$\rightarrow n$ charged particles + anything neutral, where $n = 2, 4, 6, \ldots$ and σ_2 does not include elastic scattering; $\sigma_{inel} = \sum_n \sigma_n$ is the total inelastic cross section; $f(n) = \sigma_n / \sigma_{inel}$ is the multiplicity fraction for multiplicity n; and $\langle n^q \rangle = \sum n^q f(n)$ is the qth moment of the multiplicity distribution.

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Multiplicity Distributions in High-Energy Collisions, and the Statistics of the Ideal Relativistic Bose Gas*

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Utilizing some assumptions about high-energy collisions that underlie thermodynamic and hydrodynamic models of high-energy particle production, we find simple relationships among the moments of the multiplicity distribution $\langle N^q \rangle$ that are reasonably well satisfied by recent data from the National Accelerator Laboratory (NAL) on π^- production. Using $\langle N_{ch} \rangle = 2 E_{lab}{}^{1/4}$ we obtain a reasonable one-parameter fit to all the NAL multiplicity data except f_2 and f_3 at 100, 200, and 300 GeV.

The hydrodynamical model of high-energy collisions, first proposed by Landau,[1] has recently been applied to experimental results from the National Accelerator Laboratory (NAL) and CERN Intersecting Storage Rings (ISR), with considerable success. Total multiplicities[2] and longitudinal and transverse single-particle distributions[2,3] are well predicted by the model. In this note we examine some new results on multiplicity distributions of π^- at energies of 100, 200, and 300 GeV, and show how they can be obtained from the hydrodynamic model. In fact, only two of its underlying assumptions are necessary in what follows. These are the following.

(a) Local statistical equilibrium: The fireball produced in the collision is highly inhomogeneous, but small regions of the fireball can be treated as systems in statistical equilibrium, characterized by a temperature T. Interactions between neighboring regions can be neglected, except for those implicit in determining the local temperature T (i.e., the rest of the fireball acts as a heat bath for each small region).

(b) The dynamics of each individual region (which we call "secondary fireballs" in what follows) can be described by standard statistical mechanics, in the simplest case that of an ideal relativistic gas. Particle creation and interaction between particle species are taken into account by using a grand partition function for each species.

It follows from assumption (b) that to describe pion production in each secondary fireball we need only the well-known partition function for a relativistic Bose gas[4]:

$$\ln Z = kT\sum_i \ln(1 - e^{(\mu - E_i)/kT})$$
$$\approx \frac{kTV}{h^3}\int \ln(1 - e^{(\mu - E)/kT})d^3p\,, \qquad (1)$$

where

$$E = (p^2 + m_\pi^2)^{1/2}\,.$$

Expanding the logarithm, the integration can be performed, and one obtains

$$\ln Z = \frac{V}{2\pi^2} Z_\pi^2 \left(\frac{kT}{\hbar c}\right)^3 \sum_{n=1}^{\infty} e^{\mu n/kT} K_2(nZ_\pi)/n^2\,, \qquad (2)$$

where

$$Z_\pi = m_\pi c^2/kT$$

and $K_2(z)$ is the modified Bessel function of the second kind.

From this one can obtain all properties of the pion distribution for each secondary fireball, in particular, correlations and fluctuations. Each of these quantities is a linear function of $\ln Z$, and therefore proportional to the volume V of that secondary fireball. It follows then from assumption (a) that these properties are additive, and the total particle distributions for the collision will be obtained by summing over all secondary fireballs, characterized by a certain temperature distribution T (as in Hagedorn's model[5]) and/or by a velocity distribution for the secondary fireballs (as in the hydrodynamical model) in the c.m. system of the collision.[1,2,3] In the second case, the properties in which we are interested here, such as $\langle N\rangle$, $\langle N^2\rangle - \langle N\rangle^2$, etc., do not depend on the motion of each secondary fireball, and we can consider each one at rest in its own c.m. system.

