New York, 1971), Chap. 14.

[3]While this paper was in preparation we received a preprint of R. Jackiw and K. Johnson [Phys. Rev. D 8, 2386 (1973)] dealing with the same issues in the context of a model with an axial vector. A variant of this model is the same as the Abelian model discussed here, upon suitable transformation of the fields. Some time ago the possibility of a dynamical origin to spontaneous symmetry breakdown was discussed by F. Englert and R. Brout [Phys. Rev. Lett. 13, 321 (1964)].

[4]Y. Nambu, Phys. Rev. 117, 648 (1960).

[5]K. Johnson, M. Baker, and R. Willey, Phys. Rev. 136, B1111 (1964); Phys. Rev. 163, 1699 (1967); M. Baker and K. Johnson, *ibid.* 183, 1292 (1969); Phys. Rev. D 3, 2516 (1971); 3, 2541 (1971); K. Johnson and M. Baker, *ibid.* 8, 1110 (1973); S. Adler, Phys. Rev. D 5, 3021 (1972); 7, 1948(E) (1973).

[6]This is also demonstrated by Jackiw and Johnson in Ref. 3.

[7]It has been known for some time that a bound state which communicates with an elementary particle channel does not appear in the S matrix. This was argued by C. J. Goebel and B. Sakita [Phys. Rev. Lett. 11, 293 (1963)] for nonrelativistic bound states and by Y. S. Jin and S. W. MacDowell [Phys. Rev. 137, B688 (1965)] for bound-state poles which move with the coupling constant—this does not take place for the Goldstone boson, which is fixed at $q^2=0$. A general proof of this cancellation, based on the Dyson equations, was given by J. M. Cornwall and D. J. Levy [Phys. Rev. 178, 2356 (1968)].

[8]S. Coleman and E. Weinberg, Phys. Rev. D 7, 1688 (1973).

[9]See, for example, S. Adler and W. A. Bardeen, Phys. Rev. D 4, 3045 (1971); 6, 734(E) (1972).

[10]M. Baker, K. Johnson, and B. W. Lee, Phys. Rev. 133, B209 (1964).

[11]W. A. Bardeen, Phys. Rev. 184, 1848 (1969).

[12]There would be an additional factor of $(Z_2 Z_2')^{1/2}$ on the right-hand side of (4.1) if the spinors u were normalized to unity. Although it is of no particular significance in the present context, we note that this Z_2 dependence can be included by normalizing the spinors to $\bar{u}u = Z_2$.

[13]G. Jona-Lasinio, Nuovo Cimento 34, 1790 (1964).

[14]J. M. Luttinger and J. C. Ward [Phys. Rev. 118, 1417 (1960)] demonstrated this stationarity with regard to the Dyson equation for the fermion propagator in nonrelativistic many-body theory, and C. De Dominicis and P. C. Martin [J. Math. Phys. 5, 14 (1964); 5, 31 (1964)] extended this work to include the vertex functions. The present authors, at the time unaware of the work of Dominicis and Martin, also extended these considerations to the full set of Dyson equations in relativistic field theory. The details will be published elsewhere.

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Coupled Anharmonic Oscillators. I. Equal-Mass Case

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This is the first in a series of papers on the large-order behavior of perturbation theory for coupled anharmonic oscillators. We exploit previously published dispersion techniques to convert the calculation of perturbation theory in large order into a barrier-penetration problem. We then introduce new semiclassical methods for describing tunneling through nonspherically symmetric, N-dimensional potentials. To illustrate our new methods, we calculate the large-order behavior of perturbation theory for a simple system of two equal-mass oscillators with quartic coupling. Our predictions are in complete agreement with computer calculations. We then extend our results to oscillators with x^{2N} coupling, N-oscillator systems, and some infinite-oscillator systems.

I. INTRODUCTION

In a recent paper Adler[1] argues that α, the physical charge on the electron, is an essential singularity of the Gell-Mann–Low function. Since the location of an essential singularity cannot be affected by the low-order terms in a perturbation expansion, an asymptotic study of perturbation theory for quantum electrodynamics in extremely large order seems indicated.

There has already been much work on the large-order behavior of perturbation series in quantum field theory.[2] However, these papers are concerned with determining whether diagrammatic perturbation theory converges or diverges and are satisfied with order-by-order bounds on the terms in the perturbation series. What is required now is a precise analytic study of the behavior of perturbation theory in large order and not just upper and lower bounds.

Our general approach to perturbation theory in large order is guided by recent successful studies (which we review later) of perturbation theory for the anharmonic oscillator. We hope to extend and apply this work to the perturbation series encountered in quantum field theory. A $(\phi^4)_2$ quantum field theory is a good model with which to begin because, unlike four-dimensional quantum electrodynamics, its perturbation series are finite order by order. Moreover, Glimm and Jaffe[3] have shown that, with spatial and ultraviolet cutoffs, this field theory is equivalent to a finite system of coupled anharmonic oscillators (modes). Our goal is to find the large-order behavior of perturbation theory for this system as a function of the two cutoffs, remove the ultraviolet and spatial cutoffs, and, assuming that this limit exists (we see no reason why it should not), thereby obtain the answer. We have not solved this problem, but we do report major progress toward that goal. We have developed new mathematical techniques for investigating the large-order behavior of perturbation theory for coupled oscillator systems. Specifically, we have developed very general *multidimensional* semiclassical techniques[4] for treating nonspherically symmetric potentials such as those encountered in the cutoff $(\phi^4)_2$ model.

The purpose of the first paper in this series is to explain our new methods in detail in the limited context of equal-mass oscillators. (Our second paper deals with the more difficult problem of unequal-mass oscillators.) We begin by solving the two-mode oscillator defined by

$$\left[-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+\tfrac{1}{4}x^2+\tfrac{1}{4}y^2 + \lambda(ax^4+by^4+2cx^2y^2)/4-E(\lambda)\right]\psi(x,y)=0\,, \tag{1.1}$$

where

$$\lim_{|\vec{r}|\to\infty}\psi(x,y)=0\,,\quad \vec{r}=(x,y)\,. \tag{1.2}$$

We assume that $a>0$, $b>0$, and $c\geqslant-\sqrt{ab}$, making the potential bounded below. Then it is possible to satisfy Eq. (1.2) and ψ is a bound-state wave function with $E(\lambda)$ its associated energy eigenvalue. The perturbation series for the ground-state energy is given by[5]

$$E(\lambda)=1+\sum_{n=1}^{\infty}A_n\lambda^n\,. \tag{1.3}$$

Our goal is to predict the behavior of A_n for large n as a function of a, b, and c. We solve this problem in Secs. II–VI. The solution is a discontinuous function of a, b, and c, and the discontinuities have an interesting physical explanation. In Sec. VII we hope to whet the reader's appetite by giving a partial solution of the N-mode oscillator defined by

$$\left[\sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial x_i^2}+\tfrac{1}{4}x_i^2\right) + \tfrac{1}{4}\lambda\sum_{i,j=1}^{N}a_{ij}x_i^2x_j^2-E(\lambda)\right]\psi(x_i)=0\,. \tag{1.4}$$

In addition, we exhibit some matrices $\{a_{ij}\}$ for which the limit $N\to\infty$ may be taken, and explicitly compute the large-order behavior of the ground-state energy perturbation series for those infinite systems.

Before discussing more fully the content and results of this paper, we briefly review what is known about the one-oscillator problem so that this paper will be reasonably self-contained. In early computer studies[6,7] of the ground-state energy of the anharmonic oscillator, an extraordinarily simple behavior was observed for perturbation theory in large order. In particular, for the anharmonic oscillator defined by

$$[-d^2/dx^2+x^2/4+\lambda x^4/4-E(\lambda)]\,\psi(x)=0 \tag{1.5}$$

and

$$\lim_{|x|\to\infty}\psi(x)=0\,, \tag{1.6}$$

and the ground-state energy perturbation series defined by

$$E(\lambda)=\tfrac{1}{2}+\sum_{n=1}^{\infty}A_n\lambda^n\,, \tag{1.7}$$

a numerical fit to A_n for large n was found to be

$$A_n=-\frac{\sqrt{6}}{\pi^{3/2}}(-3)^n\,\Gamma(n+\tfrac{1}{2})\left[1+O\left(\frac{1}{n}\right)\right]. \tag{1.8}$$

Later, methods were developed that explain this behavior analytically.[8,9] Indeed, two completely independent techniques were developed which predict formulas similar to Eq. (1.8) for *all* energy levels of *all* anharmonic oscillators having λx^{2N} interactions. Analogous results for anharmonic oscillators with arbitrary polynomial interactions were also obtained and the effect of Wick ordering on perturbation theory in large order was determined.[10] Moreover, higher-order corrections [$O(1/n)$ terms] were also calculated and these also agree with computer calculations.[8,9]

The anharmonic oscillator suggests how perturbation theory might behave in a $(\phi^4)_2$ field theory because it is also a quantum field theory. Equations (1.5) and (1.6) can be reinterpreted as describing a ϕ^4 field theory in one-dimensional space-time.[11] For more complicated systems such as those defined in Eqs. (1.1) and (1.2) and in Eq. (1.4), A_n behaves for large n very much like the A_n in Eq. (1.8). We would not be very surprised to find that the ground-state energy perturbation series for a $(\phi^4)_2$ field theory behaves similarly.

We now give a more detailed outline of this paper. Section II employs dispersion techniques to reduce the problem of finding the leading large-n behavior of A_n to that of solving a quantum-mechanical barrier-penetration problem. The barrier penetration is then treated semiclassically. In Secs. III and IV we develop those semiclassical methods needed to solve Eqs. (1.1) and (1.2) for the special case where $a=b=1$. In Sec. III we solve the problem approximately using geometrical optics, and in Sec. IV we find the precise leading behavior of A_n for large n using physical optics. The notions of geometrical and physical optics are discussed in detail. The results are as follows: For $-1 \leq c < 1$,

$$A_n \sim -[48c/\sin(\pi\nu)]^{1/2}(-3)^n\,\Gamma(n+\tfrac{1}{2})/\pi\,, \tag{1.9}$$

where $\nu(\nu+1)=2c$; for $c>1$,

$$A_n \sim -[48\bar{c}/\sin(\pi\bar{\nu})]^{1/2}[(-3c-6)/2]^n\,\Gamma(n+\tfrac{1}{2})/\pi\,, \tag{1.10}$$

where $\bar{\nu}(\bar{\nu}+1)=2\bar{c}=(3-c)/(1+c)$; and for $c=1$,

$$A_n \sim 2(-3)^{n+1}\,\Gamma(n+1)/\pi\,. \tag{1.11}$$

The discontinuity at $c=1$ is caused by a kind of constructive interference.

The calculation for the special spherically symmetric case ($c=1$) is done in Appendix A. Appendix A actually treats the more general case of N oscillators with spherically symmetric coupling. In Appendix B we repeat the calculations of Sec. IV for a system of two coupled oscillators with $2N$ order rather than quartic coupling.

The predictions in Eqs. (1.9)–(1.11) are compared with computer calculations in Sec. V and we observe spectacular agreement. Section VI generalizes the results in Eqs. (1.9)–(1.11) to the case where a and b are not 1. Again the theoretical and numerical calculations agree. Finally, in Sec. VII we investigate Eq. (1.4). In Appendix C we discuss some properties of equal-mass N-oscillator systems that are more general than those in Eq. (1.4).

II. DISPERSION RELATION

In this section we reduce the problem of finding the large-order behavior of the Rayleigh-Schrödinger coefficients to a semiclassical barrier-penetration problem. This reduction is necessary because, as we explain below, a direct analysis of the difference equation which generates the perturbation series is too difficult.

In general, the Rayleigh-Schrödinger coefficients [such as those defined in Eq. (1.3)] for a perturbation problem in quantum mechanics may be computed from a recursion relation which gives the nth coefficient in terms of its predecessors and integrals over all the lower-order contributions to the wave function. The special form of the unperturbed wave function for the harmonic oscillator and our choice of a polynomial perturbing potential [see Eqs. (1.1) and (1.4)] enable us to reduce the recursion relation to a single nonlinear partial difference equation. This difference equation, which we derive and discuss in detail in Secs. V and VI, may be solved on a computer.

The corresponding but much simpler difference equation for a single anharmonic oscillator can actually be solved approximately.[8,9] However, for a system of coupled anharmonic oscillators, the indices in the difference equation proliferate and the relevant approximation techniques become unwieldy, even for the two-mode oscillator. With this difficulty in mind, we have developed in a previous paper a rather indirect, but physically intuitive, method for computing the large-n behavior of A_n which generalizes immediately to the many-coupled oscillator problem. This method rests on Eq. (2.4).

