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Five-Parameter Exterior Solution of the Einstein-Maxwell Field Equations

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A five-parameter solution of the combined Einstein-Maxwell equations is given which describes a source containing mass, electric charge, magnetic dipole, higher multipole moments of all three kinds, and angular momentum. The solution is obtained by using Kinnersley's method of generating stationary Einstein-Maxwell fields from known solutions of the Einstein - Maxwell equations. We start with a two-parameter solution of a system having mass and a magnetic dipole moment discovered by Misra, Pandey, Srivastava, and Tripathi. All solutions discussed in this paper are asymptotically flat, and all have infinite red-shift surfaces that are singular. Possible relevance of these solutions to black-hole physics is remarked upon.

I. INTRODUCTION

A solution is presented of the combined Einstein-Maxwell field equations which depends on five parameters: m, e, $|c|$, c_r, β. c is a complex parameter, c_r its real value, and $|c|$ its absolute value. The first three parameters represent respectively the mass, the magnetic dipole moment, and the electric charge; the last two describe the angular momentum of a central source. The source has, in addition to these poles, a mass quadrupole, a magnetic quadrupole, an electric dipole, and higher multipole moments whose values are determined by the five parameters. The mass parameter m must have a nonzero value or the solution collapses to flat space. If m does not vanish, interesting special cases occur even when only one of the other parameters e, $|c|$, c_r, and β is not zero. If only m and e do not vanish, the system has a mass pole and higher mass multipoles, a magnetic dipole but no other magnetic poles, no electric charge or poles, and no angular momentum. When m and $|c|$ alone are nonvanishing, the system has mass and electric charge poles and higher multipoles, but no magnetic dipoles or multipoles and no angular momentum. If only m and β survive, the system has mass and angular momentum. Each solution has a singularity on a closed bounded surface as well as other singularities within this surface.

The five-parameter solution is generated by applying the solution-generating mechanism of Kinnersley[1] to a solution found by Misra, Pandey, Srivastava, and Tripathi[2] (MPST). The latter is a two-parameter solution of the Einstein-Maxwell fields describing the system mentioned above with m and e nonvanishing, while $|c|$, c_r, and β are zero. In Sec. II we briefly review the method by which MPST found their solution and write the metric they discovered. In Sec. III we briefly review Kinnersley's method of generating solutions of the Einstein-Maxwell theory from other solutions. In. Secs. IV and V we apply Kinnersley's method to the MPST solution to obtain the space-time we are discussing. We then discuss our solution and its properties in somewhat more detail.

II. THE MPST SOLUTION

A detailed derivation of the MPST solution is presented in their paper.[2] For the sake of completeness we briefly outline their method. We consider a static axially symmetric metric expressed as

$$ds^2 = f dt^2 - \frac{e^{2\kappa}}{f}[(dx^1)^2 + (dx^2)^2] - \frac{h^2}{f} d\phi^2 , \qquad (2.1)$$

and describe the electromagnetic field by the complex Maxwell tensor[1]

$$\mathfrak{F}_{\mu\nu} = F_{\mu\nu} + i\, {}^*F_{\mu\nu} . \qquad (2.2)$$

$^*F_{\mu\nu}$ is the dual of the usual Maxwell tensor $F_{\mu\nu}$. (Greek indices run from 0 to 3; Latin indices from 1 to 3.) Let $\mathcal{E}_r$ and $\mathcal{E}_i$ denote respectively the usual electrostatic and magnetic scalar potentials so that

$$F_{0j} = \frac{\partial \mathcal{E}_r}{\partial x^j} , \qquad {}^*F_{0j} = \frac{\partial \mathcal{E}_i}{\partial x^j} . \qquad (2.3)$$