The pion distribution from each secondary fireball is characterized by the following (infinite) set of quantities:

$$\langle N\rangle = kT\frac{\partial}{\partial\mu}\ln Z\bigg|_{\mu=0}$$
$$= A\sum_{1}^{\infty} K_2(nZ_\pi)/n\,, \qquad (3a)$$

$$\Delta_2 = kT\frac{\partial^2}{\partial\mu^2}\ln Z\bigg|_{\mu=0}$$
$$= A\sum K_2(nZ_\pi)$$
$$= \langle N^2\rangle - \langle N\rangle^2\,, \qquad (3b)$$

$$\Delta_3 = (kT)^3\frac{\partial^3}{\partial\mu^3}\ln Z\bigg|_{\mu=0}$$
$$= A\sum nK_2(nZ_\pi)$$
$$= \langle N^3\rangle - 3\langle N^2\rangle\langle N\rangle + 2\langle N\rangle^3\,, \qquad (3c)$$

$$\Delta_4 = (kT)^4\frac{\partial^4}{\partial\mu^4}\ln Z\bigg|_{\mu=0}$$
$$= A\sum n^2K_2(nZ_\pi)$$
$$= \langle N^4\rangle - 4\langle N^3\rangle\langle N\rangle - 3\langle N^2\rangle^2$$
$$+12\langle N^2\rangle\langle N\rangle^2 - 6\langle N\rangle^4\,, \qquad (3d)$$

$$\Delta_5 = (kT)^5\frac{\partial^5}{\partial\mu^5}\ln Z\bigg|_{\mu=0}$$
$$= A\sum n^3K_2(nZ_\pi)$$
$$= \langle N^5\rangle - 5\langle N^4\rangle\langle N\rangle - 10\langle N^3\rangle\langle N^2\rangle + 20\langle N^3\rangle\langle N\rangle^2$$
$$+30\langle N^2\rangle^2\langle N\rangle - 60\langle N^2\rangle\langle N\rangle^3 + 24\langle N\rangle^5\,, \qquad (3e)$$

where

$$A = \frac{V}{2\pi^2}\left(\frac{kT}{\hbar c}\right)^3 Z_\pi^2\,.$$

The ratios $\Delta_n/\langle N\rangle$, $n = 2, 3, \ldots$ are therefore independent of V, and turn out to be rather insensitive to the temperature T of the secondary fireball, in the neighborhood of the critical temperature $kT_c = m_\pi c^2$ (see Table I).[6] This is especially apparent for the lower moments, and as n increases, the temperature dependence becomes more important. For $n < 5$, it is sufficiently accurate to assume a constant temperature for all secondary fireballs, and fit this temperature to the transverse-momentum distributions as was done in Ref. 3. Choosing, therefore, $T = T_c$, we find

$$\Delta_2 = 1.15\langle N\rangle\,, \qquad (4a)$$
$$\Delta_3 = 1.6\langle N\rangle\,, \qquad (4b)$$
$$\Delta_4 = 3.15\langle N\rangle\,, \qquad (4c)$$
$$\Delta_5 = 10.14\langle N\rangle\,. \qquad (4d)$$

These relations, valid for each secondary fireball, will then hold also for the fireball as a whole. To confront the general assumptions (a) and (b) with experiment, we need only to fit $\langle N\rangle$, the multiplicity at a certain energy. For a parametrization of the multiplicities, we will later use the prediction of the hydrodynamical model

$$\langle N\rangle = kE_{\text{lab}}^{1/4}\,, \qquad (5)$$

fitting k at one point to explain all the data.

We rewrite Eqs. (3) as follows:

$$\langle N^2\rangle = \langle N\rangle^2 + \Delta_2\,, \qquad (6a)$$
$$\langle N^3\rangle = 3\langle N^2\rangle\langle N\rangle - 2\langle N\rangle^3 + \Delta_3\,, \qquad (6b)$$
$$\langle N^4\rangle = 4\langle N^3\rangle\langle N\rangle + 3\langle N^2\rangle^2$$
$$- 12\langle N^2\rangle\langle N\rangle^2 + 6\langle N\rangle^4 + \Delta_4\,, \qquad (6c)$$

$$\langle N^5\rangle = 5\langle N^4\rangle\langle N\rangle + 10\langle N^3\rangle\langle N^2\rangle - 20\langle N^3\rangle\langle N\rangle^2 - 30\langle N^2\rangle^2\langle N\rangle + 60\langle N^2\rangle\langle N\rangle^3 - 24\langle N\rangle^5 + \Delta_5 . \tag{6d}$$

