y^-) .$$

In our discussion we will often encounter differential constraint equations relating fields at the same x^+. Examples are

$$\partial_x^+ A(x) = B(x) \tag{2.8}$$

and

$$\partial_x^k C(x) = \partial_x^k D(x) , \tag{2.9}$$

where A, B, C, and D are all fields. It will often be necessary to solve for A in terms of B, and for C in terms of D.

First we must specify the boundary conditions. We assume that $C(x)$ and $D(x)$ have the same behavior as $|\vec{x}| \to \infty$, that is, that when C is expressed in terms of D no functions independent of $\vec{x}$ enter the expression. For fields occupying the position of A in (2.8) we choose the boundary conditions $A(x^+, \vec{x}, +\infty) = -A(x^+, \vec{x}, -\infty)$ which correspond to the Green's function $\epsilon(x^- - y^-)$. This choice of Green's function has important consequences later when we investigate the commutation relations among the field variables.

The solutions to the constraint equations are given by

$$A(x) = \tfrac{1}{4}\int dy^- \epsilon(x^- - y^-)B(x^+, \vec{x}, y^-) \equiv (\partial_x^+)^{-1}B(x) , \tag{2.10}$$

$$C(x) = D(x) . \tag{2.11}$$

Equation (2.10) defines the operator $(\partial_x^+)^{-1}$.

Often our differential constraints will involve more than one derivative (∂_x^+). The solution can then be rewritten by defining the product of distributions in terms of the Fourier transforms in momentum space.[5] For example if we have

$$F(x) = \partial_x^+\partial_x^+ E(x)$$

we can write the solution as

$$E(x) = \tfrac{1}{8}\int |x^- - z^-| F(x^+, \vec{x}, z^-) dz^- ,$$

where we have used the expression

$$2|x^- - z^-| = \int dy^- \epsilon(x^- - y^-)\epsilon(y^- - z^-)$$

for the product of the distributions.

When we calculate generators, it will be convenient to integrate by parts. In this paper we neglect all surface terms, our reason being that the field operator products in the generators are to be interpreted as distributions which have meaning only when smeared by proper test functions. This is the procedure adopted in Ref. 6.

Spin-two theories have been discussed by several authors.[14-16] In this paper the spin-two systems are described in terms of the fields and Lagrange functions used by Chang.[14] The tensor field is described by a symmetric tensor $h^{\mu\nu}$. We must also introduce a third-rank tensor ${}^\mu H^{\nu\lambda}$ which has the following properties:

$${}^\lambda H^{\mu\nu} = -{}^\lambda H^{\nu\mu} , \tag{2.12}$$

$${}^\lambda H^{\mu\nu} + {}^\mu H^{\nu\lambda} + {}^\nu H^{\lambda\mu} = 0 . \tag{2.13}$$

This additional field is employed to keep the Lagrange function linear in the gradient of the field variables as required by the Schwinger method. The Lagrangian density for a free spin-two field of mass m is then

$$\mathfrak{L}(x)=\tfrac{1}{2}(h_{\mu\nu}\partial_\lambda{}^\mu H^{\lambda\nu}-{}^\mu H^{\lambda\nu}\partial_\lambda h_{\mu\nu})-\tfrac{1}{4}({}_\mu H_{\nu\lambda}{}^\mu H^{\nu\lambda}-H_\lambda H^\lambda)-\tfrac{1}{2}m^2[h^{\mu\nu}h_{\mu\nu}-(h)^2]\,, \tag{2.14}$$

where we have used the shorthand notation $h=h^\mu_\mu$ and $H^\lambda={}^\mu H_\mu{}^\lambda$. The action is given by

$$W_{12}=\int_{\sigma_1}^{\sigma_2}dx^4\mathfrak{L}(x)\,, \tag{2.15}$$

where σ_1 and σ_2 are surfaces of constant x^+.

Application of the principle of stationary action leads to the field equations:

$$\partial_\lambda{}^{(\mu}H^{\lambda\nu)}-m^2(h^{\mu\nu}-g^{\mu\nu}h)=0\,, \tag{2.16}$$

$$2\,{}^\mu H^{\nu\lambda}-(g^{\mu\nu}H^\lambda-g^{\mu\lambda}H^\nu)+2(\partial^\nu h^{\mu\lambda}-\partial^\lambda h^{\mu\nu})=0\,, \tag{2.17}$$

where we have adopted the convention that, for indices enclosed within parentheses, the first and the last index are to be symmetrized, that is,

$$^{(\mu}H^{\lambda\nu)}=\tfrac{1}{2}({}^\mu H^{\lambda\nu}+{}^\nu H^{\lambda\mu})\,.$$

In the Schwinger interpretation of the action principle, we find the commutation relations among the fields from the freedom of changing description of the quantum system. The action must be invariant under variations $\delta h^{\mu\nu}$ and $\delta\,{}^\mu H^{\nu\lambda}$ which obey Eqs. (2.16) and (2.17). The generators of the unitary transformations which induce the variations $\delta h^{\mu\nu}$ and $\delta\,{}^\mu H^{\nu\lambda}$ are found from the surface terms in the variation of the action integral:

$$\delta W_{12}=G(x_1^+)-G(x_2^+)\,, \tag{2.18}$$

$$G(x^+)=\tfrac{1}{2}\int d\sigma_x(h^{\mu\nu}\delta\,{}_\mu H^+{}_\nu-{}^\mu H^{+\nu}\delta h_{\mu\nu})\,, \tag{2.19}$$

$$\tfrac{1}{2}i\delta h^{\mu\nu}=[h^{\mu\nu}(x),G(x^+)]\,, \tag{2.20}$$

$$\tfrac{1}{2}i\delta^\mu H^{\nu\lambda}=[{}^\mu H^{\nu\lambda},G(x^+)]\,. \tag{2.21}$$

We emphasize that already, in deducing the field equations and the generator $G(x^+)$, we have neglected total derivatives ∂^+ and ∂^i. We assume that surface terms like these can also be neglected in all subsequent calculations.

By straightforward algebraic manipulation, we can easily show that the field equations (2.16) and (2.17) are equivalent to the following set of equations, providing that $m^2\neq 0$:

$$h=0\,, \tag{2.22}$$

$$\partial_\mu h^{\mu\nu}=0\,, \tag{2.23}$$

$$^\mu H^{\nu\lambda}=\partial^\lambda h^{\mu\nu}-\partial^\nu h^{\mu\lambda}\,, \tag{2.24}$$

$$(\partial^2+m^2)h^{\mu\nu}=0\,. \tag{2.25}$$

These are the usual equations describing a spin-two field $h^{\mu\nu}$, (2.25) being the equation of motion, (2.22) and (2.23) being the usual auxiliary equations, and (2.24) being the definition of the additional field variables. These equations, (2.22)–(2.25), are all derived from the Lagrange function and the action principle, without any further input.

III. THE MASSLESS THEORY

For completeness and for future reference we present first the massless free spin-two field.[17] Gauge invariance is discussed in Sec. III A. The field equations are solved in terms of the independent variables in Sec. III B. Sections III C–III E are devoted to finding the commutation relations and verifying Lorentz invariance.