Equation (2.4) may be derived by assuming a once-subtracted dispersion relation for $E(\lambda)$:

$$F(\lambda)=\frac{1}{2\pi i}\int_{-\infty}^{0}\frac{D(x)}{x-\lambda}\,dx\,, \tag{2.1}$$

where

$$F(\lambda)\equiv[E(\lambda)-E(0)]/\lambda \tag{2.2}$$

and

$$D(\lambda)\equiv\lim_{\epsilon\to 0}[F(\lambda+i\epsilon)-F(\lambda-i\epsilon)]\,. \tag{2.3}$$

From Eq. (2.1) we obtain an exact expression for the nth Rayleigh-Schrödinger coefficient after expanding $(x-\lambda)^{-1}$ and comparing with Eq. (1.3). We find that

$$A_n=\frac{1}{2\pi i}\int_{-\infty}^{0}D(x)x^{-n}\,dx\,. \tag{2.4}$$

The proof of Eq. (2.1) for the one-mode oscillator is outlined in Sec. II of BW II. Although there is as yet no proof, it is intuitively reasonable that Eq. (2.1) is valid for the N-mode oscillator. However, a proof would require that $E(\lambda)$ be analytic in the cut λ plane. Analyticity has only been proved[12] in the region shown in Fig. 1. Fortunately, analyticity in this region is sufficient to rigorously establish Eq. (2.4) for large n.[12]

Equation (2.4) may be understood as follows. When λ is negative, the perturbing potential in Eq. (1.1) becomes repulsive. The states that were found for $\lambda > 0$ now decay. Decaying states have complex eigenvalues and the imaginary part of the eigenvalue is inversely proportional to the lifetime of the state. Equation (2.4) relates A_n to the inverse moments of the imaginary part of $E(\lambda)$ and thus to the lifetime of the decaying state. For large n, the integral is dominated by small negative values of x. Thus, the large-order behavior of A_n is related to the imaginary part of $E(\lambda)$ for small negative coupling constant.

It is interesting to note that if the perturbing potential is dominated by the unperturbed potential when λ gets small enough, all states will be bound for such values of λ. We then have $\mathrm{Im}E(\lambda) = 0$ for $|\lambda|$ sufficiently small. In this case, Eq. (2.4) tells us that $\lim_{n\to\infty} A_n = 0$, and the perturbation series converges. Thus, Eq. (2.4) quantifies Dyson's[13] old argument about the divergence of perturbation theory. We emphasize the similarity between the

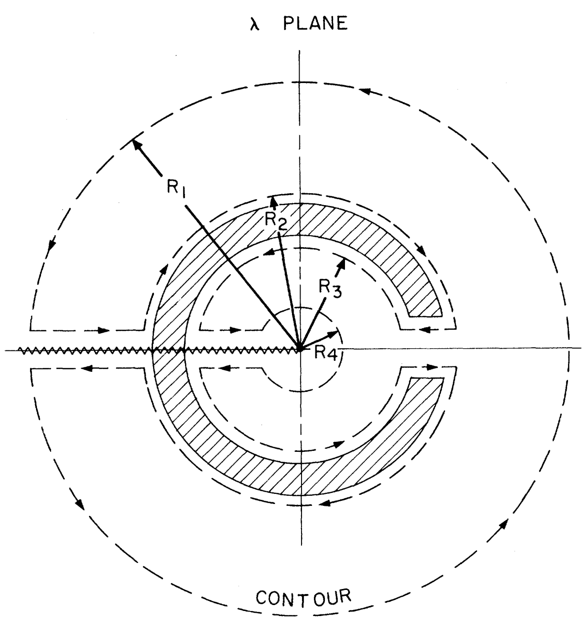

FIG. 1. The domain of analyticity of $E(\lambda)$ rigorously established by Simon. $E(\lambda)$ is analytic except on the cut along the negative real axis and in the shaded region. The contour appropriate for proving Eq. (2.4) is indicated by a dashed line.

above discussion and that given by Adler in Sec. VA of Ref. 1.

Equation (2.4) relates the computation of A_n to the familiar quantum-mechanical barrier-penetration problems as follows. Generalizing the arguments of the Appendix in Ref. 9 we can express $\mathrm{Im}E(\lambda)$ as

$$\mathrm{Im}E = \frac{\int_{S_V} \vec{J} \cdot d\vec{S}}{\int_V \psi^* \psi \, dV}, \tag{2.5}$$

where $\vec{J}$ is the probability current,

$$\vec{J} \equiv \tfrac{1}{2} i (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*), \tag{2.6}$$

and S is the boundary of volume V. We take S to lie outside the barrier so that there are no points outside of S where the probability current can reflect back toward the origin. Thus $\vec{J}$ measures decay of the state by expressing the flow of probability density out to $|\vec{r}| = \infty$.

Equations (2.4) and (2.5) relate A_n to the perturbed wave function for small negative λ.

III. GEOMETRICAL OPTICS

A. Discussion of Method of Solution

In Sec. II we reduced the calculation of the large-order behavior of perturbation theory to the familiar quantum-mechanical problem of tunneling through a nonspherical N-dimensional potential barrier. We propose a method for solving this problem which rests on a simple physical picture. A particle in an unstable state centered at the origin will ultimately penetrate the barrier and escape to infinity. The total amplitude for escape is the sum of the amplitudes over all possible paths of escape. We will show that there exist most probable escape paths (MPEP's) and that the relative amplitude to escape along other paths is exponentially small. The dominant contribution to the escape amplitude comes from regions, which we call tubes, surrounding the MPEP's. The probability current is negligible outside of these tubes during tunneling and flows outward in narrow beams. We will show that for a system without a rotational symmetry the number of tubes is finite and that they are well separated. We will then use semiclassical (WKB-related) methods to approximate the solutions to the Schrödinger equation [Eq. (1.1)] within these tubes.

We have introduced the notion of a tube in order to reduce our nonspherical multidimensional problem to one which is approximately one-dimensional. It is natural to try to solve a tunneling problem using WKB techniques.[14] There has been much work on the problem of WKB approximations to many-dimensional systems.[14] The zeroth-order WKB equation for the phase S of a wave function with energy E is[15]

$$(\vec{\nabla} S)^2 = V - E. \tag{3.1}$$

This is just the Hamilton-Jacobi equation for a classical system with Hamiltonian $\vec{p}^2 + V$. In one dimension it reduces to $(dS/dx)^2 = V - E$, whose solution is $S = \pm \int (V-E)^{1/2}$. For the general multidimensional case it is a nonlinear partial differential equation. Of course, if the Hamiltonian has a continuous symmetry, Eq. (3.1) will be separable. However, Eq. (3.1) is nontrivial in general. The new multidimensional techniques which we have discovered simplify the problem of solving Eq. (3.1) because now we need to solve it only in a small, approximately one-dimensional region. Our technique is expressly designed to deal with problems which do *not* have continuous symmetries, and is thus *complementary* to the separation of variables idea.

We briefly review the path-integral formalism. The amplitude for a particle of energy E to take a particular path P in a potential V is

$$\frac{1}{N} \exp\left[- \int_P (V-E)^{1/2} \right], \tag{3.2}$$

where $\int_P (E-V)^{1/2}$ is the classical action and the normalization factor N depends only on the end points of the path. The total amplitude is just the sum of Eq. (3.2) over all paths P.

In the tunneling region, $(V-E)$ is positive, and the amplitude is exponentially damped. Therefore, the dominant contribution to the amplitude comes from regions near the paths which minimize the action integral and thus satisfy

$$\delta \int (V-E)^{1/2} = 0. \tag{3.3}$$

The Euler-Lagrange equations following from Eq. (3.3) are[16]

$$2(V-E)\frac{d^2 x_i}{ds^2} + \frac{dx_i}{ds}\left[\sum_j \frac{dx_j}{ds}\frac{dV}{dx_j}\right] = \frac{\partial V}{\partial x_i}, \tag{3.4}$$

where s is the path length. All solutions of Eq. (3.4) are local stationary points of the action. However, we are interested in the *global* minima. This will eliminate all except a discrete[17] set of paths which are just the MPEP's.

Once we have found a set of MPEP's, we must find approximate solutions to the Schrödinger equation along these trajectories. As in any semiclassical or ray description of a wave phenomenon, we must distinguish two levels of approximation. At the first level, called geometrical optics or the eikonal approximation, the phase of the wave function is approximated by a line integral along the trajectory, while its amplitude is assumed to be constant. This is just zeroth-order WKB. The second level, called physical optics or first-order WKB, takes into account the variation of the amplitude and the spread of the wave function into the region around the trajectory. Thus, physical optics is characterized by a set of tubes through which most of the probability current flows.

These two levels of approximation are clearly distinguished in our results for the large-order behavior of perturbation theory. We find that in general for large n

$$A_n \sim K L^n \Gamma(Mn+J)[1+O(1/n)]. \tag{3.5}$$

The constants L, M, and J are determined by geometrical optics alone. Physical optics is needed to find the value of K.

B. Determination of Most Probable Escape Paths

The program we outlined in part A of this section for finding the MPEP's is of course very difficult. It involves actually finding closed-form solutions to Eq. (3.4) and explicitly selecting those solutions which minimize $\int (V-E)^{1/2}$. Fortunately, in many cases, a heuristic argument enables us to guess the most probable paths without solving Eq. (3.4) and these turn out to be straight lines. In fact, it is generally true that the MPEP's for the equal-mass oscillators defined in Eq. (1.4) are straight lines. The more difficult problem of unequal-mass oscillators, which have curved MPEP's, will be discussed in the next paper of this series.[18] It is easy to show that the straight MPEP's satisfy Eq. (3.4), but we have no way of proving that they are global minima of the action. The only convincing evidence we have for this is the excellent agreement of our results with our computer calculations.

In this section we use geometrical optics to treat the special case of Eq. (1.1) for which $a = b = 1$. This simplifies the notation without obscuring any of the important features of the problem. In Sec. IV we use physical optics to treat this same case. Equation (1.1) is solved in general in Sec. VI.

We expect a straight MPEP to satisfy certain reasonable criteria. It should be a "path of least resistance" to tunneling and thus should pass through a saddle point of the potential $V = \frac{1}{4}x^2 + \frac{1}{4}y^2 - \frac{1}{4}\epsilon(x^4 + y^4 + 2cx^2y^2)$. The saddle point should be oriented along the path (which is a radial line). A saddle point of V satisfies the equations

$$\begin{aligned} \frac{\partial V}{\partial x} &= \tfrac{1}{2}x - \epsilon(x^3 + cxy^2) = 0, \\ \frac{\partial V}{\partial y} &= \tfrac{1}{2}y - \epsilon(y^3 + cx^2y) = 0. \end{aligned} \tag{3.6}$$

Equations (3.6) have nine solutions, namely,

$$(x, y)=(0,0),$$
$$(\pm(2\epsilon)^{-1/2}, 0), \quad (0, \pm(2\epsilon)^{-1/2}), \tag{3.7}$$
and
$$\pm([2(c+1)\epsilon]^{-1/2}, \pm[2(c+1)\epsilon]^{-1/2}).$$

To identify those critical points in Eq. (3.7) which are radially oriented saddle points, we compute the Hessian matrix H (matrix of second partial derivatives). We demand that H have one positive and one negative eigenvalue at the critical point and further require that the eigenvector having negative eigenvalue must lie along the radial line connecting the critical point with the origin. We find that

$$H=\begin{bmatrix} \frac{1}{2}-\epsilon(3x^2+cy^2) & -2c\epsilon xy \\ -2c\epsilon xy & \frac{1}{2}-\epsilon(3y^2+cx^2) \end{bmatrix}. \tag{3.8}$$

It follows that $(0,0)$ is not a saddle point, that $(\pm(2\epsilon)^{-1/2}, 0)$ and $(0, \pm(2\epsilon)^{-1/2})$ are acceptable saddle points when $-1 \leqslant c < 1$, and $\pm([2(c+1)\epsilon]^{-1/2}, \pm[2(c+1)\epsilon]^{-1/2})$ are acceptable saddle points when $c>1$. When $c=1$, there are no saddle points. This is the spherically symmetric case where all straight-line paths contribute equally to the amplitude and it is treated separately in Appendix A.

It is now easy to show that radial lines through the saddle points are solutions of Eq. (3.4).[19] Equation (3.4) reduces to

$$\frac{dx}{ds}\left(\frac{dx}{ds}\frac{\partial V}{\partial x}+\frac{dy}{ds}\frac{\partial V}{\partial y}\right)=\frac{\partial V}{\partial x},$$
$$\frac{dy}{ds}\left(\frac{dx}{ds}\frac{\partial V}{\partial x}+\frac{dy}{ds}\frac{\partial V}{\partial y}\right)=\frac{\partial V}{\partial y}, \tag{3.9}$$

because our straight-line paths have the property that

$$\frac{d^2x}{ds^2}=\frac{d^2y}{ds^2}=0.$$

Equation (3.9) is satisfied by $(dx/ds, dy/ds) = (0, \pm1), (\pm1, 0)$, and $(\pm\sqrt{2}/2, \pm\sqrt{2}/2)$ because of three properties of V:

$$\left.\frac{\partial V}{\partial y}\right|_{y=0}=0,$$
$$\left.\frac{\partial V}{\partial x}\right|_{x=0}=0, \tag{3.10}$$
$$\frac{\partial V}{\partial x}=\pm\frac{\partial V}{\partial y}, \quad \text{when } x=\pm y.$$

C. Geometrical Optics

We outline here a brief and heuristic treatment. A careful and mathematically detailed approach is given in Sec. IV. We follow method 2 of the Appendix of BW II. Up to multiplicative constants the wave function $\psi(x, y)$ in the tunneling region on the MPEP is given by

$$e^{-\int_{s_0}^{(x,y)}(V-E)^{1/2}ds}.$$

We are ignoring all paths except the four MPEP's. E is the unperturbed value of the energy, namely, unity. The integral is taken along the MPEP from the inner turning point $s_0=O(1)$ (solution of $V-E=0$) to the argument of $\psi(x, y)$.

