the relation (44).

The analyzed examples show that the eigenfunctions of the vector-potential operator ψ_a form a "natural" basis for the description of the interaction of a strong electromagnetic field with quantum systems.

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PHYSICAL REVIEW D VOLUME 8, NUMBER 10 15 NOVEMBER 1973

New Two-Metric Theory of Gravity with Prior Geometry*

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(Received 14 May 1973; revised manuscript received 23 July 1973)

We present a Lagrangian-based metric theory of gravity with three adjustable constants and two tensor fields, one of which is a nondynamical "flat-space metric" η. With a suitable cosmological model and a particular choice of the constants, the "post-Newtonian limit" of the theory agrees, in the current epoch, with that of general relativity theory (GRT); consequently our theory is consistent with current gravitation experiments. Because of the role of η, the gravitational "constant" G is time-dependent and gravitational waves travel null geodesics of η rather than the physical metric g. Gravitational waves possess six degrees of freedom. The general exact static spherically-symmetric solution is a four-parameter family. Future experimental tests of the theory are discussed.

I. INTRODUCTION AND SUMMARY

Within the past few years an elegant theoretical formalism, the "parametrized post-Newtonian" (PPN) framework, has been developed[1] to analyze metric[2] theories of gravity. The PPN framework is structured around the "weak gravitational fields" and low velocities of the gravitational matter which characterize typical solar-system tests of gravity. It classifies each gravitation theory as to its form "in the post-Newtonian (PN) limit." At first it was hoped, and indeed seemed to be true, that the PN limit of each theory of gravity is unique—thus by solar-system experiments alone, one could, in principle, determine the "correct PN limit," which would then correspond to one and only one "correct theory of gravity." In addition, it was hoped and is hoped that the "correct PN limit" is that of general relativity theory (GRT) (although we try not to let this fact prejudice our investigations). To play devil's advocate, a program was initiated to attempt to formulate theories of gravity with the same PN limit (and hence PPN parameters[1]) as GRT. The aims of such a program are twofold, as one can ask the following questions: (i) If such theories exist, how complex and contrived are their formulations? (ii) Do such theories have anything in common and in what respect do they differ from GRT outside of the PN limit? The first question is primarily only of aesthetic interest. But the second has the possibility of identifying powerful new theoretical and experimental tools for testing relativistic gravity—indeed, that has been the case (see Sec. V and Refs. 3 and 4).

In this paper[5] we present and analyze a new theory of gravity—one which has the same PN limit (for the current epoch) as GRT, given a suitable cosmological model and a particular choice of the adjustable constants. Analysis of our new theory provides partial answers to questions (i) and (ii) above.

A further motivation for study of this particular theory is to analyze in detail the role of prior geometry,[2] and its influence through cosmological boundary values, in gravitation theories, a role which will be investigated in more general terms in another paper.[6]

To date the authors are aware of three other new metric theories which are candidates for sharing the property of having the same PN limit as GRT (candidates in the sense of contingency upon the existence of special but acceptable cosmological solutions and certain choices of the available adjustable constants). These theories are the Hellings-Nordtvedt theory,[7] Ni's theory,[8] and the Will-Nordtvedt theory.[9] Of these three, only Ni's theory contains prior geometric elements like our own; but no discussion of the detailed relationship between prior geometry and cosmological influences has yet been given.

A. The Lagrangian Formulation

The equations of the theory are obtained, in the usual way, by varying the dynamical variables in the Lagrangian:

$$L = \int \mathfrak{L}_G(\underline{\eta},\underline{h})d^4x + \int \mathfrak{L}_{NG}(\underline{g}, q_\lambda)d^4x\,, \tag{1a}$$

$$\underline{g} = \underline{g}(\underline{\eta},\underline{h})\,, \tag{1b}$$

$$\underline{\mathrm{Riem}}(\underline{\eta}) = 0\,, \tag{1c}$$

where $\underline{\eta},\underline{h},\underline{g}$ are second-rank symmetric tensor fields: $\underline{\eta}$ is an absolute variable[2] (not varied in L), $\underline{h}$ is dynamical, and $\underline{g}$ is constructed algebraically from $\underline{\eta}$ and $\underline{h}$. The Riemann tensor constructed out of $\underline{\eta}$ is denoted by $\underline{\mathrm{Riem}}(\underline{\eta})$, and consequently Eq. (1c) states that $\underline{\eta}$ is a "flat-space metric." It is Eq. (1c), the "field equation" for $\underline{\eta}$, that introduces geometrical structure into the theory which is independent of the matter distribution—thus the "prior geometry." The gravitational Lagrangian density is denoted by $\mathfrak{L}_G$ while the nongravitational Lagrangian density,[2] $\mathfrak{L}_{NG}$, is the same as the corresponding quantity in other metric theories (e.g., GRT), with q_λ representing the matter fields. The "physical metric," governing the response of matter to gravity, is denoted by $\underline{g}$.

Explicitly, $\mathfrak{L}_G$ and $\underline{g}$ are defined by the following:

$$\mathfrak{L}_G = -(16\pi)^{-1}\eta^{\alpha\beta}\eta^{\lambda\mu}\eta^{\rho\sigma} \times (a h_{\lambda\rho|\alpha} h_{\mu\sigma|\beta} + f h_{\lambda\mu|\alpha} h_{\rho\sigma|\beta})(-\eta)^{1/2}\,, \tag{2}$$

$$g_{\mu\nu} = (1-Kh)^2 \Delta_\mu{}^\tau \Delta_{\tau\nu}\,, \tag{3a}$$

$$\Delta^\mu{}_\nu(\delta_\mu{}^\alpha - \tfrac{1}{2}h_\mu{}^\alpha) = \delta^\alpha{}_\nu\,. \tag{3b}$$

Conventions and definitions for the above are the following:

(i) Greek indices run 0–3, Latin 1–3.

(ii) Units are chosen such that $G=c=1$ (gravitational constant today and speed of light) (see Sec. V).

(iii) Vertical slashes and semicolons denote covariant differentiation with respect to the flat-space metric $\eta_{\alpha\beta}$ and the curved-space metric $g_{\alpha\beta}$, respectively. A comma denotes a partial coordinate derivative.

(iv) η is the determinant of $\eta_{\alpha\beta}$.