Define

$$\mathcal{E} = \mathcal{E}_r + i\mathcal{E}_i . \qquad (2.4)$$

MPST consider situations such that $\mathcal{E}_r$ is proportional to $\mathcal{E}_i$ in the following way:

$$\mathcal{E}_r \equiv c \cos\beta , \qquad \mathcal{E}_i \equiv c \sin\beta , \qquad (2.5)$$

where β is a constant. Changing the value of β corresponds to performing a duality rotation[3] on the system. MPST introduce a complex function ξ defined by

$$\frac{\xi - 1}{\xi + 1} = f^{1/2} + ic . \qquad (2.6)$$

They then show that the entire set of coupled Einstein-Maxwell equations are satisfied if

$$(\xi\bar{\xi} - 1)\nabla^2\xi = 2\bar{\xi}(\vec{\nabla}\xi \cdot \vec{\nabla}\xi) . \qquad (2.7)$$

A bar over a quantity denotes complex conjugation; it is understood that the usual three-dimensional vector calculus and differential operator $\vec{\nabla}$ is used in the flat x^1, x^2, ϕ space. It is obvious that if ξ is any solution, so is $e^{i\alpha}\xi$, where α is a real constant. In the case of a constant-phase solution, one may introduce a real function ψ such that[4]

$$\xi = -e^{i\alpha}\coth\psi , \qquad (2.8)$$

providing ψ satisfies the Laplace equation

$$\nabla^2\psi = 0 . \qquad (2.9)$$

Hence any axially symmetric real solution of the Laplace equation in three-space generates static, axially symmetric solutions of the Einstein-Maxwell equations. Working in spheroidal coordinates which are related to cylindrical coordinates by

$$\rho = (\lambda^2 - 1)^{1/2}(1 - \mu^2)^{1/2} , \qquad z = \lambda\mu , \qquad (2.10)$$

MPST find the solution of Eq. (2.7):

$$\xi = \lambda \cos x + i\mu \sin x . \qquad (2.11)$$

MPST now define $\tan x \equiv e$, $\sec x \equiv m$ and change the scale so that length is measured in units of $(m^2 - e^2)^{1/2}$. When the entire metric is reconstructed, it becomes

$$ds^2 = \left(\frac{r^2 + e^2\cos^2\theta - 2mr}{r^2 + e^2\cos^2\theta}\right)^2 dt^2 - \frac{(r^2 - 2mr + e^2\cos^2\theta)^2 (r^2 + e^2\cos^2\theta)^2}{(r^2 - 2mr + e^2\cos^2\theta + m^2\sin^2\theta)^3}\left(\frac{dr^2}{r^2 - 2mr + e^2} + d\theta^2\right)$$
$$- \frac{(r^2 + e^2\cos^2\theta)^2 (r^2 - 2mr + e^2)}{(r^2 - 2mr + e^2\cos^2\theta)^2}\sin^2\theta\, d\phi^2 , \qquad (2.12)$$

where "spherical" coordinates r and θ are used, defined by

$$r \equiv \lambda(m^2 - e^2)^{1/2} + m\,, \qquad \cos\theta \equiv \mu\,. \tag{2.13}$$

The electromagnetic potential is given by

$$\mathcal{E} = \frac{i2me\cos\theta}{r^2 + e^2\cos^2\theta}. \tag{2.14}$$

It has been chosen purely imaginary so that the solution describes a magnetic dipole (as can be seen in its asymptotic form, $r \to \infty$). After a duality transformation it could equally represent an electric dipole, or both a magnetic and an electric dipole. We shall consider it to be a magnetic dipole, with $2me$ being the dipole moment.

The MPST solution is given by the metric (2.12) and the potential (2.14). From this solution, using the Kinnersley technique, we generate other solutions.

III. KINNERSLEY'S METHOD OF GENERATING STATIONARY EINSTEIN-MAXWELL FIELDS

Kinnersley[1] has shown how to generate a five-parameter family of solutions given a single solution. We apply Kinnersley's method to the two-parameter solution of MPST to obtain a new solution. It turns out that the new solution will have three additional physically interesting parameters for a total of five. We start with the line element expressed in the form

$$ds^2 = f(dt + \omega_j dx^j)^2 - \frac{h_{jk}}{f}dx^j dx^k. \tag{3.1}$$

Henceforth, $\vec\nabla$ denotes the covariant derivative operator in the three-space H with metric tensor h_{jk}. Define a twist vector

$$\vec\tau = f^2\vec\nabla\times\omega + i(\bar{\mathcal{E}}\vec\nabla\mathcal{E} - \mathcal{E}\vec\nabla\bar{\mathcal{E}}). \tag{3.2}$$