We are interested in computing the current flowing out to infinity. Since the distant turning point $s_1=O(\epsilon^{-1/2})$ is the last place where the current can be reflected back toward the origin, we compute the current at a point just beyond s_1. The total current J is the sum of the currents along each MPEP. J is proportional to

$$\exp\left[-2\int_{s_0}^{s_1}(V-E)^{1/2}ds\right]$$

for each MPEP. This reduces to

$$\exp\left\{-2\int_{s_0}^{s_1}\left[\tfrac{1}{4}s^2-\tfrac{1}{4}\epsilon s^4\alpha-1\right]^{1/2}ds\right\},$$

where $\alpha=\cos^4\theta+\sin^4\theta+2c\sin^2\theta\cos^2\theta$, and θ is the angle between the MPEP and the x axis. Computing the above integral approximately gives

$$J\propto\epsilon^{-1/2}e^{-1/(3\epsilon\alpha)}$$

for each MPEP. From Eqs. (2.4) and (2.5) we thus obtain

$$A_n\propto\int_{-\infty}^{0}d\epsilon(-\epsilon)^{-n-3/2}e^{1/(3\epsilon\alpha)}$$
$$\propto\Gamma(n+\tfrac{1}{2})(-3\alpha)^n. \tag{3.11}$$

When $-1\leqslant c<1$, the MPEP's are along the axes and $\alpha=1$. When $c>1$, the MPEP's are at 45° to the axes and $\alpha=\frac{1}{2}(c+1)$. Thus, for large n,

$$A_n\propto(-3)^n\Gamma(n+\tfrac{1}{2}), \quad -1\leqslant c<1,$$
and
$$A_n\propto[-3(c+1)/2]^n\Gamma(n+\tfrac{1}{2}), \quad c>1. \tag{3.12}$$

Equation (3.12) is continuous in c at $c=1$.

Equation (3.12) clearly illustrates the phenomenon of *decoupling* that takes place in the large-order behavior of perturbation theory. When the coupling of the oscillators is strong enough ($c>1$), A_n depends on the coupling term. But when $-1\leqslant c<1$, the system seems to behave as if the oscillators were completely uncoupled. Actually, when $c<1$ the multiplicative constant K, which we will determine in Sec. IV, still depends on c. Nevertheless, the decoupling of the *dominant* behavior of A_n for large n is quite remarkable and is typical of the simplification that we observe in large order.

IV. PHYSICAL OPTICS

In this section we use physical optics to approximate the imaginary part of the ground-state energy for the system

$$\left\{-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+\tfrac{1}{4}x^2+\tfrac{1}{4}y^2 -\tfrac{1}{4}\epsilon(x^4+y^4+2cx^2y^2)-E\right\}\psi=0\,. \quad (4.1)$$

We will solve the problem explicitly only for $-1 \leq c < 1$. When $c > 1$ we use the following symmetry transformation to reduce the problem back to the $c < 1$ case[20]:

$$x \to \bar{x}=\frac{x+y}{\sqrt{2}}\,,\qquad y\to \bar{y}=\frac{x-y}{\sqrt{2}}\,. \quad (4.2)$$

Equation (4.2) converts Eq. (4.1) into

$$\left\{-\frac{\partial^2}{\partial \bar{x}^2}-\frac{\partial^2}{\partial \bar{y}^2}+\tfrac{1}{4}\bar{x}^2+\tfrac{1}{4}\bar{y}^2 -\tfrac{1}{4}\epsilon\left(\frac{1+c}{2}\right)\left(\bar{x}^4+\bar{y}^4+2\frac{3-c}{1+c}\,\bar{x}^2\bar{y}^2\right)-E\right\}\psi=0\,. \quad (4.3)$$

We then make the additional transformations

$$c\to\bar{c}=(3-c)/(1+c)\,,\qquad \epsilon\to\bar{\epsilon}=\epsilon(1+c)/2\,, \quad (4.4)$$

and observe that $c>1$ implies that $|\bar{c}|<1$.

A. The Physical-Optics Approximation

We will solve Eq. (4.1) in a tube of thickness $O(1)$ surrounding the positive x axis. (The x axis is an MPEP when $|c|<1$.) To do so, we break the tube into two regions: Region I, where $y=O(1)$ and $x<\epsilon^{-1/4}$, and Region II, where $y=O(1)$ and $\epsilon^{-1/6}<x \ll \epsilon^{-1/2}$. Notice that the regions overlap.

In Region I we approximate Eq. (4.1) by

$$\left[-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+\tfrac{1}{4}x^2+\tfrac{1}{4}y^2-1\right]\psi=0\,, \quad (4.5)$$

whose solution is the unperturbed ground-state wave function

$$\psi_{\rm I}=e^{-(x^2+y^2)/4}\,. \quad (4.6)$$

We have freely chosen the normalization of $\psi_{\rm I}$.

In Region II, we approximate Eq. (4.1) by

$$\left[-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+\tfrac{1}{4}x^2+\tfrac{1}{4}y^2-\tfrac{1}{4}\epsilon(x^4+2cx^2y^2)-1\right]\psi=0\,, \quad (4.7)$$

where we have neglected ϵy^4 compared with y^2. As in BW II, we seek a solution which is exponentially decaying with increasing x in the tunneling region. Thus, we factor off a decreasing WKB-type function of the x variable:

$$\psi=\chi(\tfrac{1}{4}x^2-\tfrac{1}{4}\epsilon x^4-\tfrac{1}{2})^{-1/4} \times\exp\left[-\int_{\sqrt{2}}^{x}(\tfrac{1}{4}t^2-\tfrac{1}{4}\epsilon t^4-\tfrac{1}{2})^{1/2}dt\right]. \quad (4.8)$$

We have chosen the lower end point of the integral arbitrarily. The new equation satisfied by χ is

$$(x^2-\epsilon x^4)^{1/2}\chi_x-\chi_{yy}+(\tfrac{1}{4}y^2-\tfrac{1}{2}-\tfrac{1}{2}\epsilon cx^2y^2)\chi=0\,. \quad (4.9)$$

The change of scale

$$\epsilon x^2=z^2 \quad (4.10)$$

eliminates all reference to ϵ from Eq. (4.9) and gives

$$z(1-z^2)^{1/2}\chi_z-\chi_{yy}+(\tfrac{1}{4}y^2-\tfrac{1}{2}-\tfrac{1}{2}cy^2z^2)\chi=0\,. \quad (4.11)$$

In Eq. (4.8) we factored off the rapidly changing geometrical optics behavior. Equation (4.11) contains the next-order correction to this behavior which we have referred to as physical optics; that is, Eq. (4.11), when solved, will provide the multiplicative constants that were missing in Eq. (3.12). However, there are no further approximations to be made because all quantities in Eq. (4.11) are of order unity. Equation (4.11) must now be solved exactly.

The change of variables

$$w=(1-z^2)^{1/2} \quad (4.12)$$

is useful because the resulting equation,

$$(w^2-1)\chi_w-\chi_{yy}+\left[\tfrac{1}{4}y^2-\tfrac{1}{2}-\tfrac{1}{2}cy^2(1-w^2)\right]\chi=0\,, \quad (4.13)$$

no longer contains a square-root term.

One strategy for solving Eq. (4.13) is to transform the dependent variable so that a Fourier transform in the y variable gives a (hopefully soluble) first-order partial differential equation. Of course, an immediate Fourier transform of Eq. (4.13) is useless because of the y^2 term. We are thus led to the substitution

$$\chi=e^{-y^2 f(w)/4}A\,. \quad (4.14)$$

The undetermined function $f(w)$ will be chosen to eliminate the y^2 term from the differential equation for A. It will then be profitable to Fourier transform that equation because the highest power of y will be one. The above constraint on $f(w)$ takes the form of a Riccati equation:

$$-(w^2-1)f'(w)-f^2(w)+1-2c+2cw^2=0\,. \quad (4.15)$$

When Eq. (4.15) is satisfied, the equation for A simplifies to

$$(w^2-1)A_w - A_{yy} + yf(w)A_y + \tfrac{1}{2}[f(w)-1]A = 0\,. \tag{4.16}$$

First we solve Eq. (4.15). A standard substitution which linearizes the Riccati equation is

$$f(w) = (w^2-1)u'(w)/u(w)\,. \tag{4.17}$$

We obtain

$$(1-w^2)u'' - 2wu' + u\left(2c - \frac{1}{1-w^2}\right) = 0\,. \tag{4.18}$$

We gratefully recognize that Eq. (4.18) is the associated Legendre equation.[21] Solutions to this equation are

$$u(w) = P_\nu^\mu(w),\ Q_\nu^\mu(w)\,, \tag{4.19}$$

where

$$\nu(\nu+1) = 2c\,,\quad \mu^2 = 1\,. \tag{4.20}$$

For definiteness we choose[22]

$$u(w) = P_\nu^{-1}(w)\,. \tag{4.21}$$

Next, we return to Eq. (4.16) and complete its solution. It is clear that the strategy of the substitution in Eq. (4.14) has succeeded; that is, if we Fourier transform in the y variable, the resulting equation will be first order and should yield to the method of characteristics. However, we are fortunate that there is an even simpler approach. We change to new independent variables

$$(w, y) \to (w,\ s = y/u(w))\,. \tag{4.22}$$

In terms of these variables, Eq. (4.16) becomes

$$(w^2-1)u^2(w)A_w + \tfrac{1}{2}u^2(w)[f(w)-1]A = A_{ss}\,, \tag{4.23}$$

which is separable.

We now argue that the separation constant for Eq. (4.23) is 0. To justify this contention explicitly we separate

$$A(w, s) = B(w)C(s)\,. \tag{4.24}$$

For separation constant α^2, the equation for $C(s)$ is $C''(s) = \alpha C(s)$, whose solution is $C(s) \propto \cosh(\alpha s) = \cosh[\alpha y/u(w)]$. Here we have kept only the even solution in y because only an even solution can be matched to $\psi_{\rm I}$ in Eq. (4.6). However, $u(w)$ in Eq. (4.21) vanishes [see Eq. (4.29)] at $w=1$ and $w \lesssim 1$ is in the overlap of Regions I and II [see Eqs. (4.10) and (4.12)]. Therefore, there is no asymptotic match of ψ across Regions I and II unless, of course, $\alpha^2 = 0$.

Having shown that $\alpha^2 = 0$ and thus that $C(s)$ is constant, it is straightforward to solve for $B(w)$. We obtain (omitting a multiplicative constant)

$$B(w) = u^{-1/2}[(1-w)/(1+w)]^{1/4}\,. \tag{4.25}$$

Note that as $w \to 1$, $B(w) \to$ a finite constant [see Eq. (4.31)]. Thus, it is possible to match $\psi_{\rm I}$ and $\psi_{\rm II}$ asymptotically in the overlap region.

We have now solved Eq. (4.1) in Region II up to an over-all multiplicative constant β. Our final result is

$$\begin{aligned}\psi_{\rm II} = {} & \beta(\tfrac{1}{4}x^2 - \tfrac{1}{4}\epsilon x^4 - \tfrac{1}{2})^{-1/4} \\ & \times \exp\left[-\int_{\sqrt{2}}^{x} (\tfrac{1}{4}t^2 - \tfrac{1}{4}\epsilon t^4 - \tfrac{1}{2})^{1/2}dt\right] \\ & \times \exp[-y^2 f(w)/4][u(w)]^{-1/2} \\ & \times [(1-w)/(1+w)]^{1/4}\,. \end{aligned} \tag{4.26}$$

It is easy to identify the physical meaning of the three types of terms in Eq. (4.26). There is a rapidly varying term from geometrical optics and several slowly varying terms that do not depend on y. These describe the amplitude along the MPEP. Finally, the term $\exp[-y^2 f(w)/4]$ describes the falloff of probability current in the tube surrounding the x axis. A quick calculation shows that as x approaches the turning line [the line along which $V(x, y) - E = 0$] at the end of the tunnel near $x = \epsilon^{-1/2}$, w approaches 0, and, for positive c, the tube gradually widens. At the turning line the tube flares out like the bell of a trumpet. When $-1 \le c < 0$, the tube narrows as $w \to 0$. When $c = 0$, the thickness of the tube is constant along its length.

It might appear that our solution Eq. (4.26) is the result of an amazing sequence of lucky substitutions whose application is rather limited. Actually these techniques immediately generalize to *all* straight-line path problems [see Secs. V–VII and Appendix B]. Moreover, when we study an arbitrary curved-path problem in our next paper[23] we show that factoring off the geometrical optics behavior always leads to a Riccati equation whose solution is related to the thickness of the tube. So, on the contrary, the substitutions we have made are both natural and general.

B. Asymptotic Matching of Regions I and II

We now determine β by requiring that $\psi_{\rm I}$ in Eq. (4.6) and $\psi_{\rm II}$ in Eq. (4.26) become asymptotically equal in the overlap of Regions I and II. In this overlap region, x is large compared with 1 but small compared with $\epsilon^{-1/2}$. Thus, we approximate

$$(\tfrac{1}{4}x^2 - \tfrac{1}{4}\epsilon x^4 - \tfrac{1}{2})^{-1/4} \sim (\tfrac{1}{2}x)^{-1/2}\,, \tag{4.27}$$

$$-\int_{\sqrt{2}}^{x} (\tfrac{1}{4}t^2 - \tfrac{1}{4}\epsilon t^4 - \tfrac{1}{2})^{1/2} dt$$
$$\sim -\tfrac{1}{2}\int_{\sqrt{2}}^{x} (t^2-2)^{1/2}dt - \tfrac{1}{4}x^2 + \tfrac{1}{2}\ln x + \tfrac{1}{4} + O\left(\frac{1}{x^2}\right). \tag{4.28}$$

Also, in the overlap of Regions I and II, $w \sim 1$. Thus, we use[24]

$$u(w) = P_\nu^{-1}(w) \sim 2^{-1/2}(1-w)^{1/2}, \quad \text{when } w \sim 1. \tag{4.29}$$

Equation (4.29) implies that

$$f(w) = \frac{(w^2-1)u'(w)}{u(w)} \sim 1, \tag{4.30}$$

$$u^{-1/2}\left(\frac{1-w}{1+w}\right)^{1/4} \sim 1. \tag{4.31}$$

Combining Eqs. (4.26)–(4.31) gives

$$\psi_{\rm II} \sim \beta e^{-(x^2+y^2)/4} e^{1/4}\sqrt{2}. \tag{4.32}$$

Thus, comparing Eqs. (4.32) and Eq. (4.6) gives

$$\beta = 2^{-1/2} e^{-1/4}. \tag{4.33}$$

Now ψ in Region II is completely determined relative to the normalization of ψ in Region I.

C. Determination of the Probability Current $J(x)$

In the Appendix of BW II we developed a trick (referred to there as Method 2) for evaluating the probability current for values of x further from the origin than the turning line without ever doing turning-point analysis. Without further explanation we use this technique to obtain the magnitude of the probability current emerging from the end of the tube along the positive x axis:

$$J(y) = \beta^2 e^{-y^2 f(0)/2} u^{-1}(0) \times \exp\left[-\int_{\sqrt{2}}^{x_1} (t^2 - \epsilon t^4 - 2)^{1/2} dt\right]. \tag{4.34}$$

x is the distant zero of the integrand. Note that J is a function of y only.

The evaluation of the integral is given in BW II as

$$\exp\left[-\int_{\sqrt{2}}^{x_1}(t^2-\epsilon t^4-2)^{1/2}dt\right] \sim e^{-1/3\epsilon} 2(e/\epsilon)^{1/2}. \tag{4.35}$$

To compute $u(0)$ and $f(0)$ we use the formulas[25]

$$u(0) = P_\nu^{-1}(0) = \tfrac{1}{2}\pi^{-1/2}\sin(\tfrac{1}{2}\pi\nu)\Gamma(\tfrac{1}{2}\pi\nu)/\Gamma(\tfrac{3}{2}+\tfrac{1}{2}\nu),$$
$$u'(0) = \frac{d}{dw}P_\nu^{-1}(w)\Big|_{w=0} = -\pi^{-1/2}\cos(\tfrac{1}{2}\pi\nu)\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu)/\Gamma(1+\tfrac{1}{2}\nu). \tag{4.36}$$

From Eqs. (4.17) and (4.36) we have

$$\frac{f(0)}{2} = \frac{\cos(\frac{1}{2}\pi\nu)\,\Gamma(\frac{1}{2}+\frac{1}{2}\nu)\Gamma(\frac{3}{2}+\frac{1}{2}\nu)}{\sin(\frac{1}{2}\pi\nu)\,\Gamma(\frac{1}{2}\nu)\Gamma(1+\frac{1}{2}\nu)}. \tag{4.37}$$

Finally, we combine Eqs. (4.33)–(4.35) and obtain

$$J(y) = e^{-1/3\epsilon}\epsilon^{-1/2}\exp[-y^2 f(0)/2]/u(0), \tag{4.38}$$

with $u(0)$ and $f(0)/2$ given above.

D. Computation of ImE

To calculate ImE, we use Eq. (2.5) and proceed to evaluate the integrals in the denominator and numerator in turn.

The integral in the denominator is done by replacing ψ with $\psi_{\rm I}$ in Eq. (4.6) and allowing V, the region of integration, to be unbounded. This is a good approximation because the dominant contribution comes from Region I. We obtain

$$\int_V \psi^*\psi\, dV \sim 2\pi. \tag{4.39}$$

The integral in the numerator is a surface integral which reduces to an integral over y. We allow the end points of the integral to be unbounded, use Eq. (4.38), and find that

$$\int_{S_V} \vec{J}\cdot d\vec{S} \sim \frac{e^{-1/3\epsilon}\epsilon^{-1/2}}{u(0)}\int_{-\infty}^{\infty} dy\, e^{-y^2 f(0)/2} \sim e^{-1/3\epsilon}\left[\frac{2\pi}{\epsilon f(0)u^2(0)}\right]^{1/2}. \tag{4.40}$$

After using Eqs. (4.36) and (4.37), Eq. (4.40) simplifies drastically. From this result and Eqs. (2.5) and (4.39) we have finally

$$\text{Im}E = 2\sqrt{2}\, e^{-1/3\epsilon}[\nu(\nu+1)/\epsilon\sin(\pi\nu)]^{1/2}. \tag{4.41}$$

We have multiplied by an extra factor of 4 to obtain Eq. (4.41) because the contributions to the integral for ImE come from four equal tubes. As much current flows out along the positive x axis as along the negative x axis and symmetrically along the y axis in both directions.

E. Perturbation Theory in Large Order

From Eqs. (2.4) and (4.41), we have for large n

$$A_n = \left[\frac{\nu(\nu+1)\pi}{\sin(\pi\nu)}\right]^{1/2} \times \frac{2\sqrt{2}}{\pi^{3/2}}\int_0^\infty d\epsilon\, e^{-1/3\epsilon}\epsilon^{-n-3/2}(-1)^{n+1}$$
$$= 2\left[\frac{2c\pi}{\sin(\pi\nu)}\right]^{1/2}\frac{\sqrt{6}}{\pi^{3/2}}(-1)^{n+1}3^n\,\Gamma(n+\tfrac{1}{2}). \tag{4.42a}$$

Equation (4.42a) is valid for $c<1$, $\nu(\nu+1)=2c$.

When $c>1$, we use Eqs. (1.3) and (4.4) to obtain

$$A_n \sim 2\left[\frac{2\bar{c}\pi}{\sin(\pi\bar{\nu})}\right]^{1/2} \frac{\sqrt{6}}{\pi^{3/2}}(-1)^{n+1}\left(\frac{3c+6}{2}\right)^n \Gamma(n+\tfrac{1}{2}), \tag{4.42b}$$

where $\bar{\nu}(\bar{\nu}+1)=2\bar{c}=(3-c)/(1+c)$.

For $c=1$, we cite the result in Eq. (A18), to wit:

$$A_n \sim (-1)^{n+1}\frac{6}{\pi}3^n\,\Gamma(n+1). \tag{4.42c}$$

F. Discussion

We can immediately verify Eq. (4.42a) for the case $c=0 \Rightarrow \nu=0$. As $\nu \to 0$, the quantity in square brackets approaches unity and we obtain exactly twice the result in Eq. (1.8). This is because now we have two uncoupled anharmonic oscillators.

The other and more interesting limit to investigate is $c \to 1 \Rightarrow \nu \to 1$. Now the term in square brackets blows up. This singularity corresponds with the onset of spherical symmetry. Recall that our analytical procedures necessarily break down for that case because there are no isolated MPEP's—all radial paths are equally probable. The sudden increase from four to an infinite number of MPEP's allows the probability current to escape to infinity faster, reducing the lifetime of the unstable state. We thus observe a constructive interference phenomenon which causes an enhancement in the rate of divergence of perturbation theory—A_n in Eq. (4.42c) is larger than A_n in Eqs. (4.42a) and (4.42b) by a factor of $n^{1/2}$.

One encounters many similar phenomena in optics. Consider, for example, a light beam parallel to the z axis and incident upon a flat elliptical plate centered about and perpendicular to the z axis. We use ray tracing to determine the amplitude of the scattered wave at a point on the z axis behind the ellipse. Only two rays scattering off the edge of the ellipse, namely, those at the ends of the minor axes, contribute appreciably to the amplitude. However, when the lengths of the major and minor axes become equal, the scattering amplitude suffers a discontinuous jump because of constructive interference. All rays scattering off the edge of the now circular disk contribute equally to the scattered wave.

We describe the numerical verification of Eq. (4.42) in Sec. V.

V. COMPARISON WITH NUMERICAL DATA

In this section we verify the theoretical predictions in Eqs. (4.42a) and (4.42b). We have run a computer program which calculates A_n to 55th order in perturbation theory by solving a difference equation. It then fits (to six significant figures) the raw Rayleigh-Schrödinger coefficients to a formula similar to but slightly less general than that in Eq. (3.5):

$$A_n = -\frac{\sqrt{6}}{\pi^{3/2}}\alpha(-3\beta)^n\,\Gamma(n+\tfrac{1}{2})\left(1+\frac{\gamma_1}{n}+\frac{\gamma_2}{n^2}+\frac{\gamma_3}{n^3}+\frac{\gamma_4}{n^4}\right). \tag{5.1}$$

The relevant numerical techniques are discussed in BW I, Appendixes D and E.

The actual difference equation that was solved is

$$\begin{aligned}(2i+2j)C_{n,i,j} &= (i+1)(2i+1)C_{n,i+1,j} \\ &\quad + (j+1)(2j+1)C_{n,i,j+1} \\ &\quad + aC_{n-1,i-2,j} + bC_{n-1,i,j-2} \\ &\quad + 2cC_{n-1,i-1,j-1} - \sum_{k=1}^{n-1} D_{n-k}C_{k,i,j},\end{aligned} \tag{5.2}$$

where

$$C_{n,1,0}+C_{n,0,1}=D_n=(-1)^{n+1}A_n, \tag{5.3}$$

and

$$\begin{aligned}&C_{0,0,0}=1,\\ &C_{n,0,0}=0 \quad \text{for } n>0,\\ &C_{n,i,j}=0 \quad \text{for } i+j>2n,\\ &C_{n,i,j}=0 \quad \text{for } i<0 \text{ or } j<0.\end{aligned}$$

Equation (5.2) is derived by substituting

$$E(\lambda)=1-\sum_{n=1}^{\infty} D_n(-\lambda)^n, \tag{5.4}$$

$$\psi(x)=e^{-(x^2+y^2)/4}\left\{1+\sum_{n=1}^{\infty} B_n(x,y)(-\lambda)^n\right\}, \tag{5.5}$$

$$B_n(x,y)=\sum_{i,j=0}^{2n} (\tfrac{1}{2}x^2)^i(\tfrac{1}{2}y^2)^j C_{n,i,j} \tag{5.6}$$

into Eq. (1.1) and collecting powers of $(\frac{1}{2}x^2)^i$, $(\frac{1}{2}y^2)^j$, and λ^n.

The numerical predictions for α and β for various values of a, b, and c are given in Table I. The predictions in Eq. (4.42) are as follows: For $-1 \leq c<1$, $a=b=1$,

$$\begin{aligned}&\alpha=\{-8\pi c/\cos[\tfrac{1}{2}\pi(1+8c)^{1/2}]\}^{1/2},\\ &\beta=1,\end{aligned} \tag{5.7}$$

where we have eliminated ν in favor of c using Eq. (4.20). For $c>1$, $a=b=1$,

$$\begin{aligned}&\alpha=\left\{\frac{8\pi(c-3)}{(1+c)\cos[\pi(25-7c)^{1/2}(4+4c)^{-1/2}]}\right\}^{1/2},\\ &\beta=(c+1)/2,\end{aligned} \tag{5.8}$$

where we have used Eq. (4.4). For $a=b=1$ the values of α and β in Table I agree to six places with

TABLE I. Numerical values of α and β in Eq. (5.1) for various values of a, b, and c in Eq. (1.1). The theoretical predictions in Eqs. (4.42a) and (4.42b) and the numerical calculations of α and β agree to six figures, which was the available limit of computer accuracy. Some values of α were not computed.

a	b	c	α	β
1	1	−1	···	1.000 00
1	1	−0.5	···	1.000 00
1	1	−0.25	1.582 42	1.000 00
1	1	$-\frac{1}{6}$	1.705 41	1.000 00
1	1	−0.05	1.903 78	1.000 00
1	1	−0.005	1.990 02	1.000 00
1	1	0	2.000 00	1.000 00
1	1	0.005	2.010 02	1.000 00
1	1	0.05	2.104 10	1.000 00
1	1	0.1	···	1.000 00
1	1	$\frac{1}{6}$	2.383 99	1.000 00
1	1	0.25	2.623 72	1.000 00
1	1	$\frac{1}{3}$	2.907 40	1.000 00
1	1	0.5	3.672 06	1.000 00
1	1	1.5	4.338 36	1.250 00
1	1	2	2.907 40	1.500 00
1	1	2.5	2.322 11	1.750 00
1	1	3	2.000 00	2.000 00
1	1	5	1.472 28	3.000 00
1	1	33	···	17.0000
1	2	0.25	···	2.000 00
1	2	1	···	2.000 00
1	2	5	···	3.285 74
1	3	0.25	···	3.000 00
1	3	1	···	3.000 00
1	3	5	···	3.666 67
1	5	0.25	···	5.000 00
1	5	1	···	5.000 00

the expressions in Eqs. (5.7) and (5.8). Note that when the argument of the cosine function becomes imaginary, cos is replaced by cosh. The function in the curly brackets is always positive.