(v) $\delta^\alpha{}_\nu$ is the Kronecker delta.

(vi) $\Delta_\mu{}^\nu$ is defined by Eq. (3b).

(vii) Indices on $\Delta_{\alpha\beta}$ and $h_{\alpha\beta}$ *only* are raised and lowered with $\eta_{\mu\nu}$, i.e., $h^\alpha{}_\alpha = h^{\alpha\beta}\eta_{\alpha\beta} \equiv h$, and $\eta^{\alpha\beta}\eta_{\beta\gamma} = \delta^\alpha{}_\gamma$; indices on all other tensors will be raised and lowered with $g_{\alpha\beta}$.

(viii) Signatures of $\underline{\eta}$ and $\underline{g}$ are $+2$.

(ix) a, f, K are adjustable constants.

Motivation for the rather ungainly expression for the metric [Eqs. (3)] comes from an analysis[10] of the Belinfante-Swihart theory of gravity[11]—a theory which can be reformulated, at lowest order, into a metric theory with "effective metric" of the form of Eqs. (3). From that suggested algebraic form for the metric we have constructed the present full metric theory.

B. Summary

Section II includes a discussion of the field equations and a calculation of the PN limit of the theory. It is shown that there are mathematically ten degrees of freedom in the initial-value problem for $h_{\mu\nu}$ (compared with two for $g_{\mu\nu}$ in GRT). In the PN limit there are, in general, "preferred-frame effects"[1]; such effects are, however, functions of only the cosmological boundary values of $h_{\mu\nu}$. By a certain choice of the cosmological model one can make these effects vanish for the current epoch. We suspect that such time-dependent preferred-frame effects are a common property of prior-geometric gravitation theories. At any rate, the observed absence of preferred-frame effects can only place upper limits on the cosmological boundary values of $h_{\mu\nu}$.

Section III discusses the spherically symmetric, static problem. The exact exterior, static spherically symmetric solution is obtained and is found to be a four-parameter family.

Section IV discusses time-dependent solutions, conservation laws, and gravitational waves. Birkhoff's theorem[12] does not hold in this theory, i.e., the exterior geometry of a spherically symmetric and asymptotically flat spacetime need not be static—collapsing stars can radiate monopole gravitational waves. The general plane gravitational wave has six physical degrees of freedom, the maximum number possible in a metric theory of gravity.[3,4]

As the theory is Lagrangian-based, conservation laws follow and one can construct a gravitational stress-energy complex. Appropriately defined, the stress energy-density of this object is positive-definite for all possible polarizations of plane waves.

Section V discusses the time dependence of the gravitational "constant" and further possible experimental tests of the theory. In particular, a search for time delays between reception of gravitational and electromagnetic bursts and a search for "non-GRT" type polarizations of gravitational waves promise to be important future experimental tests of the theory. Such tests would also be crucial in the theories of Refs. 7, 8, 9; and their identification represents an important success in our program of "devil's advocate."

II. FIELD EQUATIONS AND POST-NEWTONIAN LIMIT

Variation of Eq. (1) with respect to the dynamical field variable $h_{\mu\nu}$ yields the following gravitational field equations:

$$(-\eta)^{1/2}(a\Box h^{\nu\mu}+f\eta^{\mu\nu}\Box h)=-4\pi T^{\alpha\beta}(-g)^{1/2}\times(\partial g_{\alpha\beta}/\partial h_{\mu\nu}), \tag{4a}$$

where

$$\Box h^{\mu\nu}\equiv\eta^{\alpha\beta}h^{\mu\nu}{}_{|\alpha|\beta}, \tag{4b}$$

$$T^{\alpha\beta}\equiv 2(-g)^{-1/2}(\delta\mathcal{L}_{NG}/\delta g_{\alpha\beta}), \tag{4c}$$

and δ is the variational derivative.

From the matter equations, obtained by variation of q_λ in Eq. (1), one can show in the usual manner (see, e.g., Ref. 13)

$$T^{\alpha\beta}{}_{;\beta}=0. \tag{5}$$

Equation (5) is the typical "matter-response equation" in metric theories.

Contraction of Eq. (4a) with $\eta_{\mu\nu}$ yields an equation for h alone, which can be substituted back into Eq. (4a) to yield

$$\Box h^{\mu\nu}=-(4\pi/a)(-g)^{1/2}(-\eta)^{-1/2}T^{\alpha\beta}\times[\theta^{\mu\nu}_{\alpha\beta}-f(a+4f)^{-1}\theta^{\gamma\tau}_{\alpha\beta}\eta_{\gamma\tau}\eta^{\mu\nu}], \tag{6a}$$

where

$$\theta^{\mu\nu}_{\alpha\beta}\equiv\partial g_{\alpha\beta}/\partial h_{\mu\nu}. \tag{6b}$$

The linearized limit of Eq. (6a) is

$$\Box h^{\mu\nu}=-(4\pi/a)T^{\alpha\beta}\times[\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}-\eta_{\alpha\beta}\eta^{\mu\nu}(f+2Ka)(a+4f)^{-1}]. \tag{7}$$

Unlike metric theories without prior geometry, the four Eqs. (5) do not follow from the gravitational field equations; they are additional equations.[6] However, there is no problem of overdetermination because all of the 10 components of $h^{\mu\nu}$ are now dynamical variables; i.e., if all of the essential coordinate freedom is used up in choosing a frame in which $\eta_{\alpha\beta}$ has a particular set of components [usually diag(−1, 1, 1, 1)], then there is no coordinate freedom left to adjust the components of $h_{\mu\nu}$.

For example, for a perfect fluid, $T^{\alpha\beta}$ is described by four matter variables once an equation of state is given (three components of the four-velocity and the energy density, for example). Thus Eqs. (5) and (6a) comprise a system of fourteen independent equations for the fourteen unknowns.

We also note that all of the ten Eqs. (6a) involve second time derivatives of $h_{\mu\nu}$. Thus in the Cauchy problem all of the $h_{\mu\nu}$ are to be regarded as dynamical variables and there are ten degrees of freedom. Once $g_{\alpha\beta}$ has been constructed from $\eta_{\alpha\beta}$ and $h_{\alpha\beta}$, however, coordinate transformations can be performed and so there can only be six "physical" degrees of freedom. This is to be contrasted with GRT in which not only can four of the $g_{\alpha\beta}$ be chosen arbitrarily by coordinate conditions, but also four of the field equations involve only first time derivatives. Thus in the corresponding Cauchy problem, the Einstein gravitational field has only two physical degrees of freedom.