A. Gauge Invariance

The Lagrangian density for the massless theory is the same as (2.14) with $m^2=0$. The field equations found from the principle of stationary action are identical to (2.16)–(2.17) with $m^2=0$:

$$\partial_\lambda{}^{(\mu}H^{\lambda\nu)}=0\,, \tag{3.1}$$

$$2\,{}^\mu H^{\nu\lambda}+(g^{\mu\lambda}H^\nu-g^{\mu\nu}H^\lambda)+2(\partial^\nu h^{\mu\lambda}-\partial^\lambda h^{\mu\nu})=0\,. \tag{3.2}$$

The action, and hence the field equations, remains unchanged by the gauge transformation induced by

$$h^{\mu\nu}\to h^{\mu\nu}-\partial^\nu\xi^\mu-\partial^\mu\xi^\nu \tag{3.3}$$

provided that $^\mu H^{\nu\lambda}$ transforms according to

$$^\mu H^{\nu\lambda}\to{}^\mu H^{\nu\lambda}+(\partial^\mu\partial^\nu-g^{\mu\nu}\partial^2)\xi^\lambda-(\partial^\mu\partial^\lambda-g^{\mu\lambda}\partial^2)\xi^\nu+(g^{\mu\nu}\partial^\lambda-g^{\mu\lambda}\partial^\nu)(\partial_\sigma\xi^\sigma)\,. \tag{3.4}$$

Incidentally, this gauge freedom assures us that, in general, the field equations (3.1) and (3.2) cannot be rewritten in the form (2.22)–(2.25) with $m^2=0$.

This invariance is very useful for finding a set of independent dynamical variables. We shall quantize the field in a gauge in which the field equations are particularly simple and the independent variables easy to find.

Since we are using the so-called infinite-momentum variables $a^\pm$, we choose the infinite-momentum gauge $h^{+\mu}=0$ for all μ. This gauge has the advantage that the gauge requirements remain unaffected by the transformations induced by four of the six generators of the homogeneous Lorentz group, namely

$$K_3=\tfrac{1}{2}J^{-+},\quad E^i=J^{+i},\quad L_3=J^{12}\,,$$

where $J^{\mu\nu}$ are the usual generators of the Lorentz group [see Eq. (3.52)]. Only under transformations induced by $F^i=J^{-i}$ are conditions on the "+" components complicated by induced gauge transformations.

For any other choice of gauge for $h^{\mu\nu}$, we reduce the problem to solving the theory in the infinite-momentum gauge by choosing the dynamical variables from a set $\tilde{h}^{\mu\nu}$ where $\tilde{h}^{+\mu}=0$, all μ. This new set of variables is related to the original fields by

$$\tilde{h}^{\mu\nu} = h^{\mu\nu} + \partial^\mu \xi^\nu + \partial^\nu \xi^\mu , \tag{3.5}$$

where

$$\xi^\mu = \tfrac{1}{2}\partial^\mu (\partial^+)^{-2} h^{++} - (\partial^+)^{-1} h^{+\mu} . \tag{3.6}$$

B. The Field Equations and the Dynamical Variables

Having selected a gauge, we may separate (3.1) and (3.2) into equations of motion, which describe the x^+ development of the system, the equations of constraint, which relate field variables at the same x^+.

Equation (3.2) is an equation of motion when $\mu = +$, $\lambda = -$, and $\nu = i$; so is Eq. (3.1) when neither μ or ν is "+". These equations can be written as

$$\partial^- \, {}^+H^{+-} = \partial^+ \, {}^-H^{+-} + 4\partial^k ({}^+H^{k-}) , \tag{3.7}$$

$$\partial^- \, {}^-H^{+-} = -2\partial^k \, {}^-H^{-k} , \tag{3.8}$$

$$\partial^- \, {}^+H^{+i} = -2\partial^+ ({}^+H^{-i}) + 4\partial^k ({}^+H^{ki}) , \tag{3.9}$$

$$\partial^- ({}^-H^{+i}) = -\tfrac{1}{2}\partial^+ \, {}^-H^{-i} + 2\partial^k ({}^-H^{ki}) , \tag{3.10}$$

$$\partial^- ({}^iH^{+j}) = -\partial^+ ({}^iH^{-j}) + 2\partial^k ({}^iH^{kj}) , \tag{3.11}$$

$$\partial^- h^{-i} = \partial^i h^{--} - {}^-H^{-i} , \tag{3.12}$$

$$\partial^- h^{ij} = \partial^i h^{-j} + {}^jH^{-i} + \tfrac{1}{2}\delta^{ij}[\tfrac{1}{2}\, {}^-H^{+-} + {}^kH^{+k}] . \tag{3.13}$$

When $\mu = +$ in (3.2) we obtain simple algebraic constraints:

$${}^+H^{+-} = -{}^kH^{+k} , \tag{3.14}$$

$${}^-H^{+i} - {}^+H^{-i} = 2\, {}^kH^{ki} , \tag{3.15}$$

$${}^+H^{+i} = 0 , \tag{3.16}$$

$${}^+H^{ij} = 0 . \tag{3.17}$$

The rest of the equations are differential constraints:

$$\partial^+ \, {}^+H^{+-} = 2\partial^k \, {}^+H^{+k} , \tag{3.18}$$

$$\partial^+ h^{--} = -\tfrac{1}{2}\, {}^-H^{+-} + {}^kH^{-k} , \tag{3.19}$$

$$\partial^+ h^{-i} = -{}^iH^{+-} , \tag{3.20}$$

$$\partial^+ h^{ij} = -{}^jH^{+i} - \tfrac{1}{2}\delta^{ij}[\tfrac{1}{2}\, {}^+H^{+-} - {}^kH^{+k}] , \tag{3.21}$$

$$\partial^+ h^{-i} = \tfrac{1}{2}[{}^-H^{i+} - {}^+H^{i-}] - {}^kH^{ki} , \tag{3.22}$$

$${}^-H^{ij} = \partial^j h^{-i} - \partial^i h^{-j} , \tag{3.23}$$

$$\partial^2 h^{11} - \partial^1 h^{12} = \tfrac{1}{2}[{}^1H^{12} + {}^+H^{-2} + {}^-H^{+2}] , \tag{3.24}$$

$$\partial^1 h^{22} - \partial^2 h^{12} = \tfrac{1}{2}[{}^2H^{21} + {}^+H^{-1} + {}^-H^{+1}] . \tag{3.25}$$

When necessary, these differential constraints may be integrated and expressed in the form (2.10) or (2.11).

From the combination of (3.14), (3.16)–(3.18), and (3.21) we can deduce

$${}^+H^{+i} = {}^+H^{ij} = {}^+H^{+-} = {}^kH^{k+} = 0 , \tag{3.26}$$

$${}^iH^{+j} = -\partial^+ h^{ij} . \tag{3.27}$$

The tensor h^{ij} is known to be symmetric and (3.27) shows it to be traceless. Thus it represents only two independent components and they are a possible set of dynamical variables. We will express all other field components in terms of h^{ij}, instead of choosing a particular two components.

Since the variables in (3.26) are now known to vanish, two equations of motion, (3.7) and (3.9), can be rewritten as equations of constraint. Note also that Eqs. (3.15), (3.20), and (3.22) lead to only two independent equations.

We can now find expressions for the remaining field variables. From Eqs. (3.9), (3.24), (3.25), (3.15), and (3.20) we find that ${}^+H^{-i}=0$, as well as finding expressions for ${}^-H^{+i}$, ${}^iH^{+-}$, ${}^1H^{12}$, ${}^2H^{21}$, and h^{-i} in terms of the dynamical variables h^{ij}. Knowing these relations, we may obtain ${}^-H^{+-}$ from (3.7) and ${}^-H^{ij}$ from (3.23). We can solve (3.11) and (3.13) for ${}^iH^{-j}$ by using the expressions already found for the other variables. Similarly we find h^{--} from Eq. (3.19).

