PHYSICAL REVIEW D VOLUME 8, NUMBER 1 1 JULY 1973

Comments and Addenda

The Comments and Addenda section is for short communications which are not of such urgency as to justify publication in **Physical Review Letters** *and are not appropriate for regular Articles. It includes only the following types of communications: (1) comments on papers previously published in* **The Physical Review** *or* **Physical Review Letters**; *(2) addenda to papers previously published in* **The Physical Review** *or* **Physical Review Letters**, *in which the additional information can be presented without the need for writing a complete article. Manuscripts intended for this section should be accompanied by a brief abstract for information-retrieval purposes. Accepted manuscripts will follow the same publication schedule as articles in this journal, and galleys will be sent to authors.*

Simple Empirical Representation of the Charged Multiplicity Distributions in Proton-Proton Collisions*

Howard Weisberg
Physics Department, University of Pennsylvania, Philadelphia, Pennsylvania 19104
(Received 26 December 1972)

The charged-prong multiplicity ratios $\sigma_n/\sigma_{\text{inel}}$, when considered as a function of the average multiplicity $\langle n\rangle$, are shown to be well represented over a wide energy range by an empirical formula having a single, energy-independent, free parameter. This representation is similar to the one used by Bozóki and co-workers in 1968. It possesses the scaling property recently discussed by Koba, Nielsen, and Olesen.

New experimental results on proton-proton charged multiplicities at high energies have recently been reported. These include data at incident momenta of 50 and 69 GeV/c from Serpukhov,[1] and at 102 (Ref. 2), 205 (Ref. 3), and 303 (Ref. 4) GeV/c from Batavia. Slattery[5,6] has found that the scaling relations[7]

$$\langle n^q\rangle = C_q\langle n\rangle^q \tag{1}$$

and

$$f(n) = \frac{1}{\langle n\rangle}\psi\left(\frac{n}{\langle n\rangle}\right), \tag{2}$$

where C_q and ψ are energy-independent, are satisfied remarkably well. He found that the factors C_q in (1) are statistically consistent with being independent of energy over the energy range covered, for $q=2,3,\ldots,10$; the energy-averaged values of C_q that he reported[5] are given in Table I. Furthermore, he found that the experimental values of $\langle n\rangle\sigma_n/\sigma_{\text{inel}}$, when plotted against $n/\langle n\rangle$, appear to lie on a universal curve, which he fitted with the empirical function[6]

$$\psi(x) = (3.79x + 33.7x^3 - 6.64x^5 + 0.332x^7)\times\exp(-3.04x).$$

This function yields $x^2 = 47$ for 50 data points and three free parameters (five parameters and two constraints introduced by the normalization conditions on the function ψ).

Equations (1) and (2) were discussed recently on theoretical grounds by Koba, Nielsen, and Olesen.[8]

The purpose of the present note is to report that the empirical formula

$$f(n) = c^{-1}n^{\beta-1}\exp(-n^2/2\alpha^2) \tag{3}$$

can reproduce the experimental high-energy C_q values very well, and simultaneously yield a reasonable fit to the multiplicity data on $f(n)$ as a function of $\langle n\rangle$, both at high and at relatively low energies.

In Eq. (3), c is a normalization constant determined by the condition

$$\sum_n f(n) = 1. \tag{4}$$

The parameter β is taken to be an energy-independent constant, and the parameter α is then determined as a function of $\langle n\rangle$ by the further normalization condition

$$\sum_n nf(n) = \langle n\rangle, \tag{5}$$

which arises because we wish to treat $\langle n\rangle$ as an independent variable. *Thus our fit contains only the single energy-independent parameter* β.