The Einstein-Maxwell equations show[5] that the curl of $\vec\tau$ vanishes, and hence a scalar "twist potential" ψ can be introduced such that

$$\vec\tau \equiv \vec\nabla\psi. \tag{3.3}$$

Now introduce the "Ernst potential"

$$\mathcal{G} \equiv f - \mathcal{E}\bar{\mathcal{E}} + i\psi. \tag{3.4}$$

We shall say that a system has twist if $\vec\nabla\times\vec\omega$ does not vanish. The Kinnersley method of generating solutions may now be described quite simply. Given a solution of the stationary Einstein-Maxwell equations, f, ω_j, and h_{jk} can be determined from the metric. The complex potential $\mathcal{E}$ can be determined from the electromagnetic field; ψ from Eqs. (3.2) and (3.3). This permits one to calculate $\mathcal{G}$ from (3.4). Kinnersley shows how to transform from $\mathcal{G}$ and $\mathcal{E}$ to a new set of Ernst and electromagnetic potentials, $\mathcal{G}'$ and $\mathcal{E}'$. From (3.4) and the new $\mathcal{G}'$ one calculates a new f' and ψ', and then from (3.3) and (3.2) a new ω'. The new solution is given by the metric f', ω', and $h'_{jk} = h_{jk}$, with the new electromagnetic field determined from $\mathcal{E}'$. Kinnersley lists five canonical classes of transformations for going from $\mathcal{G}$ and $\mathcal{E}$ to $\mathcal{G}'$ and $\mathcal{E}'$.

Under a Class-I transformation,

$$\mathcal{E} \to \mathcal{E}' = \mathcal{E} + a\,, \qquad \mathcal{G} \to \mathcal{G}' = \mathcal{G} - 2\bar{a}\mathcal{E} - a\bar{a}\,. \tag{3.5}$$

Under a Class-II transformation,

$$\mathcal{E} \to \mathcal{E}' = \mathcal{E}\,, \qquad \mathcal{G} \to \mathcal{G}' = \mathcal{G} + i\alpha\,. \tag{3.6}$$

a is complex and α is real. Transformations of Classes I and II leave both the electromagnetic field and the geometry unchanged. They correspond to electromagnetic and gravitational gauge transformations.

Under a Class-III transformation,

$$\mathcal{G} \to \mathcal{G}' = (b\bar{b})^{-1}\mathcal{G}\,, \qquad \mathcal{E} \to \mathcal{E}' = (\bar{b}b^{-2})\mathcal{E}\,. \tag{3.7}$$

When b has absolute value unity the transformation is a duality rotation which does not affect the geometry, but can change electric fields into magnetic fields or vice versa. If $b\bar{b} \neq 1$, the transformation (3.7) amounts to a rescaling:

$$ds^2 \to ds'^2 = (b\bar{b})^{-1}ds^2\,. \tag{3.8}$$

Under a Class-IV transformation,

$$\mathcal{G} \to \mathcal{G}' = \frac{\mathcal{G}}{1 + i\beta\mathcal{G}}\,, \qquad \mathcal{E} \to \mathcal{E}' = \frac{\mathcal{E}}{1 + i\beta\mathcal{G}}. \tag{3.9}$$

β is a real parameter and these transformations map static fields into stationary ones.

Under a Class-V transformation,

$$\mathcal{G} \to \mathcal{G}' = \frac{\mathcal{G}}{1 - 2\bar{c}\mathcal{E} - c\bar{c}\mathcal{G}}\,, \qquad \mathcal{E} \to \mathcal{E}' = \frac{\mathcal{E} + c\mathcal{G}}{1 - 2\bar{c}\mathcal{E} - c\bar{c}\mathcal{G}}. \tag{3.10}$$

c is a complex parameter; Class-V transformations do not preserve vacuum.

We shall apply transformation Classes IV and V to the MPST solution to obtain a new solution. Transformations of Class III will also be used for rescaling purposes and to ensure that no magnetic charge is introduced into the system.

