[16]Equation (3.4) has a classical analogy. It is formally identical to classical equations for the orbit of a particle in the potential V. Of course, for classical motion to occur we must have $E > V$, while in the quantum-mechanical tunneling problem $E < V$. Thus our most probable escape paths are analytic continuations of the classical orbits.

[17]In a problem with a continuous symmetry, such as spherical symmetry, the set of MPEP's will of course be continuous. This happens in our problem only when $c = 1$. When $c \neq 1$ (and $a = b = 1$) we have four paths because of reflection symmetry in x and y.

[18]T. Banks and C. M. Bender, following paper, Phys. Rev. D8, 3366 (1973).

[19]It is also easy to show that these are the only straight-line solutions of Eq. (3.4).

[20]This transformation is of course motivated by the direction of the MPEP's for $c > 1$ (they meet the x axis at 45° angles). See Sec. III.

[21]See the Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 1, p. 121, Eq. (1). This work is referred to hereafter as BMP.

[22]We are of course free to choose any linear combination of $P_\nu^{-1}(w)$ and $P_\nu^1(w)$. It is most convenient and simplest to make the choice in Eq. (4.21) because it is easy to argue that α^2, the separation constant for Eq. (4.23), vanishes. For choices other than that in Eq. (4.21) there may be an integral over separation constants.

[23]See Ref. 18, Sec. II.

[24]BMP, Vol. 1, p. 163, Eq. (8).

[25]BMP, Vol. 1, p. 145, Eqs. (20) and (23). There is an error in Eq. (23) which we have corrected.

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Coupled Anharmonic Oscillators. II. Unequal-Mass Case

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We develop a general formalism for calculating the large-order behavior of perturbation theory for quantized systems of unequal-mass coupled anharmonic oscillators. Our technique is based on a generalization of the semiclassical approximation which was used to study equal-mass oscillators in the first paper of this series. The unequal-mass problem is much more difficult because the path which minimizes the classical action is not a straight line. Assuming that this tunneling path is known, we derive a general expression for the physical-optics approximation to the wave function of a tunneling particle. This derivation rests on the construction of a WKB approximation in curved space. We thus completely reduce the general quantum problem to the much simpler classical one of determining the path. Then we present a perturbation scheme for finding the classical path for systems of oscillators whose masses only differ by a small amount. Finally, we illustrate our techniques by solving a two-mode unequal-mass oscillator and comparing these results with a computer calculation. Our theoretical predictions and numerical calculations agree.

I. INTRODUCTION

In the first paper of this series,[1] we developed a method for computing the large-order behavior of the perturbation series for the ground-state energy of a system of N equal-mass coupled anharmonic oscillators. The method employs an extension of the semiclassical (WKB) approximation to multidimensional systems. However, before we can make the WKB approximation we must find the particular solutions of the classical orbit equations which make the action integral $\int (V-E)^{1/2}$ both a local and global minimum.[2] In general, it is very difficult to solve the classical equations, but in BBW I it was shown that for equal-mass oscillators the MPEP's are straight lines. In the present paper we will investigate the more difficult problem of computing the large-order behavior of perturbation theory for systems whose MPEP's are *not* straight lines. To demonstrate that our techniques work in such situations, we completely reduce the calculation of the large-order behavior of perturbation theory to the classical task of finding the MPEP.

Our interest in curved-path problems is not merely academic, for we are ultimately interested in quantum field theoretic perturbation series. The $(\phi^4)_N$ field theory with spatial and ultraviolet cutoffs is equivalent[3] to a system of unequal mass anharmonic oscillators. The "masses" in this case are just the energies of the modes of the field. It therefore seems likely that any attempt

to extend our work to a real quantum field theory will have to deal with the problem of curved paths.

The work which we present here is a first step in that direction. Section II gives a formal solution of the problem, assuming that the MPEP is known. Specifically, we show how to compute the physical optics (first-order WKB) approximation to the wave function of a tunneling particle in terms of the MPEP. The expressions we derive are surprisingly simple and bear a remarkable resemblance to the expressions we found in the straight-line MPEP case treated in BBW I. As in BBW I we obtain a Riccati equation which determines the thickness of the tube surrounding the (curved) MPEP. In Sec. III we use a perturbative approach to formally attack the problem of finding the MPEP. We study potentials of the form $V=U_0+\eta U_1$, where U_0 has straight-line MPEP's and η is small. We obtain a perturbative expression for the MPEP and derive a compact form for the geometrical-optics approximation to the wave function valid to second order in η.

In Sec. IV we apply the general techniques of Secs. II and III to the specific potential

$$V=\tfrac{1}{4}(x^2+y^2)+\tfrac{1}{4}\lambda(x^4+y^4+2cx^2y^2)+\tfrac{1}{4}\eta y^2 . \tag{1.1}$$

We calculate the large-order behavior of the ground-state-energy perturbation series (a power series in λ) to second order in η. Section V gives a comparison between the theoretical predictions in Sec. IV and extensive computer calculations. The agreement is excellent.

II. TUNNELING ALONG CURVED PATHS

In this section we present a formal semiclassical treatment of tunneling along a curved path. For simplicity, we consider only two-dimensional problems defined by

$$(-\nabla^2+V-E)\psi=0 . \tag{2.1}$$

As explained in BBW I, the wave function is concentrated in a narrow region surrounding the most probable escape path (MPEP), which is the trajectory that minimizes the action integral $\int(V-E)^{1/2}$. We emphasize that although the MPEP may be curved, it is assumed to be known. Finding the MPEP is a difficult but *classical* problem. Thus, in this section we show how to reduce the quantum-mechanical problem of tunneling to a purely classical one.

We suppose the MPEP to be given parametrically by

$$\begin{aligned} x&=\phi_1(s) ,\\ y&=\phi_2(s) , \end{aligned} \tag{2.2}$$

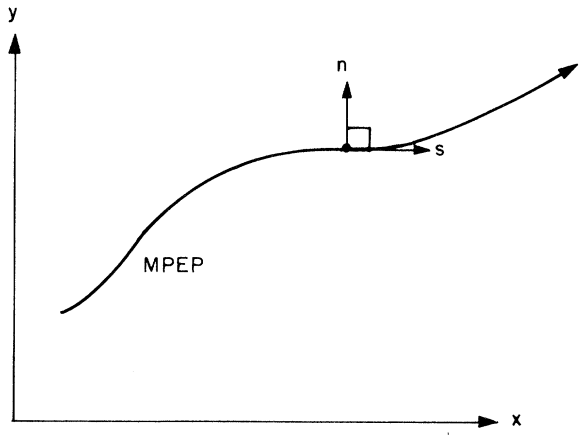

FIG. 1. The coordinate system suitable for describing the most probable escape path.

where s is the path length. Since we are interested in the region surrounding the MPEP, it is convenient to introduce a suitable curvilinear coordinate system. We take one coordinate to be the path length s and the other to be the perpendicular distance n from the curve (see Fig. 1). Of course, this is only a local definition valid for small n. It will not be necessary to describe the global nature of the coordinate system.

The tangent vector to the curve is

$$[\phi_1'(s),\phi_2'(s)] .$$

This vector is a *unit* vector because

$$\left(\frac{d\phi_1}{ds}\right)^2+\left(\frac{d\phi_2}{ds}\right)^2=\frac{(dx)^2+(dy)^2}{(ds)^2}=1 .$$

A unit vector normal to the curve (that is, to the tangent vector) is

$$[-\phi_2'(s),\phi_1'(s)] .$$

Hence the relation between the (s,n) and (x,y) coordinate systems is

$$\begin{aligned} x&=\phi_1(s)-n\phi_2'(s) ,\\ y&=\phi_2(s)+n\phi_1'(s) . \end{aligned} \tag{2.3}$$

Again we emphasize that these relations are local and must be altered for sufficiently large n.[4]

To solve Eq. (2.1), we will need an expression for the scalar product of two vectors in the (s,n) coordinate system. This is most easily obtained in terms of the metric tensor $g_{\mu\nu}$, which is given in the (s,n) coordinate system by[5]

$$\begin{pmatrix} g_{ss} & g_{sn} \\ g_{sn} & g_{nn} \end{pmatrix}=\begin{pmatrix} \left(\frac{\partial x}{\partial s}\right)^2+\left(\frac{\partial y}{\partial s}\right)^2 & \frac{\partial x}{\partial n}\frac{\partial x}{\partial s}+\frac{\partial y}{\partial n}\frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial n}\frac{\partial x}{\partial s}+\frac{\partial y}{\partial n}\frac{\partial y}{\partial s} & \left(\frac{\partial x}{\partial n}\right)^2+\left(\frac{\partial y}{\partial n}\right)^2 \end{pmatrix} . \tag{2.4}$$