The PPN framework of Nordtvedt, Will, and others can be used to analyze the predictions of all metric theories with respect to solar-system experiments (e.g., light bending, perihelion shift, gravimeter data, earth-moon separation, etc.). The reader is referred to Ref. 1 for a complete summary of the PPN framework.

We now calculate in our theory the PN limit, which will involve a perturbation solution of Eq. (6a). For calculational ease we assume a coordinate system in which $\eta_{\alpha\beta}$ takes on Minkowski values. Before we begin, a crucial point must be recognized.[14] The metric $g_{\alpha\beta}$ has the form

$$g_{\alpha\beta}=\eta_{\alpha\beta}+O(h), \tag{8}$$

and we know that far away from the solar system there is some coordinate system in which $g_{\alpha\beta}$ takes on Minkowski values. However, this coordinate system will, in general, not be the same frame in which $\eta_{\alpha\beta}$ takes on Minkowski values; there is no *a priori* reason why the boundary values of $h_{\mu\nu}$ should be zero in this coordinate system. Thus in solving Eq. (6a) we are not at liberty to set equal to zero for all time the "arbitrary constant" which may be added to $h_{\mu\nu}$; this complicates considerably the construction of the PN limit of our theory. However, we feel that this complication and its origin—the prior geometric element $\eta_{\mu\nu}$—are of sufficient educational value to warrant a detailed discussion.

Denote the nearly constant boundary values of $h_{\mu\nu}$ by $\omega_{\mu\nu}$ ($\omega_{\mu\nu}$ can only change on a cosmological time scale by definition) and the part tied directly to the solar system by $h^*_{\mu\nu}$; i.e.,

$$h_{\mu\nu}=h^*_{\mu\nu}+\omega_{\mu\nu}. \tag{9}$$

Now use the six-parameter invariance group of the Minkowski metric to pick a coordinate system in which $\omega_{\mu\nu}$ is diagonal, reducing $\omega_{\mu\nu}$ to four components. Without justification, but for simplicity, we now assume that the three spatial components of $\omega_{\mu\nu}$ are equal. Such an assumption does not affect the qualitative conclusions of this section. Further, assume that

$$|\omega_{\mu\nu}|\ll 1. \tag{10}$$

Equation (10) will turn out be an assumption consistent with the ultimate experimental limits on the $\omega_{\mu\nu}$.

Next expand Eqs. (3a) and (3b) in a power series in $h_{\mu\nu}$:

$$g_{\mu\nu}=\eta_{\mu\nu}-2Kh\,\eta_{\mu\nu}+h_{\mu\nu}+K^2h^2\eta_{\mu\nu}-2Khh_{\mu\nu}+\tfrac{3}{4}h_{\mu\tau}h^{\tau}{}_{\nu}+\cdots. \tag{11}$$

When Eq. (9) is substituted into Eq. (11) one obtains

$$g_{00}=-D_0+E_0h^*_{00}-F_0h^*-K^2h^{*2}-2Kh^*_{00}h^*-\tfrac{3}{4}h^{*2}_{00}, \tag{12a}$$

$$g_{ij}=D\delta_{ij}+Eh^*_{ij}+F\delta_{ij}h^*-2Kh^*h^*_{ij}+K^2h^{*2}\delta_{ij}+\tfrac{3}{4}h^{*2}_{ij}, \tag{12b}$$

$$g_{0k}=Hh^*_{0k}, \tag{12c}$$

where all of the constants appearing in Eqs. (12) have the form $D_0=1+O(\omega)$, etc., and are given explicitly to $O(\omega^2)$ in Appendix A, along with other constants appearing below. Using Eqs. (12) and a perfect fluid for the matter stress-energy tensor, one obtains from Eq. (6a)

$$\Box h^{*\mu\nu}=-(4\pi/a)I^{-1}\rho v^{\alpha}v^{\beta}(1+I_1h^*_{00}+I_2h^*+I_3v^2)[(1-2K\omega)\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}+L\eta_{\alpha\beta}\eta^{\mu\nu}+\tfrac{3}{2}\delta^{\mu}{}_{\alpha}\omega^{\nu}{}_{\beta}+M\eta^{\mu\nu}\omega_{\alpha\beta}+N\eta^{\mu\nu}\eta_{\alpha\beta}h^*+\tfrac{3}{2}\delta^{\mu}{}_{\alpha}h^{*\nu}{}_{\beta}-2Kh^*\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}+M\eta^{\mu\nu}h^*_{\alpha\beta}]. \tag{13}$$

In Eq. (13) I, I_1, I_2, I_3, M, N are all functions of $a, f, K, \omega_{\mu\nu}$ (see Appendix A) and

$$\omega\equiv 3\omega_{11}-\omega_{00}, \tag{14a}$$

$$v^{\alpha}\equiv dx^{\alpha}/dt, \tag{14b}$$

$\rho\equiv$ proper mass-energy density measured in the rest frame of the fluid. (14c)

To simplify an already complex presentation, we have omitted the pressure from the perfect fluid stress-energy tensor and included the internal energy in the total proper energy density ρ. (Such terms are not omitted in quoting the final PPN parameters.) We now write

$$h^{*\mu\nu}={}^{(1)}h^{*\mu\nu}+{}^{(2)}h^{*\mu\nu}+\cdots \tag{15}$$

in a perturbation expansion and obtain (see Appendix A for notation)

$$\nabla^2\,{}^{(1)}h^{*00}=-4\pi\tau\rho[(1-2K\omega)+L-\omega_0(\tfrac{3}{2}+M)]\equiv-4\pi\rho C_0, \tag{16a}$$

$$\nabla^2\,{}^{(1)}h^{*ij}=-4\pi\rho\tau(M\omega_0-L)\delta^{ij}\equiv-4\pi\rho C_1\delta^{ij}, \tag{16b}$$

$$\nabla^2\,{}^{(1)}h^{*0k}=-4\pi\tau\rho[v^k(1-2K\omega)+\tfrac{3}{2}\omega_1v^k]\equiv-4\pi\rho C_2v^k, \tag{16c}$$

$$\nabla^2\,{}^{(2)}h^{*00}=-4\pi\tau\rho(S_0\,{}^{(1)}h^{*00}+S_1\,{}^{(1)}h^*+B_0v^2)+{}^{(1)}h^{*00}_{\ ,00}, \tag{16d}$$

$$\nabla^2\,{}^{(2)}h^{*ij}=-4\pi\tau\rho[R_0v^iv^j+\delta^{ij}(R_1\,{}^{(1)}h^{*00}+R_2\,{}^{(1)}h^*+B_1v^2)]+{}^{(1)}h^{*ij}_{\ ,00}, \tag{16e}$$

where

$$\tau\equiv(aI)^{-1}. \tag{17}$$

Solutions of the equations are

$${}^{(1)}h^{*00}=C_0U, \tag{18a}$$

$${}^{(1)}h^{*ij}=\delta^{ij}C_1U, \tag{18b}$$

$${}^{(1)}h^{*0k}=C_2V_k, \tag{18c}$$

$${}^{(2)}h^{*00}=\tau[S_0C_0+S_1(3C_1-C_0)]\Phi_2+\tau B_0\Phi_1+C_0\chi_{,00}, \tag{18d}$$

$${}^{(2)}h^{*ij}=\tau R_0\mathcal{G}_3^{ij}+\tau\delta^{ij}[R_1C_0+R_2(3C_1-C_0)]\Phi_2+\tau B_1\delta^{ij}\Phi_1+C_1\delta^{ij}\chi_{,00}, \tag{18e}$$

where we have defined the five "potentials" U, V_k, Φ_1, Φ_2, $\mathcal{G}_3^{ij}$, and the "superpotential" χ as follows:

$$U(\vec{x},t)\equiv\int\rho(\vec{x}'',t)|\vec{x}-\vec{x}''|^{-1}d^3x'', \tag{19a}$$

$$V_k(\vec{x},t)\equiv\int\rho(\vec{x}'',t)|\vec{x}-\vec{x}''|^{-1}v^k\,d^3x'', \tag{19b}$$

$$\Phi_1(\vec{x},t)\equiv\int\rho(\vec{x}'',t)v^2|\vec{x}-\vec{x}''|^{-1}d^3x'', \tag{19c}$$

$$\Phi_2(\vec{x},t)\equiv\int\rho(\vec{x}'',t)|\vec{x}-\vec{x}''|^{-1}U(\vec{x}'',t)d^3x'', \tag{19d}$$

$$g_3^{ij}(\vec{x}, t) \equiv \int \rho(\vec{x}'', t)|\vec{x}-\vec{x}''|^{-1} v^i v^j d^3x'', \tag{19e}$$

$$\nabla^2 \chi = U. \tag{19f}$$

Using Eqs. (12) and our solutions, Eqs. (18), we now compute the metric:

$$g_{00} = -D_0 + K_1 U + K_2 U^2 + K_3 \Phi_2 + K_4 \Phi_1 + K_1 \chi_{,00}, \tag{20a}$$

$$g_{ij} = \delta_{ij}(D + K_5 U), \tag{20b}$$

$$g_{0k} = -HC_2 V_k. \tag{20c}$$

Notice that the metric does not approach the standard Minkowski tensor far away from the solar system (when the potentials $U, \Phi_1, \Phi_2, V_k, \chi \to 0$) because of the leading constants D_0 and D. We must therefore make a "scaling" transformation:

$$t = D_0^{-1/2} t', \tag{21a}$$

$$\vec{x} = D^{-1/2} \vec{x}'. \tag{21b}$$

In the tensor transformation law for the metric

$$\begin{aligned} g'_{\mu\nu}(x') &= g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \\ &= g_{\alpha\beta}[U(\vec{x}, t), \Phi_1(\vec{x}, t), \dots] \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}, \end{aligned} \tag{22}$$

we also need to express the potentials as functions of the new (primed) coordinates. An example of the procedure is the following: Since ρ is a scalar

$$\rho'(\vec{x}', t) = \rho(\vec{x}, t), \tag{23a}$$

$$\begin{aligned} U(\vec{x}, t) &= \int \rho(\vec{x}'', t)|\vec{x}-\vec{x}''|^{-1} d^3x'' \\ &= \int \rho'(\vec{x}''', t')|\vec{x}-\vec{x}''|^{-1} d^3x'', \\ &= D^{-1} \int \rho'(\vec{x}''', t')|\vec{x}'-\vec{x}'''|^{-1} d^3x''' \\ &= D^{-1} U'(\vec{x}', t'). \end{aligned} \tag{23b}$$

In a similar manner one finds

$$\Phi_2(\vec{x}, t) = D^{-2} \Phi_2'(\vec{x}', t'), \tag{23c}$$

$$\Phi_1(\vec{x}, t) = D_0 D^{-2} \Phi_1'(\vec{x}', t'), \tag{23d}$$

$$V_k(\vec{x}, t) = D_0^{1/2} D^{-3/2} V_k'(\vec{x}', t'), \tag{23e}$$

$$\chi_{,00} = D^{-2} D_0 \chi'_{,0'0'}. \tag{23f}$$

Making the transformation indicated in Eqs. (22) and (23) and then *dropping the primes*, $g_{\mu\nu}$ becomes

$$\begin{aligned} g_{00} = -1 &+ D_0^{-1} D^{-1} K_1 U + D_0^{-1} D^{-2} K_2 U^2 \\ &+ D_0^{-1} D^{-2} K_3 \Phi_2 + D^{-2} K_4 \Phi_1 + D^{-2} K_1 \chi_{,00}, \end{aligned} \tag{24a}$$

$$g_{ij} = \delta_{ij}(1 + D^{-2} K_5 U), \tag{24b}$$

$$g_{0k} = -HC_2 D^{-2} V_k. \tag{24c}$$

A final coordinate transformation must be made to remove the $\chi_{,00}$ term from g_{00} and reduce the metric to "standard PPN form." However, additional transformations of the form of Eqs. (23) are now negligible corrections and no distinction need be made between functions of new and old coordinates. The result of the final transformation, $t \to t + \frac{1}{2} D^{-2} K_1 \chi_{,0}$, is

$$g_{00} \to g_{00} - K_1 D^{-2} \chi_{,00}, \tag{25a}$$

$$g_{ij} \to g_{ij}, \tag{25b}$$

$$g_{0k} \to g_{0k} + \tfrac{1}{4} K_1 D^{-2}(V_k - W_k), \tag{25c}$$

where W_k is a new potential defined by

$$W_k \equiv \int \rho[\vec{v} \cdot (\vec{x}-\vec{x}'')]|\vec{x}-\vec{x}''|^{-1}(\vec{x}-\vec{x}'')_k d^3x''. \tag{26}$$