IV. CLASS-V TRANSFORMATION APPLIED TO THE MPST SOLUTION

We shall first apply a Class-V transformation to the MPST solution to get a new solution. This will introduce an electric charge and also a twist. Equation (2.12) shows that for the MPST solution

$$f=\left(\frac{r^2-2mr+e^2\cos^2\theta}{r^2+e^2\cos^2\theta}\right)^2, \tag{4.1}$$

$$\omega_j=0, \tag{4.2}$$

$$h_{rr}=\frac{(r^2-2mr+e^2\cos^2\theta)^4}{(r^2-2mr+e^2\cos^2\theta+m^2\sin^2\theta)^3}\times\frac{1}{(r^2-2mr+e^2)}, \tag{4.3}$$

$$h_{\theta\theta}=\frac{(r^2-2mr+e^2\cos^2\theta)^4}{(r^2-2mr+e^2\cos^2\theta+m^2\sin^2\theta)^3}, \tag{4.4}$$

$$h_{\phi\phi}=(r^2-2mr+e^2)\sin^2\theta. \tag{4.5}$$

Equation (2.14) gives the value of $\mathcal{E}$,

$$\mathcal{E}\equiv iE \equiv i\frac{2me\cos\theta}{r^2+e^2\cos^2\theta}. \tag{4.6}$$

The twist potential ψ vanishes and we calculate the Ernst potential

$$\mathcal{G}=f-\mathcal{E}\overline{\mathcal{E}} = \left(\frac{r^2-2mr+e^2\cos^2\theta}{r^2+e^2\cos^2\theta}\right)^2-\left(\frac{2me\cos\theta}{r^2+e^2\cos^2\theta}\right)^2. \tag{4.7}$$

$\mathcal{E}'$ is given by (3.10) with (4.6) and (4.7). The asymptotic form of $\mathcal{E}'$ as $r\to\infty$ is given by

$$\mathcal{E}'=\rho+\frac{\sigma}{r}+\frac{\lambda}{r^2}+\cdots, \tag{4.8}$$

$$\rho=\frac{c}{1-c\bar{c}}, \tag{4.9}$$

$$\sigma=-\frac{4mc}{(1-c\bar{c})^2}, \tag{4.10}$$

$$\lambda=\frac{1}{1-c\bar{c}}\left[i2me\cos\theta\left(\frac{1+c\bar{c}}{1-c\bar{c}}\right)+4m^2c\left(1+\frac{5c\bar{c}}{1-c\bar{c}}+\frac{4c^2\bar{c}^2}{(1-c\bar{c})^2}\right)\right]. \tag{4.11}$$

From $\mathcal{G}'$ as given by (3.10), from the definition of the Ernst potential $\mathcal{G}'$ as given in (3.4), from the electromagnetic potential (4.6), and calling $c\equiv c_r+c_i$, one can easily deduce

$$\psi'=\frac{2c_rE\mathcal{G}}{(1-c\bar{c}\mathcal{G}-2Ec_i)^2+(2Ec_r)^2}, \tag{4.12}$$

$$f'=\frac{f}{(1-c\bar{c}\mathcal{G}-2Ec_i)^2+(2Ec_r)^2}. \tag{4.13}$$

$\vec{\omega}'$ is now found from Eq. (3.2); explicitly

$$\vec{\nabla}\times\vec{\omega}'=\frac{1}{f'^2}[\vec{\nabla}\psi'-i(\overline{\mathcal{E}}'\vec{\nabla}\mathcal{E}'-\mathcal{E}'\vec{\nabla}\overline{\mathcal{E}}')]. \tag{4.14}$$

One sees quickly that only ω'_ϕ survives; ω'_r and ω'_θ can be taken equal to zero. Integrating over a cap of constant r over the polar axis yields

$$\omega'_\phi=\int_0^\theta\frac{1}{f'^2h_{rr}{}^{1/2}}\left[\frac{\partial\psi'}{\partial r}-i\left(\overline{\mathcal{E}}'\frac{\partial\mathcal{E}'}{\partial r}-\mathcal{E}'\frac{\partial\overline{\mathcal{E}}'}{\partial r}\right)\right]\times h_{\theta\theta}{}^{1/2}h_{\phi\phi}{}^{1/2}d\theta. \tag{4.15}$$

The determination of the new metric is complete. However, we should like to rescale to require that as $r\to\infty$, $g_{rr}\to1$, $g_{\theta\theta}\to r^2$, and $g_{\phi\phi}\to r^2\sin\theta$. Actually, as things stand, $g_{rr}\to(1-c\bar{c})^2$ since

$$f'\underset{r\to\infty}{\longrightarrow}\frac{1}{(1-c\bar{c})^2}. \tag{4.16}$$

We apply a Class-III transformation with real b and

$$b=(1-c\bar{c})^{-1}. \tag{4.17}$$

The rescaled metric and electromagnetic potential are given a subscript r and become

$$ds_r'^2=f'\left(\frac{dt}{(1-c\bar{c})}-\frac{\omega'_\phi d\phi}{(1-c\bar{c})}\right)^2-\frac{1}{(1-c\bar{c})^2f'}(h_{rr}dr^2+h_{\theta\theta}d\theta^2+h_{\phi\phi}d\phi^2), \tag{4.18}$$

$$\mathcal{E}'_r=\frac{1}{(1-c\bar{c})}\left(\rho+\frac{\sigma}{r}+\frac{\lambda}{r^2}+\cdots\right). \tag{4.19}$$

The asymptotic behavior of ω'_ϕ can be determined from (4.10), (4.7), and (4.8). From (4.7)

$$\psi'\to\frac{4mec_r\cos\theta}{(1-c\bar{c})^2r^2}. \tag{4.20}$$

Putting everything together, one calculates explicitly the asymptotic form of ω'_ϕ,

$$\omega'_\phi\to\frac{8mec_r(1-c\bar{c})}{r}(\cos^2\theta-1). \tag{4.21}$$

Hence, as $r\to\infty$, the rescaled metric (4.18) represents a flat space; the coefficient of dt^2 can be made equal to one by a rescaling of the time axis.

In order that the monopole component of the rescaled electromagnetic potential be an electric field and not a magnetic one, it is necessary to perform a duality rotation to ensure that the coefficient of r^{-1} in relation (4.19) is real. We arrive at the dually rotated rescaled potential, denoted by $\mathcal{E}'_{rd}$:

$$\mathcal{E}'_{rd} = \frac{e^{i\beta}}{(1-c\bar{c})}\left(\rho + \frac{\sigma}{r} + \frac{\lambda}{r^2} + \cdots\right),$$

$$\tan\beta \equiv -\frac{\sigma_i}{\sigma_r}, \tag{4.22}$$

$$\sigma \equiv \sigma_r + i\sigma_i .$$

The electric charge is defined by

$$Q \equiv \iint F_{01} g_{\theta\theta}{}^{1/2} g_{\phi\phi}{}^{1/2} d\theta d\phi . \tag{4.23}$$

For simplicity, the integral is taken over the surface of a spatial sphere as its radius becomes infinite. From (4.22),

$$F_{01} \to -\frac{(\sigma_r{}^2 + \sigma_i{}^2)^{1/2}}{(1-c\bar{c})r^2} = \frac{4m(c\bar{c})^{1/2}}{(1-c\bar{c})^3 r^2}. \tag{4.24}$$

Hence,

$$Q = \frac{16\pi m|c|}{(1-|c|^2)^3}. \tag{4.25}$$

The solution has four parameters, m, e, c_r, and $|c|$, whose primary roles are to determine the mass, magnetic dipole moment, twist, and electric charge, respectively. The metric is given by Eqs. (4.18), (4.13), (4.3), (4.4), (4.5), and (4.15); the twist is described by (4.15) and (4.12); the electromagnetic field potential is given asymptotically by (4.22). The metric is asymptotically flat.

V. COMBINED CLASSES-V-AND-IV TRANSFORMATION APPLIED TO THE MPST SOLUTION

Next we apply a Class-IV transformation. This could be applied directly to the metric we have just described. Instead of this we combine the transformation of Classes V and IV and apply the combined transformation directly to the MPST solution. We then rescale and perform a duality rotation and to ensure that the charge is electric and not magnetic. The combined transformation can be written as

$$\begin{aligned} \mathcal{G} \to \mathcal{G}'' &= \frac{\mathcal{G}'}{1+i\beta\mathcal{G}'} \\ &= \frac{\mathcal{G}}{1-2\bar{c}\mathcal{E} - c\bar{c}\mathcal{G} + i\beta\mathcal{G}}, \\ \mathcal{E} \to \mathcal{E}'' &= \frac{\mathcal{E}'}{1+i\beta\mathcal{G}'} \\ &= \frac{\mathcal{E} + c\mathcal{G}}{1-2\bar{c}\mathcal{E} - c\bar{c}\mathcal{G} + i\beta\mathcal{G}}. \end{aligned} \tag{5.1}$$

The potentials ψ'' and f'' become

$$\begin{aligned}\psi'' &= \mathrm{Im}\,\mathcal{G}'' \\ &= -\frac{\mathcal{G}(\beta\mathcal{G} - 2c_r E)}{(1-c\bar{c}\mathcal{G} - 2c_i E)^2 + (\beta\mathcal{G} - 2c_r E)^2},\end{aligned} \tag{5.2}$$

$$\begin{aligned}f'' &= \mathrm{Re}\,\mathcal{G}'' + \mathcal{E}''\bar{\mathcal{E}}'' \\ &= \frac{f}{(1-c\bar{c}\mathcal{G} - 2c_i E)^2 + (\beta\mathcal{G} - 2c_r E)^2}.\end{aligned} \tag{5.3}$$

The asymptotic expression for $\mathcal{E}''$ is

$$\mathcal{E}'' = \xi + \frac{\eta}{r} + \frac{\zeta}{r^2} + \cdots , \tag{5.4}$$

$$\xi = \frac{c}{1+i\beta - c\bar{c}}, \tag{5.5}$$

$$\eta = \frac{c}{1+i\beta - c\bar{c}}\left[-4m + \frac{4m(i\beta - c\bar{c})}{1+i\beta - c\bar{c}}\right], \tag{5.6}$$

$$\zeta = \frac{2m}{1+i\beta - c\bar{c}}\left[2mc + ie\cos\theta + \frac{8mc(i\beta - c\bar{c})^2}{(1+i\beta - c\bar{c})^2} - 2c\frac{(5i\beta m - 5mc\bar{c} - ie\bar{c}\cos\theta)}{(1+i\beta - c\bar{c})}\right]. \tag{5.7}$$

The asymptotic expression for ψ'' is

$$\psi'' = \frac{1}{(1-c\bar{c})^2 + \beta^2} \times \left[-\beta + \frac{8m\beta(1-c\bar{c})}{(1-c\bar{c})^2 + \beta^2}\frac{1}{r} + O\left(\frac{1}{r^2}\right)\right], \tag{5.8}$$

and

$$\frac{\partial\psi''}{\partial r} = -\frac{8m\beta(1-c\bar{c})}{[(1-c\bar{c})^2+\beta^2]^2}\frac{1}{r^2} + O\left(\frac{1}{r^3}\right). \tag{5.9}$$

To calculate the asymptotic form of ω''_ϕ, one can use (4.15) with double primes on all expressions. As $r\to\infty$, there is a finite contribution from the integration of $\partial\psi''/\partial r$ as can be seen from (5.9); there is also a finite contribution from the integration of $i(\bar{\mathcal{E}}''\partial\mathcal{E}''/\partial r - \mathcal{E}''\partial\bar{\mathcal{E}}''/\partial r)$. The asymptotic limit has the character

$$\omega''_\phi = A(m, e, c, \beta)(\cos\theta - 1). \tag{5.10}$$

A is a function of the parameters above which does not vanish except for specialized values of the parameters. Obviously $\vec{\nabla}\times\vec{\omega}''$ vanishes asymptotically [Eq. (4.14)] since $\nabla\psi''$ and $(\bar{\mathcal{E}}''\vec{\nabla}\mathcal{E}'' - \mathcal{E}''\vec{\nabla}\bar{\mathcal{E}}'')$ both go as r^{-2}. The metric is asymptotically flat, and the rescaled metric is