This empirical representation of the multiplicity data is very similar to the one introduced several years ago by Bozóki, Gombosi, Posch, and Vanicsek,[9] who obtained an excellent fit to emulsion and bubble-chamber data up to 30 GeV. The principal difference is that these authors took incident energy, rather than mean multiplicity $\langle n\rangle$,

TABLE I. Values of reduced moments $C_q = \langle n^q\rangle/\langle n\rangle^q$.

q	Data [a]	Fit [b]
2	1.2438±0.0056	1.2448
3	1.813±0.020	1.807
4	2.973±0.057	2.950
5	5.36±0.15	5.30
6	10.43±0.39	10.30
7	21.6±1.1	21.5
8	47.0±2.8	47.6
9	107.4±7.8	111.2
10	252±22	273

[a] Reference 5.
[b] Equation (6) with $\beta = 2.215$.

as the independent variable. In order to determine the parameter α, they treated α as an empirically fitted function of energy in the place of Eq. (5). The parameter α is closely related to the mean multiplicity $\langle n\rangle$, and in the present note we have chosen to avoid the question of the functional dependence of $\langle n\rangle$ on energy by treating $\langle n\rangle$ as the independent variable.

The important feature of the empirical function (3), used both in this note and in Ref. 9, is that it satisfies the scaling relations (1) and (2). In fact, as mentioned in Appendix 1 of Ref. 9, its moments are given in the high-energy limit by

$$\langle n^q\rangle = 2^{q/2}\frac{\Gamma(\frac{1}{2}(\beta+q))}{\Gamma(\frac{1}{2}\beta)}\,\alpha^q .$$

Therefore, the function (3) satisfies the condition (1) with the constants C_q given by

$$C_q = \frac{\Gamma(\frac{1}{2}(\beta+q))}{\Gamma(\frac{1}{2}(\beta+1))}\left(\frac{\Gamma(\frac{1}{2}\beta)}{\Gamma(\frac{1}{2}(\beta+1))}\right)^{q-1}. \qquad (6)$$

In particular, the function (3) [and, indeed, any parametrization of the multiplicity distribution satisfying the scaling relation (1)] has a constant value at high energy for the ratio

$$\frac{\langle n\rangle}{D} = \frac{\langle n\rangle}{(\langle n^2\rangle - \langle n\rangle^2)^{1/2}} .$$

It was first noted by Malhotra[10] in 1963 that this ratio appears to approach a constant limit near 2.0 at high energy.[11]

In Table I, the column labeled "fit" gives the values of C_q computed for Eq. (3) in the high-energy limit [i.e., from Eq. (6)]. The values are given for the value $\beta = 2.215 \pm 0.015$, which gives the best fit to the C_q ($\chi^2 = 1.8$ for nine data points and a one-parameter fit). Of course the various moments may be expected to be strongly correlated, so that once one fits the first few moments all the others fall into place automatically, but even so, the fit is certainly satisfactory. The value found for β is completely consistent with the value $\beta = 2.30$ found in Ref. 9 by fitting lower-energy data.

If, instead of fitting the moments, we fit the multiplicity data directly, varying β to give the best fit to the same data set from 50 to 300 GeV/c that was used by Slattery,[5,6,12] we find the best fit for $\beta = 2.125 \pm 0.035$, yielding $\chi^2 = 45.7$ for 50 data points and one parameter. However, we prefer to use the value, $\beta = 2.215$, that gives the best fit to the C_q and is more likely to correspond to the ultimate asymptotic value of β. It yields $\chi^2 = 52.1$.

In Fig. 1 are plots of Eq. (3) for various n values as a function of $\langle n\rangle$, using Eqs. (4) and (5) to determine c and α, for $\beta = 2.215$. Also shown are the experimental data[12] of Refs. 1–4, along with additional bubble-chamber multiplicity data for lower incident momenta: 4.00 (Ref. 13), 5.52 (Ref. 14), 10.01 (Ref. 15), and 16 (Ref. 16) GeV/c. It is remarkable that a one-parameter fit can reproduce so well a set of data covering a 100 : 1 range of incident energy and a 1000 : 1 range of $\sigma_n/\sigma_{\text{inel}}$.

The success of the fit in extrapolating *downward* in energy encourages us to extrapolate *upward* also. Therefore, Fig. 1 includes the extrapolated values of Eq. (1) in the $\langle n\rangle$ range which may be reached at the CERN Intersecting Storage Rings (ISR).