We now demand the proper *Newtonian* limit, i.e.,

$$g_{00} \approx 1 - 2U + \cdots,$$

which requires

$$K_1 D_0^{-1} D^{-1} = 2 \text{ today} \tag{27}$$

(a consequence of our choosing units in which the gravitational constant is unity today). Equation (27) expresses a constraint between the three adjustable constants a, f, and K for a given set of $\omega_{\mu\nu}$. Comparing Eqs. (24)–(25) with the definitions of the PPN parameters[1] and using Eq. (27) to simplify, one finds

$$\gamma = \tfrac{1}{2} D^{-2} K_5 \equiv \gamma'(a, f, K) + O(\omega), \tag{28a}$$

$$\beta = -\tfrac{1}{2} D_0^{-1} D^{-2} K_2 \equiv \beta'(a, f, K) + O(\omega), \tag{28b}$$

$$\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \alpha_3 = 0, \tag{28c}$$

$$\alpha_1 = 2HC_2 D^{-2} - 4\gamma - 4 = O(\omega), \tag{28d}$$

$$\alpha_2 = D_0 D^{-1} - 1 = O(\omega), \tag{28e}$$

where γ' and β' are defined implicitly by the relations

$$a = (2\gamma' + 2)^{-1}, \tag{29a}$$

$$\begin{aligned} f &= (10\beta' + 6\gamma'\beta' - 7\gamma'^2 - 8\gamma' - 6) \\ &\quad \times [2(\gamma'+1)(3\gamma' - 5 - 4\beta')^2]^{-1}. \end{aligned} \tag{29b}$$

In GRT, $\gamma = \beta = 1$ and the other seven parameters vanish. In our theory it is clear that the two adjustable constants, a and f, may be so chosen to give any value to γ and β. For example, if the $\omega_{\mu\nu}$ are all zero, one can satisfy Eq. (27) and have $\gamma = \beta = 1$ with the choice

$$(a, f, K) = (\tfrac{1}{4}, -\tfrac{5}{64}, \tfrac{1}{16}). \tag{30}$$

It has been shown[15] that the nonvanishing of α_1, α_2, or α_3 leads to noninvariance of the functional form of the metric of Eqs. (24)–(25) under post-Galilean transformations[16] (curved-space versions of Lorentz transformations). New terms, involving the velocity of the Lorentz boost with respect to the current "preferred frame" and multiplied by combinations of α_1, α_2, α_3, appear in the metric. Nordtvedt and Will[17] have calculated the experimental consequences of the resulting "preferred-frame effects" and find that they lead to periodic anomalies in such phenomena as the solid earth tides, secular perihelion shifts, etc. The reader is referred to their paper for further details and we quote here only the current experimental limits on α_1 and α_2:

$$\alpha_1 \leqslant 0.1\,, \tag{31a}$$

$$\alpha_2 \leqslant 0.02\,. \tag{31b}$$

We have calculated explicitly the quite complicated functions $\alpha_1(\omega_{\mu\nu})$, $\alpha_2(\omega_{\mu\nu})$ and have examined their numerical values over a large range of constants a and f (consistent with the experimental limits on γ and β). We find that the experimental constraints indicated in Eqs. (31) require approximately

$$|\omega_0|+|\omega_1|\lesssim 0.015\,. \tag{32}$$

Even if we had not made the simplifying assumptions about the form of $\omega_{\mu\nu}$, its individual elements presumably would still be required to satisfy roughly the constraint of Eq. (32).

Since the $\omega_{\mu\nu}$ are cosmological boundary values of $h_{\mu\nu}$, one must solve the cosmological problem for a particular cosmological model to obtain the theoretical values of the $\omega_{\mu\nu}$. Because of the absolute nature of $\eta_{\alpha\beta}$, it should be possible to construct cosmologies such that, during the current epoch, the curved and flat-space metrics approach Minkowski form, far from the solar system, in the same coordinate system. Such a cosmology would guarantee that the $\omega_{\mu\nu}$ vanish at present, although a time-dependent cosmology would certainly cause nonzero values of $\omega_{\mu\nu}$ to occur over cosmological time scales. Indeed, preliminary results from a cosmological solution[18] indicate that it is possible to make all of the $\omega_{\mu\nu}$ arbitrarily small for the current epoch—and still have a reasonable cosmological model. Thus, a consistent solution exists for which the PN limit of our theory is arbitrarily close to that of GRT in the current epoch.

Further details regarding the time dependence of the $\omega_{\mu\nu}$ are given in Sec. V.

III. THE GENERAL STATIC SPHERICALLY SYMMETRIC SOLUTION AND EQUATIONS OF STELLAR STRUCTURE

A. The General Exterior Static Spherically Symmetric Solution

Before writing down the equations of stellar structure for a static spherically symmetric star, let us construct the general static spherically symmetric exterior solution (which must then be joined onto the solution inside the star).

First of all, choose a coordinate system in which

$$\eta_{\mu\nu}=\begin{pmatrix}-1&&&\\&1&&\\&&r^2&\\&&&r^2\sin^2\theta\end{pmatrix}. \tag{33}$$

The most general form of $h_{\mu\nu}$ in this coordinate system which satisfies the symmetry requirements is[19]

$$h_{\mu\nu}=\begin{pmatrix}\varphi(r)&\mu(r)&0&0\\ \mu(r)&\psi(r)&0&0\\ 0&0&r^2\lambda(r)&0\\ 0&0&0&r^2\sin^2\theta\,\lambda(r)\end{pmatrix}. \tag{34}$$

The homogeneous field equations for $h_{\mu\nu}$ are simply

$$\eta^{\alpha\beta}h^{\mu\nu}{}_{|\alpha|\beta}=0\,. \tag{35}$$

The solutions to Eqs. (35) which are well behaved at infinity are[20]

$$h_{\mu\nu}=\begin{pmatrix}a_1/r&-2a_4/r^2&0&0\\ -2a_4/r^2&a_2/r-2a_3/r^3&0&0\\ 0&0&r^2(a_2/r+a_3/r^3)&0\\ 0&0&0&r^2\sin^2\theta(a_2/r+a_3/r^3)\end{pmatrix}, \tag{36}$$

where a_1, a_2, a_3, and a_4 are arbitrary constants. We remind the reader that the r coordinate in Eq. (36) has, at this point, no interpretation other than its relation to the group-theoretically defined assumption of spherical symmetry. Construction of $g_{\mu\nu}$ from $h_{\mu\nu}$ is purely algebraic [see Eqs. (3)], and the details will not be given here. Since $h_{\mu\nu}$ has off-diagonal terms, so will $g_{\mu\nu}$. However, having obtained $g_{\mu\nu}$, we can make the coordinate transformation

$$t \to t + \int \frac{g_{0r}}{g_{00}} dr\,, \tag{37}$$

which then diagonalizes the metric, and we finally obtain

$$g_{00} = (1-Kh)^2\gamma^2\left[\frac{a_4^2}{r^4} - \left(1 - \frac{1}{2}\frac{a_2}{r} + \frac{a_3}{r^3}\right)^2\right], \tag{38a}$$

$$g_{rr} = (1-Kh)^2\gamma^2\left\{\left(1+\frac{1}{2}\frac{a_1}{r}\right)^2 - \frac{a_4^2}{r^4} + \frac{(a_4^2/r^4)[2+\frac{1}{2}(a_1-a_2)r^{-1}+a_3r^{-3}]^2}{(a_4^2/r^4)-[1-\frac{1}{2}(a_2/r)+(a_3/r^3)]^2}\right\}, \tag{38b}$$

$$g_{\theta\theta} = (1-Kh)^2 r^2\left(1 - \frac{1}{2}\frac{a_2}{r} - \frac{1}{2}\frac{a_3}{r^3}\right)^{-2}, \tag{38c}$$

$$g_{\varphi\varphi} = \sin^2\theta g_{\theta\theta}\,, \tag{38d}$$

$$h \equiv r^{-1}(3a_2 - a_1)\,, \tag{38e}$$

$$\gamma \equiv [1 + \tfrac{1}{2}(a_1 - a_2)r^{-1} - \tfrac{1}{4}a_1a_2r^{-2} + a_3r^{-3} + (a_4^2 + \tfrac{1}{2}a_1a_3)r^{-4}]^{-1}, \tag{38f}$$

$$ds^2 = g_{00}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}d\varphi^2\,. \tag{39}$$

Equations (38) for the metric indicate a four-parameter family of solutions for the general static spherically symmetric exterior metric. One can convince himself that all four of the parameters are physical (not removable by coordinate transformations) by transforming to curvature coordinates and verifying that four arbitrary parameters remain.[21] In Sec. IV we will investigate more closely a particular member of the four-parameter family.

B. Stellar Models

The equations of stellar structure are quite complicated in this theory; and, even for a constant-density star, there is probably no analytic solution of the equations. One unusual feature of the equations is that a central pressure and equation of state do not uniquely specify a stellar model. The reader is referred to Ref. 5 for details.

IV. GRAVITATIONAL WAVES AND CONSERVATION LAWS

In the full theory (no linearized approximation) the homogeneous field equations are, as indicated previously,

$$\eta^{\alpha\beta}h^{\mu\nu}{}_{|\alpha|\beta} = 0\,, \tag{40}$$

and gravitational waves travel geodesics of η rather than g. The implication of this last fact will be explored later. The simplicity of the vacuum field equations [cf. Eq. (40)] is of great help in constructing solutions.

A. Linearized Theory and Plane Gravitational Waves

In analyzing weak gravitational waves, one should restrict one's attention to the form and behavior of the Riemann tensor, not only because it is gauge-invariant (under infinitesimal coordinate transformations) but also because it is that feature of the gravitational wave which interacts directly with test bodies. To analyze the decomposition of $R_{\alpha\beta\gamma\delta}$ into independent "wave modes" in as invariant a manner as possible, one should investigate the transformation properties of $R_{\alpha\beta\gamma\delta}$ under those Lorentz transformations which leave the wave direction fixed. With such transformations in mind, one selects a basis in which the components of $R_{\alpha\beta\gamma\delta}$ are to be computed—the quasiorthonormal tetrad basis (see, e.g., Ref. 22 for a complete discussion of the "tetrad formalism"):

$$\underline{k} = 2^{-1/2}(1,0,0,1)\,, \tag{41a}$$

$$\underline{l} = 2^{-1/2}(1,0,0,-1) \tag{41b}$$

$$\underline{m} = 2^{-1/2}(0,1,i,0)\,, \tag{41c}$$

$$\underline{\overline{m}} = 2^{-1/2}(0,1,-i,0)\,. \tag{41d}$$

Note that one of the "tetrad legs" points along the direction of the wave. In such a basis the components of the Riemann tensor are

$$R_{nkml} = R_{\alpha\beta\gamma\delta}n^\alpha k^\beta m^\gamma l^\delta\,,\ \text{etc}\,. \tag{42}$$

For waves, one can show that the only nonvanishing components of the Riemann tensor are those with two l's—thus, there are six possible degrees of freedom. Since there are no restrictions on the Riemann tensor once Eqs. (40) are satisfied, all six tetrad components will in general be nonvanishing and our theory thus has six independent gravitational wave modes. In our case, each of these modes corresponds to a degree of freedom and our theory exhibits the maximum number of gravitational wave degrees of freedom possible in a metric theory—six. In GRT, as a contrast, the field equations $R_{\alpha\beta} = 0$ imply vanishing of R_{lklk}, R_{lklm}, $R_{lkl\overline{m}}$, and $R_{lml\overline{m}}$ so that there are only two degrees of freedom—those represented by R_{lmlm} and its complex conjugate $R_{l\bar{m}l\bar{m}}$.

The reader is referred to Refs. 3 and 4 for details of the transformation properties of the objects indicated in Eq. (42). Here we quote only the results: We denote the six wave modes by Ψ_2, Ψ_3, $\bar{\Psi}_3$, Ψ_4, $\bar{\Psi}_4$, Φ_{22}; in terms of the tetrad components of the Riemann tensor and "electric" coordinate components of the Riemann tensor (those which are directly physically measurable) these are

$$\Psi_2 \equiv -\tfrac{1}{6} R_{lklk} = -\tfrac{1}{6} R_{tztz}, \tag{43a}$$

$$\Psi_3 \equiv -\tfrac{1}{2} R_{lkl\bar{m}} = -\tfrac{1}{2}(R_{txtz} - iR_{tytz}), \tag{43b}$$

$$\bar{\Psi}_3 \equiv -\tfrac{1}{2} R_{lklm} = -\tfrac{1}{2}(R_{txtz} + iR_{tytz}), \tag{43c}$$

$$\Psi_4 \equiv -R_{l\bar{m}l\bar{m}} = -R_{tyty} + R_{tztz} + 2iR_{txty}, \tag{43d}$$

$$\bar{\Psi}_4 \equiv -R_{lmlm} = -R_{tyty} + R_{tztz} - 2iR_{txty}, \tag{43e}$$

$$\Phi_{22} \equiv \tfrac{1}{2} R_{lml\bar{m}} = -R_{txtx} - R_{tyty}. \tag{43f}$$

The presence or absence of a Ψ_2 component in a gravitational wave is Lorentz-invariant. If Ψ_2 is absent in a particular wave, the presence or absence of Ψ_3 (or Ψ_3) in that wave is also Lorentz-invariant. As outlined in Refs. 3 and 4, if either Ψ_2 or Ψ_3 is present in a wave (in many theories they are always absent, but not in ours), then it is impossible to decompose the wave into states of definite helicity (spin) in a Lorentz-invariant manner: What one observer identifies as "pure spin 0" another observer will identify as "pure spin 0" plus "pure spin 1," etc. Only waves containing only Φ_{22}, Ψ_4, and $\bar{\Psi}_4$ can be decomposed into pure spins: spin 0 and spin 2. In general, then, there is no unique spin decomposition of waves in our theory and it is of class II_6 (see Refs. 3 and 4 for a complete discussion of the "classification scheme"). The physical imprints of the various modes will be discussed in Sec. V.

B. The Stress-Energy Pseudotensor for Gravitational Waves

Using the method of Noether,[23] which applies to all Lagrangian-based theories, a conserved quantity may be constructed, including a stress-energy pseudotensor for gravitational waves. The gravitational stress-energy pseudotensor has positive definite energy. We refer the reader to Ref. 5 for details.

V. THE GRAVITATIONAL CONSTANT AND FURTHER EXPERIMENTAL TESTS

A. A Time-Dependent Gravitational Constant

As discussed in Sec. II, a number of existing solar system experiments place upper limits on the cosmological boundary values of $h_{\mu\nu}$ [cf. Eqs. (31)–(32)]. These constraints can always be satisfied in a given epoch. A more relevant point is the time dependence of the $\omega_{\mu\nu}$, which is directly related to the time dependence of the gravitational constant G. With the choice of adjustable constants given in Eq. (30), and using the explicit functional forms for K_1, D_0, D, one finds from Eq. (27) and Appendix A that

$$1 - \tfrac{1}{16}(19\omega_1 + 7\omega_0) + O(\omega^2) = G. \tag{44a}$$

Thus

$$\frac{1}{G}\frac{dG}{dt} \approx -\frac{1}{16}\left(\frac{19 d\omega_1}{dt} + \frac{7 d\omega_0}{dt}\right). \tag{44b}$$

Shapiro *et al.*[24] have placed limits on the time dependence of the gravitational constant by comparing the periods of planets with the ticking rates of atomic clocks. They find

$$\left|\frac{1}{G}\frac{dG}{dt}\right| < 4\times 10^{-10}/\text{year}. \tag{45}$$

This constitutes an experimental constraint on the magnitude of the time derivatives of $\omega_{\mu\nu}$ occurring in Eq. (44b). Preliminary results from our cosmological solution[18] indicate that the time dependences of ω_0 and ω_1 satisfy Eq. (45), but an improved Shapiro experiment might still prove to be a crucial experimental test of our theory.

B. Gravitational-Wave Experiments

The analysis of Sec. IV reveals two crucial new experimental tests of our theory involving gravitational waves—two tests which have blossomed from our current program (see introductory remarks in Sec. I) and which emphasize gravitational wave detection as a powerful new tool for probing metric theories of gravity.[3,4] The two tests are: (i) time delay between simultaneously emitted gravitational and electromagnetic waves, and (ii) polarizations of gravitational waves.

Since gravitational waves travel along geodesics of the "fast metric" $\eta_{\alpha\beta}$ and electromagnetic waves travel along geodesics of the "slow metric" $g_{\alpha\beta}$, there should be a time delay in reception of the two waves—emitted, for example, in simultaneous bursts by a supernova explosion. For waves emitted at the center of the galaxy, an order-of-magnitude estimate indicates

$$\begin{aligned}\text{Time delay} &\sim (m/r)_{\text{galaxy}}(\text{light travel time})\\ &\sim (5\times 10^{-7})(3\times 10^4 \text{ light years})\\ &\approx 5 \text{ days}.\end{aligned} \tag{46}$$

Much longer delay times would hold for the Virgo Cluster.

Polarization information is also a crucial experimental test. Equations (43) indicate a purely longitudinal mode (Ψ_2), mixed longitudinal-transverse quadrupole type modes ($\Psi_3, \overline{\Psi}_3$), a purely transverse "breathing" mode (Φ_{22}), and the familiar transverse quadrupole modes of GRT ($\Psi_4, \overline{\Psi}_4$). If an observer knows the direction of the wave, he can use Eqs. (43) to unambiguously catalogue the modes. If he does not know the direction of the source, he can still draw some conclusions. For example, if displacements do occur in more than one plane, then either the longitudinal-transverse modes ($\Psi_3, \overline{\Psi}_3$) are present, or the purely longitudinal mode (Ψ_2) is mixed in with one of the purely transverse modes ($\Psi_4, \overline{\Psi}_4, \Phi_{22}$).

It is important to note that until the problem of the *generation* of the various types of waves by particular sources is solved, our theory can only be verified by the presence of—but not ruled out by the absence of—the various possible modes indicated in Eqs. (43). This is unfortunate. But new doorways have been opened in the area of experimental tests and it is clear that gravitational tests outside of the PPN formalism must be contemplated in the future.

ACKNOWLEDGMENTS

We wish to thank D. M. Eardley, W.-T. Ni, W. H. Press, K. S. Thorne, and C. M. Will for their helpful suggestions. We also thank K. S. Thorne for editing the manuscript.

APPENDIX: CONSTANTS APPEARING IN PN LIMIT (SEC. II)

The constants appearing in the PN limit calculated in Sec. II are

$$\omega_0 \equiv \omega_{00},$$
$$\omega_1 \equiv \omega_{11},$$
$$\omega \equiv 3\omega_1 - \omega_0;$$

in Eq. (12a)

$$D_0 \equiv 1 - 2K\omega + K^2\omega^2 + 2K\omega\omega_0 + \tfrac{3}{4}\omega_0{}^2 - \omega_0,$$
$$E_0 \equiv 1 - 2K\omega - \tfrac{3}{2}\omega_0,$$
$$F_0 \equiv -2K + 2K^2\omega + 2K\omega_0;$$

in Eq. (12b)

$$D \equiv 1 - 2K\omega + \omega_1 + K^2\omega^2 - 2K\omega\omega_1 + \tfrac{3}{4}\omega_1{}^2,$$
$$E \equiv 1 - 2K\omega + \tfrac{3}{2}\omega_1,$$
$$F \equiv -2K(1+\omega_1) + 2K^2\omega;$$

in Eq. (12c)

$$H \equiv 1 - 2K\omega - \tfrac{3}{4}\omega_0 + \tfrac{3}{4}\omega_1;$$

in Eq. (13)

$$I \equiv D_0{}^{1/2} D^{-3/2},$$
$$I_1 \equiv \frac{1}{2}\left(\frac{E}{D} + \frac{E_0}{D_0}\right),$$
$$I_2 \equiv \frac{1}{2}\left(\frac{3F}{D} - \frac{F_0}{D_0} + \frac{E}{D}\right),$$
$$I_3 \equiv \frac{D}{D_0},$$
$$L \equiv -(a+4f)^{-1}[f(1-2K\omega) + 2Ka(1-K\omega)],$$
$$M \equiv -(a+4f)^{-1}(2Ka + \tfrac{3}{2}f),$$
$$N \equiv 2K(f + Ka)(a+4f)^{-1};$$

in Eq. (16d)

$$S_0 \equiv I_1(1 - 2K\omega + L - \tfrac{3}{2}\omega_0 - M\omega_0) - \tfrac{3}{2} - M,$$
$$S_1 \equiv I_2(1 - 2K\omega + L - \tfrac{3}{2}\omega_0 - M\omega_0) + N - 2K,$$
$$B_0 \equiv I_3(1 - 2K\omega + L - \tfrac{3}{2}\omega_0 - M\omega_0) - L - M\omega_1;$$

in Eq. (16e)

$$R_0 \equiv 1 - 2K\omega + \tfrac{3}{2}\omega_1,$$
$$R_1 \equiv I_1(M\omega_0 - L) + M,$$
$$R_2 \equiv I_2(M\omega_0 - L) - N,$$
$$B_1 \equiv I_3(M\omega_0 - L) + L + M\omega_1;$$

in Eq. (20a)

$$K_1 \equiv E_0C_0 - F_0(3C_1 - C_0),$$
$$K_2 \equiv -[K^2(3C_1 - C_0)^2 + 2KC_0(3C_1 - C_0) + \tfrac{3}{4}C_0{}^2],$$
$$K_3 \equiv \tau[S_0C_0 + S_1(3C_1 - C_0)](E_0 + F_0) - 3\tau F_0[RC_0 + R_2(3C_1 - C_0)],$$
$$K_4 \equiv \tau[E_0B_0 - F_0(R_0 + 3B_1 - B_0)];$$

in Eq. (20b)

$$K_5 \equiv EC_1 + F(3C_1 - C_0).$$

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*Research supported in part by the National Aeronautics and Space Administration under Grant No. NGR 05-002-256 and the National Science Foundation under Grant Nos. GP-36687X and GP-28027.

†National Science Foundation Predoctoral Fellow during a portion of this work.

‡Imperial Oil Predoctoral Fellow.

[1]For a complete review of the PPN formalism, see C. M.

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PHYSICAL REVIEW D VOLUME 8, NUMBER 10 15 NOVEMBER 1973

Five-Parameter Exterior Solution of the Einstein-Maxwell Field Equations

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(Received 5 April 1973)

A five-parameter solution of the combined Einstein-Maxwell equations is given which describes a source containing mass, electric charge, magnetic dipole, higher multipole moments of all three kinds, and angular momentum. The solution is obtained by using Kinnersley's method of generating stationary Einstein-Maxwell fields from known solutions of the Einstein - Maxwell equations. We start with a two-parameter solution of a system having mass and a magnetic dipole moment discovered by Misra, Pandey, Srivastava, and Tripathi. All solutions discussed in this paper are asymptotically flat, and all have infinite red-shift surfaces that are singular. Possible relevance of these solutions to black-hole physics is remarked upon.

I. INTRODUCTION

A solution is presented of the combined Einstein-Maxwell field equations which depends on five parameters: m, e, $|c|$, c_r, β. c is a complex parameter, c_r its real value, and $|c|$ its absolute value. The first three parameters represent respectively the mass, the magnetic dipole moment, and the electric charge; the last two describe the angular momentum of a central source. The source has, in addition to these poles, a mass quadrupole, a magnetic quadrupole, an electric dipole, and higher multipole moments whose values are determined by the five parameters. The mass parameter m must have a nonzero value or the solution collapses to flat space. If m does not vanish, interesting special cases occur even when only one of the other parameters e, $|c|$, c_r, and β is not zero. If only m and e do not vanish, the system has a mass pole and higher mass multipoles, a magnetic dipole but