$$ds_r''^2 = f''\left(\frac{dt}{[(1-c\bar{c})^2+\beta^2]^{1/2}} - \frac{\omega''_\phi}{[(1-c\bar{c})^2+\beta^2]^{1/2}}d\phi\right)^2 - \frac{(h_{rr}dr^2 + h_{\theta\theta}d\theta^2 + h_{\phi\phi}d\phi^2)}{[(1-c\bar{c})^2+\beta^2]f''}. \tag{5.11}$$

Even though ω''_ϕ does not vanish asymptotically, the metric becomes asymptotically flat, as can be readily seen by looking at the metric in Cartesian

TABLE I. Physical characteristics of system that exist when specific parameters do not vanish.

	Nonzero parameters					
	m, e	$m, e, \|c\| = c_i$	m, e, c_r	m, β	$m, e, \|c\|, c_r, \beta$	$m, \|c\| = c_i$
Mass	yes	yes	yes	yes	yes	yes
Magnetic dipole moment	yes	yes	yes	no	yes	no
Electric charge	no	yes	yes	no	yes	yes
Twist	no	no	yes	yes	yes	no

coordinates. The rescaled and dually rotated electromagnetic potential is

$$\mathcal{E}''_{rd} = \frac{e^{i\gamma}}{[(1 - c\bar{c})^2 + \beta^2]^{1/2}} \left(\xi + \frac{\eta}{r} + \frac{\zeta}{r^2} + \cdots \right), \quad (5.12)$$

where

$$\tan\gamma \equiv -\frac{\eta_i}{\eta_r}, \qquad \eta \equiv \eta_r + i\eta_i. \quad (5.13)$$

The charge is given by

$$Q = -\frac{|\eta|}{[(1 - c\bar{c})^2 + \beta^2]^{1/2}} \quad (5.14)$$

This solution has five parameters, m, e, c_r, $|c|$, and β. They give essentially the mass, magnetic dipole moment, a twist, an electric charge, and a stronger twist (determined by β), respectively. If c vanishes, the electric charge disappears, as can be seen by an examination of (5.1), but the twist remains, controlled by β.

VI. DISCUSSION

A solution has been presented of the combined Einstein-Maxwell equations depending on five parameters, m, e, c_r, $|c|$, and β. These determine the following physical characteristics of the system, respectively: mass, magnetic dipole moment, twist, electric charge, and a stronger twist. Table I shows what physical characteristics the system has when the only nonvanishing parameters are given at the top of the table. For example, if m, e, and $|c|$ do not vanish, but c_r and β do, the system has mass, a magnetic dipole moment, and an electric charge, but no twist. Of course, higher multipole moments of all kinds also exist.

All solutions found are singular on the surface described by f (or f' or f'') equal to zero. Hence one might conclude that the solutions have no relevance to black-hole physics. However, a different possibility suggests itself. The solutions described here are the only exact solutions we know that contain magnetic dipole moments. When the dipole moment parameter e goes to zero, the solutions do not become the charged or uncharged Schwarzschild or Kerr solutions. They belong to an altogether different family of solutions depending analytically on parameters. Since astrophysical systems do have magnetic moments, it seems appropriate to start the study of a collapsing physical situation with an exact solution that describes the prevailing magnetism. It would be nice to discover an exact solution with a magnetic moment that goes to the Kerr solution when the magnetic moment disappears. If no such solution exists, a different starting point for the study of the collapse problem is necessary.