$g_{\mu\nu}$ in Eq. (2.4) may be simplified using the relations

$$\frac{\partial x}{\partial s} = \phi_1' - n\phi_2'' ,$$

$$\frac{\partial y}{\partial s} = \phi_2' + n\phi_1'' ,$$

$$\frac{\partial x}{\partial n} = -\phi_2' ,$$

$$\frac{\partial y}{\partial n} = \phi_1' .$$

We obtain

$$\begin{aligned} g_{ss} &= 1 + 2n(\phi_2'\phi_1'' - \phi_1'\phi_2'') + n^2\left[(\phi_1'')^2 + (\phi_2'')^2\right] \\ &= (1+n\rho)^2 , \\ g_{nn} &= (\phi_1')^2 + (\phi_2')^2 = 1 , \\ g_{sn} &= 0 , \end{aligned} \tag{2.5}$$

where ρ, the curvature of the path, is given by $\phi_2'\phi_1'' - \phi_1'\phi_2''$. The off-diagonal elements of the metric tensor vanish because Eq. (2.3) describes an orthogonal coordinate system. Finally, we observe that the reciprocal of the metric tensor is given by

$$\begin{aligned} g^{ss} &= (1+n\rho)^{-2} , \\ g^{nn} &= 1 , \\ g^{ns} &= 0 . \end{aligned} \tag{2.6}$$

Having established Eq. (2.6), we proceed to find the WKB approximation to ψ in Eq. (2.1). Substituting

$$\psi = Ae^{-S} \tag{2.7}$$

into Eq. (2.1) gives

$$-\nabla^2 A + 2\vec{\nabla}A\cdot\vec{\nabla}S + A\nabla^2 S - A(\vec{\nabla}S)^2 + A(V-E) = 0 . \tag{2.8}$$

The WKB approximation follows from the assumption that S^2 and $V-E$ are large and of the same order of magnitude. Equating powers of S gives the eikonal equation

$$(\vec{\nabla}S)^2 = V - E \tag{2.9}$$

and the transport equation

$$2\vec{\nabla}A\cdot\vec{\nabla}S + A\nabla^2 S = 0 . \tag{2.10}$$

We have disregarded the term $\nabla^2 A$. We simplify Eq. (2.10) by multiplying by A:

$$\vec{\nabla}\cdot(A^2\vec{\nabla}S) = 0 . \tag{2.11}$$

The n-dimensional eikonal equation is very hard to solve in general. However, we need only solve it in the neighborhood of the MPEP, which is an approximately one-dimensional region. We expand $V-E$, S, and A as power series in n:

$$V - E = V_0(s) + nV_1(s) + n^2V_2(s) + \ldots , \tag{2.12}$$

$$S = S_0(s) + nS_1(s) + n^2S_2(s) + n^3S_3(s) + \ldots , \tag{2.13}$$

$$A = A_0(s) + \ldots . \tag{2.14}$$

We will assume that the linear term S_1 in the expansion of S vanishes. This is the first place where we use the assumption that $\vec{\phi}(s)$ describes an MPEP. We expect the amplitude to reach a maximum on the MPEP and to fall off exponentially on either side. This is not true if $S_1 \neq 0$.

Using the expansion in Eq. (2.13) and Eq. (2.6), we express the scalar product $(\vec{\nabla}S)^2$ as

$$\begin{aligned} g^{\mu\nu}\nabla_\mu S\nabla_\nu S &= (1+n\rho)^{-2}S_s^2 + S_n^2 \\ &= S_0'^2 - 2\rho n S_0'^2 \\ &\quad + n^2(3\rho^2 S_0'^2 + 2S_0'S_2' + 4S_2^2) \\ &\quad + O(n^3) . \end{aligned} \tag{2.15}$$

Matching powers of n reduces the eikonal equation [Eq. (2.9)] to three equations:

$$\begin{aligned} S_0'^2 &= V_0 , \\ -2\rho S_0'^2 &= V_1 , \\ 3\rho^2 S_0'^2 + 2S_0'S_2' + 4S_2^2 &= V_2 . \end{aligned} \tag{2.16}$$

The first of these equations has the familiar solution $S_0' = \pm\sqrt{V_0}$, when

$$S_0 = +\int^s \sqrt{V_0}\, ds . \tag{2.17}$$

We choose the plus sign because we are describing tunneling. The wave function ψ should decrease with increasing path length.

The second equation may then be rewritten as

$$\rho = -\frac{V_1}{2V_0} . \tag{2.18}$$

This equation relates the path directly to the potential and makes no reference to ψ. We therefore view Eq. (2.18) as a consistency condition for our approximation scheme. This condition arises because we have assumed that $S_1 = 0$. In the Appendix we show that Eq. (2.18) is a consequence of the classical equations of motion [the Euler-Lagrange equations of $\delta\int(V-E)^{1/2} = 0$]. Thus, our approximation is valid along classical paths.

The third equation is a Riccati equation:

$$S_2' + 2V_0^{-1/2}S_1^2 = \tfrac{1}{2}V_0^{-1/2}(V_2 - 3\rho^2 V_0) . \tag{2.19}$$

To convert Eq. (2.19) into a linear second-order differential equation, we substitute

$$S_1 = \tfrac{1}{2}V_0^{1/2}u'/u . \tag{2.20}$$

Then,

$$u'' + \tfrac{1}{2}(V_0'/V_0)u' = u(V_2 - 3\rho^2 V_0)/V_0 , \tag{2.21}$$

or in self-adjoint form

$$(u' V_0^{1/2})' + u(3\rho^2 V_0^{1/2} - V_2 V_0^{-1/2}) = 0 . \tag{2.22}$$

This equation can be further simplified to Schrödinger form by introducing the new independent variable

$$t(s) = \int^s ds\, V_0^{-1/2} . \tag{2.23}$$

We obtain

$$-\frac{d^2}{dt^2}u + u(V_2 - 3\rho^2 V_0) = 0 . \tag{2.24}$$

We have thus reduced the computation of S_2, which describes the thickness of the beam of probability current, to the solution of a second-order *ordinary* differential equation. In BBW I, where we considered only straight-line ($\rho = 0$) MPEP's, Eq. (2.22) was an associated Legendre equation.

This completes our study of the eikonal equation. The transport equation [Eq. (2.11)] need only be solved to zeroth order in n. In this order it is

$$\frac{d}{ds}[A_0^2 S_0'] + 2S_2 A_0^2 = 0$$

or (2.25)

$$\frac{dB}{dS} + 2S_1 V_0^{-1/2} B = 0 ,$$

where

$$B = A_0^2 V_0^{1/2} . \tag{2.26}$$

Hence,

$$B = (\text{const}) \exp\left[-2\int^s S_1 V_0^{-1/2} ds\right],$$

and

$$A_0 = (\text{const}) V_0^{-1/4} U^{-1/2} , \tag{2.27}$$

where u is defined in Eq. (2.20).

Combining Eqs. (2.7), (2.13), (2.17), (2.20), and (2.27), our final WKB expression for the wave function is

$$\psi = (\text{const}) V_0^{-1/4} u^{-1/2} e^{-\int^s V_0^{1/2}} e^{-n^2 V_0^{1/2} u'/2u} , \tag{2.28}$$

where u satisfies Eq. (2.21). To define u (and thus ψ) uniquely, boundary conditions must be imposed on Eq. (2.21). These are determined by matching ψ asymptotically to the solutions of Eq. (2.1), which are valid outside the tunneling region. The matching procedure is exactly analogous to the one for straight-line MPEP's described in Sec. IV of BBW I.

Equation (2.28) is remarkably similar to the wave function we found in BBW I [Eq. (4.26)]. As in BBW I, we can identify three physically distinct terms in the wave function ψ. Along the MPEP, ψ is given by a rapidly varying exponential term from geometrical optics and a slowly varying term independent of n. The term $\exp(-\frac{1}{2}n^2 V_0^{1/2} u'/u)$ describes the spread of probability current into the area surrounding the MPEP.

There are, of course, two difficulties present in Eq. (2.28) which were not encountered in the straight-line case. Equation (2.21) is a second-order differential equation and cannot always be solved. Moreover, the functions V_0, V_2, and ρ appearing in Eqs. (2.21) and (2.28) are undetermined until we have found the MPEP.

The first difficulty is not as bad as it might seem. Because Eq. (2.21) contains a small parameter (the anharmonic coupling constant) for the class of problems in which we are interested, it may be possible to treat it perturbatively. Furthermore, even if we cannot find u, Eq. (2.8) still enables us to compute the dominant large-order behavior of perturbation theory. We demonstrated in BBW I that the leading behavior (the factorial and power growth) could be computed directly from the geometrical optics approximation to the wave function. This approximation is given by

$$\psi \sim e^{-\int^s \sqrt{V_0}\, ds}$$

and requires no knowledge of u.

The second difficulty, that of actually determining the MPEP, is more serious and we have not found a general way to avoid it. It might at first appear that the classical equations may be treated perturbatively for small λ. However, as we show in Sec. III, this is not true. A simple scaling of the dependent and independent variables completely eliminates the anharmonic coupling constant from the classical equations, leaving a complicated system of coupled nonlinear equations which must be solved exactly. In Sec. III we show how to solve these equations when the MPEP is very nearly a straight line.

III. SLIGHTLY CURVED MPEP's

In this section we show how to obtain the MPEP perturbatively when it differs only slightly from a straight line.

An MPEP is a solution of the classical equations which makes the action $\int (V-E)^{1/2} ds$ a global minimum. The classical equations are the Euler-Lagrange equations obtained from

$$\delta \int (V-E)^{1/2} ds = 0 , \tag{3.1}$$

where s is the path length. Equation (3.1) is a *constrained* variational problem because the path always satisfies

$$\left[\frac{d}{ds}\vec{\phi}(s)\right]^2 = 1 . \tag{3.2}$$

We can proceed by introducing either a Lagrange multiplier or a dummy parameter t. Following the second method, we derive the Euler-Lagrange equations of the new variation problem

$$\delta\int (V-E)^{1/2}\left[\left(\frac{d}{dt}\vec{\phi}\right)^2\right]^{1/2} dt = 0 , \tag{3.3}$$

and then set $t = s$. Either way we obtain

$$2(V-E)\frac{d^2\vec{\phi}}{ds^2} + \frac{d\vec{\phi}}{ds}\left(\frac{d\vec{\phi}}{ds}\cdot\vec{\nabla}V\right) = \vec{\nabla}V , \tag{3.4}$$

where $(d\vec{\phi}/ds)^2 = 1$ and $\vec{\nabla}V$ means $\partial V/\partial\vec{\phi}$.

For the specific potential V in Eq. (1.1), Eq. (3.4) is a pair of equations, the first of which is

$$\begin{aligned}[\tfrac{1}{2}(x^2+y^2+\eta y^2) - \tfrac{1}{2}\epsilon(x^4+y^4+2cx^2y^2) - 2E]x''(s)\\ + [x'(s)]^2(\tfrac{1}{2}x - \epsilon x^3 - \epsilon cxy^2)\\ + x'(s)y'(s)(\tfrac{1}{2}y - \epsilon y^3 - \epsilon cyx^2)\\ = \tfrac{1}{2}x - \epsilon x^3 - \epsilon cxy^2 ,\end{aligned} \tag{3.5}$$

where we must have $\lambda = -\epsilon < 0$ for tunneling to occur. The second equation of the pair is similar. Equation (3.5) is very difficult to solve in general, and the following scaling argument shows that it cannot even by solved perturbatively for small ϵ. Simply letting $x \to x\epsilon^{-1/2}$, $y \to y\epsilon^{-1/2}$, $s \to s\epsilon^{-1/2}$, and neglecting E compared with x^2 in the tunneling region gives a new equation, almost identical to Eq. (3.5), in which all reference to ϵ has vanished. Fortunately, we do not have to find all solutions to Eq. (3.5). The MPEP we seek is a special solution and has in the past been relatively easy to find. In BBW I we found straight-line MPEP's for the equal-mass case ($\eta = 0$) even though the general solution remains unknown. Here we show how to solve equations like Eq. (3.5) for the MPEP as a perturbation series in η.

We proceed formally by assuming a potential of the form

$$V - E = U_0 + \eta U_1 , \tag{3.6}$$

where U_0 has a straight-line MPEP and η is small. U_0 and U_1 are functions of $\vec{\phi}(s)$ and thus implicitly functions of s.

The straight-line solution of the unperturbed problem is just

$$\vec{\phi}_0(s) = s\vec{\phi}_0' . \tag{3.7}$$

$\vec{\phi}_0'$ is a constant vector pointing along the path. Plugging this result into Eq. (3.4) gives

$$\vec{\phi}_0'\{\vec{\phi}_0'\cdot\vec{\nabla}U_0[\vec{\phi}_0(s)]\} = \vec{\nabla}U_0[\vec{\phi}_0(s)] , \tag{3.8}$$

in which $(\vec{\phi}_0')^2 = 1$. Equation (3.8) may be written more simply as

$$\vec{\nabla}U_0[\vec{\phi}_0(s)] = \frac{d}{ds}U_0[\vec{\phi}_0(s)]\vec{\phi}_0' . \tag{3.9}$$

From here on we simplify our notation by suppressing the argument of any function which is evaluated at $\vec{\phi}_0(s)$. This will shorten many of the formulas which are to follow, but we caution that it can lead to some confusion. $\vec{\nabla}W$ means compute $\partial W/\partial\vec{\phi}$ and evaluate the result at $\vec{\phi} = \phi_0$. Thus, $a(s)\cdot\vec{\nabla}W = 0$ does not imply that $(a(s)\cdot\vec{\nabla})^2W = 0$.

We now expand $\vec{\phi}(s)$ in a power series in η:

$$\vec{\phi} = \vec{\phi}_0 + \eta\vec{\phi}_1 + \eta^2\vec{\phi}_2 + \cdots . \tag{3.10}$$

The constraint [Eq. (3.2)] gives

$$1 = (\vec{\phi}_0')^2\sum_{n=1}^{\infty}\eta^n\sum_{m=0}^{n}\vec{\phi}_m'\cdot\vec{\phi}_{n-m}' ,$$

when

$$\sum_{m=0}^{n}\vec{\phi}_m'\cdot\vec{\phi}_{n-m}' = 0 . \tag{3.11}$$

Also, we have

$$U_{0,1}(\vec{\phi}) = \sum_{n=0}^{\infty}\frac{1}{n!}\left\{\left[\left(\sum_{m=1}^{\infty}\eta^m\vec{\phi}_m\right)\cdot\vec{\nabla}\right]^n U_{0,1}\right\}(\vec{\phi}_0) . \tag{3.12}$$

Plugging these expressions into Eq. (3.4) and keeping terms to first order in η gives

$$\vec{\phi}_1'\cdot\vec{\phi}_0' = 0 \tag{3.13}$$

and

$$\begin{aligned}2U_0\vec{\phi}_1'' + \vec{\phi}_1'\frac{d}{ds}U_0(s)\\ + \vec{\phi}_0'[\vec{\phi}_1'\cdot\vec{\nabla}U_0 + (\vec{\phi}_0'\cdot\vec{\nabla})(\vec{\phi}_1\cdot\vec{\nabla}U_0) + \vec{\phi}_0'\cdot\vec{\nabla}U_1]\\ = \vec{\nabla}(\vec{\phi}_1\cdot\vec{\nabla}U_0 + U_1) .\end{aligned} \tag{3.14}$$

Equation (3.13) implies that $\vec{\phi}_1\cdot\vec{\phi}_0'$ is a constant and we choose the constant to be 0 without loss of generality. Using Eq. (3.9) we then have

$$\vec{\phi}_1\cdot\vec{\nabla}U_0 = 0 . \tag{3.15}$$

Furthermore, since the component of $\vec{\phi}_1$ parallel to $\vec{\phi}_0'$ vanishes, we need only consider the components of Eq. (3.14) which are perpendicular to $\vec{\phi}_0'$. The equation for these components simplifies to

$$\begin{aligned}2U_0(\vec{\delta}\cdot\vec{\phi}_1)'' + \frac{d}{ds}U_0(s)(\vec{\delta}\cdot\vec{\phi}_1)' - \vec{\delta}\cdot\vec{\nabla}U_1 - (\vec{\delta}\cdot\vec{\nabla})(\vec{\phi}_1\cdot\vec{\nabla})U_0\\ = 0 ,\end{aligned} \tag{3.16}$$

where $\vec{\delta}$ is a unit vector perpendicular to $\vec{\phi}_0'$.

We have not yet specified the boundary conditions that $\vec{\phi}_1$ must satisfy. We will see later that the requirements that $\vec{\phi}_1$ be finite and have a finite derivative are sufficient to completely define the solution of Eq. (3.16).

A. Calculation to First Order in η

It is surprising that we can calculate the wave function in the geometrical optics approximation [see Eq. (2.17)],

$$\psi = e^{-\int \sqrt{V_0}\, ds} ,$$

to first order in η without even solving Eq. (3.15). The solution of Eq. (3.16) is needed to compute ψ to second order in η.

To first order in η we have

$$V_0 = U_0 + \eta U_1 + \eta \vec{\phi}_1 \cdot \vec{\nabla} U_0 , \tag{3.17}$$

but Eq. (3.15) eliminates the last term from this equation. Hence

$$\psi = \exp\left\{ - \int [(U_0 + \eta U_1)(\vec{\phi}_0'(s))] ds \right\} . \tag{3.18}$$

This expression is indeed independent of $\vec{\phi}_1$.

B. Calculation to Second Order in η

In BBW I we showed that the large-order behavior of perturbation theory was determined up to a multiplicative constant by the geometrical optics approximation to the wave function. In particular, the barrier penetration factor P, given by

$$P = \exp\left[-2 \int_{s_0}^{s_1} V_0^{1/2}(s) ds \right], \tag{3.19}$$

is required. [s_0 and s_1, the nearby and distant turning points, are zeros of $V_0(s)$.]

We now calculate $\ln P$ to second order in η. We have

$$V_0 = U_0 + \eta U_1 + \eta^2 [\vec{\phi}_2 \cdot \vec{\nabla} U_0 + \tfrac{1}{2} (\vec{\phi}_1 \cdot \vec{\nabla})^2 U_0 + \vec{\phi}_1 \cdot \vec{\nabla} U_1] . \tag{3.20}$$

Equation (3.9) gives

$$\vec{\phi}_2 \cdot \vec{\nabla} U_0 = \vec{\phi}_2 \cdot \vec{\phi}_0' \frac{dU_0}{ds} . \tag{3.21}$$

When $n = 2$, Eq. (3.11) gives

$$\vec{\phi}_2' \cdot \vec{\phi}_0' = -\tfrac{1}{2} \vec{\phi}_1'^2 , \tag{3.22}$$

from which we have

$$\vec{\phi}_2 \cdot \vec{\phi}_0' = -\tfrac{1}{2} \int_{\bar{s}_0}^{s} [\vec{\phi}_1'(s)]^2 ds . \tag{3.23}$$

$\bar{s}_0$ is the nearby zero of $U_0[\vec{\phi}_0(s)]$ and it differs from s_0, the nearby zero of V_0, by terms of order η.