No claim is made that Eq. (3) is uniquely determined by the data. In fact, it is clear that any function having similar behavior in the $n/\langle n\rangle$ range of existing data, no matter what its behavior at large or small $n/\langle n\rangle$, will work. A simple example is the "stretched Poisson distribution"

$$\psi\left(\frac{n}{\langle n\rangle}\right) = \frac{2a^{a(n/\langle n\rangle)+1}e^{-a}}{\Gamma(a(n/\langle n\rangle)+1)} .$$

If the parameter a is set to 4.0, the C_q values are fitted satisfactorily ($\chi^2 = 5.3$), and the 50–300-GeV/c data are fitted with $\chi^2 = 85$. The fit to the lower-energy data is not as impressive as that in Fig. 1, but the general trends are still reproduced. Yet, the behavior for large and small $n/\langle n\rangle$ is quite different from that of Eq. (3).

Clearly, new experimental measurements of multiplicity distributions are needed to check the validity for the scaling relations (1) and (2) over a wider range, and for other than proton-proton collisions. If the relations indeed hold, the new measurements will help elucidate further the nature of the function $\psi(n/\langle n\rangle)$ and the validity of the empirical fit (3), both at small $n/\langle n\rangle$ (two prongs at ISR energies) and at large $n/\langle n\rangle$ [counter