It may be that the physical system to be examined is described adequately by a solution given in Sec. IV or Sec. V above. Suppose it is and suppose collapse begins. We would expect the end point of collapse to be the Kerr solution. It seems there is only one way to go from one of our solutions to the Kerr solution. This is by changing the character of the physical system "explosively." If we imagine collapse to be taking place "slowly" we imagine that the parameters of our solution are changing smoothly; the system may be radiating, the magnetic moment may be disappearing, but the parameters nevertheless change smoothly. The system cannot in this fashion ever be described by the Kerr solution since no smooth variation of parameters will yield the Kerr solution. However, we can imagine that as the system collapses, much material approaches the singular surface, $f = 0$. We can further imagine that tidal forces dramatically disrupt the physical situation; a supernovalike explosion takes place; shock waves propagate; the residual central physical object can now be described by a solution that changes smoothly to a Kerr solution.

It seems to us quite reasonable to expect that many collapsing objects undergo explosions; the system before the explosion and after the explosion can be given by two solutions depending on parameters which cannot be obtained one from the other by an analytic variation of the parameters.

Note added in proof. The solution we have attributed to MPST was actually first discovered by Bonnor.[6] It should have been referred to as the Bonnor solution.

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Gravitational-Wave Observations as a Tool for Testing Relativistic Gravity*

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Gravitational-wave observations can be powerful tools in the testing of relativistic theories of gravity—perhaps the only tools for distinguishing between certain extant theories in the foreseeable future. In this paper we examine gravitational radiation in the far field using a formalism that encompasses all "metric theories of gravity." There are *six* possible modes of polarization, which can be completely resolved by feasible experiments. We set forth a theoretical framework for classification of waves and theories, based on the Lorentz transformation properties of the six modes. We also show in detail how the six modes may be experimentally identified and to what extent such information limits the "correct" theory of gravity.

I. INTRODUCTION

Within the past few years, as experimental tests of gravity have been analyzed and refined, and as gravitation theories have been systematically compared,[1] most extant theories have been ruled out. Indeed, analysis of data from existing "solar system" experiments promises to distinguish more and more clearly between the theories that today remain viable. [For example, within the next two years, a search for the Nordtvedt effect[2] in lunar laser-ranging data[3] should either rule out general-relativity theory (GRT),[4] or place a limit of $\omega > 30$ on the Dicke coupling constant of Dicke-Brans-Jordan theory.[5]] An elegant theoretical formalism, the "parametrized post-Newtonian" (PPN) framework,[6] exists for analysis of metric theories[7] in the limit of weak gravitation and slow motion. All gravitation experiments that have played key roles in ruling out theories, except the Eötvös-Dicke experiment,[8] fall within the PPN framework. The Eötvös-Dicke experiment itself probably forces the "correct" theory of gravity to be a metric theory[7,9] and, in fact, there are no known complete[7] nonmetric theories which do not violate the Eötvös-Dicke experiment.

But the PPN framework has fundamental limitations. In the last year or so, new metric theories of gravity,[10-13] with widely varying structures, have been invented which are virtually indistinguishable from one another and from GRT in the post-Newtonian limit. Existing and proposed solar-system experiments cannot hope to distinguish between such theories in the foreseeable future. There is, however, a strong element of hope: that new theories[10-13] and GRT differ markedly in the observable properties of their gravitational waves. With this motivation, we have embarked upon a program to develop a theoretical foundation for the analysis of gravitational waves in arbitrary metric theories of gravity—a foundation which is theory-independent and analogous to the PPN framework. (Gravitational-wave phenomena fall outside of the PPN framework.) We feel that experiments to detect gravitational waves from astronomical sources can prove to be a powerful experimental tool, in the foreseeable future, for ruling out gravitation theories.

The idea of building a theory-independent framework for analyzing gravitational-wave experiments was first conceived of by Wagoner.[14] At about the same time, and independently, our group was analyzing the gravitational-wave properties of a particular metric theory—one that two of us had recently invented.[13] When our analysis was near completion (several months *after* we learned of Wagoner's ideas), we suddenly realized that our theory exhibits the most general type of gravitational wave admitted by any metric theory—and that, therefore, with a mere change of viewpoint, our analysis would become the general framework that Wagoner had proposed constructing. Upon contacting Wagoner we discovered that he and Will had already proceeded a long way toward the construction of this same framework. We therefore published a brief account of the framework jointly with them.[15] This paper presents a more detailed account of our "Caltech"