We now approximate the expression for $\ln P$ as follows:

$$\begin{aligned} \ln P &= -2 \int_{s_0}^{s_1} \{ U_0 + \eta U_1 + \eta^2 [\vec{\phi}_2 \cdot \vec{\nabla} U_0 + \tfrac{1}{2} (\vec{\phi}_1 \cdot \vec{\nabla})^2 U_0 + \vec{\phi}_1 \cdot \vec{\nabla} U_1] \}^{1/2} ds \\ &\sim -2 \int_{s_0}^{s_1} ds\, (U_0 + \eta U_1)^{1/2} - \eta^2 \int_{s_0}^{s_1} ds \frac{-\tfrac{1}{2}(dU_0/ds) \int_{\bar{s}_0}^{s} \vec{\phi}_1'^2 + \tfrac{1}{2} (\vec{\phi}_1 \cdot \vec{\nabla})^2 U_0 + (\vec{\phi}_1 \cdot \vec{\nabla}) U_1}{(U_0 + \eta U_1)^{1/2}} . \end{aligned} \tag{3.24}$$

Here we have used Eqs. (3.20)–(3.23). At this point in the computation all reference to ϕ_2 has been eliminated. Because the second integral in Eq. (2.24) is multiplied by η^2, we can change its present limits s_0 and s_1 to $\bar{s}_0$ and $\bar{s}_1$, the zeros of U_0. Also, we can neglect the U_1 term in the denominator. Hence,

$$\ln P = -2 \int_{s_0}^{s_1} ds (U_0 + \eta U_1)^{1/2} - \eta^2 \int_{\bar{s}_0}^{\bar{s}_1} ds\, U_0^{-1/2} \left\{ -\frac{1}{2} \frac{dU_0}{ds} \int_{\bar{s}_0}^{s} \vec{\phi}_1'^2 + \tfrac{1}{2} (\vec{\phi}_1 \cdot \vec{\nabla})^2 U_0 + (\vec{\phi}_1 \cdot \vec{\nabla}) U_1 \right\} + O(\eta^3) . \tag{3.25}$$

Some manipulation of integrals must be done to simplify Eq. (3.25). We define

$$J = - \int_{\bar{s}_0}^{\bar{s}_1} \tfrac{1}{2} ds\, U_0^{-1/2} \left\{ \frac{dU_0}{ds} \int_{\bar{s}_0}^{s} (\vec{\phi}_1')^2 \right\} . \tag{3.26}$$

Integrating by parts gives

$$J = -U_0^{1/2} \int_{\bar{s}_0}^{s} (\vec{\phi}_1')^2 ds \Big|_{\bar{s}_0}^{\bar{s}_1} + \int_{\bar{s}_0}^{\bar{s}_1} ds\, U_0^{1/2} (\vec{\phi}_1')^2 .$$

Integrating by parts again gives

$$J = -U_0^{1/2} \int_{\bar{s}_0}^{s} (\vec{\phi}_1')^2 ds \Big|_{\bar{s}_0}^{\bar{s}_1} + U_0^{1/2} \vec{\phi}_1 \cdot \vec{\phi}_1' \Big|_{\bar{s}_0}^{\bar{s}_1} - \int_{\bar{s}_0}^{\bar{s}_1} \tfrac{1}{2} ds U_0^{-1/2} \frac{dU_0}{ds} \vec{\phi}_1' \cdot \vec{\phi}_1 - \int_{\bar{s}_0}^{\bar{s}_1} ds\, U_0^{1/2} \vec{\phi}_1'' \cdot \vec{\phi}_1 . \tag{3.27}$$

Next, we use the differential equation for $\vec{\phi}_1$ [Eq. (3.14)] to evaluate the last integral:

$$-\int_{\bar{s}_0}^{\bar{s}_1} ds\, U_0^{1/2}\vec{\phi}_1''\cdot\vec{\phi}_1 = -\int_{\bar{s}_0}^{\bar{s}_1}\tfrac{1}{2}ds\, U_0^{-1/2}\left\{\vec{\phi}_1\cdot\vec{\nabla}U_1 + (\vec{\phi}_1\cdot\vec{\nabla})^2U_0 - \vec{\phi}_1\cdot\vec{\phi}_1'\frac{dU_0}{ds}\right\}. \tag{3.28}$$

We have used the result that $\vec{\phi}_1\cdot\vec{\phi}_0' = 0$ to simplify Eq. (3.28).

Plugging Eq. (3.28) into Eq. (3.27) gives

$$J = -U_0^{1/2}\int_{\bar{s}_0}^{s}(\vec{\phi}_1')^2 ds\Big|_{\bar{s}_0}^{\bar{s}_1} + U_0\vec{\phi}_1'\cdot\vec{\phi}_1\Big|_{\bar{s}_0}^{\bar{s}_1} - \int_{\bar{s}_0}^{\bar{s}_1}\tfrac{1}{2}ds\, U_0^{-1/2}\{\vec{\phi}_1\cdot\vec{\nabla}U_1 + (\vec{\phi}_1\cdot\vec{\nabla})^2U_0\}. \tag{3.29}$$

Finally, we substitute Eq. (3.29) into Eq. (3.25):

$$\ln P = -2\int_{s_0}^{s_1}(U_0+\eta U_1)^{1/2}ds - \eta^2\left\{\int_{\bar{s}_0}^{\bar{s}_1}\tfrac{1}{2}ds\, U_0^{-1/2}(\vec{\phi}_1\cdot\vec{\nabla})U_1 - U_0^{1/2}\int_{s_0}^{s}(\vec{\phi}_1')^2 ds\Big|_{\bar{s}_0}^{\bar{s}_1} + U_0^{1/2}\vec{\phi}_1'\cdot\vec{\phi}_1\Big|_{\bar{s}_0}^{\bar{s}_1}\right\} + O(\eta^3). \tag{3.30}$$

The linear differential equation for $\vec{\phi}_1$ [Eq. (3.14)] has singular points at the zeros of U_0, namely, $\bar{s}_0$ and $\bar{s}_1$. Therefore, the boundary terms in Eq. (3.30) do not obviously vanish even though $U_0^{1/2}$ vanishes at both $\bar{s}_0$ and $\bar{s}_1$. Fortunately, we are not interested in the general solution of Eq. (3.14) because $\vec{\phi}_1$ must be interpreted as a real path. Consequently, it is required that $\vec{\phi}_1$ be everywhere finite. Furthermore, since $(\vec{\phi}_1')^2 = 1$ all along the path, $\vec{\phi}_1'$ must also be finite. Thus, we disregard surface terms and obtain

$$\ln P = -2\int_{s_0}^{s_1} ds(U_0+\eta U_1)^{1/2} - \eta^2\int_{\bar{s}_0}^{\bar{s}_1}\tfrac{1}{2}ds\, U_0^{-1/2}(\vec{\phi}_1\cdot\vec{\nabla})U_1 + O(\eta^3). \tag{3.31}$$

This is our final expression for $\ln P$, which we will evaluate for the specific potential in Eq. (1.1) in Sec. IV. We will, at that time, pursue the question of the boundary conditions for $\vec{\phi}_1$ in greater detail.

C. Comments on the nth-Order Calculation

Unfortunately, the nonlinearity of Eq. (3.4) makes the perturbation calculation that we have outlined quite complex as the order n increases. However, we will show that there are some features of the calculation that are true in all orders.

First, we emphasize that Eq. (3.31) is correct to second order in η. The first-order term is independent of the perturbed path, while the second-order term depends on the first-order correction to the path. In general, the constraint in Eq. (3.11) allows us to express the nth-order contribution to P in terms of $\vec{\phi}_0, \vec{\phi}_1, \ldots, \vec{\phi}_{n-1}$, but not $\vec{\phi}_n$.

Second, in every order, the equation comparable to Eq. (3.4) is a linear inhomogenous differential equation. The differential operator is the *same* in all orders; only the inhomogeneous term changes from order to order. To verify these assertions we introduce the following notation:

$$U_0[\vec{\phi}(s)] = \sum_{n=0}^{\infty}\eta^n U_0^{(n)}(\vec{\phi}_0, s),$$

$$U_1[\vec{\phi}(s)] = \sum_{n=0}^{\infty}\eta^n U_1^{(n)}(\vec{\phi}_0, s),$$

where $U_i^{(n)}(\vec{\phi}_0, s)$ is the coefficient of η^n in

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left\{\left[\sum_{m=1}^{\infty}\eta^m\vec{\phi}_m(s)\right]\cdot\vec{\nabla}\right\}^n U_i\Big|_{\vec{\phi}_0(s)}.$$

$U_1^{(n)}$ can of course be written in terms of multinomial coefficents but the resulting expression is not very illuminating. We now express the nth order equation as

$$2\sum_{j=1}^{n}[U_0^{(n-j)} + U_1^{(n-j-1)}]\vec{\phi}_j''(s) + \sum_{j=0}^{n}\left[\frac{d}{ds}U_0^{(n-j)} + \frac{d}{ds}U_1^{(n-j-1)}\right]\vec{\phi}_j'(s) = \vec{\nabla}[U_0^{(n)} + U_1^{(n-1)}], \tag{3.32}$$

in which the relation

$$\vec{\phi}(s)\cdot\vec{\nabla}U[\vec{\phi}(s)] = \frac{d}{ds}U[\vec{\phi}(s)]$$

and the convention

$$U_i^{(n)} = 0, \quad n<0$$

have been used.