It is important to notice that (6) follows simply from statistical mechanics, regardless of the specific form of the partition function. If interactions between particles are included in Z, this will only modify the last term in each of Eqs. (6). Therefore, in evaluating the right-hand side of (6) for each moment, we use the *experimental* values of the lower moments. In Table II, we give the experimental values of the left-hand side of various energies, and the corresponding right-hand side evaluated as explained, with Δ_i from Eq. (4). Data are from Refs. 7–9.[10]

We remark that in each case, Δ_q is a small fraction of $\langle N^q\rangle$, so that the relations are quite well obeyed regardless of the specific form of Z. We regard this as a striking confirmation of the assumption of local statistical equilibrium and its corollary, additivity for all Δ_q.

Next we examine correlations. The parameters of interest are

$$\Delta_2 = \langle N^2\rangle - \langle N\rangle^2 = 1.15\langle N\rangle , \tag{7a}$$

$$f_2 = \Delta_2 - \langle N\rangle = 0.15\langle N\rangle , \tag{7b}$$

$$\langle N\rangle / (\Delta_2)^{1/2} = 0.93(\langle N\rangle)^{1/2} , \tag{7c}$$

$$g_2 = \langle N(N-1)\rangle = 1.15\langle N\rangle + \langle N\rangle^2 , \tag{7d}$$

$$f_3 = \Delta_3 - 3\Delta_2 + 2\langle N\rangle = 0.15\langle N\rangle , \tag{7e}$$

$$g_3 = \langle N(N-1)(N-2)\rangle = f_3 + 3g_2\langle N\rangle - 2\langle N\rangle^3 . \tag{7f}$$

Using Eq. (4) and the experimental value of $\langle N\rangle$ at each energy, we find the values of Table III, under "Fit (a)."

We see that g_2, g_3, and $\langle N\rangle/(\Delta_2)^{1/2}$ are quite accurately described by the dynamics of an ideal

TABLE I. Temperature dependence of $\Delta_q/\langle N\rangle$ for an ideal relativistic Bose gas, from Eq. (3).

kT (MeV)	$\Delta_2/\langle N\rangle$	$\Delta_3/\langle N\rangle$	$\Delta_4/\langle N\rangle$	$\Delta_5/\langle N\rangle$
280	1.18	1.74	3.88	14.69
200	1.17	1.66	3.46	11.91
140	1.15	1.60	3.15	10.14
100	1.14	1.55	2.96	9.18
70	1.13	1.52	2.84	8.63

TABLE II. Moments of the π^- multiplicity distribution. The experimental data are from Refs. 7–9. The moments are calculated from $\sum n^q\sigma_{n-}/\sigma_{\text{inelastic}}$. Theoretical values are obtained from Eq. (6), using in each case the experimental values of lower moments to calculate higher ones.

E_{lab} (GeV)		$\langle N_-^2\rangle$	$\langle N_-^3\rangle$	$\langle N_-^4\rangle$	$\langle N_-^5\rangle$
102	Exp.	7.24	29.5	137.8	713.7
	Th.	7.18	30.2	143.9	744.9
205	Exp.	11.74	58.8	336.4	2128.6
	Th.	11.2	58.9	344.4	2183
303	Exp.	16.55	96.7	649	4840
	Th.	15.7	95.1	653	4907

Bose gas. Our values for f_2 have the right order of magnitude but the energy dependence is not well reproduced. This seems to indicate that correlations due to interactions are present, and that the temperature of the fireball may be slightly energy-dependent. It will be interesting to see whether the extra correlations can be obtained by including specific π-π interactions in the partition function. It is to be noticed that energy and charge conservation laws also impose correlations on the particle distributions, but these are of the opposite sign, indicating that π-π interactions may be even more important.

The experimental values of f_3 are quite inaccurate (a typical value is -0.4 ± 0.7), but seem to be negative. This might mean that at these energies, f_3 is determined mostly by these conservation laws.