Finally, the last variable remaining to be found in terms of the dynamical variables is ${}^-H^{-i}$, and it can be extracted from Eqs. (3.12) and (3.10). Equation (3.8) provides no new information; however, it is consistent with all the other equations.

The expressions for the covariant field variables in terms of the dynamical variables h^{ij} are

$$h^{+\mu} = 0 \text{ for all } \mu , \tag{3.28}$$

$$h^{-i} = 2(\partial^+)^{-1}(\partial^k h^{ki}) , \tag{3.29}$$

$$h^{--} = 4(\partial^+)^{-2}(\partial^k \partial^l h^{kl}) , \tag{3.30}$$

$${}^+H^{\mu\nu} = 0 \text{ for all } \mu, \nu , \tag{3.31}$$

$${}^-H^{+-} = -4(\partial^+)^{-1}(\partial^k \partial^l h^{kl}) , \tag{3.32}$$

$${}^-H^{-i} = 2(\partial^+)^{-2}[2\partial^i(\partial^k\partial^l h^{kl}) - \nabla^2(\partial^k h^{ki})] , \tag{3.33}$$

$${}^-H^{ij} = 2(\partial^+)^{-1}[\partial^j(\partial^k h^{ki}) - \partial^i(\partial^k h^{kj})] , \tag{3.34}$$

$${}^iH^{+-} = {}^-H^{+i} = -2(\partial^k h^{ki}) , \tag{3.35}$$

$${}^iH^{+j} = -\partial^+ h^{ij} , \tag{3.36}$$

$$\begin{aligned} {}^iH^{jk} &= \delta^{ik}(\partial^l h^{lj}) - \delta^{ij}(\partial^l h^{lk}) \\ &= \partial^k h^{ij} - \partial^j h^{ik} , \end{aligned} \tag{3.37}$$

$${}^iH^{-j} = (\partial^+)^{-1}[\delta^{ij}(\partial^k\partial^l h^{kl}) - \partial^i(\partial^k h^{kj}) + \partial^j(\partial^k h^{ki})] . \tag{3.38}$$

The equation of motion of the dynamical variables was found when we solved above for ${}^iH^{-j}$. It is

$$\partial^- h^{ij} = (\partial^+)^{-1}\nabla^2 h^{ij}\,. \tag{3.39}$$

Equation (3.39) implies that, for all field variables,

$$\partial^2 h^{\mu\nu} = \partial^2\, {}^\mu H^{\nu\lambda} = 0\,. \tag{3.40}$$

Inspection of the solutions also reveals that the following equations are satisfied:

$${}^\mu H^{\nu\lambda} = \partial^\lambda h^{\mu\nu} - \partial^\nu h^{\mu\lambda}\,, \tag{3.41}$$

$$h = 0\,, \tag{3.42}$$

$$\partial^\mu h_{\mu\nu} = 0\,. \tag{3.43}$$

Hence, in this particular gauge, the field variables obey Eqs. (2.22)–(2.25) with $m^2=0$. We keep in mind, however, that these equations can hold only in a few special gauges. [The other gauges for which (3.40)–(3.43) hold differ by a gauge transformation which obeys $\partial_\mu \xi^\mu = 0$ and $\partial^2\xi^\mu = 0$.]

C. Commutation Relations Among the Dynamical Variables

The light-front commutation relations among the fields are determined by relations (2.20) and (2.21). The commutators involving independent variables are the easiest to determine. In terms of the dynamical variables the generator (2.19) is simply

$$G(x^+) = \tfrac{1}{2}\int d\sigma_x [h^{kl}\,\delta^k H^{+l} - {}^k H^{+l}\,\delta h^{kl}]$$
$$= \int d\sigma_x [(\partial^+ h^{kl})\delta h^{kl}]\,. \tag{3.44}$$

Then, using

$$\tfrac{1}{2} i\,\delta h^{kl} = [h^{kl}(x), G(x^+)]\,, \tag{3.45}$$

we find for $x^+ = y^+$

$$[h^{kl}(x), \partial_y^+ h^{mn}(y)] = \tfrac{1}{2} i[\delta^{km}\delta^{ln} + \delta^{kn}\delta^{lm} - \delta^{kl}\delta^{mn}]$$
$$\times \delta^3(x-y)\,, \tag{3.46}$$

where we have assumed that δh^{kl} is a commuting c number since $h^{\mu\nu}$ represents a field obeying Bose statistics. We may rewrite (3.46) by using (2.10) as

$$[h^{kl}(x), h^{mn}(y)] = -\tfrac{1}{2} i[\delta^{km}\delta^{ln} + \delta^{kn}\delta^{lm} - \delta^{kl}\delta^{mn}]$$
$$\times \tfrac{1}{4}\epsilon(x^- - y^-)\delta^2(x-y)\,. \tag{3.47}$$

We now see the importance of using the Green's function $\frac{1}{4}\epsilon(x^- - y^-)$. Had we used another Green's function denoted by $K(x^- - y^-)$, then $K(y^- - x^-)$ would replace $-\frac{1}{4}\epsilon(x^- - y^-)$ on the right-hand side of (3.47). However, the left-hand side is antisymmetric under the interchange of x and y; thus $K(x^- - y^-)$ must also be antisymmetric. Hence we are led back to the choice of $K(x^- - y^-) = \frac{1}{4}\epsilon(x^- - y^-)$.

The commutation relations among the field variables may be derived from (3.47) and expressions (3.28)–(3.38). We need only consider the commutation relations among the fields $h^{\mu\nu}$ in order to compare our quantized field with the field quantized on the equal-time plane with the gauge $h^{+\mu}=0$. The commutators are expressed conveniently in the form

$$[h^{\mu\nu}(x), h^{\sigma\tau}(y)] = -\int \frac{d^4k}{(2\pi)^3}\delta(k^2)\epsilon(k^+)$$
$$\times e^{ik(x-y)} P^{\mu\nu\sigma\tau}\,,$$

where, in terms of the vector $n^\mu = g^{+\mu}$, we have

$$P^{\mu\nu\sigma\tau} = \tfrac{1}{2}(g^{\sigma\mu}g^{\nu\tau} + g^{\mu\tau}g^{\nu\sigma} - g^{\mu\nu}g^{\sigma\tau}) + \frac{1}{(n\cdot k)^2}\{n^\mu n^\nu k^\sigma k^\tau + n^\sigma n^\tau k^\mu k^\nu\}$$
$$- \frac{1}{2(n\cdot k)}\{g^{\nu\tau}(n^\mu k^\sigma + n^\sigma k^\mu) + g^{\mu\sigma}(n^\nu k^\tau + n^\tau k^\nu) + g^{\nu\sigma}(n^\mu k^\tau + n^\tau k^\mu)$$
$$+ g^{\mu\tau}(n^\nu k^\sigma + n^\sigma k^\nu) - g^{\mu\nu}(n^\sigma k^\tau + n^\tau k^\sigma) - g^{\sigma\tau}(n^\mu k^\nu + n^\nu k^\mu)\}\,.$$

It is easy to verify that $P^{\mu\nu\sigma\tau}$ equals the sum of possible polarization tensors in this gauge;

$$P^{\mu\nu\sigma\tau} = \sum_{\lambda=1}^{2} \epsilon^{\mu\nu}(k,\lambda)\epsilon^{\sigma\tau}(k,\lambda)\,,$$

where $\epsilon^{\mu\nu}(k,\lambda)$ are traceless and symmetric and obey

$$n_\mu \epsilon^{\mu\nu}(k,\lambda) = 0\,,$$
$$\epsilon_{\mu\nu}(k,\lambda)\epsilon^{\mu\nu}(k,\lambda') = \delta^{\lambda\lambda'}\,,$$
$$k_\mu \epsilon^{\mu\nu}(k,\lambda) = 0\,.$$

Thus our quantization procedure gives the same results we would obtain by writing the massless spin-two field in the gauge $h^{+\mu}=0$ in terms of operators which create and destroy particles of momentum k^μ, helicity λ, and spin 2.