Next we observe that in Eq. (3.32) the only term in $U_i^{(n)}$ that contains $\vec{\phi}_n$ is $\vec{\phi}_n\cdot\vec{\nabla}U_i$. Also, $\vec{\phi}_n$ does not appear in $U_i^{(m)}$ for $m < n+i$. Thus, we rewrite Eq. (3.32) as

$$2U_0^{(0)}\vec{\phi}_n'' + \frac{d}{ds}U_0^{(0)}\vec{\phi}_n' + \vec{\phi}_0'\frac{d}{ds}[\vec{\phi}_n\cdot\vec{\nabla}U_0^{(0)}] - \vec{\nabla}[\vec{\phi}_n\cdot\vec{\nabla}U_0^{(0)}] = \vec{h}_n, \tag{3.33}$$

where $\vec{h}_n$ does not depend on $\vec{\phi}_n$. Hence, only the

inhomogeneous term of Eq. (3.33) changes from order to order. Since Eq. (3.33) determines $\vec{\phi}_n\cdot\vec{\phi}_0'$ in terms of $\vec{\phi}_1,\ldots,\vec{\phi}_{n-1}$, we need only solve it for directions perpendicular to $\vec{\phi}_0'$. Hence, if $\vec{\delta}\cdot\vec{\phi}_0'=0$, we have

$$2U_0^{(0)}\vec{\delta}\cdot\vec{\phi}_n''+\frac{dU_0^{(0)}}{ds}\vec{\phi}_n'\cdot\vec{\delta}-\vec{\delta}\cdot\vec{\nabla}(\vec{\phi}_n\cdot\vec{\nabla})U_0=\vec{\delta}\cdot\vec{h}_n\,. \tag{3.34}$$

Observe the similarity between Eqs. (3.16) and (3.34). The computation of $\vec{\phi}_n$ to all orders is possible in principle if we can solve the homogeneous equation corresponding to Eq. (3.34) exactly. If we cannot, further approximations must be made.

IV. ILLUSTRATIVE CALCULATION

Our presentation to this point has been extremely formal and general. In this section we illustrate the techniques we have developed in Secs. II and III by specializing to a particular oscillator. We consider the unequal-mass version of the two-mode oscillator that we studied in Secs. III and IV of BBW I:

$$\begin{aligned}V-E&=\tfrac{1}{4}(x^2+y^2)-\tfrac{1}{4}\epsilon(x^4+2cx^2y^2+y^4)+\tfrac{1}{4}\eta y^2-E\\&\equiv U_0+\eta U_1\,.\end{aligned} \tag{4.1}$$

The specific calculation that we perform here is a geometrical optics determination of the large-order behavior of perturbation theory to second order in η for the potential in Eq. (4.1). We will do this by evaluating the expression for $\ln P$ given in Eq. (3.31). As is the case in BBW I [see Eq. (3.5)] geometrical optics gives the factorial and power growths of the Rayleigh-Schrödinger coefficients. At the end of this section, we show that our results compare favorably with a computer calculation.

A. Discussion of Saddle Points

Because we work to leading order in ϵ, we may neglect E in Eq. (4.1). Recall that when $\eta=0$, U_0 has radially directed saddle points at $x=\pm(2\epsilon)^{-1/2}$, $y=0$, and $y=\pm(2\epsilon)^{-1/2}$, $x=0$ when $c<1$ and at $x=\pm y$, $y=\pm[2\epsilon(c+1)]^{-1/2}$ when $c>1$. The analysis of BBW I tells us that the MPEP's are straight lines through these saddle points.

When $\eta\neq 0$, we have saddle points at

$$x=\pm(2\epsilon)^{-1/2},\ y=0,\quad 1+\eta>c\,, \tag{4.2a}$$

$$y=\pm(2\epsilon)^{-1/2},\ x=0,\quad 1+\eta<\frac{1}{c}\,, \tag{4.2b}$$

$$x=\pm\left(\frac{c-1+c\eta}{2\epsilon c^2-2\epsilon}\right)^{1/2},\quad y=\pm\left(\frac{c-1-c\eta}{2\epsilon c^2-2\epsilon}\right)^{1/2},\quad \frac{1}{c}<1+\eta<c\,. \tag{4.2c}$$

The first two sets of saddle points face the origin but the off-axis saddle points do not; we have straight-line MPEP's if $1+\eta>c$ or $1+\eta<1/c$, but otherwise we must solve a curved-path problem. To solve this more difficult problem we assume that for small η the MPEP's are perturbations of the four straight lines through $x=\pm y$, $y=\pm[2\epsilon(c+1)]^{-1/2}$. By symmetry we need only perturb about one of these lines.

B. Perturbative Determination of the Path to First Order in η

In the notation of Sec. III we have $\vec{\phi}_0=\frac{1}{2}\sqrt{2}\,s(1,1)$. This is the expression for the MPEP to lowest order in η and is all that is necessary to evaluate the first integral in Eq. (3.31) for $\ln P$. However, we defer this calculation until part C of this section. We proceed to compute the first-order correction to the MPEP because this result is needed to evaluate the second integral in $\ln P$.

The equation for $\vec{\phi}_0$ allows us to express the following quantities as functions of s:

$$\begin{aligned}&U_0(\vec{\phi}_0)=\tfrac{1}{4}s^2[1-\epsilon\alpha s^2]\,,\\&U_1(\vec{\phi}_0)=\tfrac{1}{8}s^2\,,\\&\vec{\nabla}U_1(\vec{\phi}_0)=\tfrac{1}{4}\sqrt{2}\,s(0,1)\,,\end{aligned} \tag{4.3}$$

where

$$\alpha\equiv\tfrac{1}{2}(c+1)\,.$$

Equation (3.15) implies that $\vec{\phi}_1$ has the form

$$\vec{\phi}_1=(\phi_1,-\phi_1)\,. \tag{4.4}$$

Hence, Eqs. (4.1), (4.3), and (4.4) give

$$\begin{aligned}&(\vec{\phi}_1\cdot\vec{\nabla})\vec{\nabla}U_0\\&=\vec{\phi}_1\cdot\vec{\nabla}(\tfrac{1}{2}x-\epsilon(x^3+cxy^2),\tfrac{1}{2}y-\epsilon(y^3+cyx^2))\big|_{x=y=\sqrt{2}\,s/2}\\&=\vec{\phi}_1(\tfrac{1}{2}-\tfrac{1}{2}\epsilon(3-c)s^2,-\tfrac{1}{2}+\epsilon(3-c)s^2)\,.\end{aligned}$$

Thus, $(\vec{\phi}_0'\cdot\vec{\nabla})(\vec{\phi}_1\cdot\vec{\nabla})U_0=0$ and

$$(\vec{\phi}_0'\cdot\vec{\nabla})U_1=\tfrac{1}{4}s\,. \tag{4.5}$$

Plugging the above expressions into the formal result in Eq. (3.16), we finally obtain the differential equation for $\phi_1(s)$:

$$s^2[1-\epsilon\alpha s^2]\frac{d^2\phi_1(s)}{ds^2}+s[1-2\epsilon\alpha s^2]\frac{d\phi_1(s)}{ds}+(\epsilon\omega s^2-1)\phi_1(s)=-\tfrac{1}{4}\sqrt{2}\,s\,, \tag{4.6}$$

where $\omega \equiv 3 - c$.

We convert Eq. (4.6) to a more familiar form by introducing the variables

$$z = (1 - \epsilon \alpha s^2)^{1/2} \tag{4.7}$$

and

$$f(z) = \epsilon^{-1/2} \phi_1(s) . \tag{4.8}$$

In terms of these variables, Eq. (4.6) becomes

$$(1 - z^2) f''(z) - 2z f'(z) + \left[\frac{\omega}{\alpha} - (1 - z^2)^{-1} \right] f(z) = -\tfrac{1}{2} [2\alpha (1 - z^2)]^{-1/2} . \tag{4.9}$$

We recognize this equation, from which all ϵ dependence has disappeared, as an inhomogeneous Legendre equation.[6] Homogeneous solutions are associated Legendre functions of the form

$$P_\nu^{\pm 1}, Q_\nu^{\pm 1} ,$$

where $\nu(\nu + 1) = \omega/\alpha$.