We will now use Eq. (5), a specific result of the hydrodynamic model, to obtain one-parameter fits to these quantities. Normalizing Eq. (5) to the charged multiplicity at 100 GeV, we find $K=2$. The number of π^- is obtained from charge conservation and the assumption that all negative prongs are pions:

$$N_- = \tfrac{1}{2}(N_{\text{ch}} - 2) .$$

This leads to

$$\langle N_-\rangle = E_{\text{lab}}{}^{1/4} - 1 . \tag{8a}$$

Then,

$$D^2 = \Delta_2 = 1.15(E^{1/4} - 1) , \tag{8b}$$

$$f_2 = 0.15(E^{1/4} - 1) , \tag{8c}$$

$$\langle N_-\rangle/D = 0.97(E^{1/4} - 1)^{1/2} , \tag{8d}$$

$$g_1 = \langle N_-\rangle , \tag{8e}$$

$$g_2 = E^{1/2} - 1.85E^{1/4} + 0.85 , \tag{8f}$$

$$f_3 = 0.15(E^{1/4} - 1) = f_2 . \tag{8g}$$

In terms of these,

TABLE III. Correlation parameters for π^- multiplicity distributions. The experimental data are from Refs. 7–9. Fit (a) is obtained from Eq. (7) and the experimental value of $g_1 = \langle N_- \rangle$ at each energy. Fit (b) is obtained from the parametrization given by Eq. (8). The over-all parameter is adjusted to g_1 at 102 GeV.

E_{lab} (GeV)		g_1	$\Delta_2 = D^2$	$\langle N_- \rangle / D$	f_2	g_2	f_3	g_3
102	Exp.	2.16	2.55	1.35	0.38	5.07	−0.5	12.08
	Fit (a)	···	2.49	1.37	0.32	4.92	0.32	11.98
	Fit (b)	···	2.51	1.37	0.32	5.02	0.32	10.48
205	Exp.	2.82	3.77	1.45	0.95	8.91	−1.2	29.2
	Fit (a)	···	3.24	1.53	0.42	8.37	0.42	26.4
	Fit (b)	2.78	3.22	1.55	0.41	8.16	0.4	26.6
303	Exp.	3.42	4.79	1.57	1.36	13.12	−0.4	54
	Fit (a)	···	3.94	1.7	0.51	12.3	0.51	49.4
	Fit (b)	3.17	3.65	1.66	0.48	10.5	0.48	36.9

$$g_3 = f_3 + 3g_1g_2 - 2g_1^3. \quad (8h)$$

These functions are compared with experiment in Table III, under "Fit (b)." Data are from Refs. 7–9. Typical experimental errors are 15–20%.

We see that this fit to all present data is quite reasonable, except for the interaction-sensitive parameters f_2 and f_3.

The picture we have presented, that thermal fluctuations alone cause correlations, cannot possibly be the whole story at NAL energies. At finite energies when one integrates

$$\frac{d^2\sigma}{dy_1 dy_2} - \frac{1}{\sigma}\frac{d\sigma}{dy_1}\frac{d\sigma}{dy_2}$$

to get $\langle n(n-1)\rangle - \langle n \rangle^2$, one gets contributions not only from the two-particle correlation functions inside a single fireball, but also from the fact that the integrals over the fireball velocities will have different upper limits due to energy conservation. Some crude attempt to do this in the thermodynamic model has been done by Ranft and Ranft.[11] Second, the ideal-Bose-gas correlation is extremely short-range in rapidity (i.e., a δ function[12]) and thus this model will suffer from the same failures as the short-range correlation (SRC) in rapidity models. Since Landau's model (just as the multiperipheral model) neglects the leading proton effects, it also neglects the small diffractive contributions. Thus it leads to $f_N \approx C_N N$ just as the SRC models do. This problem can be "solved" in the manner of Frazer *et al.* by allowing for some small leading particle (diffraction) effect.[13] The other possibility is that some Bose condensation effects are occurring which prevent the assumption used in Eq. (1) that $\sum_i \approx \int (d^3p/h^3)V$. These would lead to $f_2 \sim \langle n \rangle^2/k + C_2\langle n \rangle$, where k is the number of cells of phase space available.[14]

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