D. The Lorentz Generators

We could demonstrate Lorentz invariance by reverting to the generators defined as integrals over on equal-time plane; however, we prefer to express the Lorentz generators as integrals over the light front. We will need these new expressions when we consider the massless limit of the massive theory.

The ten Lorentz generators are constructed from the stress tensor density $T^{\mu\nu}$, which is defined as a measure of the response of the system to a spacetime displacement.

$$x^\mu \to x^\mu + \delta x^\mu , \tag{3.48}$$

$$\delta W = -\int d^4x \; T^{\mu\nu}\partial_\mu(\delta x_\nu) . \tag{3.49}$$

Defined in this manner, $T^{\mu\nu}$ can be chosen to be symmetric and it is conserved. Using standard procedures we find for the massless field[14]

$$T^{\mu\nu} = -\mathcal{L}g^{\mu\nu} - {}_\lambda H^{(\mu\sigma}\partial^{\nu)}h^\lambda{}_\sigma + \partial_\lambda[{}^\lambda H^{(\mu\sigma}h^{\nu)}{}_\sigma + {}^\sigma H^{(\mu\lambda}h^{\nu)}{}_\sigma - {}^{(\mu}H^{\nu)\sigma}h^\lambda{}_\sigma] , \tag{3.50}$$

where we understand that the bilinear products are always properly symmetrized; that is, $h^{\mu\nu}{}_\mu H_{\lambda\nu}$ means $\frac{1}{2}[h^{\mu\nu}{}_\mu H_{\lambda\nu} + {}_\mu H_{\lambda\nu}h^{\mu\nu}]$.

The definitions of the energy-momentum and angular momentum operators, written in terms of the stress tensor densities, are

$$P^\mu(x^+) = \int d\sigma_x T^{+\mu} , \tag{3.51}$$

$$J^{\mu\nu}(x^+) = \int d\sigma_x (x^\mu T^{+\nu} - x^\nu T^{+\mu}) . \tag{3.52}$$

Since $T^{\mu\nu}$ is conserved, it is easy to show that the Lorentz generators (3.51) and (3.52) are constants of the motion. Thus they may be defined on any surface where $x^+ =$ constant. For convenience we choose $x^+ = 0$. Then the generators normally associated with an infinite-momentum frame are P^μ and

$$K_3 = \tfrac{1}{2}J^{-+} = \int d\sigma_x(\tfrac{1}{2}x^- T^{++}) , \tag{3.53}$$

$$E^i = J^{+i} = -\int d\sigma_x (x^i T^{++}) , \tag{3.54}$$

$$L_3 = J^{12} = \int d\sigma_x (x^1 T^{+2} - x^2 T^{+1}) , \tag{3.55}$$

$$F^i = J^{-i} = \int d\sigma_x (x^- T^{+i} - x^i T^{+-}) . \tag{3.56}$$

In order to calculate commutators, the Lorentz generators are rewritten in terms of the dynamical variables

$$P^\mu = \int d\sigma_x (\partial^+ h^{kl})(\partial^+ h^{kl}) , \tag{3.57}$$

$$E^i = -\int d\sigma_x x^i (\partial^+ h^{kl})(\partial^+ h^{kl}) , \tag{3.58}$$

$$K_3 = \tfrac{1}{2}\int d\sigma_x (x^- \partial^+ h^{kl})(\partial^+ h^{kl}) , \tag{3.59}$$

$$L_3 = \int d\sigma_x [x^1(\partial^+ h^{kl})(\partial^+ h^{kl}) - x^2(\partial^+ h^{kl})(\partial^+ h^{kl}) + 4h^{k1}\partial^+ h^{k2}] , \tag{3.60}$$

$$F^i = \int d\sigma_x [x^-(\partial^+ h^{kl})(\partial^i h^{kl}) - x^i(\partial^+ h^{kl})(\partial^- h^{kl}) - 4(\partial^k h^{kl})h^{il}] , \tag{3.61}$$

where we have taken advantage of the known expression (3.39) for $\partial^- h^{kl}$.

For a ground-state vacuum to exist, the x^+ Hamiltonian operator, P^-, must be positive definite. It indeed satisfies this criterion:

$$P^- = \int d\sigma_x [-h^{kl}\nabla^2 h^{kl}] = \int d\sigma_x (\partial^m h^{kl})(\partial^m h^{kl}) \geq 0 . \tag{3.62}$$

There is a similar requirement on P^+, since $P^2 > 0$ for all states, and it is also fulfilled:

$$P^+ = \int d\sigma_x (\partial^+ h^{kl})(\partial^+ h^{kl}) \geq 0 . \tag{3.63}$$

Another requirement is that the vacuum expectation value of the equal-x^+ commutator $[i\partial^- A, A]$, for an arbitrary operator A, be positive definite.[18] For those $A(x^+)$ which are linear in the dynamical variables, the relation is easily verified.

Let

$$A(x^+) = \int d\sigma_x f^{kl}(x) h^{kl}(x) ,$$

where $f^{kl}(x)$ is a numerical function which can be chosen to be both symmetric and traceless. Then we find the operator equation

$$[i\partial^- A(x^+), A(y^+)] = -\int d\sigma_x \int d\sigma_y [f^{kl}(x)f^{kl}(y) - \tfrac{1}{2}f^{kk}(x)f^{ll}(y)](\partial_x^+)^{-1}(\nabla_x^2 - m^2)(\partial_y^+)^{-1}\delta^3(x-y)$$
$$= \int d\sigma_x \{((\partial_x^+)^{-1}\partial^m f^{kl}(x))((\partial_x^+)^{-1}\partial^m f^{kl}(x)) + m^2[(\partial_x^+)^{-1}f^{kl}(x)][(\partial_x^+)^{-1}f^{kl}(x)]\} > 0 .$$

E. Lorentz Invariance

For our theory to be consistent, we must verify that the generators defined in (3.51)–(3.56) have the commutation relations that the Lorentz generators must satisfy. To that end, we first calculate the commutators of the dynamical variables with the generators:

$$[h^{kl}(x), P^\mu] = i\partial^\mu h^{kl}(x)\,, \tag{3.64}$$

$$[h^{kl}(x), E^i] = -ix^-\partial^+ h^{kl}(x)\,, \tag{3.65}$$

$$[h^{kl}(x), K_3] = \tfrac{1}{2} ix^-\partial^+ h^{kl}(x)\,, \tag{3.66}$$

$$\begin{aligned}[h^{kl}(x), L_3] = {} & i[x^1\partial^2 - x^2\partial^1]h^{kl}(x) \\ & + i[\delta^{k2}h^{l1}(x) + \delta^{l2}h^{k1}(x)] \\ & - i[\delta^{k1}h^{l2} + \delta^{l1}h^{k2}]\,,\end{aligned} \tag{3.67}$$

$$\begin{aligned}[h^{kl}(x), F^i] = {} & i[x^-\partial^i - x^i\partial^-]h^{kl}(x) \\ & + i[\delta^{ik}h^{-l}(x) + \delta^{il}h^{-k}(x)] \\ & - 2i(\partial_x^+)^{-1}[\partial^k h^{il} + \partial^l h^{ik}]\,.\end{aligned} \tag{3.68}$$