A particular solution of Eq. (4.9) is found by noticing that

$$\left[(1 - z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} - \frac{1}{1 - z^2} \right] (1 - z^2)^{-1/2} = 0 , \tag{4.10}$$

so that

$$\left[(1 - z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} - \frac{1}{1 - z^2} + \omega/\alpha \right] \sigma (1 - z^2)^{-1/2} = \frac{\sigma \omega}{\alpha} (1 - z^2)^{-1/2} .$$

Therefore, if $\sigma = \sqrt{2\alpha}/(4\omega)$, then

$$\sigma (1 - z^2)^{-1/2} \tag{4.11}$$

is a particular solution of Eq. (4.9).

The general solution is then

$$f(z) = \beta P_\nu^1(z) + \gamma Q_\nu^1(z) - \frac{\sqrt{2\alpha}}{4\omega} (1 - z^2)^{-1/2} . \tag{4.12}$$

β and γ are still arbitrary but, as we argued in Sec. III, $f(s)$ and $f'(s)$ must be finite along the path. In particular, f and its derivative must be finite at the endpoints of the path, which in the notation of Sec. III, are the points s_1 and s_0. s_1 and s_0 are the distant and nearby turning points of $V_0(s)$ [see Eq. (3.20)]. Actually, we will require finiteness at $\bar{s}_1$ and $\bar{s}_0$, the turning points of U_0, because it is simplest to work at these points. [Choosing to use $\bar{s}_0$ and $\bar{s}_1$ instead of s_0 and s_1 can only affect the evaluation of $\ln P$ to third order in η because $s_0 - \bar{s}_0 = O(\eta)$ and $s_1 - \bar{s}_1 = O(\eta)$.]

As $s \to \bar{s}_0$, $z \to 1$ and[7]

$$\lim_{z \to 1} f(z) = \lim_{z \to 1} \left\{ -\frac{\gamma}{\sqrt{2}} (1 - z)^{-1/2} - \frac{\sqrt{\alpha}}{4\omega} (1 - z)^{-1/2} \right\} .$$

Thus, we must choose

$$\gamma = -\sqrt{2\alpha}/(4\omega) . \tag{4.13}$$

The distant zero of U_0 is $\bar{s}_1 = (\epsilon \alpha)^{-1/2}$, which corresponds to $z = 0$. $f(z)$ is finite at $z = 0$. However, Eq. (4.7) gives

$$\frac{df}{ds} = -\frac{\epsilon \alpha s}{(1 - \epsilon \alpha s^2)^{1/2}} f'(z) ,$$

and thus the derivative of the path at $\bar{s}_1$ is finite only if $f'(0) = 0$. But

$$f'(0) = \beta \frac{d}{dz} P_\nu^1(z) \Big|_{z=0} + \gamma \frac{d}{dz} Q_\nu^1(z) \Big|_{z=0} .$$

This expression reduces to[8]

$$f'(0) = \beta \left[\frac{4\Gamma(\frac{1}{2}\nu + \frac{3}{2}) \sin \pi(\nu + 1)/2}{\sqrt{\pi}\, \Gamma(\frac{1}{2}\nu)} \right] + \gamma \left[\frac{2\sqrt{\pi}\, \Gamma(\frac{1}{2}\nu + \frac{3}{2}) \cos \pi(\nu + 1)/2}{\Gamma(\frac{1}{2}\nu)} \right] .$$

The vanishing of $f'(0)$ thus requires that

$$\beta = \tfrac{1}{2} \gamma \pi \tan(\tfrac{1}{2}\pi\nu) . \tag{4.14}$$

Combining Eqs. (4.12)–(4.14) completely determines $f(z)$:

$$f(z) = -\frac{\sqrt{2\alpha}}{4\omega} [\tfrac{1}{2}\pi \tan(\tfrac{1}{2}\pi\nu) P_\nu^1(z) + Q_\nu^1(z) + (1 - z^2)^{-1/2}] . \tag{4.15}$$

This is our final result for the MPEP to first order in η. To review, the first-order correction to the straight-line path is given by $\vec{\phi}_1$ in Eq. (4.14); $f(z)$ is related to $\vec{\phi}_1$ though Eqs. (4.7) and (4.8). The finiteness conditions on $f(s)$ assure us that the boundary terms in Eq. (3.30) vanish.

C. Evaluation of $\ln P$. First Integral in Eq. (3.31)

The evaluation of the first term of $\ln P$ in Eq. (3.31) is relatively easy. The integral is formally given by

$$-2 \int_{s_0}^{s_1} ds (U_0 + \eta U_1)^{1/2} .$$

For the case of the potential in Eq. (4.1) this integral is just

$$-\int_{s_0}^{s_1} s (1 + \tfrac{1}{2}\eta - \epsilon \alpha s^2)^{1/2} ds = -\frac{1}{3\epsilon\alpha} (1 + \tfrac{1}{2}\eta - \epsilon \alpha s^2)^{3/2} \Big|_{s_0}^{s_1} . \tag{4.16}$$

s_1 is the larger zero of $V_0(s) = U_0 + \eta U_1 + O(\eta^2)$ [see Eq. (3.20)]. Therefore, $(1 + \frac{1}{2}\eta - \epsilon \alpha s_1^2)^{3/2} = [O(\eta^2)]^{3/2} = O(\eta^3)$. Furthermore, s_0 is $O(1)$. Thus, $\epsilon \alpha s_0^2$ is negligible relative to $1 + \frac{1}{2}\eta$. This implies that we can approximate Eq. (4.16) by

$$\frac{1}{3\epsilon\alpha}(1+\tfrac{1}{2}\eta)^{3/2}=\frac{1}{3\epsilon\alpha}(1+\tfrac{3}{4}\eta+\tfrac{3}{32}\eta^2)+O(\eta^3)\,. \quad (4.17)$$

Equation (4.17) is the desired expression for the first contribution to $\ln P$. Observe that it was not necessary to know $f(z)$ to obtain Eq. (4.17).

D. Evaluation of $\ln P$. Second Integral in Eq. (3.31)

The second integral contributing to $\ln P$ is formally

$$-\int_{\bar{s}_0}^{\bar{s}_1}\frac{ds}{2\sqrt{U_0}}(\vec{\phi}_1\cdot\vec{\nabla})U_1\,.$$

But, according to Eq. (4.3),

$$\frac{(\vec{\phi}_1\cdot\vec{\nabla})U_1}{2\sqrt{U_0}}=-\frac{1}{\sqrt{2\epsilon}}\frac{f(z)}{z}\,.$$

Furthermore, from Eq. (4.7),

$$ds=-\frac{1}{\sqrt{\epsilon\alpha}}\frac{z}{(1-z^2)^{1/2}}\,dz\,.$$

Hence, the integral in Eq. (4.16) reduces to

$$\frac{1}{2\epsilon\sqrt{2\alpha}}\int_0^1\frac{f(z)dz}{(1-z^2)^{1/2}}\,. \quad (4.18)$$

A glance at Eq. (4.15) shows that each of the three contributions to Eq. (4.18) is separately divergent at $z=1$. To extract a finite answer we must integrate up to $x<1$ and then let x tend to unity at the end of the calculation. The third term gives

$$\int_0^x\frac{dz}{(1-z^2)^{1/2}}=\tfrac{1}{2}\ln\frac{1+x}{1-x}$$
$$\sim\tfrac{1}{2}\ln 2-\tfrac{1}{2}\ln(1-x)\,, \quad (4.19)$$

for $x\sim 1$.

The first and second terms are slightly more complicated. Using the differential equations satisfied by P_ν^1, Q_ν^1, and $(1-z^2)^{-1/2}$ [Eqs. (4.9) and (4.10)], it is easy to show that[9]

$$\int_0^x\frac{W(z)dz}{(1-z^2)^{1/2}}=\frac{1}{\nu(\nu+1)}(1-z^2)^{1/2}\left[\frac{zW}{(1-z^2)^{1/2}}-W'\right]\Bigg|_{z=0}^{z=x}\,,$$

where $W=P_\nu^1$ or Q_ν^1. A substantial amount of algebra now gives

$$\int_0^x\frac{dz}{(1-z^2)^{1/2}}[\tfrac{1}{2}\pi\tan(\tfrac{1}{2}\pi\nu)P_\nu^1(z)+Q_\nu^1(z)]=-\frac{1}{\nu(\nu+1)}\frac{d}{dx}\{\tfrac{1}{2}\pi\tan(\tfrac{1}{2}\pi\nu)P_\nu^1(x)(1-x^2)^{1/2}+Q_\nu^1(x)(1-x^2)^{1/2}\}. \quad (4.20)$$

We must evaluate this expression for x near 1. $P_\nu^1(x)$ is finite near $x=1$, so it is sufficient to consider its leading behavior there[10]: $P_\nu^1(x)\sim-\nu(\nu+1)[\tfrac{1}{2}(1-x)]^{1/2}$. Thus,

$$\frac{d}{dx}[P_\nu^1(x)(1-x^2)^{1/2}]\sim\nu(\nu+1)\,. \quad (4.21)$$

We must be more careful with the second term in Eq. (4.20). We use an expression for $Q_\nu^1(x)$ in terms of hypergeometric functions[11]:

$$Q_\nu^1(x)=\frac{2\pi^{3/2}}{(1-x^2)^{1/2}}\left\{\frac{x\tan(\tfrac{1}{2}\pi\nu)}{\Gamma(\tfrac{1}{2}\nu)\Gamma(-\tfrac{1}{2}-\tfrac{1}{2}\nu)}F(-\tfrac{1}{2}\nu,\tfrac{1}{2}+\tfrac{1}{2}\nu;\tfrac{3}{2};x^2)+\frac{\cot(\tfrac{1}{2}\pi\nu)}{2\Gamma(-\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2})}F(-\tfrac{1}{2}-\tfrac{1}{2}\nu,\tfrac{1}{2}\nu;\tfrac{1}{2};x^2)\right\}, \quad (4.22)$$

and an asymptotic expansion of F near $x=1$[12]:

$$F(a,b;a+b+1;x^2)\sim\frac{\Gamma(a+b+1)}{\Gamma(a)\Gamma(b)}\left\{\frac{1}{ab}+(1-x^2)[\ln(1-x^2)-h_{ab}]\right\}+O((1-x^2)^2\ln(1-x^2))\,. \quad (4.23)$$

In Eq. (4.23),

$$h_{ab}=\psi(1)+\psi(2)-\psi(a+1)-\psi(b+1)\,, \quad (4.24)$$

where ψ is the logarithmic derivative of Γ.