We have done one further and rather amusing numerical calculation which does not appear in Table I. We computed A_n for $c=-5$. This problem has no apparent physical significance because the Hamiltonian is not bounded below, and therefore has no discrete eigenvalues. Nevertheless, the perturbation series is still well defined and we found that $\beta=-2.000\,00$. This result agrees with β in Eq. (5.8).

In Sec. VI we obtain theoretical values for α and β when $a, b\neq 1$ and compare these with the data in Table I.

VI. THE CASE $a, b \neq 1$

The generalization of the discussion of Secs. III and IV to the case where $a, b\neq 1$ in Eq. (1.1) is entirely straightforward. As before, we find the saddle points of V by solving

$$\frac{\partial V}{\partial x}=\tfrac{1}{2}x-\epsilon(ax^3+cxy^2)=0\,,$$

$$\frac{\partial V}{\partial y}=\tfrac{1}{2}y-\epsilon(by^3+cx^2y)=0$$

and requiring that the Hessian matrix

$$H=\begin{bmatrix}\tfrac{1}{2}-\epsilon(3ax^2+cy^2) & -2cxy\epsilon\\ -2cxy\epsilon & \tfrac{1}{2}-\epsilon(3by^2+cx^2)\end{bmatrix}$$

have negative determinant.

The critical points are $(0,0)$, $(0,\pm(2b\epsilon)^{-1/2})$, $(\pm(2a\epsilon)^{-1/2},0)$, and $\pm[2\epsilon(c^2-ab)]^{-1/2}((c-b)^{1/2}, \pm(c-a)^{1/2})$. $(0,\pm(2b\epsilon)^{-1/2})$ are saddle points if $c/b<1$ and $(\pm(2a\epsilon)^{-1/2},0)$ are saddle points if $c/a<1$. If $c>\max(a,b)$ then the off-axis critical points are saddle points. (Recall that for the Hamiltonian to be bounded below we must have $a\geq 0$, $b\geq 0$, and $c\geq -\sqrt{ab}$.) All of the saddle points are radially directed.

As in Sec. IV, the off-axis case can be reduced to the case $c/a<1$ by a rotation. To simplify the algebra we introduce the following notation:

$$D=ab-c^2=\mathrm{Det}\begin{vmatrix}a & c\\ c & b\end{vmatrix}, \tag{6.1a}$$

$$D_1=b-c=\mathrm{Det}\begin{vmatrix}1 & c\\ 1 & b\end{vmatrix}, \tag{6.1b}$$

$$D_2=a-c=\mathrm{Det}\begin{vmatrix}a & 1\\ c & 1\end{vmatrix}, \tag{6.1c}$$

$$S=D_1+D_2=a+b-2c\,. \tag{6.1d}$$

Then a suitable rotation is

$$\begin{aligned}x&=(D_1/S)^{1/2}\bar{x}\pm(D_2/S)^{1/2}\bar{y}\,,\\ y&=\mp(D_2/S)^{1/2}\bar{x}+(D_1/S)^{1/2}\bar{y}\,.\end{aligned} \tag{6.2}$$

In terms of the new variables the potential is

$$V=\tfrac{1}{4}(\bar{x}^2+\bar{y}^2)-\tfrac{1}{4}\epsilon(\bar{a}\bar{x}^4+\bar{b}\bar{y}^4+2\bar{c}\,\bar{x}^2\bar{y}^2)\,, \tag{6.3}$$

where

$$\begin{aligned}\bar{a}&=D/S\,,\\ \bar{b}&=[(D_1-D_2)^2+D]/S\,,\\ \bar{c}&=3D/S-2c\,.\end{aligned} \tag{6.4}$$

Observe that

$$\bar{c}-\bar{a}=2D/S-2c=\frac{2D_1D_2}{S}<0\,,$$

when $a>0$, $b>0$, and $c>\max(a,b)$.

Without loss of generality, then, we assume $c<a$. We will solve the Schrödinger equation in a narrow tube surrounding the x axis. In Region I where $0\leq x<\epsilon^{-1/4}$ we have

$$\psi_{\rm I}\sim e^{-(x^2+y^2)/4}\,. \tag{6.5}$$

In Region II, $\epsilon^{-1/8}<x<\epsilon^{-1/2}$ and we can approximate the differential equation by

$$\left[-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+\tfrac{1}{4}x^2-\tfrac{1}{4}\epsilon(ax^4+2cx^2y^2)-E\right]\psi_{\rm II}=0\,. \tag{6.6}$$

The substitutions

$$a\epsilon\to\epsilon\,,\qquad c/a\to c \tag{6.7}$$

reduce Eq. (6.6) to Eq. (4.7), which has already been solved.

We therefore immediately deduce that the large-order behavior of perturbation theory is

$$A_n=-\frac{\sqrt{6}}{\pi^{3/2}}\alpha(-3\beta)^n\,\Gamma(n+\tfrac{1}{2})\,, \tag{6.8}$$

where

$$\alpha=\left\{\frac{-2\pi c}{a\cos[\frac{1}{2}\pi(1+8c/a)^{1/2}]}\right\}^{1/2},\qquad \beta=a\,, \tag{6.9}$$

for the case $a>b>0$, $a>c\geq-\sqrt{ab}$.

There is a factor of 2 missing from the expression for α in Eq. (6.9) relative to Eq. (5.7) because the contribution from the tube along the y axis is negligible when $a>b$. When $b>a$, we have similar results:

$$\alpha=\left\{\frac{-2\pi c}{b\cos[\frac{1}{2}\pi(1+8c/b)^{1/2}]}\right\}^{1/2},\qquad \beta=b\,, \tag{6.10}$$

where $b>a>0$, $b>c\geq-\sqrt{ab}$.

For the off-axis case we use Eq. (6.4) to deduce that

$$\alpha=\left\{\frac{8\pi(2ac+2bc-3ab-c^2)}{(ab-c^2)\cos[\frac{1}{2}\pi(25-16c(a+b-2c)/(ab-c^2))^{1/2}]}\right\}^{1/2},\qquad \beta=\frac{ab-c^2}{a+b-2c}\,, \tag{6.11}$$

where $a>0$, $b>0$, $c>\max(a,b)$.

The results in Eqs. (6.9)–(6.11) agree to six places with the numbers in Table I.

VII. GENERALIZATION TO N DIMENSIONS

It is natural to try to extend the techniques we have developed for two-mode oscillators to more complicated problems. In this section we will show that such an extension is possible for a large class of N-mode equal-mass oscillator systems. Our aim will be to present a brief overview of what can be accomplished in N dimensions, and we will not dwell on algebraic or numerical details.

We begin by studying systems having potentials of the form

$$V=\sum_{i=1}^N\tfrac{1}{4}x_i^{\,2}+\tfrac{1}{4}\lambda\sum_{i,j=1}^N a_{ij}x_i^{\,2}x_j^{\,2}\,, \tag{7.1}$$

where a_{ij} is a real symmetric matrix. The Hamiltonian in (7.1) must be bounded below, and therefore a must satisfy

$$\sum_{i,j}a_{ij}y_iy_j\geq 0\ \text{for}\ y_i\geq 0\,. \tag{7.2}$$

When $\lambda=-\epsilon(\epsilon>0)$ the critical points of V are the solutions of

$$0=\frac{\partial V}{\partial x_i}=\tfrac{1}{2}x_i-\epsilon x_i\sum_{j=1}^N a_{ij}x_j^{\,2}\,. \tag{7.3}$$

Thus

$$x_i=0\,,\quad\text{or}\quad \frac{1}{2\epsilon}=\sum_{j=1}^N a_{ij}x_j^{\,2}\,. \tag{7.4}$$

We will first consider the case where all $x_i\neq 0$. Then Eq. (7.4) has a solution if a is nonsingular. We define D_i to be the determinant of the matrix obtained from a by replacing each element of the ith column by one. Then $\sum_j a_{ij}D_j=\det(a)$ for all i, and

$$x_j^{\,2}=\frac{1}{2\epsilon}\frac{D_j}{\det(a)}\,. \tag{7.5}$$

If this is to correspond to a point in real space we must have

$$\frac{D_i}{\det(a)}>0\,,\quad 1\leq i\leq N\,. \tag{7.6}$$

Of course there are 2^{N-1} vectors which satisfy Eq. (7.5) because we can choose the sign of each component of x_i independently.

The Hessian matrix at the critical point is

$$H_{ij}=\tfrac{1}{2}\delta_{ij}-\epsilon\delta_{ij}\sum_k a_{ik}x_k^{\,2}-2\epsilon x_ix_ja_{ij}\,. \tag{7.7}$$

Using Eq. (7.5) we can rewrite this as

$$H_{ij} = -2\epsilon x_i x_j a_{ij}$$
$$= -\frac{\sigma_i\sigma_j}{\det(a)}(D_i D_j)^{1/2} a_{ij}\,, \tag{7.8}$$

where σ_i is the sign of x_i. Note that $D_i D_j$ is always positive [see Eq. (7.6)]. Equations (7.3) and (7.8) imply that

$$\sum_j H_{ij}\sigma_j (D_j)^{1/2} = -\sigma_i (D_i)^{1/2}\,, \tag{7.9}$$

which means that the radial line through each critical point is one of the critical point's principal axes. Furthermore, the minus sign in Eq. (7.9) implies that along this radial line, V has a maximum at the critical point. Thus if all the other eigenvalues of H at the critical point are positive, we have a radially directed saddle point.

A necessary condition for H to have one negative and $N-1$ positive eigenvalues is

$$\det H < 0\,. \tag{7.10}$$

But,

$$\det H = [-\det(a)]^{-N} \det[\sigma_i\sigma_j (D_i D_j)^{1/2} a_{ij}]$$
$$= -[-\det(a)]^{-N-1} \prod_{i=1}^{N} D_i\,.$$

Using Eq. (7.6) we see that

$$\det H < 0 \Longleftrightarrow (-1)^N \det(a) < 0\,. \tag{7.11}$$

In two dimensions, Eq. (7.11) is indeed satisfied by the off-axis saddle points that we discussed in Sec. VI. In fact this condition and Eq. (7.6) imply that $c > \max(a, b)$, which we have shown to be a necessary and *sufficient* condition for an off-axis saddle point in two dimensions. In higher dimensions however, we can have $\det H < 0$ without having one negative and $N-1$ positive eigenvalues. It is difficult to give a simple necessary and sufficient condition for saddle points in the general N-dimensional case.

If several of the x_i are zero the procedure for finding a saddle point is slightly more complicated. We choose to label the axes so that the first $M x_i$ are zero. Then, the condition for a critical point becomes

$$x_i = 0\,, \quad i = 1, \ldots, M$$
$$\frac{1}{2\epsilon} = \sum_{j=M+1}^{N} a_{ij} x_j^2\,, \quad i = M+1, \ldots, N. \tag{7.12}$$

The discussion proceeds as before in the subspace of nonzero components. We find that the Hessian matrix is given by

$$\begin{bmatrix} \frac{1}{2} - \epsilon \sum_{k=M+1}^{N} a_{1k} x_k^2 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \frac{1}{2} - \epsilon \sum_{k=M+1}^{N} a_{Mk} x_k^2 & 0 \\ 0 & \cdots & 0 & -2\epsilon x_i x_j a_{ij} \end{bmatrix},$$

where the x_i's are the solutions of Eq. (7.12). Thus, in addition to the usual conditions on the N-M dimensional matrix $x_i x_j a_{ij}$, we must have

$$\tfrac{1}{2} - \epsilon \sum_{M+1}^{N} a_{ik} x_k^2 > 0\,, \quad i = 1, \ldots, M.$$

The special case of an on-axis critical point, where all but one of the x_i vanish, is important because any other configuration can be reduced to this one by a rotation. Here the condition for a saddle point becomes

$$\tfrac{1}{2} - \epsilon a_{iN} x_N^2 > 0\,, \quad i = 1, \ldots, N-1$$

where

$$x_N^2 = \frac{1}{2\epsilon a_{NN}}\,. \tag{7.13}$$

It is now easy to generalize the arguments of Secs. IV and VI to compute the contribution to the large-order behavior of perturbation theory from this saddle point. We observe that in the tube where $\epsilon^{-1/6} < x_N < \epsilon^{-1/2}$, $x_i = O(1)$, $i \neq N$, we can make the approximation

$$\tfrac{1}{4}\epsilon \sum_{i,j} a_{ij} x_i^2 x_j^2 \sim \tfrac{1}{4}\epsilon x_N^2 \sum_{j=1}^{N-1} a_{Nj} x_j^2 + \tfrac{1}{4}\epsilon a_{NN} x_N^4\,.$$

Following the procedure of Sec. IV, we factor a rapidly varying WKB function of the x_N variable [see Eq. (4.8)] out of the wave function and neglect all terms in the resulting equation which vanish as $\epsilon \to 0$. After a simple change of variables [see Eq. (4.12)], we obtain a partial differential equation which must be solved exactly:

$$(w^2 - 1)\chi_w + \sum_{i=1}^{N-1} \left[-\frac{\partial^2}{\partial x_i^2} + \tfrac{1}{4} x_i^2 - \tfrac{1}{2} - \frac{a_{iN}}{2a_{NN}} x_i^2 (1 - w^2) \right] \chi = 0\,.$$

The ansatz

$$\chi = A\exp\left[-\tfrac{1}{4}\sum_{i=1}^{N-1} f_i(w)x_i^2\right]$$

generates $N-1$ Riccati equations whose solutions govern the thickness of the tube of probability current in the directions perpendicular to the MPEP. Then the change of variables

$$w = w\,,$$

$$s_i = x_i/u_i(w)\,, \quad 1 \le i \le N-1$$

with

$$f_i = (w^2-1)u_i'(w)/u_i(w)$$

reduces the equation for A to one that is separable:

$$(w^2-1)A_w + \tfrac{1}{2}\sum_{i=1}^{N-1}[f_i(w)-1]A = \sum_{i=1}^{N-1}\frac{1}{u_i^2(w)}\frac{\partial^2}{\partial x_i^2}A\,.$$

Finally, we require that the wave function in the tunneling region match to a harmonic-oscillator wave function (the solution of the Schrödinger equation near the origin). As in Sec. IV, this implies that A is a function only of w. It is then easy to determine the probability current and evaluate the dispersion integral. The resulting contribution to the large-order behavior of perturbation theory is

$$A_n = -\left\{\prod_{i=1}^{N-1}\left[\frac{8\pi a_{iN}}{a_{NN}\sin(\pi\nu_i)}\right]^{1/2}\right\} \times\sqrt{6}\,\pi^{-3/2}(-3a_{NN})^n\,\Gamma(n+\tfrac{1}{2})\,, \tag{7.14}$$

where $\nu_i(\nu_i+1) = 2a_{iN}/a_{NN}$.