Thus h^{kl} transforms like a second-rank tensor under P^μ, E^i, K_3, and L_3. The generator F^i does not preserve the gauge requirements, so that transformations of h^{kl} induced by F^i must be accompanied by a gauge transformation

$$h^{kl} \to h^{kl} - \partial^k \xi^l - \partial^l \xi^k\,, \tag{3.69}$$

with

$$\xi^k = 2i(\partial^+)^{-1}h^{ik}\,. \tag{3.70}$$

We now complete the tedious task of verifying the Lorentz invariance of this formulation of the massless spin-two field by showing that the generators have the correct commutation relations among themselves. They are

$$[P^\mu, P^\nu] = 0\,, \tag{3.71}$$

$$i[P^\alpha, J^{\mu\nu}] = -g^{\alpha\mu}P^\nu + g^{\alpha\nu}P^\mu\,, \tag{3.72}$$

$$i[J^{\mu\nu}, J^{\sigma\lambda}] = -g^{\mu\sigma}J^{\lambda\nu} + g^{\mu\lambda}J^{\sigma\nu} - g^{\nu\sigma}J^{\mu\lambda} + g^{\nu\lambda}J^{\mu\sigma}. \tag{3.73}$$

Our massless theory is thus shown to be Lorentz-invariant, and to satisfy the positivity requirements.

IV. THE MASSIVE THEORY

The field equations for the massive theory are easy to solve in terms of five independent variables. However, most choices of independent variables are impractical for evaluating commutators. In Sec. IV A we find a set of dynamical variables which have diagonal commutation relations, and we use them to find the commutation relations among the fields. The expressions for the Lorentz generators in terms of the independent variables are derived in Sec. IV B, and Lorentz invariance is verified.

A. Commutation Relations Among the Field Variables

When the spin-two field has mass, the field equations can immediately be recast in the form

$$h^\mu_\mu = 0\,, \tag{4.1}$$

$$\partial^\mu h^{\mu\nu} = 0\,, \tag{4.2}$$

$${}^\mu H^{\nu\lambda} = \partial^\lambda h^{\mu\nu} - \partial^\nu h^{\mu\lambda}\,, \tag{4.3}$$

$$(\partial^2 + m^2)h^{\mu\nu} = 0\,. \tag{4.4}$$

We can treat (4.4) as the equation of motion:

$$\partial^- h^{\mu\nu} = (\partial^+)^{-1}(\nabla^2 - m^2)h^{\mu\nu}\,. \tag{4.5}$$

The third-rank tensor ${}^\mu H^{\nu\lambda}$ also obeys the same equation of motion. Thus we may choose any five independent fields from $h^{\mu\nu}$ and ${}^\mu H^{\nu\lambda}$ as our dynamical variables. All the other variables are then defined by the equations of constraint (4.1)–(4.3). It is easy to solve these equations to find the covariant field variables in terms of any independent variables, but most sets of dynamical variables have complicated commutators among themselves. Since we have a large number of commutators to evaluate, we require a set of variables which has simple commutation relations. A set which diagonalizes the generator matrix is ideal.

The simplest method of finding this ideal set of variables is by trial and error. A set with the required properties is h^{++}, ${}^+H^{+k}$, and C^{kl}, where

$$\begin{aligned}C^{kl} \equiv {} & [\partial^{+\,(k}H^{+l)} - \partial^{\,(k\,+}H^{+l)}] \\ & - \tfrac{1}{2}\delta^{kl}[\partial^{+\,m}H^{+m} - \partial^{m\,+}H^{+m}]\,.\end{aligned} \tag{4.6}$$

Note that C^{kl} is both symmetric and traceless, as h^{kl} is in the massless theory. In order to minimize the number of "+" derivatives which appear in the commutation relations and in the Lorentz generators, it is preferable to use the following independent variables:

$$\alpha = (\partial^+)^{-2}h^{++}\,, \tag{4.7}$$

$$\beta^k = (\partial^+)^{-2}\,{}^+H^{+k}\,, \tag{4.8}$$

$$\gamma^{kl} = (\partial^+)^{-2}C^{kl}\,. \tag{4.9}$$

All the field variables can be rewritten in terms of these variables. These expressions for $h^{\mu\nu}$ and ${}^\mu H^{\nu\lambda}$ are given in Appendix A.

The generator is, as promised, diagonal when expressed in these variables. It is given by

$$\begin{aligned}G(x^+) = {} & \tfrac{1}{8}\int d\sigma_x[h^{++}\delta\,{}^-H^{+-} + h^{+-}\delta\,{}^+H^{+-} - 4h^{+k}\delta\,^{(-}H^{+k)} - 2h^{-k}\delta\,{}^+H^{+k} + 4h^{kl}\delta\,^{(k}H^{+l)} - {}^-H^{+-}\delta h^{++} - {}^+H^{+-}\delta h^{+-} \\ & + 4\,^{(-}H^{+k)}\delta h^{+k} + 2\,{}^+H^{+k}\,\delta h^{-k} - {}^{(k}H^{+l)}\,\delta h^{kl}] \\ = {} & \int d\sigma_x[\tfrac{3}{2}m^4(\partial^+\alpha)(\delta\alpha) + 2m^2(\partial^+\beta^k)(\delta\beta^k) + (\partial^+\gamma^{kl})(\delta\gamma^{kl})]\,.\end{aligned} \tag{4.10}$$

From the expression of the generator in terms of the independent variables, the commutation relations among the dynamical variables follow from

$$\tfrac{1}{2}i\delta f(x)=[f(x),G(x^+)]\,, \tag{4.11}$$

where $f(x)$ is any of the dynamical variables, and $\delta f(x)$ is treated as a commuting c number.

The commutation relations are found to be (for $x^+=y^+$)

$$[\alpha(x),\alpha(y)]=-\frac{2i}{3m^4}\,\tfrac{1}{4}\epsilon(x^- - y^-)\delta^2(\vec{x}-\vec{y})\,, \tag{4.12}$$

$$[\beta^k(x),\beta^l(y)]=-\frac{i}{2m^2}\,\delta^{kl}\tfrac{1}{4}\epsilon(x^- - y^-)\delta^2(\vec{x}-\vec{y})\,, \tag{4.13}$$

$$[\gamma^{kl}(x),\gamma^{mn}(y)]=-\tfrac{1}{2}i[\delta^{km}\delta^{ln}+\delta^{kn}\delta^{lm}-\delta^{kl}\delta^{mn}] \times\tfrac{1}{4}\epsilon(x^- - y^-)\delta^2(\vec{x}-\vec{y})\,, \tag{4.14}$$

$$[\alpha(x),\beta^k(y)]=[\alpha(x),\gamma^{kl}(y)]=[\beta^k(x),\gamma^{mn}(y)]=0\,. \tag{4.15}$$