We combine Eqs. (4.22)–(4.24) with the second term of Eq. (4.20) and simplify. The leading divergence drops out leaving one that is only logarithmic:

$$-\frac{d}{dx}\left[\frac{(1-x^2)^{1/2}Q_\nu^1(x)}{\nu(\nu+1)}\right]=\frac{\sin^2(\tfrac{1}{2}\pi\nu)}{\nu(\nu+1)}+\tfrac{1}{2}+\tfrac{1}{2}\ln 2+\tfrac{1}{2}\ln(1-x)-\tfrac{1}{2}[h_1\sin^2(\tfrac{1}{2}\pi\nu)+h_2\cos^2(\tfrac{1}{2}\pi\nu)]+O((1-x)\ln(1-x))\,, \quad (4.25)$$

where

$$h_1=\psi(1)+\psi(2)-\psi(1-\tfrac{1}{2}\nu)-\psi(\tfrac{3}{2}+\tfrac{1}{2}\nu)\,,$$
$$h_2=\psi(1)+\psi(2)-\psi(\tfrac{1}{2}-\tfrac{1}{2}\nu)-\psi(1+\tfrac{1}{2}\nu)\,. \quad (4.26)$$

Observe that the divergent terms in Eqs. (4.19) and (4.25) just cancel.

Next, we combine all the expressions in Eqs. (4.15) and (4.18)–(4.25) and obtain a finite result:

$$-\frac{1}{8\epsilon\omega}\left[\ln 2+\tfrac{1}{2}+\frac{\sin^2(\frac{1}{2}\pi\nu)}{\nu(\nu+1)}-\tfrac{1}{2}\pi\tan(\tfrac{1}{2}\pi\nu)\right.$$
$$\left.-\tfrac{1}{2}h_1\sin^2(\tfrac{1}{2}\pi\nu)-\tfrac{1}{2}h_2\cos^2(\tfrac{1}{2}\pi\nu)\right]. \quad (4.27)$$

This expression can be simplified using Eq. (4.26) and some well-known identities for the ψ function.[13] Our final result for the second contribution to $\ln P$ is

$$-\frac{1}{8\epsilon\omega}[\gamma+\psi(\nu+1)-\tfrac{1}{2}\pi\tan(\tfrac{1}{2}\pi\nu)], \quad (4.28)$$

where γ is Euler's constant ($=0.5772156\ldots$).

E. The Rayleigh-Schrödinger Coefficients

Adding together the results in Eqs. (4.17) and (4.28) completes the evaluation of $\ln P$:

$$\ln P=-\frac{1}{3\epsilon\alpha}\left\{1+\tfrac{3}{4}\eta+\frac{3\alpha}{8\omega}\left[\frac{\omega}{4\alpha}+\gamma+\psi(\nu+1)\right.\right.$$
$$\left.\left.-\tfrac{1}{2}\pi\tan(\tfrac{1}{2}\pi\nu)\right]\eta^2+O(\eta^3)\right\}, \quad (4.29)$$

where $\omega=3-c$, $\alpha=\frac{1}{2}(c+1)$, and $\nu(\nu+1)=\omega/\alpha$.

It is more convenient to invert the expression in curly brackets, to wit:

$$\ln P=\{-3\epsilon\alpha[1-\tfrac{3}{4}\eta+I\eta^2+O(\eta^3)]\}^{-1}, \quad (4.30)$$

where we have introduced

$$I=\tfrac{3}{8}\left[\tfrac{5}{4}-\frac{\alpha}{\omega}\left(\gamma+\psi(\nu+1)-\tfrac{1}{2}\pi\tan\frac{\pi\nu}{2}\right)\right]. \quad (4.31)$$

The barrier penetration factor P is just the geometrical optics contribution to the imaginary part of the energy. Thus, inserting P into the dispersion representation for A_n [Eq. (2.4) of BBW I], we find that the large-n behavior of the perturbation series for the potential in Eq. (4.1) is

$$A_n\propto-[-3\alpha(1-\tfrac{3}{4}\eta+\eta^2 I)]^n\Gamma(n+\tfrac{1}{2}). \quad (4.32)$$

This result is correct up to a multiplicative factor independent of n.

F. Computer Verification

To check Eq. (4.32) we calculated the first 65 A_n on a CDC 6400 computer using a difference equation. Generalizing the arguments of Sec. V of BBW I slightly, we find that for the potential in Eq. (4.1) the Rayleigh-Schrödinger coefficients are given by

$$A_n=(-1)^n(C_{n,1,0}+C_{n,1,0}),$$

where

$$[2j+2k(1+\eta)^{1/2}]C_{n,j,k}=(j+1)(2j+1)C_{n,j+1,k}$$
$$+(k+1)(2k+1)C_{n,j,k+1}$$
$$+C_{n-1,j,k-2}+C_{n-1,j-2,k}$$
$$+2cC_{n-1,j-1,k-1}$$
$$-\sum_{p=1}^{n-1}(C_{p,1,0}+C_{p,0,1})\,C_{n-p,j,k}. \quad (4.33)$$

We solved Eq. (4.33) for $c=2$ and a range of values of $(1+\eta)^{1/2}$ and fit the values of A_n to the formula[14]

$$A_n\propto-\Gamma(n+\tfrac{1}{2})(-3K)^n. \quad (4.34)$$

The values of K as a function of η are recorded in Table I.

Several remarks should be made about the entries in Table I. First, when $(1+\eta)^{1/2}>\sqrt{c}=\sqrt{2}$, $k=1$ (to within the expected accuracy) and when $(1+\eta)^{1/2}<1/\sqrt{c}=1/\sqrt{2}$, $k=(1+\eta)^{-3/2}$. These are just the values that we obtain from *straight-line* MPEP's passing through the saddle points in Eqs. (4.2a) and (4.2b). That is, the curved-path region lies, as it should, for values of $(1+\eta)^{1/2}$ between $\sqrt{2}/2$ and $\sqrt{2}$. Second, when $\eta=0$ we have an equal-mass potential and the results of BBW I as well as those in Eq. (4.32) imply that $k=1.5$. This suggests that near $(1+\eta)^{1/2}=1$ our numerical results in Table I

TABLE I. Numerical values of K as a function of η in Eq. (4.34). These values are in good agreement with the theoretical predictions in Eq. (4.35).

$(1+n)^{1/2}$	η	K
0.4	−0.84	15.625 004
0.5	−0.75	8.000 003
0.75	−0.4375	2.534 419
0.8	−0.36	2.216 049
0.9	−0.19	1.777 643
0.98	−0.0396	1.546 811
0.99	−0.0199	1.522 942
1.00	0.0	1.499 996
1.01	0.0201	1.477 928
1.02	0.0404	1.456 695
1.1	0.21	1.312 745
1.2	0.44	1.183 367
1.3	0.69	1.092 641
1.4	0.96	1.038 811
$\sqrt{2}$	1.0	1.032 995
1.5	1.25	1.004 064
1.7	1.89	0.999 954
1.8	2.24	0.999 994
1.9	2.61	0.999 999
2.0	3.00	1.000 000
2.1	3.41	1.000 000
2.9	7.41	1.000 000
3.0	8.0	1.000 000
3.1	8.61	1.000 000

are low by four parts in 1.5×10^6.

With this in mind, we proceed to compare our theoretical and numerical data. Our theoretical predictions from Eq. (4.32) take the form of a Taylor series valid near $\eta=0$:

$$K=\tfrac{3}{2}-\tfrac{9}{8}\eta+1.378\eta^2+O(\eta^3)\,. \tag{4.35}$$

Equation (4.35) is obtained by evaluating the expression for I in Eq. (4.31) using $c=2$, $\alpha=\frac{3}{2}$, $\omega=1$, and $\nu=-0.5\pm(\frac{11}{12})^{1/2}$.