This expression will be equal to the true large-order behavior if the x_N axis is the MPEP. However, as we mentioned above, we can use Eq. (7.14) even if the dominant saddle point does not lie on this axis. As an example, let us consider the case of a dominant critical point whose coordinates are nonvanishing. Then the following rotation will align the radial line through the saddle point with the $\bar{x}_N$ axis:

$$x_i = \sum_{j=1}^{N} R_{ij}\,\bar{x}_j\,,$$

$$R = \begin{bmatrix} a_{11} & \cdots & a_{N-1\,1} & \left(D_1\Big/\sum_{i=1}^{N} D_i\right)^{1/2} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ a_{1N} & \cdots & a_{N-1\,N} & \left(D_N\Big/\sum_{i=1}^{N} D_i\right)^{1/2}\end{bmatrix}. \tag{7.15}$$

We choose the first $N-1$ column vectors of R to lie along the other principal axes of the saddle point. In the new coordinate system V will no longer have the simple form in Eq. (7.1). It will contain terms like $\bar{x}_i{}^3\bar{x}_j$. However, it is easy to see that there are no terms of the form $\bar{x}_N{}^3\bar{x}_j$ $(j\neq N)$ or $\bar{x}_N{}^2\bar{x}_i\,\bar{x}_j$ $(i,j\neq N)$. Such terms would give a nonvanishing contribution to $\partial^2V/\partial\bar{x}_N\partial\bar{x}_j$ or to $\partial^2V/\partial\bar{x}_i\,\partial\bar{x}_j$ at the critical point. (The coordinates of the critical point are $\bar{x}_j=0$, $1\le j\le N-1$, $\bar{x}_N\neq 0$.) Thus V has the form

$$V = \sum_{i=1}^{N}\tfrac{1}{4}\bar{x}_i{}^2 - \tfrac{1}{4}\epsilon\left(\bar{a}_{NN}\bar{x}_N{}^4 + \sum_{i=1}^{N-1}\bar{a}_{iN}x_i{}^2x_N{}^2 + b\right),$$

where b depends at most linearly on $\bar{x}_N$. In the tube where $\bar{x}_i = O(1)$, $1\le i\le N-1$, and $\epsilon^{-1/6}\le\bar{x}_N \le\epsilon^{-1/2}$, we can clearly approximate V by neglecting b entirely.

We then use Eq. (7.14) to compute the large-order behavior of perturbation theory as before. The computation of $\bar{a}_{iN}$ for all i is a tedious algebraic problem. Still it is easy to find the value of $\bar{a}_{NN}$, the coefficient of $\bar{x}_N{}^4$ in $\sum_{i,j}a_{ij}\,x_i{}^2x_j{}^2$. Using Eq. (7.15) it is

$$\bar{a}_{NN} = \sum_{i,j} a_{ij}\frac{D_iD_j}{(\sum_j D_j)^2} = \frac{1}{(\sum_j D_j)^2}\sum_j D_j\det(a) = \frac{\det(a)}{\sum_j D_j}\,.$$

Thus, the large-order behavior of perturbation theory is

$$A_n \sim -\sqrt{6}\,\pi^{-3/2}2^{N-1}K_N\left[\frac{-3\det(a)}{\sum_j D_j}\right]^n\Gamma(n+\tfrac{1}{2})\,. \tag{7.16}$$

The constant K_N may be determined from Eq. (7.14) once we have computed $\bar{a}_{iN}$ for $i\neq N$. The factor 2^{N-1} reflects the possible choices of the sign of x_j in Eq. (7.5). Similar formulas exist for the case where some of the x_i vanish at the dominant saddle point.

Equation (7.14) may also be used to find the large-order behavior of the perturbation series for systems having an infinite number of degrees of freedom. As an example, consider the sequence of potentials

$$V^{(N)} = \tfrac{1}{4}\sum_{i=1}^{N}x_i{}^2 - \tfrac{1}{4}\epsilon\sum_{\substack{i,j=1\\ i\neq j}}^{N}x_i{}^2x_j{}^2\,. \tag{7.17}$$

The critical points of $V^{(N)}$ are given by

$$0 = \frac{\partial V^{(N)}}{\partial x_i} = \tfrac{1}{2}x_i - \tfrac{1}{4}\epsilon x_i\sum_{j\neq i}x_j{}^2\,, \tag{7.18}$$

whose solutions are

$$x_i = 0 \text{ or } x_i{}^2 = [2\epsilon(M-1)]^{-1}. \qquad (7.19)$$

M is the number of nonzero x_i. Note that M can never be one, so Eq. (7.19) always makes sense. The Hessian matrix is

$$H_{ij}^{(N)} = -\frac{\delta_{ij}}{2(M-1)} - 2\epsilon x_i x_j + 3\epsilon\delta_{ij}(x_i{}^2). \qquad (7.20)$$

Observe from Eq. (7.20) that we cannot have a saddle point if any of the x_i vanish because Eq. (7.13) is violated. Hence, all $x_i \neq 0$ and

$$H_{ij}^{(N)} = \frac{\delta_{ij}}{2(N-1)} - \frac{\sigma_i\sigma_j}{(N-1)},$$

where σ_i is the sign of x_i. The eigenvalues of this matrix are $\frac{1}{2} - N/(N-1)$, which is negative, and $2/(N-1)$, the latter having multiplicity $N-1$. Therefore, we have a saddle point. There is a different saddle point for each of the 2^{N-1} choices of sign for x_i.

To determine the large-order behavior of perturbation theory for this system, we must compute $D_i^{(N)}$ and $\det[a^{(N)}]$ [see Eq. (7.16)]. The matrix $a^{(N)}$ is given by

$$\begin{bmatrix} 0 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 1 & 1 & \cdots \\ 1 & 1 & 0 & 1 & \cdots \\ 1 & 1 & 1 & 0 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

Thus,

$$\det[a^{(N)}] = (N-1)(-1)^{N-1}$$

and

$$D_i^{(N)} = (-1)^{N-1}, \quad 1 \le i \le N.$$

We can also compute $\bar{a}_{iN}^{(N)}$ because the potential is so symmetric. We find that

$$\begin{aligned} \bar{a}_{iN}^{(N)} &= \frac{N-3}{N}, \quad 1 \le i \le N-1 \\ \bar{a}_{NN}^{(N)} &= \frac{N-1}{N}. \end{aligned} \qquad (7.21)$$

Therefore, from Eqs. (7.14) and (7.21), the large-order behavior of perturbation theory is

$$A_n^{(N)} = -\sqrt{6}\,\pi^{-3/2}\left[\frac{-8\pi(N-3)/(N-1)}{\cos\frac{3}{2}\pi(1-16/9(N-1))^{1/2}}\right]^{(N-1)/2}\left(-3\frac{N-1}{N}\right)^n \Gamma(n+\tfrac{1}{2}). \qquad (7.22)$$

Now consider the limit as $N\to\infty$. This limit defines an infinite-mode oscillator system which, strictly speaking, is some nonlocal field theory. The leading contribution to Eq. (7.22) which comes from geometrical optics remains finite in this limit:

$$\left(-3\frac{N-1}{N}\right)^n \Gamma(n+\tfrac{1}{2}) \to (-3)^n\,\Gamma(n+\tfrac{1}{2}).$$

However, the constant from physical optics blows up. The divergence of this constant derives from two sources. The factor $(\sqrt{8\pi})^{N-1}$ occurs in any N-mode problem in which no axis passes through the dominant saddle point. The vanishing of the cosine term as $N\to\infty$ is a more singular divergence of the form $N^{N/2}$. It reflects the disappearance of the saddle point. The extreme symmetry of the potential makes the saddle become flat as N becomes large. This kind of symmetry is not present in potentials arising from $(\phi^N)_2$ quantum field theories.

It is amusing that we can eliminate this divergence by a mass renormalization. We will argue that by adding a lower-order mass term to the potential we can ensure that the $N\to\infty$ limit of $A_n^{(N)}$ exists. Consider the effect of adding a term of the form

$$\epsilon R^{(N)} \sum_{i=1}^{N} \tfrac{1}{4} x_i{}^2.$$

Because this term is at most $O(1)$ in the tunneling region, it cannot affect the determination of the MPEP. This term is merely a correction of $O(\epsilon)$ to the mass, and thus can actually be viewed as a mass renormalization. Following the scaling arguments of Sec. VII, Ref. 9, we find that this term contributes an over-all multiplicative constant (independent of n) to the large-order growth of perturbation theory, namely,

$$\exp\left[\frac{-NR^{(N)}}{2(N-1)}\right].$$

On the other hand, we easily determine from Eq. (7.22) that the two large-N divergences which we discussed give precisely

$$[6(N-3)]^{(N-1)/2}.$$

Hence, if we choose

$$R^{(N)} = \frac{(N-1)^2}{N}\ln(6N-18), \qquad (7.23)$$

then the limit $N\to\infty$ may be taken. We obtain the remarkable result that the large-order behavior of the Rayleigh-Schrödinger coefficients A_n for the ground-state energy perturbation series of the infinite-mode system described by the potential

$$V=\lim_{N\to\infty}\left\{\tfrac{1}{4}[1-\lambda R^{(N)}]\sum_{i=1}^{N}x_i^{\,2}+\tfrac{1}{4}\lambda\sum_{\substack{i,j=1\\ i\neq j}}^{\infty}x_i^{\,2}x_j^{\,2}\right\}\tag{7.24}$$

is

$$A_n\sim-\sqrt{6}\,\pi^{-3/2}(-3)^n\,\Gamma(n+\tfrac{1}{2}),\tag{7.25}$$

where $N\to\infty$ in such a way that λN is small. This is precisely the result in Eq. (1.8) for the one-mode anharmonic oscillator. Of course, it can be argued that the choice of the potential in Eq. (7.24) is somewhat contrived. Nonetheless, we feel that Eq. (7.25) is just one more example of the extraordinary simplification that takes place in the large-order limit of perturbation theory.

APPENDIX A: OSCILLATORS WITH SPHERICAL SYMMETRY

Here we investigate the ground-state-energy perturbation series for spherically coupled oscillators. This special case obtains when we choose $c=1$ in Eq. (1.4). However, a spherically symmetric configuration of oscillators is so easy to treat that we immediately generalize from the two-mode problem of Eq. (1.4) to the N-mode problem, which we define by the equation

$$\left\{\sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial x_i^{\,2}}+\tfrac{1}{4}x_i^{\,2}\right)+\tfrac{1}{4}\lambda\left(\sum_{i=1}^{N}x_i^{\,2}\right)^2-E(\lambda)\right\}\psi(x_i)=0,\tag{A1}$$

where $\lim_{|x_i|\to\infty}\psi=0$.

We use spherical symmetry to transform Eq. (A1) to N-dimensional spherical coordinates. Moreover, we seek a wave function ψ which depends only on r, the radial coordinate, because the ground-state wave function has no angular dependence. We thus reduce Eq. (A1) to the ordinary differential equation

$$\left[-\frac{d^2}{dr^2}-\frac{N-1}{r}\frac{d}{dr}+\tfrac{1}{4}r^2+\tfrac{1}{4}\lambda r^4-E(\lambda)\right]\psi(r)=0,\tag{A2}$$

where

$$r^2=\sum_{i=1}^{N}x_i^{\,2}.$$

Our problem is to compute the eigenvalue $E(\lambda)$ perturbatively in large order. We expand $E(\lambda)$ into the perturbation series

$$E(\lambda)=\tfrac{1}{2}N-\sum_{n=1}^{\infty}(-\lambda)^n C_n.\tag{A3}$$

We could solve this problem by removing the first derivative term from Eq. (A2) by making a suitable transformation and then using WKB in the same manner as in the body of this paper. However, we prefer a much simpler approach. We will convert Eq. (A2) into a partial difference equation which has already been solved asymptotically in BW II.

To transform Eq. (A2) into a partial difference equation, we substitute the expression

$$\psi(r)=e^{-r^2/4}\left\{1+\sum_{n=1}^{\infty}(-\lambda)^n\sum_{j=1}^{2n}(\tfrac{1}{2}r^2)^j C_{n,j}\right\}\tag{A4}$$

and Eq. (A3) into Eq. (A2) and collect powers of $r^2/2$ and $-\lambda$. The coefficient of $(\tfrac{1}{2}r^2)^j$ and $(-\lambda)^n$ is the desired difference equation:

$$2jC_{n,j}=(j+1)(2j+N)C_{n,j+1}+C_{n-1,j-2}-\sum_{p=1}^{n-1}C_{p,1}C_{n-p,j},\tag{A5}$$

with initial value $C_{0,0}=1$ and boundary condition $C_{n,j}\neq 0$ for $n>1$ and $1\leq j\leq 2n$; $C_{n,j}=0$ otherwise. C_n is related to $C_{n,j}$ by

$$C_n=NC_{n,1}.\tag{A6}$$

Following Sec. VI A of BW II, we approximate Eq. (A5) by dropping the nonlinear convolution term. As was argued there, the neglected term does not affect the leading asymptotic behavior of $C_{n,1}$ for large n. Thus, the equation to be solved is

$$2jC_{n,j}=(j+1)(2j+N)C_{n,j+1}+C_{n-1,j-2}.\tag{A7}$$

We put Eq. (A7) into a more useful form by substituting

$$C_{n,j}=D_{n,j}/[j\Gamma(j+\tfrac{1}{2}N)].\tag{A8}$$

The equation satisfied by $D_{n,j}$ is

$$D_{n,j}=D_{n,j+1}+\frac{(j+\tfrac{1}{2}N-1)(j+\tfrac{1}{2}N-2)}{2(j-2)}D_{n-1,j-2}.\tag{A9}$$

Next, we replace Eq. (A9) by a new approximate equation satisfied by a new dependent variable $E_{n,j}$:

$$E_{n,j}=E_{n,j+1}+\tfrac{1}{2}(j+N-1)E_{n-1,j-2}.\tag{A10}$$

Equation (A10) is derived by approximating the coefficient of $D_{n-1,j-2}$ in Eq. (A9) for large j, keeping terms of orders j and 1 and neglecting terms of order j^{-1}.

We must introduce an extra condition which fixes the multiplicative scale of $E_{n,j}$ because Eq. (A10) is homogeneous; to wit, we require that

$$\lim_{n\to\infty}E_{n,2n}/D_{n,2n}=1.\tag{A11}$$

From Eq. (A10) we easily deduce that

$$E_{n,2n}=E_{0,0}\Gamma(n+\tfrac{1}{2}(N+1))/\Gamma(\tfrac{1}{2}(N+1)). \quad (A12)$$

$E_{0,0}$ is the multiplicative factor which adjusts the scale of $E_{n,j}$. Also, from Eqs. (A7) and (A8) we have

$$D_{n,2n}=2n\Gamma(2n+\tfrac{1}{2}N)4^{-n}/n!\ . \quad (A13)$$

Combining Eqs. (A11)–(A13) gives

$$E_{0,0}=\Gamma(\tfrac{1}{2}(N+1))2^{N/2}\pi^{-1/2}. \quad (A14)$$

Finally, we recall that Sec. VI H of BW II gives a complete treatment of the asymptotic behavior of solutions for difference equations like that in Eq. (A10). It is shown there that for large n,

$$E_{n,1}\sim E_{0,0}\frac{3^{n+N/2}\Gamma(n+\frac{1}{2}N)}{2\sqrt{\pi}\,\Gamma(\frac{1}{2}(N+1))}. \quad (A15)$$

Thus, we combine Eqs. (A6), (A8), and (A15) to obtain

$$C_n\sim\Gamma(n+\tfrac{1}{2}N)3^n\,6^{N/2}\pi^{-1}/\Gamma(\tfrac{1}{2}N). \quad (A16)$$

This is the precise leading asymptotic behavior of C_n for large n and is the general result we have sought.

Two special cases of this equation are noteworthy. For the one-mode oscillator ($N=1$), the coefficients of the ground-state-energy perturbation series grow like

$$C_n\sim 6^{1/2}\pi^{-3/2}3^n\,\Gamma(n+\tfrac{1}{2}), \quad (A17)$$

which agrees with Eq. (1.8).

Second, the coefficients for the two-mode oscillator ($N=2$) diverge like

$$C_n\sim\frac{2}{\pi}\,3^{n+1}n!\ , \quad (A18)$$

as in Eqs. (1.11) and (4.42c).

Observe also that the rate of divergence of perturbation theory increases with increasing N: $\Gamma(n+\frac{1}{2}N)\sim\Gamma(n)n^{N/2}$. This is a phenomenon characteristic only of spherically symmetrically coupled oscillators. As is shown in Sec. IV, it results physically from a kind of constructive interference.

APPENDIX B: NONQUARTIC COUPLING

To demonstrate that the analytical methods of Sec. IV are easily generalized to more difficult problems, we treat the two-mode, nonquartically coupled oscillator. This system is defined by the equation

$$\left[-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+\tfrac{1}{4}x^2+\tfrac{1}{4}y^2+\lambda 2^{-N}(x^{2N}+cx^{2N-2}y^2+dx^{2N-4}y^4+\cdots+dx^4y^{2N-4}+cx^2y^{2N-2}+y^{2N})-E(\lambda)\right]\psi=0. \quad (B1)$$

Corresponding to Eq. (4.1) we have

$$\left[-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+\tfrac{1}{4}x^2+\tfrac{1}{4}y^2-\epsilon^{N-1}2^{-N}(x^{2N}+cx^{2N-2}y^2+\cdots)-1\right]\psi=0. \quad (B2)$$

As in Sec. IV, we solve Eq. (B2) in a tube surrounding the positive x axis. In Region I we obtain the same result as in Eq. (4.6) and in Region II we make a substitution similar to that in Eq. (4.8):

$$\psi=\chi(\tfrac{1}{4}x^2-\epsilon^{N-1}2^{-N}x^{2N}-\tfrac{1}{2})^{-1/4}\exp\left[-\int_{\sqrt{2}}^{x}(\tfrac{1}{4}t^2-\epsilon^{N-1}2^{-N}t^{2N}-\tfrac{1}{2})^{1/2}dt\right]. \quad (B3)$$

In the resulting differential equation for χ we let

$$\epsilon x^2=z^2 \quad (B4)$$

and obtain

$$z(1-2^{2-N}z^{2N-2})^{1/2}\chi_z-\chi_{yy}+\left(\tfrac{1}{4}y^2-\frac{c}{2^N}z^{2N-2}y^2-\tfrac{1}{2}\right)\chi=0. \quad (B5)$$

Equation (B5), like Eq. (4.11), must be solved *exactly*—there are no further approximations.

To simplify Eq. (B5), we let

$$\omega=(1-2^{2-N}z^{2N-2})^{1/2}, \quad (B6)$$

which gives

$$(N-1)(\omega^2-1)\chi_\omega-\chi_{yy}+[\tfrac{1}{4}y^2-\tfrac{1}{2}-\tfrac{1}{4}c(1-\omega^2)y^2]\chi=0. \quad (B7)$$

The substitution

$$\chi=e^{-y^2f(\omega)/4}A \quad (B8)$$

leads to a Riccati equation that is almost identical to that in Eq. (4.15):

$$-(N-1)(\omega^2-1)f'(\omega)-f^2(\omega)+1-c+c\omega^2=0. \quad (B9)$$

To linearize Eq. (B9), we let

$$f(\omega)=(N-1)(\omega^2-1)u'(\omega)/u(\omega). \quad (B10)$$

As in Sec. IV, $u(\omega)$ satisfies a Legendre equation:

$$u(\omega)=P_\nu^\mu(\omega)\,, \tag{B11}$$

where $\mu=-1/(N-1)$ and $\nu(\nu+1)=c/(N-1)^2$.

When Eq. (B9) is satisfied, A solves a much simpler equation than Eq. (B7), namely,

$$(N-1)(\omega^2-1)A_\omega-A_{yy}+yf(\omega)A_y+\tfrac{1}{2}[f(\omega)-1]A=0\,. \tag{B12}$$

The substitution $(\omega, y)\to(\omega, s=y/u(\omega))$ reduces Eq. (B12) to one that is separable:

$$(N-1)(\omega^2-1)u^2(\omega)A_\omega+\tfrac{1}{2}u^2(\omega)[f(\omega)-1]A=A_{ss}\,. \tag{B13}$$

The argument given in Sec. IV that the separation constant for Eq. (4.23) vanishes also holds for Eq. (B13). It follows that, up to an arbitrary multiplicative constant β,

$$A=\beta[u(\omega)]^{-1/2}[(1-\omega)/(1+\omega)]^{1/(4N-4)}\,. \tag{B14}$$

This completes the solution of Eq. (B2).

We determine β by asymptotically matching ψ in Region II to ψ in Region I. In the matching region, $\omega\sim1$, and we have[24]

$$P_\nu^{-1/(N-1)}(\omega)\sim(\tfrac{1}{2}-\tfrac{1}{2}\omega)^{(N-1)/2}/\Gamma(N/(N-1))\,. \tag{B15}$$

From Eq. (B15) we have for $\omega\sim1$

$$\begin{aligned}&A\sim\beta\Gamma^{1/2}(N/(N-1))\,,\\&f(\omega)\sim1\,.\end{aligned} \tag{B16}$$

In the matching region we also have formulas similar to those in Eqs. (4.27) and (4.28) from which we conclude that [see Eq. (B3)]

$$\psi/\chi\sim2^{1/2}e^{1/4}e^{-x^2/4}\,. \tag{B17}$$

From Eqs. (B16) and (B17) and Eq. (4.6), we obtain the generalization of Eq. (4.33):

$$\beta=e^{-1/4}\{2\Gamma(N/(N-1))\}^{-1/2}\,. \tag{B18}$$

Continuing to follow Sec. IV, we use the formulas[25]

$$P_\nu^\mu(0)=2^\mu\pi^{-1/2}\cos\frac{\pi(\nu+\mu)}{2}\,\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)}{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu)}\,, \tag{B19}$$

$$\frac{d}{dx}P_\nu^\mu(x)\bigg|_{x=0}=2^{\mu+1}\pi^{-1/2}\times\sin\frac{\pi(\nu+\mu)}{2}\,\frac{\Gamma(1+\frac{1}{2}\nu+\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)}\,, \tag{B20}$$

from which it follows that

$$f(0)=2(1-N)\times\frac{\sin[\pi(\nu+\mu)/2]\Gamma(1+\frac{1}{2}\nu+\frac{1}{2}\mu)\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu)}{\cos[\pi(\mu+\nu)/2]\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)}\,. \tag{B21}$$

Equations (B19) and (B21) are now used to determine $J(x)$, and then ImE, and finally the behavior of A_n for large n. We jump immediately to the final answer because the calculation is relatively easy and the techniques are described in Sec. IV. Our result is that for large n,

$$A_n\sim-(2K)\,\frac{N-1}{\pi^{3/2}}(-2)^{-n}\,\Gamma(nN-n+\tfrac{1}{2})\times\left[\frac{\Gamma(2N/(N-1))}{\Gamma^2(N/(N-1))}\right]^{nN-n+1/2}, \tag{B22}$$

where

$$K=\left[\frac{\Gamma(1+\nu-\mu)\Gamma(-\nu-\mu)}{\Gamma(1-\mu)\Gamma(-\mu)}\right]^{1/2}. \tag{B23}$$

Finally, we check (B22) and (B23) in three important limiting cases. First, we let $\nu\to0$. In this limit K in Eq. (B23) is 1. Thus, A_n in Eq. (B22) is exactly *twice* the theoretical prediction for the large-order behavior of perturbation theory for the x^{2N} oscillator.[8] This is reasonable because when $\nu=0$, $c=0$, and we have two noninteracting x^{2N} oscillators.

Second, we pick $N=2$, when $\mu=-1$. In this case

$$K=[\Gamma(2+\nu)\Gamma(1-\nu)]^{1/2}\,,$$

which further simplifies to

$$K=[\nu(\nu+1)\pi/\sin(\pi\nu)]\,.$$

Here we recover the same result as in Eq. (4.42a).

Third, we expect to find a singularity in K when Eq. (B1) has spherical symmetry because of constructive interference. Hence we investigate the limit $C\to N$. Solving $\nu(\nu+1)=N/(N-1)^2$ gives

$$\nu_{1,2}=1/(N-1),\ -N/(N-1)\,.$$

But, $\mu=-1/(N-1)$, so

$$u+\nu_1=0\,,\quad 1+\nu_2-\mu=0\,.$$

Thus, one of the two Γ functions in Eq. (B23) blows up and K is indeed singular.

We conclude that the results in Eqs. (B22) and (B23) are quite reasonable and attest to the generalizability of the methods introduced in Sec. IV.

APPENDIX C

Our discussion of N-dimensional oscillators in Sec. VII was incomplete because we did not show that straight lines through radially directed saddle points were solutions of the classical orbit equations, and we did not give criteria for the existence of saddle points. The purpose of this appendix is to remedy these omissions.