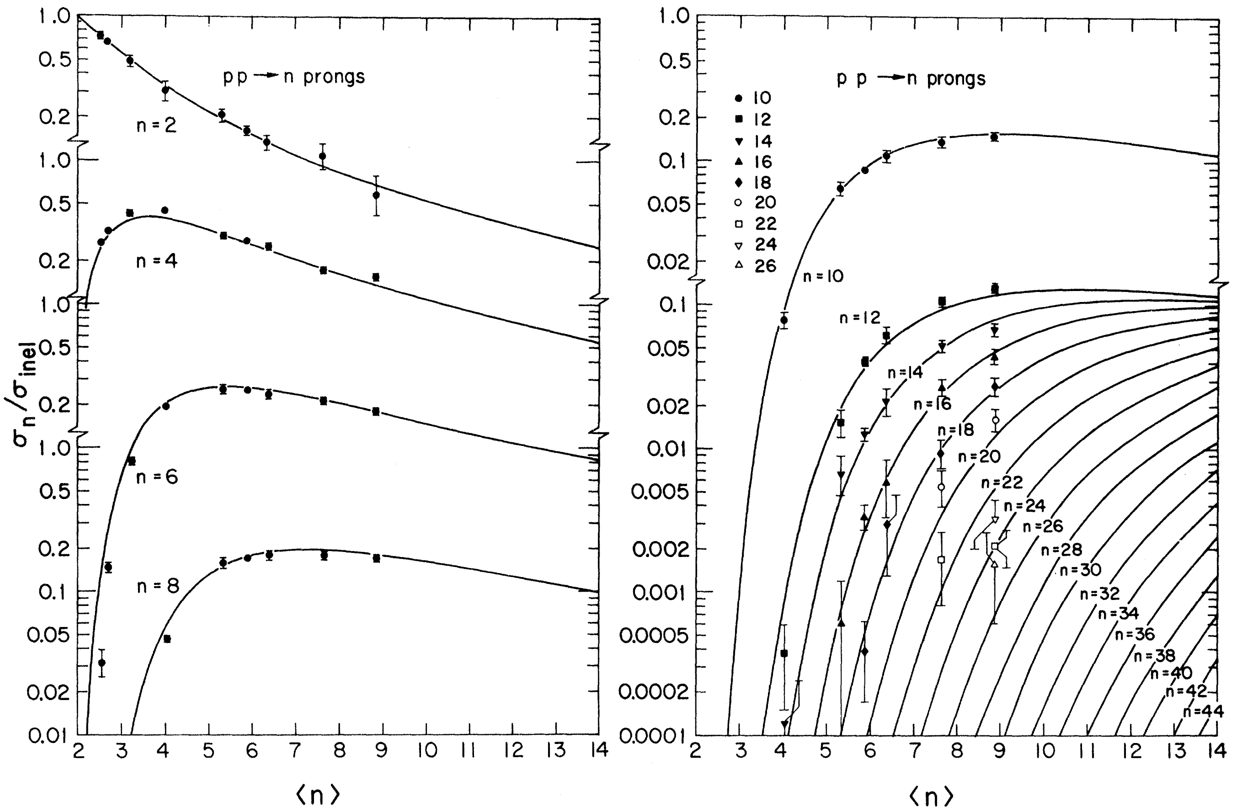

FIG. 1. Values of σ_n/σ_{inel} plotted against $\langle n\rangle$. The solid lines are obtained from Eq. (3) with $\beta=2.215$ and with c and α determined by Eqs. (4) and (5). Note that the data and curves for $n=2,4,6,8$, and 10 have each displaced upward by one decade from the data below.

or wire-chamber experiments at ISR, National Accelerator Laboratory (NAL), or lower energies].

If Eqs. (1) and (2) hold, and the guess (3) is a good one, then it follows that at high energies

$$\frac{\sigma_n}{\sigma_{inel}}=3.70\frac{1}{\langle n\rangle}\left(\frac{n}{\langle n\rangle}\right)^{1.215}\exp\left[-\frac{1.78}{2}\left(\frac{n}{\langle n\rangle}\right)^2\right] . \tag{7}$$

In any event, it appears to the author that the multiplicity distributions are exhibiting the following behavior, at least for energies and multiplicities near the range which has been studied experimentally so far:

(1) At fixed energy and large n, σ_n falls faster than any power of n; and

(2) at fixed n, and large enough energy, σ_n falls approximately as a small inverse power of $\langle n\rangle$.

I am happy to thank G. Bozóki for introducing me to the work of Refs. 9, 10, and 11, and for several helpful discussions.

---

*Research supported by the U. S. Atomic Energy Commission.

[1]Soviet-French collaboration, contribution submitted to the Sixteenth International Conference on High Energy Physics, Batavia, Ill., 1972 (unpublished), quoted in Refs. 5 and 6.

[2]J. W. Chapman *et al*. (Michigan-Rochester collaboration), contribution submitted to the Sixteenth International Conference on High Energy Physics, Batavia, Ill., 1972 (unpublished); Phys. Rev. Lett. 29, 1686 (1972), quoted in Refs. 5 and 6.

[3]G. Charlton *et al*., Phys. Rev. Lett. 29, 515 (1972), quoted in Refs. 5 and 6.

[4]F. T. Dao *et al*., NAL-UCLA collaboration, report submitted to the Sixteenth International Conference on High Energy Physics, Batavia, Ill., 1972 (unpublished), quoted in Refs. 5 and 6.

[5]P. Slattery, Phys. Rev. D 7, 2073 (1973).

[6]P. Slattery, Phys. Rev. Lett. 29, 1624 (1972); P. Olesen, Phys. Lett. 41B, 602 (1972).

[7]We use the conventional notation that σ_n represents the partial cross section for the process proton + proton

$\to n$ charged particles + anything neutral, where $n = 2, 4, 6, \ldots$ and σ_2 does not include elastic scattering; $\sigma_{\text{inel}} = \sum_n \sigma_n$ is the total inelastic cross section; $f(n) = \sigma_n / \sigma_{\text{inel}}$ is the multiplicity fraction for multiplicity n; and $\langle n^q \rangle = \sum n^q f(n)$ is the qth moment of the multiplicity distribution.

[8]Z. Koba, H. B. Nielsen, and P. Olesen, Nucl. Phys. B40, 317 (1972)

[9]G. Bozóki, E. Gombosi, M. Posch, and L. Vanicsek, Nuovo Cimento 64A, 881 (1969).

[10]P. K. Malhotra, Nucl. Phys. 46, 559 (1963).