There is no simple formula that fits the numerical data in Table I for $\sqrt{2}>(1+\eta)^{1/2}>\sqrt{2}/2$. We can, however, compute the first three terms in the Taylor series of K about $(1+\eta)^{1/2}=1$. Using the five values for K associated with $(1+\eta)^{1/2}=0.98$, 0.99, 1.00, 1.01, and 1.02, and assuming that the four parts in 1.5×10^6 discrepancy mentioned above hold for all these values of $(1+\eta)^{1/2}$, we obtain the expansion $K=1.5-\frac{9}{4}[(1+\eta)^{1/2}-1]+4.388[(1+\eta)^{1/2}-1]^2+O([(1+\eta)^{1/2}-1]^3)$. Rewriting this as a power series in η, we obtain

$$K=\tfrac{3}{2}-\tfrac{9}{8}\eta+1.378\eta^2\,,$$

in complete agreement with the theoretical predictions in Eq. (4.35).

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APPENDIX

In this appendix we show that Eq. (2.18) is valid whenever the path $\vec{\phi}(s)$ satisfies the classical equations of motion.

The classical equations are given in Sec. III as

$$2\{V[\vec{\phi}(s)]-E\}\vec{\phi}''(s)+\vec{\phi}'(s)\{\vec{\phi}'(s)\cdot\vec{\nabla}V[\vec{\phi}(s)]\}=\vec{\nabla}V[\vec{\phi}(s)]\,. \tag{A1}$$

The functions $V_0(s)$ and $V_1(s)$ which appear in Eq. (2.18) are defined in Eq. (2.12) as

$$V[\vec{\phi}(s)+n\vec{\chi}(s)]-E=V_0(s)+nV_1(s)+O(n^2)\,, \tag{A2}$$

where $\vec{\chi}(s)$ is the unit normal vector

$$\vec{\chi}=(-\phi_2',\,\phi_1')\,.$$

Expanding the left-hand side of Eq. (A2) as a power series in n, we find that

$$\begin{aligned} V_0(s)&=V[\vec{\phi}(s)]-E\,,\\ V_1(s)&=\vec{\chi}\cdot\vec{\nabla}V[\vec{\phi}(s)]\,. \end{aligned} \tag{A3}$$

Now we take the scalar product of (A1) with $\vec{\chi}$ and use $\vec{\chi}\cdot\vec{\phi}'=0$ to obtain

$$2V_0(s)\vec{\chi}\cdot\vec{\phi}''=\vec{\chi}\cdot\vec{\nabla}V[\vec{\phi}(s)]=V_1(s)\,. \tag{A4}$$

But,

$$\vec{\chi}\cdot\vec{\phi}''=\phi_2''\phi_1'-\phi_1''\phi_2'=-\rho\,. \tag{A5}$$

Hence

$$\rho=-\frac{V_1}{2V_0}\,, \tag{A6}$$

which is Eq. (2.18). It is clear then that Eq. (2.18) is just the perpendicular component of the equation of motion. The component of Eq. (A1) parallel to the path (that is, parallel to $\vec{\phi}'$) is a trivial identity because $\vec{\phi}'^2=1$ and $\vec{\phi}'\cdot\vec{\phi}''=0$. Thus, Eq. (2.18) is valid if and only if $\vec{\phi}$ is a classical path.

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[1]T. Banks, C. M. Bender, and T. T. Wu, preceding paper, Phys. Rev. D 8, 3346 (1973). Hereafter we refer to this paper as BBW I.

[2]As in BBW I, we call these paths "most probable escape paths" (MPEP's).

[3]See Ref. 3 of BBW I.

[4]Equation (2.3) is *exactly* true so long as the lines of constant s, which are normal to the MPEP, do not cross. Thus Eq. (2.3) is valid for $|n|$ less than the radius of curvature ρ^{-1} [see Eq. (2.5)].

[5]Recall that the path length $(dl)^2$ is given by $dx^\nu dx^\mu g_{\mu\nu}$. $g_{\mu\nu}$ in the $(x,\,y)$ coordinate system is the 2×2 unit matrix. Thus $dx^\nu dx^\mu g_{\mu\nu}=(dx)^2+(dy)^2$. In the (n,s) system $(dl)^2=g_{ss}(ds)^2+g_{nn}(dn)^2+2g_{ns}\,ds\,dn$. Equation (2.4) follows from Eq. (2.2).

[6]See *Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 1, p. 121, Eq. (1). This work is referred to hereafter as BMP.

[7]BMP, Vol. 1, p. 163, Eqs. (8) and (10).

[8]BMP, Vol. 1, p. 145, Eqs. (22) and (23). Equation (23) has a misprint which we have corrected in the text.

[9]See BMP, Vol. 1, p. 169, Eq. (1).

[10]See Ref. 7.

[11]BMP, Vol. 1, p. 144, Eq. (12). There is an error in this equation which we have corrected in the text.

[12]BMP, Vol. 1, p. 110, Eq. (12).

[13]BMP, Vol. 1, Sec. (1.7.1).

[14]The number of entries in the matrix $C_{n,j,k}$ grows as n^3 and the convolution term in Eq. (4.33) requires

$C_{n,j,k}$ for all n less than the order of perturbation theory computed. The limitations of the core memory prevent a calculation of A_n for $n \gtrsim 20$. To go to order 65 in perturbation theory we retain only the first and last terms in the convolution sum. This is a good approximation for n large and is justified in Ref. 9 of BBW I. This approximation accounts for the four parts in 1.5×10^6 error in Table I.

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Nonuniqueness of the Energy-Momentum Tensor of a Spin-One Field*

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A first-order Lagrangian formulation is given for a spin-one field in which the usual antisymmetric field tensor is replaced by a symmetrical one. Despite this seemingly major modification, the construction is such that the equations of motion and the canonical commutation relations are identical to those which pertain to the usual description if one requires that only the four-vector part of the field participate in the coupling. Differences arise, however, when one considers more general interactions. Thus, in the case of minimal coupling to an electromagnetic field it is found that the g factor is opposite in sign to the usual one for a vector meson. More surprising is the fact that the energy-momentum tensor for the free field contains additional terms which violate the minimal form of the Dirac-Schwinger covariance condition. Although Schwinger has shown that fields of spin greater than unity generally require the nonminimal form, this paper demonstrates for the first time the fact that even a lower-spin field need not satisfy the covariance condition in its simplest form.

I. THE SYMMETRIC-TENSOR THEORY

The conventional theory of the spin-one field (particularly in the zero-mass limit) is doubtless the best-known and most widely discussed of all field theories. In its first-order form it is described by the Lagrangian [1]

$$L = \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}F^{\mu\nu}(\partial_\mu B_\nu - \partial_\nu B_\mu) - \tfrac{1}{2}\mu^2 B^\mu B_\mu + j^\mu B_\mu, \tag{1.1}$$

where j^μ is a (not necessarily conserved) current and all operator products are assumed to be symmetrized. The resulting equations of motion are readily seen to be

$$F^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu,$$
$$\partial_\nu F^{\mu\nu} = -\mu^2 B^\mu + j^\mu. \tag{1.2}$$

The elimination of $F^{\mu\nu}$ from the set (1.2) leads to the familiar second-order form

$$(-\partial^2 + \mu^2)B^\mu + \partial^\mu(\partial B) = j^\mu, \tag{1.3}$$

appropriate to a current-coupled vector-meson field. Although it is doubtful that there exists a compelling alternative to Eq. (1.3) as a description of a spin-one object, the first-order form (1.2) appears to be considerably more open to doubt, thereby suggesting the possibility of a somewhat different description.

There does in fact exist another formulation in which the antisymmetric tensor $F^{\mu\nu}$ is replaced by a symmetric one $G^{\mu\nu}$. This is therefore a fourteen-component theory in contrast to the ten components which appear in the Lagrangian (1.1). The relevant Lagrangian in this case (including a coupling to j^μ) is

$$L = \tfrac{1}{4}G^{\mu\nu}G_{\mu\nu} - \tfrac{1}{2}G^{\mu\nu}(\partial_\mu B_\nu + \partial_\nu B_\mu) + G\,\partial_\mu B^\mu - \tfrac{1}{4}G^2 - \tfrac{1}{2}\mu^2 B^\mu B_\mu + j^\mu B_\mu, \tag{1.4}$$

where we have included a scalar G for convenience. It will be seen that the equations of motion imply that G is the trace of $G^{\mu\nu}$ and thus the whole theory could equally well be written in an explicit fourteen-component form as

$$L = \tfrac{1}{4}G^{\mu\nu}G_{\mu\nu} - \tfrac{1}{2}G^{\mu\nu}(\partial_\mu B_\nu + \partial_\nu B_\mu) + g^{\mu\nu}G_{\mu\nu}\partial_\lambda B^\lambda - \tfrac{1}{4}(g_{\mu\nu}G^{\mu\nu})^2 - \tfrac{1}{2}\mu^2 B^\mu B_\mu + j^\mu B_\mu,$$

rather than the fifteen-component form (1.4). The equations of motion implied by (1.4) are

$$G^{\mu\nu} = \partial^\mu B^\nu + \partial^\nu B^\mu,$$
$$\partial_\nu G^{\mu\nu} = \mu^2 B^\mu + \partial^\mu G + j^\mu,$$
$$G = 2\,\partial_\mu B^\mu,$$