Consider an N-dimensional potential of the form

$$V=\sum_i \tfrac{1}{4}x_i^{\,2}+\frac{\lambda}{2M}\sum_{i_1,\ldots,i_{2M}} A_{i_1\cdots i_{2M}}x_{i_1}\cdots x_{i_{2M}}. \tag{C1}$$

The interaction term is the most general $2M$th-order homogeneous polynomial, subject to the restriction that V be bounded below for $\lambda>0$. Then we have the following theorem: If $\lambda=-\epsilon$, $\epsilon>0$, then the radial line through any critical point of V is a solution of the classical equations [Eq. (3.4)]. Furthermore, this radial line is a principal axis of the critical point and V reaches a maximum at the critical point along this line.

Proof: Let $a=(a_1,\ldots,a_n)$ be a critical point of V. Then,

$$\left.\frac{\partial V}{\partial x_i}\right|_{x_i=a_i}=\tfrac{1}{2}a_i-\epsilon\sum_{i_2,\ldots,i_{2M}} A_{ii_2\cdots i_{2M}}a_{i_2}\cdots a_{i_{2M}}=0. \tag{C2}$$

The radial line through a_i is

$$x_i(s)=\frac{a_i}{|a|}\,s. \tag{C3}$$

Using Eqs. (C2) and (C3) we can evaluate the expressions in Eq. (3.4) in terms of s:

$$\begin{aligned}\nabla_i V(x_i(s))&=\frac{a_i}{2|a|s}-\epsilon\sum_{i_2,\ldots,i_{2M}}A_{ii_2\cdots i_{2M}}\left(\frac{s}{|a|}\right)^{2M-1}\\&=\frac{a_i s}{2|a|s}-\frac{a_i}{s}\left(\frac{s}{|a|}\right)^{2M-1}\end{aligned} \tag{C4}$$

and

$$\sum_i x_i'(s)\nabla_i x_i(s)=\tfrac{1}{2}s-\frac{s^{2M-1}}{|a|^{2M-2}}. \tag{C5}$$

Thus

$$x_i'(s)\sum_j x_j'(s)\nabla_j x_i(s)=\nabla_i V(x_i(s)),$$

and $x_i(s)$ satisfies Eq. (3.4), the equation of motion.

Next, to show that the radial line $x_i(s)$ is a principal axis of the critical point, we compute the Hessian matrix H_{ij}:

$$\begin{aligned}H_{ij}&=\frac{\partial^2 V}{\partial x_i\,\partial x_j}(a)\\&=\tfrac{1}{2}\delta_{ij}-\epsilon^{(2M-1)}\sum_{i_3,\ldots,i_{2M}}A_{iji_3\cdots i_{2M}}a_{i_3}\cdots a_{i_{2M}}.\end{aligned} \tag{C6}$$

Thus,

$$\sum_j H_{ij}\,x_j(s)=\frac{sa_i}{|a|}(1-M);$$

that is, $x_i(s)$ is an eigenvector of H with eigenvalue $1-M$ and by definition it is a principal axis of the critical point. Because $1-M$ is less than 0, the critical point is a maximum along the x_i direction. This completes the proof.

Next we will show that for almost all values of the parameters of V in Eq. (C1) there is at least one radially directed saddle point.

Let us first review the results that we obtained in two dimensions. For the potential $V=\frac{1}{4}(x^2+y^2)-\frac{1}{4}\epsilon(ax^4+2cx^2y^2+by^4)$, there are three distinct possibilities:

$$c\neq a,\quad c\neq b, \tag{C7a}$$

$$a=b=c, \tag{C7b}$$

$$a=c>b. \tag{C7c}$$

In the first case we found that V always has a radial saddle point. The second case is spherically symmetric and is treated by separation of variables in Appendix A. The third case was not treated in Sec. VI, but it is easy to see that there is a degenerate critical point along the x axis (the Hessian has a zero eigenvalue). We will see that a similar trichotomy occurs for the general potential in Eq. (C1).

Let us consider an arbitrary unit vector $\hat{u}$ and study the variation of V along the radial line through $\hat{u}$:

$$V=\tfrac{1}{4}r^2-\frac{\epsilon}{2M}r^{2M}T(\hat{u}), \tag{C8}$$

where $x_j=r\hat{u}_j$ and

$$T(\hat{u})=\sum_{i_1,\ldots,i_{2M}}A_{i_1\cdots i_{2M}}\hat{u}_{i_1}\cdots\hat{u}_{i_{2M}}.$$

Then

$$\frac{\partial V}{\partial r}=\tfrac{1}{2}r-\epsilon r^{2M-1}T(\hat{u}), \tag{C9}$$

and

$$\frac{\partial^2 V}{\partial r^2}=\tfrac{1}{2}-\epsilon(2M-1)r^{2M-2}T(\hat{u}). \tag{C10}$$

Thus V has a maximum on this radial line at

$$r=\left(\frac{1}{2\epsilon T(\hat{u})}\right)^{1/(2M-2)} \tag{C11}$$

Equation (C11) defines an $N-1$ dimensional hypersurface B. It is easy to show that V attains a minimum on B. If $T(u)$ does not vanish anywhere, then the hypersurface is compact and, because V is continuous, a minimum exists. If T does vanish, it can only do so for a finite number of vectors $\hat{u}$. Therefore consider a small cone of solid angles around each such $\hat{u}$. These cones will intersect B in a finite number of small patches, and the complement of these patches B' is compact. Thus V has a minimum on B'. Since we can choose the opening angle of the cones to be as small as we wish, the point where V has a minimum is separated from the patches by a finite distance. (Otherwise the minimum would lie along one of the directions where T vanishes; but this would be absurd because $V\to+\infty$ in such a direction.)

There are now two possibilities. Either the minimum of V is isolated or else V attains its minimum on some connected subset of V. The latter case corresponds to Eq. (C7b) and is exceptional in the sense that we can make the minimum isolated by changing the interaction term $A_{i_1\cdots i_{2M}}$ infinitesimally and we will therefore not consider it further.

If the minimum is isolated, then we have a radially directed saddle point because, by construction, V has a maximum along the radial direction and a minimum in all other directions. As we have shown above, the radial direction is one of the principal axes of the saddle point. The saddle point may of course be degenerate as in Eq. (C7c). (The WKB methods which we have introduced depend on the existence of a nondegenerate saddle point.) However, since degeneracy is also extremely exceptional, there is almost always a saddle point.

We conclude finally that the straight-line WKB methods which we have introduced in this paper are applicable to almost all potentials of the form in Eq. (C1).

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[1]S. L. Adler, Phys. Rev. D 5, 3021 (1972); 7, 1948 (1973).

[2]See A. M. Jaffe, Commun. Math. Phys. 1, 127 (1965). This paper gives references to the works of C. Hurst, W. Thirring, A. Petermann, and E. R. Caianiello, A. Campolattaro, and M. Marinaro.

[3]See, for example, A. M. Jaffe, in *Proceedings of the Conference on the Mathematical Theory of Elementary Particles*, edited by R. Goodman and I. Segal (M.I.T. Press, Cambridge, Mass., 1966), p. 45.

[4]Adler also suggests the possibility of using semiclassical methods to determine α. See Ref. 1.

[5]The coefficients A_n are called Rayleigh-Schrödinger coefficients and Eq. (1.3) is called a Rayleigh-Schrödinger perturbation series.

[6]C. M. Bender and T. T. Wu, Phys. Rev. Lett. 21, 406 (1968).

[7]C. M. Bender and T. T. Wu, Phys. Rev. 184, 1231 (1969), Appendix E. This paper is referred to hereafter as BWI.

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[9]C. M. Bender and T T. Wu, Phys. Rev. D 7, 1620 (1973). This paper is referred to hereafter as BWII.

[10]T. I. Banks and C. M. Bender, J. Math. Phys. 13, 1320 (1972).

[11]The equivalence is demonstrated in BWI, Appendix A. In addition to the usual quantum-mechanical perturbative methods, Feynman rules and diagrams can be used to calculate the terms in the series in Eq. (1.7). A_n is the sum of all connected bubble diagrams having n vertices (see BWI, Appendix B). The topology of diagrams for ϕ^4 field theories in any space-time dimension is the same.

[12]To prove Eq. (2.4) we use two rigorous results of B. Simon as given in Ann. Phys. (N.Y.) 58, 79 (1970). Simon proved that $E(\lambda)$ is analytic in the domain shown in Fig. 1 and that the Rayleigh-Schrödinger series is asymptotic in the cut plane for sufficiently small λ. We can use the Cauchy theorem to write

$$F(\lambda)=\frac{1}{2\pi i}\oint_C\frac{F(z)}{z-\lambda}\,dz,$$

where C is the dashed contour shown in Fig. 1. A simple scaling argument (the Symanzik transformation) shows that $F(\lambda)\sim\lambda^{-2/3}$ as $|\lambda|\to\infty$. Thus as $R_1\to\infty$, the contribution from the large circle vanishes. Furthermore, since the Rayleigh-Schrödinger series is asymptotic, the integral along the small circle $\to 0$ as $R_4\to 0$. Expanding the denominator as in the text gives the formula

$$A_n=\frac{1}{2\pi i}\int_{-\infty}^{-R_2}D(x)x^{-n}\,dx+\frac{1}{2\pi i}\int_B F(z)z^{-n}\,dz+\frac{1}{2\pi i}\int_{-R_3}^{0}D(x)x^{-n}\,dx,$$

where B is the contour around the boundary of the open annulus. We now let n get large and observe that the first two integrals contributing to A_n are overwhelmed by the third, assuming that $D(x)\neq 0$ for $-R_3<x<0$. This establishes Eq. (2.4) for large n.

[13]F. J. Dyson, Phys. Rev. 85, 631 (1952).

[14]A discussion of the WKB approach to tunneling problems may be found in L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Nonrelativistic Theory* (Pergamon, London, 1958), p. 171 et seq. For multidimensional WKB see V. P. Maslov, *Théorie des Perturbations et Méthodes Asymptotiques* (Dunod, Gauthier-Villars, Paris, 1972) and A. Sommerfeld, *Optics: Lectures on Theoretical Physics* (Academic, New York, 1964), Vol. 4.

[15]See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1949), p. 179.

[16]Equation (3.4) has a classical analogy. It is formally identical to classical equations for the orbit of a particle in the potential V. Of course, for classical motion to occur we must have $E > V$, while in the quantum-mechanical tunneling problem $E < V$. Thus our most probable escape paths are analytic continuations of the classical orbits.

[17]In a problem with a continuous symmetry, such as spherical symmetry, the set of MPEP's will of course be continuous. This happens in our problem only when $c=1$. When $c \neq 1$ (and $a=b=1$) we have four paths because of reflection symmetry in x and y.

[18]T. Banks and C. M. Bender, following paper, Phys. Rev. D8, 3366 (1973).

[19]It is also easy to show that these are the only straight-line solutions of Eq. (3.4).

[20]This transformation is of course motivated by the direction of the MPEP's for $c > 1$ (they meet the x axis at 45° angles). See Sec. III.

[21]See the Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 1, p. 121, Eq. (1). This work is referred to hereafter as BMP.

[22]We are of course free to choose any linear combination of $P_\nu^{-1}(w)$ and $P_\nu^{1}(w)$. It is most convenient and simplest to make the choice in Eq. (4.21) because it is easy to argue that α^2, the separation constant for Eq. (4.23), vanishes. For choices other than that in Eq. (4.21) there may be an integral over separation constants.

[23]See Ref. 18, Sec. II.

[24]BMP, Vol. 1, p. 163, Eq. (8).

[25]BMP, Vol. 1, p. 145, Eqs. (20) and (23). There is an error in Eq. (23) which we have corrected.

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Coupled Anharmonic Oscillators. II. Unequal-Mass Case

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We develop a general formalism for calculating the large-order behavior of perturbation theory for quantized systems of unequal-mass coupled anharmonic oscillators. Our technique is based on a generalization of the semiclassical approximation which was used to study equal-mass oscillators in the first paper of this series. The unequal-mass problem is much more difficult because the path which minimizes the classical action is not a straight line. Assuming that this tunneling path is known, we derive a general expression for the physical-optics approximation to the wave function of a tunneling particle. This derivation rests on the construction of a WKB approximation in curved space. We thus completely reduce the general quantum problem to the much simpler classical one of determining the path. Then we present a perturbation scheme for finding the classical path for systems of oscillators whose masses only differ by a small amount. Finally, we illustrate our techniques by solving a two-mode unequal-mass oscillator and comparing these results with a computer calculation. Our theoretical predictions and numerical calculations agree.

I. INTRODUCTION

In the first paper of this series,[1] we developed a method for computing the large-order behavior of the perturbation series for the ground-state energy of a system of N equal-mass coupled anharmonic oscillators. The method employs an extension of the semiclassical (WKB) approximation to multidimensional systems. However, before we can make the WKB approximation we must find the particular solutions of the classical orbit equations which make the action integral $\int (V-E)^{1/2}$ both a local and global minimum.[2] In general, it is very difficult to solve the classical equations, but in BBW I it was shown that for equal-mass oscillators the MPEP's are straight lines. In the present paper we will investigate the more difficult problem of computing the large-order behavior of perturbation theory for systems whose MPEP's are *not* straight lines. To demonstrate that our techniques work in such situations, we completely reduce the calculation of the large-order behavior of perturbation theory to the classical task of finding the MPEP.

Our interest in curved-path problems is not merely academic, for we are ultimately interested in quantum field theoretic perturbation series. The $(\phi^4)_N$ field theory with spatial and ultraviolet cutoffs is equivalent[3] to a system of unequal mass anharmonic oscillators. The "masses" in this case are just the energies of the modes of the field. It therefore seems likely that any attempt