It is only a matter of algebra to find the commutation relations among the field variables with definite transformation properties. Once found, these relations can be rewritten in the simple covariant form:

$$\begin{aligned}[h^{\mu\nu}(x),h^{\lambda\sigma}(y)] &= i\Big[\tfrac{1}{2}(g^{\mu\lambda}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\lambda}-\tfrac{2}{3}g^{\mu\nu}g^{\lambda\sigma})\\ &\quad+\frac{1}{2m^2}(g^{\mu\lambda}\partial^\nu\partial^\sigma+g^{\mu\sigma}\partial^\nu\partial^\lambda+g^{\nu\lambda}\partial^\mu\partial^\sigma+g^{\nu\sigma}\partial^\mu\partial^\lambda)\\ &\quad-\frac{1}{3m^2}(g^{\mu\nu}\partial^\lambda\partial^\sigma+g^{\lambda\sigma}\partial^\mu\partial^\nu)+\frac{2}{3m^4}\partial^\mu\partial^\nu\partial^\lambda\partial^\sigma\Big]\\ &\quad\times\Delta(x-y)\,,\end{aligned} \tag{4.16}$$

where we have used the invariant function

$$\Delta(x-y,m^2)=i\int\frac{d^4k}{(2\pi)^3}\,\delta(k^2-m^2)\epsilon(k^+)e^{ik(x-y)}\,, \tag{4.17}$$

which has the following properties:

$$\begin{aligned}&\partial_x^+\Delta(x-y)|_{x^+=y^+}=-\delta^3(x-y)\,,\\ &\Delta(x-y)|_{x^+=y^+}=-\tfrac{1}{4}\epsilon(x^- - y^-)\delta(\vec{x}-\vec{y})\,,\end{aligned} \tag{4.18}$$

$$\partial^-\Delta(x-y,m^2)=(\partial_x^+)^{-1}(\nabla_x{}^2-m^2)\Delta(x-y)\,. \tag{4.19}$$

The second property shows that the expression (4.18) has the proper δ-function character on the light front, while the last expression indicates that the equations of motion for the fields and for $\Delta(x-y,m^2)$ are the same. Hence we can express the commutator in terms of $\Delta(x-y)$ for all space.

At this point we can identify our quantized field as being the same as the one found from quantizing on an equal-time surface since the covariant commutation relations are identical.

Similar expressions involving ${}^\mu H^{\nu\lambda}$ can easily be derived by using (4.3); they will not be reproduced here.

B. The Lorentz Generators and Their Commutation Relations

The stress tensor density used to define the ten Lorentz generators is found by the standard method[14] to be

$$\begin{aligned}T^{\mu\nu}&=\tfrac{1}{2}g^{\mu\nu}\partial_\lambda({}^\alpha H^{\lambda\beta}h_{\alpha\beta})-{}_\lambda H^{(\mu\sigma}\partial^{\nu)}h^\lambda{}_\sigma\\ &\quad+\partial_\lambda[{}^{(\mu}H^{\sigma\nu)}h^\lambda{}_\sigma+{}^\lambda H^{(\mu\sigma}h^{\nu)}{}_\sigma+{}^\sigma H^{(\mu\lambda}h^{\nu)}{}_\sigma]\,.\end{aligned} \tag{4.20}$$

It is a lengthy calculation to express the generators of the Lorentz group in terms of the dynamical variables. The expressions are simplified, however, by defining the function

$$\begin{aligned}R^\mu&=\tfrac{3}{2}m^4(\partial^+\alpha)(\partial^\mu\alpha)+2m^2(\partial^+\beta^k)(\partial^\mu\beta^k)\\ &\quad+(\partial^+\gamma^{kl})(\partial^\mu\gamma^{kl})\,.\end{aligned} \tag{4.21}$$

Then the generators are given by

$$P^\mu=\int d\sigma_x R^\mu(x)\,, \tag{4.22}$$

$$E^i=J^{+i}=\int d\sigma_x x^i R^+(x)\,, \tag{4.23}$$

$$K_3=\tfrac{1}{2}J^{-+}=\int d\sigma_x x^- R^+(x)\,, \tag{4.24}$$

$$L_3=J^{12}=\int d\sigma_x[x^1R^2(x)-x^2R^1(x) +4m^2\beta^1\partial^+\beta^2+4\gamma^{k1}\partial^+\gamma^{k2}]\,, \tag{4.25}$$

$$\begin{aligned}F^i=J^{-i}=\int d\sigma_x[&x^-R^i(x)-x^iR^-(x)\\ &+12m^4\alpha\beta^i-8m^2(\partial^k\beta^k)\beta^i\\ &-8m^2\beta^l\gamma^{li}-8\gamma^{li}(\partial^k\gamma^{kl})]\,.\end{aligned} \tag{4.26}$$

We must verify that these expressions do indeed have the properties required of the Lorentz generators. First we note that, as required, both P^+ and P^- are positive definite. Likewise, the commutator $\langle 0|[i\partial^- A,A]|0\rangle$, where A is an arbitrary operator, can easily be shown to be positive for operators linear in the dynamical variables. It remains to show that P^μ and $J^{\mu\nu}$ do in fact generate Lorentz transformations.

For calculational purposes, we first evaluate the commutation relations of the dynamical variables with the Lorentz generators, and express them entirely in terms of the dynamical variables. These results are given in Appendix B.

In order to verify Lorentz covariance, we must find the commutation relations of the field variables and the Lorentz generators and express them in terms of the covariant field variables. These relations for the field variables with definite transformation properties, $h^{\mu\nu}$ and ${}^{\mu}H^{\nu\lambda}$, can be obtained from the expressions in Appendixes A and B. They are found to be the relations required for the interpretation of P^{μ} and $J^{\mu\nu}$ as the Lorentz generators. These relations are expressed most easily in the covariant form as

$$[{}^{\mu}H^{\sigma\nu}, P^{\lambda}] = i\partial_x^{\lambda}\, {}^{\mu}H^{\sigma\nu}(x)\,, \tag{4.27}$$

$$[h^{\mu\nu}(x), P^{\lambda}] = i\partial_x^{\lambda} h^{\mu\nu}(x)\,, \tag{4.28}$$

$$\begin{aligned}[h^{\mu\nu}(x), J^{\sigma\tau}] = {} & i(x^{\sigma}\partial^{\tau} - x^{\tau}\partial^{\sigma})h^{\mu\nu}(x) \\ & + i[h^{\mu\tau}(x)g^{\nu\sigma} + g^{\mu\sigma}h^{\tau\nu}(x)] \\ & - i[g^{\mu\tau}h^{\sigma\tau}(x) + g^{\nu\tau}h^{\mu\sigma}(x)]\,,\end{aligned} \tag{4.29}$$

$$\begin{aligned}[{}^{\mu}H^{\lambda\nu}(x), J^{\sigma\tau}] = {} & i(x^{\sigma}\partial^{\tau} - x^{\tau}\partial^{\sigma})\, {}^{\mu}H^{\lambda\nu}(x) \\ & + i[{}^{\tau}H^{\lambda\nu}(x)g^{\mu\sigma} + {}^{\mu}H^{\tau\nu}(x)g^{\lambda\sigma} + {}^{\mu}H^{\lambda\tau}(x)g^{\nu\sigma}] \\ & - i[g^{\mu\tau}\, {}^{\sigma}H^{\lambda\nu}(x) + g^{\lambda\tau}\, {}^{\mu}H^{\tau\nu}(x) + g^{\nu\tau}\, {}^{\mu}H^{\lambda\sigma}(x)]\,.\end{aligned} \tag{4.30}$$

By the use of (4.20)–(4.23), it is straightforward to obtain the commutation relations among the generators themselves. They are indeed the ones demanded by Lorentz invariance:

$$[P^{\mu}, P^{\nu}] = 0\,, \tag{4.31}$$

$$[P^{\mu}, J^{\lambda\nu}] = i[g^{\mu\lambda}P^{\nu} - g^{\mu\nu}P^{\lambda}]\,, \tag{4.32}$$

$$[J^{\mu\nu}, J^{\sigma\tau}] = i[g^{\mu\sigma}J^{\tau\nu} + g^{\nu\sigma}J^{\mu\tau} - g^{\mu\tau}J^{\sigma\nu} - g^{\nu\tau}J^{\mu\sigma}]\,. \tag{4.33}$$

This completes the discussion of the massive spin-two field. We have omitted such topics as finding the covariant Green's function and showing that the system represented is indeed a spin-two system. These calculations can be done without reference to the frame of quantization,[14] so their inclusion here would be redundant.

V. THE MASSLESS LIMIT OF THE MASSIVE FIELD

It is easy, in the foregoing theory, to take the limit as $m^2 \to 0$.[9] We must, however, be careful to use dynamical variables which correctly describe the 5 degrees of freedom of the massive field in this limit. To this end, some of the dynamical variables must be rescaled in order to avoid singularities in their commutation relations caused by the vanishing of the mass. We define the new set of independent dynamical variables as follows:

$$\varphi = (\tfrac{3}{2})^{1/2} m^2 \alpha\,, \tag{5.1}$$

$$\chi^k = \sqrt{2}\, m\beta^k\,, \tag{5.2}$$

$$\xi^{kl} = \gamma^{kl}\,. \tag{5.3}$$

The commutation relations among these variables are (for $x^+ = y^+$)

$$[\varphi(x), \partial_y^+\varphi(y)] = i\delta^3(x-y)\,, \tag{5.4}$$

$$[\chi^k(x), \partial_y^+\chi^k(y)] = i\delta^{kl}\delta^3(x-y)\,, \tag{5.5}$$

$$[\xi^{kl}(x), \partial_y^+\xi^{mn}(y)] = \tfrac{1}{2}i[\delta^{km}\delta^{ln} + \delta^{kn}\delta^{lm} - \delta^{kl}\delta^{mn}] \times \delta^3(x-y)\,, \tag{5.6}$$

$$[\varphi(x), \chi^k(y)] = [\varphi(x), \xi^{mn}(y)] = [\chi^k(x), \xi^{mn}(y)] = 0\,. \tag{5.7}$$

There are no mass terms on the right-hand sides of these relations; thus they remain unchanged as we let $m^2 \to 0$.

Inspection of the commutation relations (5.4)–(5.6) reveals that φ, χ^k, and ξ^{kl} have commutation relations identical to those that the dynamical variables of massless scalar, vector, and tensor fields, respectively, have when they are quantized in the infinite-momentum gauge. Equation (5.7) shows that the three systems are independent. Since the dynamical variables all obey, in the limit of vanishing mass, the same equations of motion as the independent variables of the massless theories quantized in the infinite-momentum frame, that is,

$$\partial^2\varphi = \partial^2\chi^k = \partial^2\xi^{kl} = 0\,, \tag{5.8}$$

we may construct from them the three massless theories. The auxiliary fields used in the covariant formulation are defined in terms of the dynamical variables as the solutions of the usual constraint equations. For example, we define the auxiliary fields used in the massless spin-two theory through Eqs. (3.28)–(3.38) with h^{kl} replaced by ξ^{kl}. The stress tensor and the ten Lorentz generators are constructed similarly from the dynamical variables. We shall refer to these generators of the Lorentz transformations as P^{μ}_{φ} and $J^{\mu\nu}_{\varphi}$ for the scalar field, P^{μ}_{χ} and $J^{\mu\nu}_{\chi}$ for the vector field, and P^{μ}_{ξ} and $J^{\mu\nu}_{\xi}$ for the tensor field.

The fact that many of the field components of ${}^{\mu}H^{\nu\lambda}$ and $h^{\mu\nu}$ diverge as $m \to 0$ does not concern us. The degrees of freedom are correctly described in this limit by the rescaled dynamical variables, and the auxiliary fields necessary to make the formulation covariant can easily be defined as mentioned above.

We now turn to the massless limit of the Lorentz generators of our massive theory. We find that, as the mass vanishes, all the generators decouple into three independent parts. They are given by

$$\lim_{m^2\to 0} P^{\mu} = P^{\mu}_{\varphi} + P^{\mu}_{\chi} + P^{\mu}_{\xi}\,, \tag{5.9}$$

$$\lim_{m^2\to 0} J^{\mu\nu} = J^{\mu\nu}_{\varphi} + J^{\mu\nu}_{\chi} + J^{\mu\nu}_{\xi}\,, \tag{5.10}$$

where $(P^\mu_\varphi, J^{\mu\nu}_\varphi)$, $(P^\mu_\chi, J^{\mu\nu}_\chi)$, and $(P^\mu_\xi, J^{\mu\nu}_\xi)$ are the generators of the massless theories mentioned above.

We thus find that our massive theory splits into three totally independent massless theories when the mass vanishes; each of them is identical to quantizing one of the usual massless theories in the infinite-momentum gauge. Since the systems are decoupled, the contribution to the generators of the lower-spin systems can be subtracted, leaving only a massless spin-two theory identical to the one we described in Sec. III. This decoupling of the additional degrees of freedom as $m^2 \to 0$ is similar to the decoupling of the longitudinal and transverse degrees of freedom in the massless limit of the massive vector field.

VI. COMMENTS ON LIGHT-FRONT QUANTIZATION

The foregoing study illustrates the well-known fact that only half as many independent components are needed to describe a system on the light front. For a field theory with several independent variables, the reduction is enormously helpful. We also verified that the Schwinger approach is valid for spin two, despite the added complexity of the field.

The study of spin two illustrates a new advantage of quantization on the light front — the limit of vanishing mass is particularly transparent in the variables suggested by our quantization technique. The light front thus appears to be the natural frame in which to quantize massless particles. It should be pointed out, however, that the generators defined in our discussion are the same as one would find in the usual equal-time theory. Our assumed boundary conditions and the definitions of the generators in terms of the conserved stress tensor are sufficient to guarantee the equivalence of these two theories. However, the light-front formulation does help us to find the set of variables — the ones which diagonalize the generator matrix — for which the Lorentz generators have simple decompositions as the mass vanishes.

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APPENDIX A

In this appendix we list the expressions for the covariant field variables in terms of the dynamical variables α, β^k, and γ^{kl}. These relations are used in the tedious calculations to find the Lorentz generators in terms of the dynamical variables. We therefore write expressions for the symmetric combination ${}^{(\mu}H^{\nu\lambda)}$, since this is the form that appears in the generators. The expressions are

$$h^{++} = (\partial^+)^2\alpha\,, \tag{A1}$$

$$h^{+-} = [(\nabla^2 + m^2)\alpha - 2\partial^k\beta^k]\,, \tag{A2}$$

$$h^{+i} = \partial^+[\partial^i\alpha - \beta^i]\,, \tag{A3}$$

$$h^{-i} = (\partial^+)^{-1}[(\nabla^2 + 2m^2)\partial^i\alpha - 2\partial^i\partial^k\beta^k - (\nabla^2 + m^2)\beta^i - 2\partial^k\gamma^{ki}]\,, \tag{A4}$$

$$h^{ij} = [\partial^j\partial^i\alpha + \tfrac{1}{2}m^2\delta^{ij}\alpha - \gamma^{ij} - \partial^i\beta^j - \partial^j\beta^i]\,, \tag{A5}$$

$$h^{--} = (\partial^+)^{-2}[(\nabla^4 + 4m^2\nabla^2 + m^4)\alpha - 4(\nabla^2 + m^2)\partial^k\beta^k - 4\partial^k\partial^l\gamma^{kl}]\,, \tag{A6}$$

$${}^+H^{+i} = (\partial^+)^2\beta^i\,, \tag{A7}$$

$${}^+H^{+-} = 2(\partial^+)[\partial^k\beta^k - m^2\alpha]\,, \tag{A8}$$

$${}^{(m}H^{+n)} = \partial^+[\gamma^{mn} + \tfrac{1}{2}(\partial^m\beta^n + \partial^n\beta^m) - \tfrac{1}{2}\delta^{mn}m^2\alpha]\,, \tag{A9}$$

$${}^{(i}H^{+-)} = [\partial^i\partial^k\beta^k - 2m^2\partial^i\alpha + 2\partial^k\gamma^{ki} + (3m^2 + \nabla^2)\tfrac{1}{2}\beta^i]\,, \tag{A10}$$

$${}^-H^{+-} = (\partial^+)^{-1}[2(3m^2 + \nabla^2)\partial^k\beta^k + 4\partial^k\partial^l\gamma^{kl} - 2m^2(m^2 + 2\nabla^2)\alpha]\,, \tag{A11}$$

$${}^{(i}H^{-j)} = (\partial^+)^{-1}[3m^2\partial^i\partial^j\alpha + \tfrac{1}{2}(\nabla^2 - 3m^2)(\partial^i\beta^j + \partial^j\beta^i) - 2\partial^j\partial^i\partial^k\beta^k - \tfrac{1}{2}(\nabla^2 - m^2)\delta^{ij}m^2\alpha + (\nabla^2 - m^2)\gamma^{ij} - \partial^i\partial^k\gamma^{kj} - \partial^j\partial^k\gamma^{ki}]\,, \tag{A12}$$

$${}^{(+}H^{ij)} = (\partial^+)^{-1}[\tfrac{1}{4}\delta^{ij}m^2\alpha + \tfrac{1}{2}\partial^i\beta^j - \partial^j\beta^i - \tfrac{1}{2}\gamma^{ij}]\,, \tag{A13}$$

$${}^{(-}H^{ij)} = (\partial^+)^{-1}[-\tfrac{3}{2}m^2\partial^i\partial^j\alpha + \tfrac{1}{4}\delta^{ij}(\nabla^2 - m^2)m^2\alpha + \partial^i\partial^j\partial^k\beta^k - \tfrac{1}{2}(\nabla^2 - m^2)\gamma^{ij} - \nabla^2\partial^j\beta^i + \tfrac{1}{2}(3m^2 + \nabla^2)\partial^i\beta^j - \partial^j\partial^k\gamma^{ki} + 2\partial^k\partial^i\gamma^{kj}]\,, \tag{A14}$$

$${}^{(+}H^{i-)} = \partial^i\partial^k\beta^k - \tfrac{1}{2}m^2\partial^i\alpha - \nabla^2\beta^i - \partial^k\gamma^{ki}\,, \tag{A15}$$

$${}^{(+}H^{-i)} = [\tfrac{5}{2}m^2\partial^i\alpha - 2\partial^i\partial^k\beta^k + \tfrac{1}{2}(\nabla^2 - 3m^2)\beta^i - \partial^k\gamma^{ki}]\,, \tag{A16}$$

$${}^-H^{-i} = (\partial^+)^{-2}[3m^2(\nabla^2 + m^2)\partial^i\alpha - 2(\nabla^2 + 3m^2)\partial^i\partial^k\beta^k + (\nabla^4 - m^4)\beta^i + 2(\nabla^2 - m^2)\partial^k\gamma^{ki} - 4\partial^i\partial^k\partial^l\gamma^{kl}]\,, \tag{A17}$$

$${}^{(k}H^{il)} = \partial^i\gamma^{kl} - \tfrac{1}{2}(\partial^l\gamma^{ik} + \partial^k\gamma^{il}) + \tfrac{1}{4}m^2[\partial^l\delta^{ki} + \partial^k\delta^{li} - 2\partial^i\delta^{kl}]\alpha + \tfrac{1}{2}\partial^k\partial^i\beta^l + \tfrac{1}{2}\partial^l\partial^i\beta^k - \partial^l\partial^k\beta^i. \tag{A18}$$

APPENDIX B

We record in this appendix the commutation relations of the dynamical variables with the Lorentz generators. We express the commutation relations in terms of the dynamical variables rather than the field variables with definite transformation properties since they are the easiest to calculate with. The transformation properties in terms of field variables can easily be recovered from Eqs. (4.27)–(4.30), if they are desired.

The commutation relations are found to be

$$[f(x), P^\mu] = i\partial^\mu f(x)\,, \tag{B1}$$

$$[f(x), E^i] = -ix^i\partial^+ f(x)\,, \tag{B2}$$

$$[f(x), K_3] = ix^- \tfrac{1}{2}\partial^+ f(x)\,, \tag{B3}$$

where $f(x)$ may be any of α, β^k, and γ^{kl}.

The remaining relations are

$$[\alpha(x), L_3] = i[x^1\partial^2 - x^2\partial^1]\alpha(x)\,, \tag{B4}$$

$$[\beta^k(x), L_3] = i[x^1\partial^2 - x^2\partial^1]\beta^k(x) - i\delta^{k1}\beta^2(x) - i\delta^{k2}\beta^1(x)\,, \tag{B5}$$

$$[\gamma^{kl}(x), L_3] = i[x^1\partial^2 - x^2\partial^1]\gamma^{kl}(x) - i[\delta^{k1}\gamma^{l2} + \delta^{l1}\gamma^{k2}] + i[\delta^{k2}\gamma^{l1} + \delta^{l2}\gamma^{k1}]\,, \tag{B6}$$

$$[\alpha(x), F^i] = i(x^-\partial^i - x^i\partial^-)\alpha(x) - 4i(\partial^+)^{-1}\beta^i(x)\,, \tag{B7}$$

$$[\beta^k(x), F^i] = i(x^-\partial^i - x^i\partial^-)\beta^k(x) - i3m^2\delta^{ki}(\partial^+)^{-1}\alpha(x) + 2i\delta^{ki}(\partial^+)^{-1}\partial^m\beta^m(x) - 2i(\partial^+)^{-1}\partial^k\beta^i + 2i(\partial^+)^{-1}\gamma^{ki}\,, \tag{B8}$$

$$[\gamma^{kl}(x), F^i] = i(x^-\partial^i - x^i\partial^-)\gamma^{kl}(x) + 2im(\partial^+)^{-1}[\delta^{ik}\beta^l + \delta^{il}\beta^k - \delta^{kl}\beta^i] + 2i(\partial^+)^{-1}[\partial^m\gamma^{ml}\delta^{ki} + \delta^{li}\partial^m\gamma^{mk} - \partial^l\gamma^{ki} - \partial^k\gamma^{li}]\,. \tag{B9}$$

From relations (B1)–(B9) and (A1)–(A18) expressing $h^{\mu\nu}$ and ${}^\mu H^{\nu\lambda}$ in terms of the dynamical variables, we obtain the commutation relations between the covariant field operators and the Lorentz generators given in the text by Eqs. (4.27)–(4.30).

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†Present address: Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139.

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