[11]The empirical scaling properties of multiplicity distributions have also been studied by O. Czyżewski and K. Rybicki [Nucl. Phys. B47, 633 (1972); Cracow Report No. 703/PH, 1970 (unpublished)] who have emphasized the fact that the data when properly presented fall on a universal curve, consistent with Eq. (2).

[12]We use the error values recalculated by Slattery in Ref. 5 for the data of Refs. 1–4 and 16. These errors are smaller than those quoted by the original authors, sometimes by a factor of as much as 2. The errors for the other data (which appear in Fig. 1, but are not used in the numerical fits) were obtained from the original papers without taking correlations or systematic errors into account.

[13]L. Bodini, L. Casè, J. Kidd, L. Mandelli, V. Pelosi, S. Ratti, V. Russo, L. Tallone, C. Caso, F. Conte, M. Dameri, and G. Tomasini, Nuovo Cimento 58A, 475 (1968).

[14]G. Alexander, O. Benary, G. Czapek, B. Haber, N. Kidron, B. Reuter, A. Shapira, E. Simopoulou, and G. Yekutieli, Phys. Rev. 154, 1248 (1967).

[15]S. P. Almeida, J. G. Rushbrooke, J. H. Scharenguivel, M. Behrens, V. Blobel, I. Borecka, H. C. Dehne, J. Díaz, G. Knies, A. Schmitt, K. Strömer, and W. P. Swanson, Phys. Rev. 174, 1638 (1968).

[16]H. Bøggild, E. Dahl-Jensen, K. H. Hansen, J. Johnstad, E. Lohse, M. Suk, L. Veje, V. J. Karismake, K. V. Laurikainen, E. Rlipinen, T. Jacobsen, S. O. Sorensen, J. Allan, G. Blomquist, O. Danielsen, G. Ekspong, L. Granstrøm, S. O. Holmgren, S. Nilsson, B. E. Ronne, U. Svedin, and N. K. Yamdagni, Nucl. Phys. B27, 285 (1971).

PHYSICAL REVIEW D VOLUME 8, NUMBER 1 1 JULY 1973

Multiplicity Distributions in High-Energy Collisions, and the Statistics of the Ideal Relativistic Bose Gas*

Fred Cooper and Edmond Schonberg

Belfer Graduate School of Science, Yeshiva University, New York, New York 10033

(Received 19 January 1973)

Utilizing some assumptions about high-energy collisions that underlie thermodynamic and hydrodynamic models of high-energy particle production, we find simple relationships among the moments of the multiplicity distribution $\langle N^q \rangle$ that are reasonably well satisfied by recent data from the National Accelerator Laboratory (NAL) on π^- production. Using $\langle N_{\text{ch}} \rangle = 2 E_{\text{lab}}^{1/4}$ we obtain a reasonable one-parameter fit to all the NAL multiplicity data except f_2 and f_3 at 100, 200, and 300 GeV.

The hydrodynamical model of high-energy collisions, first proposed by Landau,[1] has recently been applied to experimental results from the National Accelerator Laboratory (NAL) and CERN Intersecting Storage Rings (ISR), with considerable success. Total multiplicities[2] and longitudinal and transverse single-particle distributions[2,3] are well predicted by the model. In this note we examine some new results on multiplicity distributions of π^- at energies of 100, 200, and 300 GeV, and show how they can be obtained from the hydrodynamic model. In fact, only two of its underlying assumptions are necessary in what follows. These are the following.

(a) Local statistical equilibrium: The fireball produced in the collision is highly inhomogeneous, but small regions of the fireball can be treated as systems in statistical equilibrium, characterized by a temperature T. Interactions between neighboring regions can be neglected, except for those implicit in determining the local temperature T (i.e., the rest of the fireball acts as a heat bath for each small region).

(b) The dynamics of each individual region (which we call "secondary fireballs" in what follows) can be described by standard statistical mechanics, in the simplest case that of an ideal relativistic gas. Particle creation and interaction between particle species are taken into account by using a grand partition function for each species.

It follows from assumption (b) that to describe pion production in each secondary fireball we need only the well-known partition function for a relativistic Bose gas[4]: